Editors' Suggestion

Gravity interpretation for the Bethe Ansatz expansion of the $\mathcal{N}=4$ SYM index

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The superconformal index of the $\mathcal{N}=4$ $SU(N)$ supersymmetric Yang-Mills theory counts the 1/16-BPS (Bogomol'nyi-Prasad-Sommerfield) states in this theory, and has been used via the AdS/CFT correspondence to count black hole microstates of 1/16-BPS black holes. On one hand, this index may be related to the Euclidean partition function of the theory on $S^3 \times S^1$ with complex chemical potentials, which maps by the AdS/CFT correspondence to a sum over Euclidean gravity solutions. On the other hand, the index may be expressed as a sum over solutions to Bethe Ansatz (BA) equations. We show that the solutions to the BA equations that are known to have a good large N limit, for the case of equal chemical potentials for the two angular momenta, have a one-to-one mapping to (complex) Euclidean black hole solutions on the gravity side. This mapping captures both the leading contribution from the classical gravity action (of order N^2), as well as nonperturbative corrections in $1/N$, which on the gravity side are related to wrapped D3-branes. Some of the BA solutions map to orbifolds of the standard Euclidean black hole solutions (which obey exactly the same boundary conditions as the other solutions). *A priori* there are many more gravitational solutions than Bethe Ansatz solutions, but we show that, by considering the nonperturbative effects, the extra solutions are ruled out, leading to a precise match between the solutions on both sides.

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I. INTRODUCTION AND SUMMARY

The AdS/CFT correspondence [1–4] maps black holes in asymptotically anti–de Sitter (AdS) spacetimes to coarse-grained descriptions of states in conformal field theories (CFTs), such that the Bekenstein-Hawking entropy of the black holes may be given a statistical mechanics interpretation as a counting of CFT microstates. For general nonsupersymmetric black holes we do not have good methods to count these states in the strongly coupled theories which map under the AdS/CFT correspondence to weakly curved backgrounds (where the Bekenstein-Hawking computation is valid). In the last few years, considerable progress has been made [5–28] in performing this counting for 1/16-BPS (Bogomol'nyi-Prasad-Sommerfield) black holes in type IIB string theory on $\mathrm{AdS}_5 \times S^5$, which map to 1/16-BPS states in the $\mathcal{N}=4$ $SU(N)$ supersymmetric Yang-Mills (SYM) theory on S^3 (this followed related results in other backgrounds, starting with [29] and including [30–48]).

The counting of 1/16-BPS states is based on computing the superconformal index [49,50], which is a sum over these states with chemical potentials for four of the five charges carried by the black holes [the black holes carry three global symmetry charges $R_{1,2,3}$ in the Cartan algebra of the $SU(4)_R$ symmetry, and two angular momenta $J_{1,2}$ in AdS_5]. It is not possible to directly relate the index to the number of BPS states with given charges, both because the index is insensitive to one of the charges, and because it counts states with a factor of $(-1)^F$, so there can be cancellations between bosonic and fermionic states. Indeed, when the index was first computed in the large N limit with real chemical potentials [50], a mismatch was found between the value of the index and the expectation from black holes. However, this discrepancy does not arise for generic complex chemical potentials [6,7],[1] and in various limits the index was successfully matched

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[1]It was already shown in AdS_4 [32] how the index with complex chemical potentials captures the Bekenstein-Hawking entropy of BPS black holes.

(at leading order in the large N limit) with the expectation from the black hole solutions.

More precisely, the superconformal index is a grand-canonical partition function, and one would expect its Legendre transform [51] to agree—at leading order—with the black hole entropy. This is indeed the case if one performs the Legendre transform with respect to the four available chemical potentials. This was shown both directly in the large N limit [7], as well as going through a Cardy-like limit that retains states with charges much larger than the central charge [6].

In this paper we do not discuss the black hole microstate counting, but instead we aim to understand better the interpretation (on the gravity side) of the superconformal index as a Euclidean partition function, hoping that this better understanding will also be useful in the future for the microstate counting problem. In general, as we review in Sec. II, the superconformal index is related to the Euclidean partition function of the theory on $S^3 \times S^1$, with specific background fields related to the chemical potentials appearing in the index (which can be real or complex), which ensure that supersymmetry is preserved [52–54]. The AdS/CFT correspondence tells us that this partition function should be described in the large N limit on the gravity side in terms of a sum over all asymptotically $AdS_5 \times S^5$ solutions of type IIB string theory that satisfy appropriate boundary conditions (related to the background fields). Up to now, only specific solutions, which give the leading contribution to the partition function for certain chemical potentials, were considered; however, the full answer on the gravity side includes all of these solutions, and also the fluctuations around these solutions, including both perturbative contributions in $1/N$ (from supergravity fields) and nonperturbative contributions (from wrapped D-branes). And moreover, different solutions dominate for different values of the chemical potentials.

We would like to ask whether the sum over Euclidean gravity solutions and the fluctuations around them can be reproduced by a direct computation of the index. There are several different methods that have been used to compute the index in the large N limit. One is the so-called Cardy limit [6,55–57], a sort of "high-temperature" limit taken on the chemical potentials in which the integral expression of the index considerably simplifies, and that can then be easily followed by the large N limit. Another one is the Bethe Ansatz method [7], described below. A third method is the saddle-point approximation applied to a nonanalytic extension of the index integral formula [15,17,24]. A fourth method is an expansion of the index integral formula around a complexified Gross-Witten-Wadia model [22,26].

In this paper we focus on the Bethe Ansatz method [7], which is valid when the chemical potentials for the angular momenta obey specific relations; we discuss in this paper only the simplest case when this method applies, which is for equal chemical potentials for the two angular momenta. In this method the index is written as a sum over solutions to Bethe Ansatz equations (BAEs) (at least when these solutions are discrete). Some of the known solutions to these equations give contributions which in the large N limit resemble our expectations from the gravity side; there is a leading term (in the logarithm of the contribution to the partition function) of order N^2, then power-law corrections in $1/N$, and then nonperturbative corrections of order e^{-N}. This raises a natural question—can we identify the sum over Bethe Ansatz solutions in the CFT computation of the index, with the sum over gravitational solutions? *A priori* there is no reason for such a matching of individual terms in the sum (as opposed to the full partition function); in particular, it has not been shown on the gravity side that the partition function localizes to a discrete sum over solutions, analogous to the Bethe Ansatz result. However, we will show that there is in fact a precise matching between the two sides.

In one direction, we show that every solution to the Bethe Ansatz equations which is known to have a good large N limit, may be identified with a specific Euclidean solution on the gravity side. These solutions are either Euclidean black holes (belonging to one of the two branches of known supersymmetric black hole solutions), or $\mathbb{Z}_m$ orbifolds of Euclidean black holes which satisfy the same boundary conditions. The matching involves both the leading contribution of order N^2 to the logarithm of the partition function, as well as the form of the nonperturbative corrections, which on the gravity side arise from wrapped D3-branes.

In the other direction, we note that *a priori* there are many more gravitational solutions with the correct boundary conditions than Bethe Ansatz solutions, and that if we naïvely sum over all solutions on the gravity side we obtain divergences, since the real part of their actions is not bounded from below. However, our supersymmetric Euclidean solutions are all complex valued, and it is not obvious what the rules for including such solutions in the gravitational theory are. We use (as previous works do) a specific prescription for evaluating the action of the complex solutions (at leading order in $1/N$) by analytic continuation. However, it is not clear when this analytic continuation is valid, since various poles may be encountered when one tries to explicitly continue the relevant contours in the gravitational path integral (to the extent that such a path integral description makes sense to begin with). And moreover, when we analytically continue only the leading contribution to some solution, we risk having subleading terms become dominant over the leading ones [for instance, a nonperturbative correction of the form e^{-Na^2} is negligible for real a, but becomes dominant when $\mathbb{R}\mathrm{e}(a^2) < 0$]. We suggest a specific prescription for which solutions should be included in the path integral, namely that solutions for which the nonperturbative contributions from wrapped D3-branes are exponentially large (rather

than exponentially suppressed) should not be included. With this (reasonable but not rigorously derived) prescription, we find a precise one-to-one match between the solutions on both sides.

The wrapped D3-brane solutions mentioned above, which give nonperturbative corrections on the gravity side matching the Bethe Ansatz solutions, wrap an S^3 inside the S^5 and an S^1 in the AdS_5 coordinates. In addition to these, there are also other Euclidean wrapped D3-branes that preserve supersymmetry, which wrap an S^1 inside the S^5 and an S^3 in the AdS_5 coordinates. These additional D3-branes are not related to nonperturbative terms in the Bethe Ansatz contribution of the Hong-Liu solutions, but they still give corrections to the gravitational partition function. The interpretation of these corrections in the Bethe Ansatz expansion will be discussed elsewhere [58]. In this paper, whenever we mention wrapped D3-branes without indicating their type, we will always be referring to the first type of D3-brane solutions.

Our analysis in this paper focuses on the Eucllidean path integral, but along the way, we have also found new supersymmetric Lorentzian probe D3-brane configurations in the BPS black hole backgrounds (and complexification thereof) that generalize the giant and dual giant graviton configurations in AdS_5 [59–61]. It would be interesting to analyze how these configurations affect the counting of black hole microstates, and in particular if they may be related to any multicenter black hole solutions in AdS, but this lies beyond the scope of this work.

After reviewing the superconformal index, the partition function, and their relation in Sec. II, we review in Sec. III the Bethe Ansatz method and the known solutions (found in [33,62]), and we compute their perturbative and nonperturbative contributions to the index in the large N limit. In Sec. IV we review the known black hole solutions using the consistent truncation to a 5D supergravity theory, and in particular the Euclidean supersymmetric black hole solutions. We show that the action of some of these solutions agrees (at leading order in $1/N$) with the Bethe Ansatz results, but that more solutions exist on the gravity side. In Sec. V we lift the Euclidean black hole solutions to 10D, and compute the nonperturbative contributions to the partition function from wrapped Euclidean D3-branes in these backgrounds. In Sec. VI we describe the prescription mentioned above for keeping only the solutions whose nonperturbative corrections from wrapped D3-branes are small, and we show that these solutions match precisely to a subclass of the Bethe Ansatz solutions, including both the leading term and the form of the nonperturbative corrections. Finally, in Sec. VII we show that some specific orbifolds of Euclidean black hole solutions also contribute to the same Euclidean partition function (since they have the same boundary conditions), and that the orbifold solutions which satisfy our criterion precisely match with the remaining Bethe Ansatz solutions. Appendix A contains reference material, Appendices B–D contain details on field theory and gravity computations, while in Appendix E we present the new Lorentzian D3-brane giant graviton configurations.

A. Future directions

There are many open questions left by our analysis. Some of them are as follows:

(1) The rules of the AdS/CFT correspondence in the presence of complex sources (such as the background metric or gauge fields) are not clear. A commonly used method is to take results for the classical action on the gravity side computed for real sources, and to analytically continue them to complex sources, where the result may also be interpreted as related to a complex gravity solution (with complex boundary conditions related to the complex sources). Analytic continuations of this type may be dangerous, both because poles or other features may be encountered when shifting the contours where the gravitational fields are valued, and because subleading contributions for real sources may become dominant for some complex-valued sources. In this paper we discuss in detail an example of the latter problem, and suggest a prescription in which only (complex) gravitational solutions which do not have any instabilities related to nonperturbative contributions becoming large should be included. It would be useful to find a rigorous justification for this prescription by carefully analyzing the analytic continuations that are involved.

(2) Our results suggest that, as the Bethe Ansatz prescription does on the CFT side, the gravitational computation of the supersymmetric partition function on $S^3 \times S^1$ may perhaps be localized to a discrete sum over supersymmetric gravity solutions. It would be interesting to perform such a localization on the gravity side, at least in the supergravity approximation and perhaps along the lines of [63–65], and to see if it reproduces the results we found in this paper.

(3) In this paper we only compared the leading exponents in the contributions of various solutions to the path integral—both for the classical gravity solutions (contributing to the logarithm of the partition function at order N^2) and for the nonperturbative corrections coming from D3-branes. From the Bethe Ansatz we can also find the subleading contributions to the logarithm of the partition function, of order $\log(N)$ (already considered in [21]) and of order 1, and we seem to find no further perturbative corrections in $1/N$. It would be interesting to reproduce these corrections from one-loop fluctuations around our gravitational solutions,

and to confirm that higher-loop corrections are absent on the gravity side. In the case of the D3-branes, we can again compute on the field theory side also the coefficients of the exponentials that we matched, and it would be interesting to match those with multiplicities of D3-brane solutions on the gravity side.

(4) So far there is no classification of all solutions to the Bethe Ansatz equations for the $SU(N)$ $\mathcal{N}=4$ SYM theories, and for finite N there are certainly additional solutions beyond the ones we analyzed, including continuous families of solutions [16,66,67]. It would be interesting to classify all the solutions, and to understand whether some of the other solutions might have an interpretation as classical gravity solutions as well.

There are also many possible generalizations of our analysis:

(1) We analyzed only the case of equal chemical potentials for the two angular momenta, since in this case the Bethe Ansatz computation of the index is the simplest. The Bethe Ansatz computation works whenever the ratio between the two angular momenta is rational [20,68], but for general rational values, extra parameters appear in the sum over Bethe Ansatz solutions, and the number of these parameters is of order N, so that in the large N limit there is an exponentially large number of terms appearing in the sum. On the other hand, on the gravity side nothing drastic seems to change when the chemical potentials are not equal (or even when their ratio is not rational), though some of the details of our analysis may be modified. This suggests that the extra parameters should be summed over before matching to the gravity side in order to still have a matching between the two sides. It would be interesting to perform these sums and to match the Bethe Ansatz results to a sum over gravity solutions also for other ratios of chemical potentials. Note that it is not obvious if this should be possible or not; even if the gravity computation localizes as mentioned above, it is not guaranteed that the same sum over solutions will appear on the gravity side as on the field theory side (only the full partition function has to match). However, our results suggest that such a mapping may be possible also for general ratios of the angular momenta chemical potentials.

(2) Many of our computations, both on the field theory side (using the Bethe Ansatz) and on the gravity side, may be generalized to general theories with 4D $\mathcal{N}=1$ superconformal symmetry. Indeed, the leading contributions from both sides have been matched in some other cases [14,20]. It would be interesting to check if also in these other cases, the full sum over Bethe Ansatz solutions may be matched in the large N limit to a sum over gravitational solutions, as we found for the $\mathcal{N}=4$ SYM theories. It may also be possible to generalize the analysis to supersymmetric $S^{d-1}\times S^1$ partition functions of d-dimensional superconformal field theories in other dimensions (perhaps starting from [44,69,70]).

(3) Nonperturbative corrections to the Euclidean partition function coming from wrapped branes, similar to the ones we found, presumably exist for all black hole solutions in any dimension. It would be interesting to evaluate these contributions and, where possible, to match them to the field theory side.[2] Of course, this requires lifting these black hole solutions—which are often found in truncations of the gravitational theory to a lower-dimensional theory—to 10D or 11D (depending on the case). It would also be interesting to understand if there are other cases where these contributions "destabilize" some complexified solutions of the gravity equations of motion, as we found.

Last but not least, it would be interesting to understand the implications of our analysis (which is purely in Euclidean signature) to the counting of 1/16-BPS black hole microstates.

II. THE SPHERE PARTITION FUNCTION AND THE INDEX

A. The sphere thermal partition function

In this paper we would like to match the partition function of $\mathcal{N}=4$ $SU(N)$ SYM on $S^3\times S^1$, which is closely related to the superconformal index, with the gravitational partition function of its holographic dual, type IIB string theory on a spacetime which is asymptotically $AdS_5\times S^5$. We will begin in this section with the field theory side by describing the precise relation between the partition function and the superconformal index.

Using $\mathcal{N}=1$ notation, the field content of the $\mathcal{N}=4$ SYM theory consists of a vector multiplet and three chiral multiplets X, Y, Z, all in the adjoint representation of the gauge group, with a superpotential proportional to $W=\mathrm{Tr}(X[Y,Z])$. The R symmetry is $SU(4)_R$: going to the Cartan $U(1)^3$, we choose a basis of generators $R_{1,2,3}$ each giving R charge 2 to a single chiral multiplet and zero to the other two, in a symmetric way. In addition, local operators (or, equivalently, states in the theory on S^3) are labeled by two angular momenta $J_{1,2}$, which are half-integer, and each rotates an $\mathbb{R}^2\subset\mathbb{R}^4$ in which S^3 is embedded. We define the fermion number as $F=2J_1$. Note that all fields in the theory (and thus all states) have integer charges under $R_{1,2,3}$ and obey

[2]Nonperturbative corrections to the contribution to the superconformal index (or other supersymmetric partition functions) coming from AdS_5 have been studied, e.g., in [71–73].

$$F = 2J_{1,2} = R_{1,2,3} \pmod 2. \tag{2.1}$$

We can add chemical potentials for each of these five charges, and the thermal partition function of the theory on S^3 can then be expressed as a trace over the Hilbert space of the theory as

$$Z = \mathrm{Tr}(e^{-\beta H + \sum_i \beta \Omega_i J_i + \frac{1}{2}\sum_a \beta \Phi_a R_a}), \tag{2.2}$$

where β is the inverse temperature, H is the Hamiltonian on S^3 in the absence of background fields (related to the operator dimension by the state/operator correspondence), and Ω_i (with $i = 1, 2$) and Φ_a (with $a = 1, 2, 3$) are chemical potentials. The $\frac{1}{2}$ for the R charges comes from our convention that the J_i are half-integers, while the R_a are integers. With this convention, given (2.1), the partition function (2.2) is periodic under shifts

$$\{\Omega_1, \Omega_2, \Phi_1, \Phi_2, \Phi_3\} \to \left\{\Omega_1 + \frac{2\pi i n_1}{\beta}, \Omega_2 + \frac{2\pi i n_2}{\beta}, \Phi_1 + \frac{2\pi i m_1}{\beta}, \Phi_2 + \frac{2\pi i m_2}{\beta}, \Phi_3 + \frac{2\pi i m_3}{\beta}\right\} \tag{2.3}$$

for any integers n_1, n_2, m_1, m_2, m_3 whose sum is even.

As usual, we can think of (2.2) as a Euclidean partition function[3] on $S^1_\beta \times S^3$ where the Euclidean time direction t_E has periodicity β (up to a zero-point energy that we will discuss below). The chemical potentials Ω_i and Φ_a can be thought of as background holonomies (on the thermal circle) of gauge fields coupled to the conserved currents (whose charges are the angular momenta and the R charges). For example, the Φ_a are equivalent to background fields $A^a_{t_\mathrm{E}} = \frac{i}{2}\Phi_a$ for the gauge fields coupled to the currents of R_a. In principle, the chemical potentials in (2.2) can be real or complex, as long as the partition function (2.2) converges; the inverse temperature β can also be complex, though then the geometrical Euclidean interpretation is less clear.

For the chemical potentials related to the angular momenta, the background fields in question are really off-diagonal components of the metric; for an angular momentum involving a shift in an angular coordinate ϕ of S^3, we can think of them as $g_{t_\mathrm{E}\phi}$ components of this metric, proportional to $(i\Omega_i)$. This can also be thought of as a different complex structure for the fibration of the Euclidean time circle over the sphere [54]. When Ω is purely imaginary, the modification in the metric is real, while otherwise the metric becomes complex.

In this case we can also implement the chemical potentials in a different way. Since wave functions of states with angular momentum J have angular dependence $e^{iJ\phi}$ on an angular variable ϕ, we can alternatively keep the original S^3 metric on the sphere, but we use a different identification around the time circle,

$$(t_\mathrm{E}, \phi) \sim (t_\mathrm{E} + \beta, \phi - i\beta\Omega), \tag{2.4}$$

which leads to the very same factor of $e^{\beta\Omega J}$ in the trace (2.2). We will call this the "coordinate shift realization" of the chemical potentials, while the previous implementation is the "metric realization."

When Ω is purely imaginary, using (2.4) is a reasonable alternative description, equivalent to modifying the metric, but otherwise we need to be careful in how to interpret the identification (2.4) [together with the standard (2π) periodicity of ϕ]. A consistent way to implement this realization for general complex Ω is to define a new angular coordinate

$$\tilde{\phi} = \phi + i\Omega t_\mathrm{E} \tag{2.5}$$

such that the new coordinate $\tilde{\phi}$ still has a 2π periodicity, and it can be taken to be real (even when Ω is complex), while the other identification is simply $(t_\mathrm{E}, \tilde{\phi}) \sim (t_\mathrm{E} + \beta, \tilde{\phi})$. This defines the coordinate range of ϕ and t_E for the "coordinate shift realization" of the chemical potentials. Note that in general the original coordinate ϕ of the S^3 is complex now (in particular since β may also be complex, so that t_E which runs from 0 to β is also complex), but $\tilde{\phi}$ is real. The metric in the new coordinate $\tilde{\phi}$ is generally complex. Indeed, the metric in this new coordinate is essentially the same as the metric in the first approach discussed above. In this framework we know the range of the coordinates even when Ω and β are complex, and this will be useful when we discuss holography below.

B. The superconformal index

The superconformal index is defined for general 4D $\mathcal{N} = 1$ superconformal field theories; in the $\mathcal{N} = 4$ SYM theory it counts (with sign) 1/16 BPS states on S^3 preserving one complex supercharge $\mathcal{Q}$, which we choose to be associated with a specific $U(1)_R$ symmetry with charge $r = (R_1 + R_2 + R_3)/3$. The superconformal index keeps track of some combinations of the R charges and the two angular momenta $J_{1,2}$. It is useful to introduce two flavor generators $\mathfrak{q}_{1,2} = \frac{1}{2}(R_{1,2} - R_3)$ that commute with $\mathcal{Q}$ and the $\mathcal{N} = 1$ superconformal subalgebra containing it.

[3]Actually, there could be a proportionality constant $Z_{S^1\times S^3} = e^{-C(\beta,\Omega_i,\Phi_a)}Z$, where Z is as in (2.2), coming from background dependent counterterms. For our purposes this distinction can be ignored, as explained below.

The superconformal index [49,50] is then defined by the trace[4]

$$\mathcal{I}(p,q,y_1,y_2)=\mathrm{Tr}'[(-1)^F e^{-\beta\{\mathcal{Q},\bar{\mathcal{Q}}\}} p^{J_1+\frac{1}{2}R_3} q^{J_2+\frac{1}{2}R_3} y_1^{\mathfrak{q}_1} y_2^{\mathfrak{q}_2}] \tag{2.6}$$

over the Hilbert space on S^3, where by Tr' we mean the trace in the convention where the contribution of the vacuum is 1. Here p, q, $y_{1,2}$ are fugacities, and it is convenient to introduce chemical potentials σ, τ, $\Delta_{1,2}$ such that

$$p=e^{2\pi i\sigma},\qquad q=e^{2\pi i\tau},\qquad y_{1,2}=e^{2\pi i\Delta_{1,2}}. \tag{2.7}$$

The index is well-defined for $|p|$, $|q|<1$, namely for $\mathbb{I}\mathrm{m}(\sigma)$, $\mathbb{I}\mathrm{m}(\tau)>0$. Given (2.1), the index is a single-valued function of the fugacities (2.7); i.e., it is periodic under integer shifts of the chemical potentials σ, τ, Δ_1, and Δ_2. By standard arguments [74], the index only counts states annihilated by $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ and is thus independent of β.

For the purpose of relating the partition function (2.2) to the superconformal index, note that, in order to preserve the specific supercharge $\mathcal{Q}$ mentioned above on $S^1_\beta\times S^3$, the chemical potentials in (2.2) have to satisfy [5]

$$\beta(1+\Omega_1+\Omega_2-\Phi_1-\Phi_2-\Phi_3)=2\pi i n,\quad n\in\mathbb{Z}. \tag{2.8}$$

In addition, in order for (2.2) to have the interpretation of computing a thermal partition function, we must have antiperiodic boundary conditions for the fermions (and in particular for the supercharge) around the thermal cycle, and this requires n to be odd [5]. We will choose $n=1$, but this does not affect anything because of the periodicity (2.3) of the chemical potentials.

Our specific conserved supercharge has the algebra [50]

$$\{\mathcal{Q},\bar{\mathcal{Q}}\}=H-J_1-J_2-\frac{1}{2}(R_1+R_2+R_3). \tag{2.9}$$

Thus, we can write the partition function (2.2) as

$$Z=\mathrm{Tr}(e^{-\beta\{\mathcal{Q},\bar{\mathcal{Q}}\}+\sum_i\beta(\Omega_i-1)J_i+\frac{1}{2}\sum_a\beta(\Phi_a-1)R_a}), \tag{2.10}$$

and using the relation (2.8) between the chemical potentials one finds

$$\begin{aligned}Z=\mathrm{Tr}(&e^{-\beta\{\mathcal{Q},\bar{\mathcal{Q}}\}}e^{\pi i R_3}e^{\sum_i\beta(\Omega_i-1)(J_i+\frac{1}{2}R_3)}\\&\times e^{\beta(\Phi_1-1)\frac{R_1-R_3}{2}}e^{\beta(\Phi_2-1)\frac{R_2-R_3}{2}}).\end{aligned} \tag{2.11}$$

For the charge assignments of our theory, the factor $e^{\pi i R_3}$ is equivalent to $(-1)^F$. Note also that in the new expression, since (R_1-R_3) and (R_2-R_3) are even and $J_i+\frac{1}{2}R_3$ is an integer, the periodicities of the remaining four chemical potentials are reduced to $(2\pi i/\beta)$. Defining new chemical potentials for the four independent charges that, given (2.8), we have access to, namely

$$\sigma=\frac{\beta(\Omega_1-1)}{2\pi i},\qquad \tau=\frac{\beta(\Omega_2-1)}{2\pi i},\qquad \Delta_a=\frac{\beta(\Phi_a-1)}{2\pi i}\quad \text{for } a=1,2, \tag{2.12}$$

one finally finds that

$$\begin{aligned}Z&=\mathrm{Tr}((-1)^F e^{-\beta\{\mathcal{Q},\bar{\mathcal{Q}}\}}e^{2\pi i\sigma(J_1+\frac{1}{2}R_3)}e^{2\pi i\tau(J_2+\frac{1}{2}R_3)}\\&\quad\times e^{2\pi i\Delta_1\mathfrak{q}_1}e^{2\pi i\Delta_2\mathfrak{q}_2}),\\&=e^{-\beta E_0}\mathcal{I}(\sigma,\tau,\Delta_1,\Delta_2),\end{aligned} \tag{2.13}$$

where $\mathcal{I}$ is the superconformal index (2.6) defined above, and E_0 (which is related to the charges of the ground state on S^3) is called here the Casimir energy.[5] This relation between the thermal partition function (2.2) [when (2.8) holds] and the index will enable us to relate the index that we compute in the field theory, to computations of the $S^1\times S^3$ partition function using the gravitational dual.

Note that, in general, E_0 is scheme dependent. The authors of [75] argued that supersymmetry dictates (if a supersymmetric regularization is used) a specific choice of renormalization scheme which fixes the value of E_0 to a specific value E_{SUSY} depending on the chemical potentials, which is then called the supersymmetric Casimir energy. In [5] a localization computation verified (2.13) when (2.8) applies and found the supersymmetric Casimir energy.

Both of these papers relied on coupling the theory to curved space using the R multiplet. Recently, it was argued that for theories with an anomalous $U(1)_R$ symmetry (in the sense of nonconservation of the current in the presence of a background field, as in $\mathcal{N}=4$ SYM) this coupling induces supercurrent anomalies, see [76–82]. They suggest that additional counterterms should be added to the action in order to restore supersymmetry. These counterterms could change the value of E_0. A similar argument was made in [83], where it was suggested that in order to regularize the theory in a supersymmetric and diffeomorphism invariant way one should add an additional counterterm to the action. This changes (2.13) to $Z=e^{-\beta W}e^{-\beta E_{\mathrm{SUSY}}}\mathcal{I}$, where W is the additional counterterm and E_{SUSY} is the same as before.

We will avoid this discussion by considering the ratio between the partition function and the contribution of the vacuum, which equals the index $\mathcal{I}$ in every consistent scheme. On the gravity side we realize this through

[4]Often the powers in the index are written as $p^{J_1+\frac{1}{2}r}q^{J_2+\frac{1}{2}r}$. Compared to this convention, we have swallowed a power of pq into y_1 and y_2 in order to obtain a single-valued function. The relation of our variables to those of [50] is $p=t^3x|_{\mathrm{there}}$, $q=t^3/x|_{\mathrm{there}}$, $y_1=t^2v|_{\mathrm{there}}$, $y_2=t^2w/v|_{\mathrm{there}}$.

[5]If our renormalization scheme requires additional counterterms, they would appear similarly to E_0.

"background subtraction," namely regularizing the gravitational action by subtracting the contribution coming from empty thermal AdS, and we compare the result of this regularization to the computation of the index.[6]

III. FIELD THEORY ANALYSIS

We are interested in the superconformal index (2.6) of the four-dimensional $\mathcal{N}=4$ supersymmetric Yang-Mills theory with gauge group $SU(N)$.

Bethe Ansatz formulation.—The index is independent of the gauge coupling, and thus can be computed exactly [49,50,85,86] in terms of a certain contour integral. However, in order to extract the large N limit, it is convenient to consider the so-called Bethe Ansatz formula [68,87]. In this paper we will mainly be interested in the case of two equal angular fugacities:

$$p=q \quad \Leftrightarrow \quad \sigma=\tau. \tag{3.1}$$

In this case the Bethe Ansatz formula reads[7]

$$\mathcal{I}(q,y_1,y_2)=\kappa_N \sum_{\hat{u}\in \mathrm{BAEs}} \mathcal{Z}(\hat{u};\Delta,\tau)H(\hat{u};\Delta,\tau)^{-1}. \tag{3.2}$$

The sum is over the solution set $\{\hat{u}\}$ to a system of transcendental equations, dubbed here BAEs. Defining the Bethe Ansatz (BA) operators

$$Q_i(u;\Delta,\tau)\equiv e^{6\pi i\sum_j u_{ij}}\prod_{j=1}^{N}\frac{\theta_0(u_{ji}+\Delta_1;\tau)\theta_0(u_{ji}+\Delta_2;\tau)\theta_0(u_{ji}-\Delta_1-\Delta_2;\tau)}{\theta_0(u_{ij}+\Delta_1;\tau)\theta_0(u_{ij}+\Delta_2;\tau)\theta_0(u_{ij}-\Delta_1-\Delta_2;\tau)} \tag{3.3}$$

for $i=1,\ldots,N$, the BA equations are given by

$$1=\frac{Q_i}{Q_N} \quad \text{for } i=1,\ldots,N-1. \tag{3.4}$$

The quasielliptic function θ_0 is defined in Appendix A, while $u_{ij}=u_i-u_j$. The unknowns are the "complexified $SU(N)$ holonomies" u_i living on a torus of modular parameter τ, namely with identifications

$$u_i\sim u_i+1\sim u_i+\tau \quad \text{for } i=1,\ldots,N-1, \tag{3.5}$$

while u_N is fixed by the constraint $\sum_{i=1}^{N}u_i=0$. The BAEs (3.4) are invariant under such shifts. In fact, a stronger property holds: Q_i are invariant under shifts of the components of the antisymmetric tensor u_{ij} by 1 or τ, even relaxing the condition that $u_{ij}=u_i-u_j$.

It was proven in [68] that only the solutions that are not invariant under any nontrivial element of the Weyl group of $SU(N)$ (namely, only solutions with all u_i different on the torus) actually contribute to the sum in (3.2). Moreover, it was shown in [67] that only the solutions to a more restrictive set of equations,

$$Q_i=(-1)^{N-1} \quad \text{for } i=1,\ldots,N \tag{3.6}$$

and dubbed "reduced Bethe Ansatz equations," actually contribute to the index. Notice that $\prod_{i=1}^{N}Q_i=1$ identically. We stress that from each solution in terms of u_{ij} on the torus, one gets up to N^2 solutions in terms of u_i, related by a shift of the "center of mass" of the first $N-1$ components. Some of those, however, could be equivalent up to the Weyl group action, and thus the exact multiplicity should be determined by a case-by-case analysis.

The prefactor in (3.2) is

$$\kappa_N=\frac{1}{N!}\left(\frac{(q;q)_\infty^2\tilde{\Gamma}(\Delta_1;\tau,\tau)\tilde{\Gamma}(\Delta_2;\tau,\tau)}{\tilde{\Gamma}(\Delta_1+\Delta_2;\tau,\tau)}\right)^{N-1}, \tag{3.7}$$

where $(z;q)_\infty$ is the q-Pochhammer symbol while $\tilde{\Gamma}(u;\sigma,\tau)$ is the elliptic gamma function, defined in Appendix A. The function $\mathcal{Z}$ is given by

$$\mathcal{Z}(u;\Delta,\tau)=\prod_{i\neq j}^{N}\frac{\tilde{\Gamma}(u_{ij}+\Delta_1;\tau,\tau)\tilde{\Gamma}(u_{ij}+\Delta_2;\tau,\tau)}{\tilde{\Gamma}(u_{ij}+\Delta_1+\Delta_2;\tau,\tau)\tilde{\Gamma}(u_{ij};\tau,\tau)} \tag{3.8}$$

and it coincides [using (A16)] with the integrand of the standard integral formula for the superconformal index. Finally, H is a Jacobian defined as

$$H=\det\left[\frac{1}{2\pi i}\frac{\partial\log(Q_i/Q_N)}{\partial u_j}\right]_{i,j=1,\ldots,N-1}\equiv\det(\mathcal{A}_{ij}). \tag{3.9}$$

Here we introduced the $(N-1)\times(N-1)$ Jacobian matrix $\mathcal{A}_{ij}$ for later convenience.

Notice that Q_i, κ_N, $\mathcal{Z}$, and H are all invariant under integer shifts of τ, Δ_1, and Δ_2, implying that the superconformal index (3.2) is a single-valued function of the

[6]Note that background subtraction is believed to give the same results for the ratio we compute as a more precise computation using holographic renormalization in the bulk, as in [69] (see also [84] for an alternative scheme). With the specific regularization used there, E_0 matches the usual Casimir energy in a free CFT, see also Appendix A of [75] (and our footnote 19).

[7]This expression was derived in [68] assuming that the Bethe Ansatz equations have isolated solutions. It has been noticed in [16,66,67] that for the $\mathcal{N}=4$ SYM theory, the equations can also have continuous families of solutions. In this case the expression of their contributions should likely be modified. In this paper we will ignore the contributions of those continuous families, leaving their study to future work.

fugacities. This is also apparent from the Hamiltonian definition (2.6), as already noted.

The superconformal index of the $\mathcal{N}=4$ SYM theory with gauge group $U(N)$ is related to the one for gauge group $SU(N)$ by the simple relation

$$\mathcal{I}_{U(N)} = \mathcal{I}_{U(1)}\mathcal{I}_{SU(N)}, \tag{3.10}$$

where

$$\mathcal{I}_{U(1)}(p,q,y_1,y_2) = (p;p)_\infty (q;q)_\infty \frac{\tilde{\Gamma}(\Delta_1;\sigma,\tau)\tilde{\Gamma}(\Delta_2;\sigma,\tau)}{\tilde{\Gamma}(\Delta_1+\Delta_2;\sigma,\tau)} \tag{3.11}$$

is the index of the free $\mathcal{N}=4$ $U(1)$ theory.

Hong-Liu solutions.—The full set of solutions to (3.6) is not known (see also footnote 7); however, a large set was found in [33,62] and we will refer to them as Hong-Liu (HL) solutions. They are labeled by three integers:

$$\{m,n,r\} \quad \text{such that} \quad N = m\cdot n, \quad 0 \le r < n. \tag{3.12}$$

The solutions are

$$u_j \equiv u_{\hat{j}\hat{k}} = \bar{u} + \frac{\hat{j}}{m} + \frac{\hat{k}}{n}\left(\tau + \frac{r}{m}\right). \tag{3.13}$$

Here we have decomposed the index $j = 0,\ldots,N-1$ into the indices $\hat{j} = 0,\ldots,m-1$ and $\hat{k} = 0,\ldots,n-1$. Moreover, $\bar{u}$ is a constant chosen in such a way to solve the $SU(N)$ constraint (but recall that what enters in all formulas are the differences $u_{ij} = u_i - u_j$). Notice that these solutions are in one-to-one correspondence with subgroups of $\mathbb{Z}_N \times \mathbb{Z}_N$ of order N. It turns out that each HL solution has multiplicity N (besides the $N!$ coming from permutations of u_is), corresponding to the inequivalent solutions to the $SU(N)$ constraint.

The BAEs (3.6) are invariant under $SL(2,\mathbb{Z})$ modular transformations of the torus, namely under the generators

$$T:\begin{cases}\tau \mapsto \tau+1\\ u\mapsto u\end{cases}, \qquad S:\begin{cases}\tau\mapsto -1/\tau\\ u\mapsto u/\tau\end{cases}, \qquad C:\begin{cases}\tau\mapsto\tau\\ u\mapsto -u\end{cases}. \tag{3.14}$$

It follows that the HL solutions form orbits under $PSL(2,\mathbb{Z})$, completely classified by the integer $d = \gcd(m,n,r)$. The action of $PSL(2,\mathbb{Z})$ is given by

$$\begin{aligned} &T:\{m,n,r\}\mapsto\{m,n,r+m\},\\ &S:\{m,n,r\}\mapsto\left\{\gcd(n,r),\frac{mn}{\gcd(n,r)},\frac{m(n-r)}{\gcd(n,r)}\right\}, \end{aligned} \tag{3.15}$$

where the last entry of $\{m',n',r'\}$ is understood mod n'.

A. Contributions of Hong-Liu solutions

We will compute the contribution of each of the HL solutions (3.13) to the sum in (3.2). While H will remain somehow implicit, we will be able to obtain a very explicit expression for $\mathcal{Z}$.

It is convenient to rewrite u_{ij} as

$$u_{ij} \equiv u_{(j_1k_1)(j_2k_2)} = v_{j_1j_2} + w_{k_1k_2}, \tag{3.16}$$

with $j_1,j_2 = 0,\ldots,m-1$ and $k_1,k_2 = 0,\ldots,n-1$, as well as

$$\begin{aligned} v_{j_1j_2} &= v_{j_1} - v_{j_2} = \frac{j_1-j_2}{m},\\ w_{k_1k_2} &= w_{k_1} - w_{k_2} = \frac{k_1-k_2}{n}\left(\tau+\frac{r}{m}\right). \end{aligned} \tag{3.17}$$

We also define

$$\xi_j = e^{2\pi i v_j}, \qquad \zeta_k = e^{2\pi i w_k}. \tag{3.18}$$

1. Elliptic gamma functions

To evaluate the contribution to the index, we need to compute

$$\begin{aligned} \gamma_\Delta &= \sum_{i\neq j}^{N} \log\left(\tilde{\Gamma}(u_{ij}+\Delta;\tau,\tau)\right)\big|_{(3.16)},\\ &= \sum_{j_1,j_2=0}^{m-1}\sum_{k_1\neq k_2}^{n-1}\log\left(\tilde{\Gamma}(v_{j_1j_2}+w_{k_1k_2}+\Delta;\tau,\tau)\right)\\ &\quad + n\sum_{j_1\neq j_2}^{m-1}\log\left(\tilde{\Gamma}(v_{j_1j_2}+\Delta;\tau,\tau)\right) \end{aligned} \tag{3.19}$$

for $\Delta \in \{0,\Delta_1,\Delta_2,\Delta_1+\Delta_2\}$. Notice that while $\exp(\gamma_\Delta)$ is well defined, γ_Δ is only defined modulo $2\pi i$. We set

$$y = e^{2\pi i\Delta}. \tag{3.20}$$

When Δ is a generic complex number, one easily verifies the shift property

$$\gamma_{\Delta+\frac{1}{m}} - \gamma_\Delta = -N\log\left(\frac{\tilde{\Gamma}(\Delta+\frac{1}{m};\tau,\tau)}{\tilde{\Gamma}(\Delta;\tau,\tau)}\right). \tag{3.21}$$

We will use this relation in the following. It follows that $\gamma_{\Delta+1} = \gamma_\Delta$.

The case $\Delta \neq 0$.—Our strategy is to expand the functions appearing in (3.19) inside a common domain of convergence, and then manipulate and resum the expansions. As long as we obtain exact expressions, the latter will be valid on the full domain of analyticity, possibly up to ambiguities by $2\pi i$ from the logarithms [such ambiguities disappear when considering $\exp(\gamma_\Delta)$]. The details of the computation are in Appendix B 1. Defining

$$\check{\tau} = m\tau + r \quad \text{and} \quad \check{q} = e^{-2\pi i/\check{\tau}}, \tag{3.22}$$

we obtain the exact expression[8]

$$\gamma_\Delta = \pi i \frac{N^2}{m} \frac{(\check{\tau} - m\Delta)(\check{\tau} - m\Delta - \frac{1}{2})(\check{\tau} - m\Delta - 1)}{3\check{\tau}^2} - \frac{\pi i m}{6}\left(\check{\tau} - m\Delta - \frac{1}{2}\right) - N \log\left(\tilde{\Gamma}(\Delta;\tau,\tau)\right) - N \log\left(\theta_0\left(\frac{N\Delta}{\check{\tau}}; -\frac{n}{\check{\tau}}\right)\right) + m \sum_{k=0}^{\infty} \log\left(\frac{\psi(\frac{k+1+m\Delta}{\check{\tau}/n})}{\psi(\frac{k-m\Delta}{\check{\tau}/n})}\right). \tag{3.23}$$

The function $\psi(u)$ appearing here is defined in Appendix A.

The expression we obtained will be convenient when taking the large N limit; however, if we are interested in an explicit exact expression, even the remaining infinite sum can be computed exploiting the modular formula (A19). We obtain the following[9]:

$$\gamma_\Delta = \pi i \frac{N^2}{m} \frac{(\check{\tau} - m\Delta)(\check{\tau} - m\Delta - \frac{1}{2})(\check{\tau} - m\Delta - 1)}{3\check{\tau}^2} - \frac{\pi i m}{6}\left(\check{\tau} - m\Delta - \frac{1}{2}\right) + \pi i m \mathcal{Q}(m\Delta; \check{\tau}/n, \check{\tau}/n) - (N-m)\log\left(\theta_0\left(\frac{N\Delta}{\check{\tau}}; -\frac{n}{\check{\tau}}\right)\right) - N \log\left(\tilde{\Gamma}(\Delta;\tau,\tau)\right) + m \log\left(\tilde{\Gamma}\left(m\Delta; \frac{\check{\tau}}{n}, \frac{\check{\tau}}{n}\right)\right). \tag{3.24}$$

Here $\mathcal{Q}(u;\tau,\tau)$ is a cubic polynomial in u, defined in (A20).

The case $\boldsymbol{\Delta} = \mathbf{0}$.—This case requires a separate treatment, which can be found in Appendix B 2. We obtain

$$\gamma_0 = \pi i \frac{N^2}{m} \frac{\check{\tau}(\check{\tau} - \frac{1}{2})(\check{\tau} - 1)}{3\check{\tau}^2} - \frac{\pi i m \check{\tau}}{6} + N \log\left(\frac{\check{\tau}}{N}\right) + 2N \log\left[\frac{(q;q)_\infty}{(\check{q}^n;\check{q}^n)_\infty}\right]. \tag{3.25}$$

Notice that we could have obtained this expression directly as the $\Delta \to 0$ limit of (3.23). To do that, we need the following asymptotic behaviors, easy to derive:

$$\theta_0\left(\frac{N\Delta}{\check{\tau}}; -\frac{n}{\check{\tau}}\right) \overset{\Delta\to 0}{\sim} (1 - e^{2\pi i N\Delta/\check{\tau}})(\check{q}^n;\check{q}^n)_\infty^2$$
$$\tilde{\Gamma}(\Delta;\tau,\tau) \overset{\Delta\to 0}{\sim} \frac{1}{1 - e^{2\pi i \Delta}} \frac{1}{(q;q)_\infty^2}. \tag{3.26}$$

It follows that

$$-N \lim_{\Delta\to 0} \log\left[\tilde{\Gamma}(\Delta;\tau,\tau)\theta_0\left(\frac{N\Delta}{\check{\tau}}; -\frac{n}{\check{\tau}}\right)\right] = 2N \log\left[\frac{(q;q)_\infty}{(\check{q}^n;\check{q}^n)_\infty}\right] + N \log\left(\frac{\check{\tau}}{N}\right). \tag{3.27}$$

The last sum in (3.23) is regular at $\Delta = 0$: most terms cancel out, and one uses $\psi(0) = e^{\pi i/12}$.

[8]Taking the exponential of the right-hand side, one verifies that it is invariant under $m\Delta \to m\Delta + 1$, performed in all terms but the first one in the second line [while if we do not take the exponential, we are left with $-\pi i m n^2 - N\log(-1)$ plus possible multiples of $2\pi i$ from the branch cuts of the logs]. It follows that the right-hand side satisfies the shift property (3.21).

[9]One can verify that under $r \to r + n$, which corresponds to $\check{\tau} \to \check{\tau} + n$, the quantity γ_Δ gets shifted by $m\pi i \frac{n(n+1)(n-1)}{3}$ (which is always a multiple of $2\pi i$) plus possible multiples of $2\pi i$ from the branch cuts of the logs. It follows that $\exp(\gamma_\Delta)$ is invariant, as it should.

2. The Jacobian

The Jacobian was defined in (3.9). It is convenient to introduce an auxiliary chemical potential Δ_3 defined by

$$\sum_{a=1}^{3}\Delta_a-\sigma-\tau\in\mathbb{Z} \tag{3.28}$$

(and then set $\sigma=\tau$), in order to make manifest the action of the Weyl group of $SU(3)$—the global symmetry contained in $SU(4)_R$ that commutes with the preserved supercharges $\mathcal{Q},\bar{\mathcal{Q}}$. We can then rewrite the BA operators (3.3) as

$$Q_i=e^{-2\pi i\sum_j u_{ij}}\prod_{a=1}^{3}\prod_{j=1}^{N}\frac{\theta_0(u_{ji}+\Delta_a;\tau)}{\theta_0(u_{ij}+\Delta_a;\tau)}. \tag{3.29}$$

With a few manipulations that we report in Appendix B 3, the $(N-1)\times(N-1)$ Jacobian matrix $\mathcal{A}_{ij}$ can be compactly written as

$$\mathcal{A}_{ij}=Nb_N\mathcal{B}_{ij}+\mathcal{C}_{ij}. \tag{3.30}$$

Here

$$\begin{aligned}
b_N&=-4+\frac{1}{\check{\tau}}\sum_{a=1}^{3}\left[2m\Delta_a+1+\mathcal{G}\left(0,\frac{N\Delta_a}{\check{\tau}};-\frac{n}{\check{\tau}}\right)\right],\\
\mathcal{B}_{ij}&=\delta_{ij}+1,\\
\mathcal{C}_{ij}&=\sum_{a=1}^{3}[\mathcal{G}(u_{iN},\Delta_a;\tau)+\mathcal{G}(u_{jN},\Delta_a;\tau)\\
&\quad-\mathcal{G}(u_{ij},\Delta_a;\tau)-\mathcal{G}(0,\Delta_a;\tau)],
\end{aligned} \tag{3.31}$$

with $i,j=1,\ldots,N-1$, where the function $\mathcal{G}(u,\Delta;\tau)$ is defined in Appendix A. Notice that $\det(\mathcal{B})=N$ and $(\mathcal{B}^{-1})_{ij}=\delta_{ij}-1/N$. To compute the determinant, we factorize $\mathcal{A}$ and obtain

$$\begin{aligned}
\log(H)\equiv\log(\det(\mathcal{A}))&=N\log(N)+(N-1)\log(b_N)\\
&\quad+\log\left[\det\left(\mathbb{1}+\frac{b_N^{-1}}{N}\mathcal{B}^{-1}\mathcal{C}\right)\right].
\end{aligned} \tag{3.32}$$

The last term of this expression is still rather implicit, however this will be good enough to compute the large N limit.

3. Total contribution

We denote the total contribution of the HL solution $\{m,n,r\}$ to the index as

$$\mathcal{I}_{\{m,n,r\}}=N\cdot N!\cdot\kappa_N\mathcal{Z}(u;\Delta,\tau)H(u;\Delta,\tau)^{-1}|_{(3.16)}. \tag{3.33}$$

The factor $N\cdot N!$ takes into account the multiplicity of the solution $\{m,n,r\}$: indeed $N!$ comes from the Weyl group action that permutes the eigenvalues u_i, while N comes from the inequivalent choices of sets of points $\{u_i\}$ on the torus that give rise to the same u_{ij}s, as discussed after (3.13). Such N inequivalent choices correspond to shifts of the center of mass of the first $N-1$ holonomies, and can be parametrized, for $(\hat{\jmath},\hat{k})\neq(m,n)$, as

$$u_{\hat{\jmath}\hat{k}}^{(\ell)}=\bar{u}+\frac{\hat{\jmath}}{m}+\frac{\hat{k}}{n}\left(\tau+\frac{r}{m}\right)+\frac{\ell_1\tau+\ell_2}{N}, \tag{3.34}$$

with $\ell_1=0,\ldots,m-1$ and $\ell_2=0,\ldots,n-1$. As discussed before, all these solutions give exactly the same contribution to $\mathcal{Z}$ and H.

Using that $\tilde{\Gamma}(u+\Delta_1+\Delta_2;\sigma,\tau)^{-1}=\tilde{\Gamma}(-u+\Delta_3;\sigma,\tau)$ and putting everything together, we eventually find

$$\begin{aligned}
\log(\mathcal{I}_{\{m,n,r\}})&=\mathcal{P}(m\Delta,\check{\tau})+\log(N)+2\log\left[\frac{(\check{q}^n;\check{q}^n)_\infty^N}{(q;q)_\infty}\right]-N\log(\check{\tau})-(N-1)\log(b_N)\\
&\quad+\sum_{a=1}^{3}\left[-N\log\left[\theta_0\left(\tfrac{N\Delta_a}{\check{\tau}};-\tfrac{n}{\check{\tau}}\right)\right]-\log(\tilde{\Gamma}(\Delta_a;\tau,\tau))+m\sum_{k=0}^{\infty}\log\left(\frac{\psi(\frac{k+1+m\Delta_a}{\check{\tau}/n})}{\psi(\frac{k-m\Delta_a}{\check{\tau}/n})}\right)\right]\\
&\quad+\tfrac{\pi i m}{6}\left(\sum_a m\Delta_a-2\check{\tau}+\tfrac{3}{2}\right)-\log\left[\det\left(\mathbb{1}+\tfrac{b_N^{-1}}{N}\mathcal{B}^{-1}\mathcal{C}\right)\right].
\end{aligned} \tag{3.35}$$

Recall that $\check{\tau}=m\tau+r$ and $\check{q}=e^{-2\pi i/\check{\tau}}$. Here $\mathcal{P}(m\Delta,\check{\tau})$ is the contribution of order N^2:

$$\mathcal{P}(m\Delta,\check{\tau})=\frac{\pi i N^2}{3m}\left[\sum_{a=1}^{3}\frac{(\check{\tau}-m\Delta_a)(\check{\tau}-m\Delta_a-\frac{1}{2})(\check{\tau}-m\Delta_a-1)}{\check{\tau}^2}-\frac{\check{\tau}(\check{\tau}-\frac{1}{2})(\check{\tau}-1)}{\check{\tau}^2}\right]. \tag{3.36}$$

B. The large N limit

We proceed to compute the large N limit of (3.35). Clearly, this depends on how we scale m, n, r with N. In this paper[10] we consider BA solutions $\{m,n,r\}$ with m, r fixed and $n\to\infty$.

There are various terms in (3.35) that are exponentially suppressed in the limit and go to zero. For instance, $N\log\left((\check{q}^n;\check{q}^n)_\infty\right)\sim -N\check{q}^n$ for $n\to\infty$, and thus this term is of order $\mathcal{O}(Ne^{-N})$ and goes to zero. It turns out that when the electric chemical potentials $\Delta_{a=1,2,3}$ are in a particular range, namely

$$0<\mathbb{I}\mathrm{m}\left(\frac{m\Delta_a}{\check{\tau}}\right)<\mathbb{I}\mathrm{m}\left(-\frac{1}{\check{\tau}}\right), \tag{3.37}$$

many other terms are exponentially suppressed as $\mathcal{O}(Ne^{-N})$. We have the following:

$$\begin{aligned}
&N\log\left[\theta_0\left(\frac{N\Delta_a}{\check{\tau}};-\frac{n}{\check{\tau}}\right)\right]\sim -N(\tilde{y}_a^n+(\check{q}/\tilde{y}_a)^n)\to 0,\\
&m\sum_{k=0}^{\infty}\log\left(\frac{\psi(\frac{k+1+m\Delta_a}{\check{\tau}/n})}{\psi(\frac{k-m\Delta_a}{\check{\tau}/n})}\right)\sim -N\sum_{k=0}^{\infty}\left[\frac{k+1+m\Delta_a}{\check{\tau}}(\check{q}/\tilde{y}_a)^n\right.\\
&\qquad\left.-\frac{k-m\Delta_a}{\check{\tau}}\tilde{y}_a^n\right]\check{q}^{nk}\to 0,\\
&N\mathcal{G}\left(0,\frac{N\Delta_a}{\check{\tau}};-\frac{n}{\check{\tau}}\right)\sim 2N(\tilde{y}_a^n-(\check{q}/\tilde{y}_a)^n)\to 0.
\end{aligned} \tag{3.38}$$

On the other hand, the quantity $\log[\det(\mathbb{1}+\mathcal{B}^{-1}\mathcal{C}/b_N N)]$ is of order $\mathcal{O}(1)$. To see that, notice from (3.31) that the matrix $\mathcal{C}$ has entries of order 1, the matrix $\mathcal{B}^{-1}\mathcal{C}$ has entries of order 1, b_N is of order 1, and hence the matrix $\mathcal{E}\equiv\mathcal{B}^{-1}\mathcal{C}/b_N N$ has entries of order $1/N$. It follows that all traces $\mathrm{Tr}(\mathcal{E}^\ell)$ are of order 1 (or higher powers of $1/N$), and so is $\log[\det(1+\mathcal{E})]=\sum_{\ell=1}^{\infty}\frac{1}{\ell}(-1)^{\ell+1}\mathrm{Tr}(\mathcal{E}^\ell)$. We will study the Jacobian in more detail in Sec. III D below.

We conclude that in the range (3.37) it is relatively simple to take the large N limit—and the only term that contributes at order $\mathcal{O}(N^2)$ is $\mathcal{P}$. The range (3.37) corresponds to $m\Delta_a$ lying within an open strip in the complex plane, bounded by the line $\mathcal{L}_{\check{\tau}}$ through 0 and $\check{\tau}$ on the right, and by the line through -1 and $\check{\tau}-1$ on the left (see Fig. 1). From (3.23) and as noticed in footnote 8, γ_Δ is invariant if we shift $m\Delta\to m\Delta+1$ in all terms but $N\log(\tilde{\Gamma}(\Delta;\tau,\tau))$; similarly b_N in (3.31) is invariant under that shift. We can exploit this fact to push $m\Delta_a$ towards the range (3.37).

[10]Other large N limits may also be considered, and we have not checked all the possibilities. However, our matching to the gravitational side suggests that any other limit does not correspond to black hole solutions with an action of order N^2.

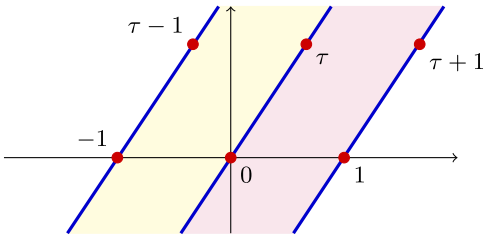

FIG. 1. Fundamental strips for $[\Delta]_\tau$ and $[\Delta]'_\tau$. The function $[\Delta]_\tau$ is the restriction of Δ mod 1 to the region $\mathbb{I}\mathrm{m}(-1/\tau)>\mathbb{I}\mathrm{m}(\Delta/\tau)>0$ (in yellow, on the left), while $[\Delta]'_\tau$ is the restriction of Δ mod 1 to the region $0>\mathbb{I}\mathrm{m}(\Delta/\tau)>\mathbb{I}\mathrm{m}(1/\tau)$ (in purple, on the right). We dubbed $\mathcal{L}_\tau$ the blue line through 0 and τ.

To that purpose, we introduce the periodic discontinuous function $[\Delta]_\tau$ defined as

$$\begin{aligned}
[\Delta]_\tau=\Big(&z|z=\Delta \mod 1,\mathbb{I}\mathrm{m}\left(-\frac{1}{\tau}\right)>\mathbb{I}\mathrm{m}\left(\frac{z}{\tau}\right)>0\Big)\\
&\text{for }\mathbb{I}\mathrm{m}\left(\frac{\Delta}{\tau}\right)\notin\mathbb{Z}\times\mathbb{I}\mathrm{m}\left(-\frac{1}{\tau}\right).
\end{aligned} \tag{3.39}$$

This gives the image of Δ under an integer shift that sets it between $\mathcal{L}_\tau-1$ on the left and $\mathcal{L}_\tau$ on the right (see Fig. 1 for a picture of this domain, in yellow), while it remains undefined if $\Delta\in\mathcal{L}_\tau+\mathbb{Z}$. In other words, $[\Delta]_\tau$ is defined by the conditions

$$[\Delta]_\tau=\Delta \mod 1,\qquad \mathbb{I}\mathrm{m}\left(-\frac{1}{\tau}\right)>\mathbb{I}\mathrm{m}\left(\frac{[\Delta]_\tau}{\tau}\right)>0. \tag{3.40}$$

We also define

$$[\Delta]'_\tau=[\Delta]_\tau+1, \tag{3.41}$$

which gives the image of Δ under an integer shift that sets it between $\mathcal{L}_\tau$ on the left and $\mathcal{L}_\tau+1$ on the right, whenever Δ does not lie on $\mathcal{L}_\tau+\mathbb{Z}$. These functions satisfy

$$\begin{aligned}
&[\Delta+1]_\tau=[\Delta]_\tau,\qquad [\Delta+\tau]_\tau=[\Delta]_\tau+\tau,\\
&[-\Delta]_\tau=-[\Delta]_\tau-1=-[\Delta]'_\tau.
\end{aligned} \tag{3.42}$$

In particular, $\sum_{a=1}^3\Delta_a-2\tau\in\mathbb{Z}$ implies $[\Delta_3]_\tau=2\tau-1-[\Delta_1+\Delta_2]_\tau$, and multiplying by m also implies

$$[m\Delta_3]_{\check{\tau}}=2\check{\tau}-1-[m\Delta_1+m\Delta_2]_{\check{\tau}}. \tag{3.43}$$

The strategy is thus to perform integer shifts of the $m\Delta_a$s in the arguments of γ_Δ and b_N in such a way that we land inside the range (3.37). This is always possible as long as none of the $m\Delta_a$ is exactly on the line $\mathcal{L}_{\check{\tau}}$ or one of its images under integer shifts. This allows us to write the contribution from the HL solution $\{m,n,r\}$ as

$$\log(\mathcal{I}) = \mathcal{P}([m\Delta]_{\check\tau},\check\tau) + \log(N) + 2\log\left[\frac{(\check q^n;\check q^n)_\infty^N}{(q;q)_\infty}\right] - N\log(\check\tau) - (N-1)\log[b_N([m\Delta]_{\check\tau},\check\tau)]$$
$$+\sum_{a=1}^{3}\left[-N\log\left[\theta_0\left(\frac{n[m\Delta_a]_{\check\tau}}{\check\tau};-\frac{n}{\check\tau}\right)\right] - \log\left(\tilde\Gamma(\Delta_a;\tau,\tau)\right) + m\sum_{k=0}^{\infty}\log\left(\frac{\psi(\frac{k+1+[m\Delta_a]_{\check\tau}}{\check\tau/n})}{\psi(\frac{k-[m\Delta_a]_{\check\tau}}{\check\tau/n})}\right)\right]$$
$$+\frac{\pi i m}{6}\left(\sum_a[m\Delta_a]_{\check\tau} - 2\check\tau + \frac{3}{2}\right) - \log\left[\det\left(\mathbb{1} + \frac{b_N([m\Delta]_{\check\tau},\check\tau)^{-1}}{N}\mathcal{B}^{-1}\mathcal{C}\right)\right]. \tag{3.44}$$

Note that no shift has been done within the matrix $\mathcal{C}$ in the last term. Because of the constraint (3.28) between Δ_a and τ that defines Δ_3, the functions $\mathcal{P}$ and b_N simplify. Notice that, in general, $\mathbb{I}\mathrm{m}(-2/\check\tau) > \mathbb{I}\mathrm{m}(([m\Delta_1]_{\check\tau} + [m\Delta_2]_{\check\tau})/\check\tau) > 0$, and therefore one should distinguish between two cases in parameter space:

$$\begin{aligned}&\text{1st case:}\quad [m\Delta_1]_{\check\tau} + [m\Delta_2]_{\check\tau} = [m\Delta_1 + m\Delta_2]_{\check\tau}\\ &\text{2nd case:}\quad [m\Delta_1]_{\check\tau} + [m\Delta_2]_{\check\tau} = [m\Delta_1 + m\Delta_2]_{\check\tau} - 1.\end{aligned} \tag{3.45}$$

The second case is equivalent to $[m\Delta_1]'_{\check\tau} + [m\Delta_2]'_{\check\tau} = [m\Delta_1 + m\Delta_2]'_{\check\tau}$. In a more permutation-invariant way, they can be written as

$$\begin{aligned}&\text{1st case:}\quad \sum_{a=1}^{3}[m\Delta_a]_{\check\tau} - 2\check\tau + 1 = 0\\ &\text{2nd case:}\quad \sum_{a=1}^{3}[m\Delta_a]'_{\check\tau} - 2\check\tau - 1 = 0.\end{aligned} \tag{3.46}$$

The second case is also equivalent to $\sum_{a=1}^{3}[m\Delta_a]_{\check\tau} - 2\check\tau + 2 = 0$.

In the first case, with a little bit of algebra we find the large N limit

$$\boxed{\begin{aligned}\log(\mathcal{I}_{\{m,n,r\}}) = &-\tfrac{\pi i N^2}{m}\tfrac{[m\Delta_1]_{\check\tau}[m\Delta_2]_{\check\tau}[m\Delta_3]_{\check\tau}}{\check\tau^2} + \log(N) - \log(\mathcal{I}_{U(1)}) \qquad \text{1st case}\\ &-\log(\check\tau) + \tfrac{\pi i m}{12} - \log\left[\det\left(\mathbb{1} + \tfrac{b_N^{-1}}{N}\mathcal{B}^{-1}\mathcal{C}\right)\right] + \mathcal{O}(Ne^{-N}).\end{aligned}} \tag{3.47}$$

Here $\mathcal{I}_{U(1)}$ is the index of the free $U(1)$ theory (3.11) specialized to $\sigma = \tau$, that we can also write as

$$\mathcal{I}_{U(1)}|_{\sigma=\tau} = (q;q)_\infty^2\prod_{a=1}^{3}\tilde\Gamma(\Delta_a;\tau,\tau). \tag{3.48}$$

In (3.47) the first term is of order $\mathcal{O}(N^2)$, the second term is of order $\mathcal{O}(\log(N))$, all other terms[11] are of order $\mathcal{O}(1)$, and the corrections are exponentially small. In the second case we similarly find[12]

$$\boxed{\begin{aligned}\log(\mathcal{I}_{\{m,n,r\}}) = &-\tfrac{\pi i N^2}{m}\tfrac{[m\Delta_1]'_{\check\tau}[m\Delta_2]'_{\check\tau}[m\Delta_3]'_{\check\tau}}{\check\tau^2} + \log(N) - \log(\mathcal{I}_{U(1)}) \qquad \text{2nd case}\\ &-\log(\check\tau) + \pi i - \tfrac{\pi i m}{12} - \log\left[\det\left(\mathbb{1} + \tfrac{b_N^{-1}}{N}\mathcal{B}^{-1}\mathcal{C}\right)\right] + \mathcal{O}(Ne^{-N}).\end{aligned}} \tag{3.49}$$

Here we used the $2\pi i$ ambiguity of the logarithm and the fact that $e^{\pi i m n^2 - \pi i m n} = 1$.

[11] In this paper we have not explicitly computed the determinant, which we leave for future work, however see a partial analysis in Sec. III D.

[12] Notice that both in (3.47) and (3.49), b_N stands for $b_N([m\Delta]_{\check\tau},\check\tau)$.

Our result is compatible with previous works on the subject. In particular, the leading contribution from special subfamilies of HL solutions already appeared in [15,16], while the correction at order $\mathcal{O}(\log(N))$ was already found in [21], both for the basic solution as well as for the $\{m,n,0\}$ subfamily of HL solutions.

C. Nonperturbative corrections

The terms that we reported in the large N limit expressions (3.47) and (3.49) are the leading $\mathcal{O}(N^2)$ contribution and the perturbative corrections (notice that the determinant gives rise to both perturbative and nonperturbative corrections). Here we would like to estimate the nonperturbative corrections $\mathcal{O}(Ne^{-N})$. More precisely, we will compute the leading exponent—or instanton action—that controls the nonperturbative corrections, while leaving a more detailed analysis for future work.

In (3.47) and (3.49), the missing terms are

$$\log(\mathcal{I}_{\{m,n,r\}}) = \begin{pmatrix} 3.47 \\ 3.49 \end{pmatrix} + 2N\log((\check{q}^n;\check{q}^n)_\infty) - (N-1)\log\left[1 \pm \sum_{a=1}^{3}\mathcal{G}\left(0,\frac{[m\Delta_a]_{\check{\tau}}}{\check{\tau}/n};-\frac{n}{\check{\tau}}\right)\right] - \sum_{a=1}^{3}\left[N\log\left(\theta_0\left(\frac{[m\Delta_a]_{\check{\tau}}}{\check{\tau}/n};-\frac{n}{\check{\tau}}\right)\right) - m\sum_{k=0}^{\infty}\log\left(\frac{\psi(\frac{k+1+[m\Delta_a]_{\check{\tau}}}{\check{\tau}/n})}{\psi(\frac{k-[m\Delta_a]_{\check{\tau}}}{\check{\tau}/n})}\right)\right], \tag{3.50}$$

where the two equation numbers and the $\pm$ signs in the first line correspond to the first and second case, respectively. These nonperturbative corrections are needed in order to make the right-hand sides of (3.47) and (3.49) an analytic function of the chemical potentials for finite N, even though $[m\Delta_a]_{\check{\tau}}$ are not analytic functions. By expanding the various functions above using the formulas in Appendix A, we find that the nonperturbative corrections have three kinds of exponents:

$$\exp\left(\frac{2\pi i N}{m}k\frac{[m\Delta_a]_{\check{\tau}}-\ell}{\check{\tau}}\right), \qquad \exp\left(\frac{2\pi i N}{m}k\frac{-[m\Delta_a]_{\check{\tau}}-1-\ell}{\check{\tau}}\right), \qquad \exp\left(-\frac{2\pi i N}{m}\frac{k}{\check{\tau}}\right), \tag{3.51}$$

for integers $k \geq 1$ and $\ell \geq 0$. All these exponents can be written as products of

$$\exp\left(\frac{2\pi i N}{m}\frac{[m\Delta_a]_{\check{\tau}}}{\check{\tau}}\right), \qquad \exp\left(-\frac{2\pi i N}{m}\frac{[m\Delta_a]_{\check{\tau}}+1}{\check{\tau}}\right), \qquad \exp\left(-\frac{2\pi i N}{m}\frac{1}{\check{\tau}}\right). \tag{3.52}$$

In the first case of parameter space, taking into account the relation (3.46) among the chemical potentials, one finds that in fact all exponents can be written as products of

$$\exp\left(\frac{2\pi i N}{m}\frac{[m\Delta_a]_{\check{\tau}}}{\check{\tau}}\right) \quad \text{for } a = 1,2,3. \tag{3.53}$$

Notice that because of the property (3.40) of the function $[]_\tau$, all nonperturbative corrections are exponentially small in the large N limit. In the second case of parameter space, instead, all exponents can be written as products of

$$\exp\left(-\frac{2\pi i N}{m}\frac{[m\Delta_a]'_{\check{\tau}}}{\check{\tau}}\right) \quad \text{for } a = 1,2,3. \tag{3.54}$$

Again, all nonperturbative corrections are exponentially small.

D. Perturbative corrections

The perturbative corrections (i.e., the perturbative expansion in $1/N$) to the leading $\mathcal{O}(N^2)$ contribution to $\log(\mathcal{I}_{\{m,n,r\}})$ come from the terms in (3.47) and (3.49). We were not able to evaluate the determinant explicitly, however we collected some evidence that—at least in the case $m = 1$—the determinant is of order $\mathcal{O}(1)$ plus nonperturbative corrections of order $\mathcal{O}(e^{-N})$, but without further perturbative corrections of order $\mathcal{O}(1/N)$ or smaller. We thus conjecture that the series of perturbative corrections stops at order $\mathcal{O}(1)$.

Our strategy is to expand

$$-\log\left[\det\left(\mathbb{1}+\frac{\mathcal{B}^{-1}\mathcal{C}}{b_N N}\right)\right] = \sum_{\ell=1}^{\infty}\frac{(-1)^\ell}{\ell}\mathrm{Tr}\left[\left(\frac{\mathcal{B}^{-1}\mathcal{C}}{b_N N}\right)^\ell\right] \tag{3.55}$$

and compute the first few terms on the right-hand side. The details of the computation are in Appendix B 4. At first order, in the large N limit we find

$$\frac{\mathrm{Tr}(\mathcal{B}^{-1}\mathcal{C})}{b_N N} = 1 \pm \check{\tau}\left(1-\sum_{a=1}^{3}\mathcal{G}(0,\Delta_a;\tau)\right) + \mathcal{O}(e^{-N}), \tag{3.56}$$

where the $\pm$ signs refer to the first and second case of parameter space, respectively. At second order the computation becomes considerably more involved, and we only performed it in the case $m = 1$, i.e., for the T-transformed solutions. Setting $\check{\tau} = \tau + r$, in the large N limit we find

$$\frac{\mathrm{Tr}(\mathcal{B}^{-1}\mathcal{C}\mathcal{B}^{-1}\mathcal{C})}{b_N^2 N^2} = \frac{1}{2}\sum_{a,b=1}^{3}\left[\tilde{\gamma}\left(\frac{[\Delta_a]_{\check{\tau}} + [\Delta_b]_{\check{\tau}}}{\check{\tau}}; -\frac{1}{\check{\tau}}\right) - \tilde{\gamma}\left(\frac{[\Delta_a]_{\check{\tau}} - [\Delta_b]'_{\check{\tau}}}{\check{\tau}}; -\frac{1}{\check{\tau}}\right)\right] + \mathcal{O}(e^{-N}). \tag{3.57}$$

Here we introduced the function

$$\tilde{\gamma}(u;\tau) \equiv \frac{1}{2\pi i}\partial_u \log(\tilde{\Gamma}(u;\tau,\tau)), \tag{3.58}$$

see also (B49). Both at first and second order (for $m = 1$), the perturbative series in $1/N$ stops at order $\mathcal{O}(1)$, the only large N corrections being nonperturbative.

E. The universal superconformal index

It is interesting to consider also a special case of the superconformal index (2.6), which is a simpler and more universal object. This is constructed by only using charges in the $\mathcal{N} = 1$ superconformal subalgebra,[13] and it is defined as

$$\mathcal{J}(p,q) = \mathrm{Tr}[e^{-\pi i r} e^{-\beta\{\mathcal{Q},\bar{\mathcal{Q}}\}} e^{2\pi i\sigma(J_1+\frac{r}{2})} e^{2\pi i\tau(J_2+\frac{r}{2})}], \tag{3.59}$$

where r is the superconformal R symmetry in the $\mathcal{N} = 1$ subalgebra. Since different states in the same supermultiplet have R charges that differ by integers and the supercharge $\mathcal{Q}$ has charge $r = 1$, $\mathcal{J}$ has all the good properties of a supersymmetric index. When the R charges are rational, this index will have some periodicity in the σ, τ plane.

For $\mathcal{N} = 4$ SYM, this special case corresponds to choosing

$$\Delta_1 = \Delta_2 = \Delta_3 \equiv \Delta = \frac{\sigma + \tau - 1}{3}. \tag{3.60}$$

To simplify even further we will take $\sigma = \tau$, which means $2\tau - 3\Delta = 1$. The corresponding index

$$\mathcal{J}(q) = \mathrm{Tr}[e^{-\pi i r} e^{-\beta\{\mathcal{Q},\bar{\mathcal{Q}}\}} e^{2\pi i\tau(J_1+J_2+r)}] \tag{3.61}$$

is periodic under $\tau \to \tau + 3$, and it is defined for $\mathbb{I}\mathrm{m}(\tau) > 0$.

As in Sec. III B we are interested in the contribution[14] from HL solutions $\{m, n, r\}$, in a large N limit in which m, r are kept fixed while $n \to \infty$. This means applying the restriction (3.60) to the large N expressions (3.47) and (3.49) in the first and second region of parameter space, respectively. One easily computes

$$[m\Delta]_{\check{\tau}} = \begin{cases} \text{undefined} & \text{if } m + 2r = 0 \mod 3 \\ \frac{2\check{\tau}-1}{3} & \text{if } m + 2r = 1 \mod 3 \\ \frac{2\check{\tau}-2}{3} & \text{if } m + 2r = 2 \mod 3 \end{cases}. \tag{3.62}$$

The undefined case arises because for these chemical potentials $m\Delta$ happens to sit precisely on one of the images of $\mathcal{L}_{\check{\tau}}$ under integer shifts. For $m + 2r = 1 \mod 3$ the solution sits in the first case of parameter space, according to (3.45), while for $m + 2r = 2 \mod 3$ it sits in the second case. It follows that the leading behavior of the index is

$$\log(\mathcal{I}_{\{m,n,r\}}) \simeq \begin{cases} \text{undefined} & \text{if } m + 2r = 0 \mod 3 \\ -\frac{\pi i N^2}{27m}\frac{(2\check{\tau}-1)^3}{\check{\tau}^2} & \text{if } m + 2r = 1 \mod 3 \quad (1\text{st case}) \\ -\frac{\pi i N^2}{27m}\frac{(2\check{\tau}+1)^3}{\check{\tau}^2} & \text{if } m + 2r = 2 \mod 3 \quad (2\text{nd case}) \end{cases}. \tag{3.63}$$

The undefined case should be interpreted as the fact that there are two exponentially large competing contributions to $\mathcal{I}_{\{m,n,r\}}$ (coming from the first and second case solutions when we slightly shift the chemical potentials in one direction or the other) with the same absolute value, but with a relative phase that oscillates very rapidly.

IV. FIVE-DIMENSIONAL BLACK HOLES

In this section we review the gravitational solutions that contribute to the $S^1 \times S^3$ partition function of the $\mathcal{N} = 4$ SYM theory, for supersymmetric values of the chemical potentials, and compare their contributions to what we obtained from the index. Since we work at large N on the field theory side, the gravitational side is semiclassical and its partition function should be a sum over all the classical solutions that satisfy the appropriate boundary conditions for our chemical potentials.[15] Our field theory computation is independent of the 't Hooft coupling constant; on the gravity side it will be useful to work at large 't Hooft coupling, where we can approximate type IIB string theory by

[13] All choices are physically equivalent.

[14] The contribution from the T-transformed solutions $\{1, N, r\}$ was already analyzed in [7].

[15] With perturbative corrections given by the fluctuations around them, which we will not consider in this work, and with nonperturbative corrections that we will discuss in the next section.

type IIB supergravity,[16] and just look for appropriate solutions of this theory, which are asymptotically $AdS_5 \times S^5$.

The rules of the AdS/CFT correspondence tell us how to find gravitational solutions when the chemical potentials in (2.2) correspond to real boundary conditions for the bulk fields in Euclidean space. Unfortunately, as discussed below, for generic values of the chemical potentials this is not consistent with (2.8), so for most supersymmetric configurations some of the chemical potentials are complex, and we do not know what are the rules for constructing solutions dual to such complex chemical potentials. We will begin by dealing with this issue in the most naïve possible way, by assuming that one can analytically continue the results computed for real fields (with real boundary conditions) to complex fields (with complex boundary conditions).[17] We will find that this will lead to some problems, and we will suggest how to resolve them in the following sections.

In general, finding ten-dimensional black hole solutions is complicated. Luckily, type IIB supergravity has several known consistent truncations, with known black hole solutions that satisfy our boundary conditions. The most general black hole solution has two angular momenta J_1, J_2 and three electric charges Q_1, Q_2, Q_3, where we defined $Q_a = R_a/2$. The simplest consistent truncation is to 5D $\mathcal{N} = 2$ minimal gauged supergravity, which contains just a single gauge field, and which can be used to describe solutions with $Q_1 = Q_2 = Q_3 \equiv Q$ (or, equivalently, with equal chemical potentials $\Phi_1 = \Phi_2 = \Phi_3$). In this section we use this truncation to consider these solutions, further limiting ourselves, for simplicity, to the case of equal angular momenta $J_1 = J_2 \equiv J$ (or, equivalently, $\Omega_1 = \Omega_2$). A detailed discussion of generalizations to black holes with more generic charges (or chemical potentials) is left to Appendices C and D. The analysis of this section mostly follows [5].

A. Black hole solutions

The bosonic fields of 5D $\mathcal{N} = 2$ minimal gauged supergravity are the metric $g_{\mu\nu}$ and a $U(1)$ gauge field A_μ. The bosonic Lagrangian is[18]

$$\mathcal{L} = (R + 12g^2) * 1 - \frac{2}{3} F \wedge *F + \frac{8}{27} F \wedge F \wedge A, \tag{4.1}$$

such that the radius of the vacuum AdS_5 solution is $1/g$. In the following we will set the dimensionful constant g to 1. Specializing to the case of equal angular momenta $a = b$, the black hole solutions found in [88] in asymptotically global AdS_5 (with boundary $\mathbb{R} \times S^3$) are given by

$$\begin{aligned} ds^2 &= -\frac{(1+r^2)}{\Xi_a} dt^2 - \frac{2q}{\Xi_a \rho^2} \nu \varpi + \frac{f_t}{\Xi_a^2 \rho^4} \varpi^2 + \frac{\rho^2}{\Delta_r} dr^2 \\ &\quad + \frac{\rho^2}{\Xi_a} (d\theta^2 + \sin^2(\theta) d\phi^2 + \cos^2(\theta) d\psi^2), \\ A &= \frac{3q}{2\rho^2 \Xi_a} \varpi - \alpha dt, \end{aligned} \tag{4.2}$$

where we defined

$$\begin{aligned} &\nu = a(\sin^2(\theta) d\phi + \cos^2(\theta) d\psi), \qquad \varpi = dt - \nu, \\ &\Delta_r = \frac{\rho^4(1+r^2) + q^2 + 2a^2 q}{r^2} - 2m, \qquad \rho^2 = r^2 + a^2, \\ &f_t = 2(m + a^2 q)\rho^2 - q^2, \qquad \Xi_a = 1 - a^2. \end{aligned} \tag{4.3}$$

The coordinates ϕ, ψ have period 2π (with the fermions antiperiodic under this shift), while $\theta \in [0, \pi/2]$. The solution depends on three parameters (a, q, m) (and on α, which is arbitrary in Lorentzian signature).

The black hole has an outer horizon at the largest positive root of $\Delta_r(r_+) = 0$, denoted by r_+. One can then solve for m and find

$$m = \frac{(r_+^2 + a^2)^2 (1 + r_+^2) + q^2 + 2a^2 q}{2 r_+^2}, \tag{4.4}$$

which can be used to eliminate m and use r_+ instead.

The horizon is a Killing horizon generated by the Killing vector field

$$V = \frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} + \Omega \frac{\partial}{\partial \psi}, \tag{4.5}$$

where

$$\Omega = \frac{a(r_+^2 + a^2)(1 + r_+^2) + aq}{(r_+^2 + a^2)^2 + a^2 q}. \tag{4.6}$$

Evaluating the surface gravity, the Hawking temperature is

$$T \equiv \frac{1}{\beta} = \frac{r_+^4 [1 + 2(r_+^2 + a^2)] - (a^2 + q)^2}{2\pi r_+ [(r_+^2 + a^2)^2 + a^2 q]}. \tag{4.7}$$

The electrostatic potential at the horizon is

[16] The field theory analysis tells us that the result should be the same for any value of the 't Hooft coupling, and thus it should be independent of the string scale, but we do not know how to see this directly on the gravity side.

[17] More precisely, the results are computed for a real metric, but for Euclidean charged black holes there is an imaginary gauge field. This still leads to a real action.

[18] The Lagrangian used in [88,89] is related to the one used here by $\lambda_{\text{there}} = -g^2$ and $A_{\text{there}} = \frac{2}{\sqrt{3}} A$. The normalization used here has the advantage that A couples canonically to the superconformal R-symmetry current, however recall that the R charges of scalar gauge-invariant operators are multiples of $2/3$.

$$\Phi \equiv \iota_V A|_{r_+} - \iota_V A|_\infty = \frac{3qr_+^2}{2[(r_+^2+a^2)^2+a^2q]}. \tag{4.8}$$

For our purposes we would like to continue these solutions to Euclidean signature and to compactify the Euclidean time direction, such that the boundary is $S^1_\beta \times S^3$. Following [5], these solutions can be analytically continued to real Euclidean solutions by rotating $t \to -it_{\rm E}$, if we also take $a \to i\tilde{a}$. In terms of the original a, which is now purely imaginary, and the real-valued m and q, one finds that the Euclidean metric is real, A is purely imaginary (if $\alpha \in \mathbb{R}$), $\iota_V A$ is real, β is real, and Φ is real, while Ω is purely imaginary. We assume that we can rotate the integration over the gauge fields from real to imaginary fields without encountering any problems.

Expansion near the horizon.—In the Euclidean solutions we need to periodically identify the Euclidean time direction $t_{\rm E}$, and make sure that the solutions are smooth when this cycle shrinks at the horizon. We expand the metric around $r \gtrsim r_+$, setting $R^2 = r - r_+$. The near-horizon metric takes the following form:

$$\begin{aligned} ds^2 = {} & h_{RR}\left(dR^2 + R^2\left(\frac{2\pi}{\beta}dt_{\rm E}\right)^2\right) + h_{\theta\theta}d\theta^2 \\ & + h_{\phi\phi}(d\phi + i\Omega dt_{\rm E})^2 + h_{\psi\psi}(d\psi + i\Omega dt_{\rm E})^2 \\ & + 2h_{\phi\psi}(d\phi + i\Omega dt_{\rm E})(d\psi + i\Omega dt_{\rm E}). \end{aligned} \tag{4.9}$$

Here h_{RR}, $h_{\theta\theta}$, $h_{\phi\phi}$, $h_{\psi\psi}$, $h_{\phi\psi}$, Ω are functions of R and θ with nonvanishing limits for $R \to 0$, which depend on the parameters a, q, r_+. The limit of Ω is the value given in (4.6).

We see that the geometry is smooth around r_+ if we make the following identifications of the coordinates:

$$\begin{aligned} (t_{\rm E}, \phi, \psi) & \cong (t_{\rm E} + \beta, \phi - i\Omega\beta, \psi - i\Omega\beta) \\ & \cong (t_{\rm E}, \phi + 2\pi, \psi) \cong (t_{\rm E}, \phi, \psi + 2\pi). \end{aligned} \tag{4.10}$$

Moreover, the Killing vector field V in (4.5) is the one that generates rotations of the circle that shrinks. The identifications (4.10) take the same form as (2.4), so we anticipate that the chemical potentials for the angular momenta will be realized, in our coordinate system for the black hole solutions, by modifying the coordinate identifications rather than by changing the CFT metric.

Regularity of the gauge field at $r = r_+$ requires (in a gauge that is regular at the horizon)

$$\iota_V A|_{r_+} = 0 \quad \Rightarrow \quad \alpha = \Phi, \tag{4.11}$$

because the Wilson line around the shrinking circle should be trivial at $r = r_+$.

Now that we have smooth Euclidean solutions with the identification (4.10), we can compute their on shell action. As usual in holography, this requires a regularization for large r, and the computation was done in [5] using background subtraction[19] (which assigns vanishing action to AdS_5). It has the form

$$I_{\rm SUGRA} = \frac{\pi\beta}{4\Xi_a^2}\left[m - (r_+^2+a^2)^2 - \frac{q^2r_+^2}{(r_+^2+a^2)^2+a^2q}\right]. \tag{4.12}$$

Note that $I_{\rm SUGRA}$ satisfies the so-called quantum statistical relation

$$I_{\rm SUGRA} = \beta E - S - 2\beta\Omega J - \beta\Phi Q, \tag{4.13}$$

where E is the energy and S is the entropy of the black hole.

As mentioned above, for supersymmetric solutions we generally need the action for complex values of the chemical potentials, and we assume that we can extend the results above by analyticity to the full space of complex parameters (a, q, r_+). As discussed in Sec. II A, in the continued solutions ϕ and ψ are complex, but their tilded versions (2.5) obey standard identifications and remain real (but with a complex metric). *A priori* it is not clear whether the path integral with complex chemical potentials indeed receives contributions from such analytically continued bulk solutions or not, and we will return to this issue below.

Expansion near the boundary.—In order to precisely identify which chemical potentials the Euclidean solutions above correspond to, we need to analyze the behavior of the solutions near the boundary. The asymptotic $r \to \infty$ limit of the metric (4.2) can be written as

$$ds^2 = \frac{d\hat{r}^2}{\hat{r}^2} + \hat{r}^2 ds^2_{\rm bdry} + \mathcal{O}(\hat{r}^0), \tag{4.14}$$

where $\Xi_a \hat{r}^2 = r^2$ and

$$ds^2_{\rm bdry} = dt_{\rm E}^2 + d\theta^2 + \sin^2(\theta)d\phi^2 + \cos^2(\theta)d\psi^2 \tag{4.15}$$

is the round metric on $S^1_\beta \times S^3$ but still with the twisted identifications (4.10). So, as expected from the discussion of Sec. II A on the "coordinate shift realization" of the chemical potentials, the CFT metric remains the same, and the parameter Ω affects the boundary only through the identifications (4.10). In addition, we find that the gauge field near the boundary $r \to \infty$ is

$$A_{\rm bdry} = i\Phi dt_{\rm E}, \tag{4.16}$$

[19]The on shell action was computed more carefully using holographic renormalization in [69], and was found to have an extra constant term $I_0 = \beta E_0 = \frac{3\pi\beta}{32g^2G_5}$, where I_0 is the action of thermal AdS_5.

corresponding to a chemical potential for the $U(1)$ global symmetry.

Shifted chemical potentials.—Note that shifts of Ω by $\frac{2\pi i}{\beta}\mathbb{Z}$ do not change the global identification:

$$\begin{aligned}(t_E,\phi,\psi) &\cong \left(t_E+\beta,\phi-i\left(\Omega+\frac{2\pi i\mathbb{Z}}{\beta}\right)\beta,\right.\\ &\quad \left.\psi-i\left(\Omega+\frac{2\pi i\mathbb{Z}}{\beta}\right)\beta\right),\\ &= (t_E+\beta,\phi-i\Omega\beta+2\pi\mathbb{Z},\psi-i\Omega\beta\\ &\quad +2\pi\mathbb{Z}) \cong (t_E+\beta,\phi-i\Omega\beta,\psi-i\Omega\beta),\end{aligned} \tag{4.17}$$

and thus they do not change the behavior of our solutions near the boundary. Thus, we actually found a family of black hole backgrounds with the same boundary metric and topology but with different bulk metrics (corresponding to the shifted values of Ω). Noting that solutions with angular momenta J_1 and J_2 have wave functions $e^{-iJ_1\phi-iJ_2\psi}$ and thus acquire a factor $e^{J_1\Omega\beta+J_2\Omega\beta}$ under the identification (4.17), and, comparing to (2.2), we see that our black hole configurations contribute to the partition function with

$$\Omega_1=\Omega_2=\Omega. \tag{4.18}$$

The CFT partition function itself is periodic under shifts of Ω by $\frac{2\pi i}{\beta}$, and this arises on the gravity side from summing over all these shifted solutions.

Similarly, if we consider a configuration with some charges R_1, R_2, R_3, recalling that the gauge field couples to $r=(R_1+R_2+R_3)/3$, the boundary condition (4.16) gives us a factor of $e^{\Phi\beta(R_1+R_2+R_3)/3}$ when we go around the circle. So these black holes contribute to the partition function (2.2) with

$$\Phi_1=\Phi_2=\Phi_3=\frac{2}{3}\Phi. \tag{4.19}$$

Note that since scalar operators in the bulk (which are periodic under the shift around the Euclidean circle) have R charges which are multiples of $2/3$, shifts of Φ by $\frac{3\pi i}{\beta}\mathbb{Z}$ correspond to large gauge transformations near the boundary (this is consistent also with the behavior of fermions, which are antiperiodic under the shift). So, as in our discussion above of the chemical potentials for angular momenta, configurations that differ by these shifts all contribute to the same partition function (2.2).

All in all, if we consider the partition function (2.2) with these values of Ω_i and Φ_a, it gets contributions from all the solutions with $\Omega=\Omega_1+\frac{2\pi i}{\beta}\mathbb{Z}$ and with $\Phi=\frac{3}{2}\Phi_1+\frac{3\pi i}{\beta}\mathbb{Z}$. In Appendices C and D we consider more general bulk solutions, with not necessarily equal angular momenta or $U(1)$ charges (and corresponding chemical potentials). In this context, contributions to (2.2) for specific values (equal or not) of the chemical potentials come from a 5-parameter infinite series of gravitational solutions, in which each of Ω_1, Ω_2, Φ_1, Φ_2, and Φ_3 may be independently shifted by $\frac{2\pi i}{\beta}\mathbb{Z}$. Assuming that all of these complexified solutions contribute to the partition function, in the supergravity approximation we can write

$$Z_{S^1\times S^3}(\Omega_1,\Omega_2,\Phi_1,\Phi_2,\Phi_3)=\sum_{n_1,n_2,m_1,m_2,m_3\in\mathbb{Z}} e^{-I_{\mathrm{SUGRA}}(\Omega_1+\frac{2\pi i}{\beta}n_1,\Omega_2+\frac{2\pi i}{\beta}n_2,\Phi_1+\frac{2\pi i}{\beta}m_1,\Phi_2+\frac{2\pi i}{\beta}m_2,\Phi_3+\frac{2\pi i}{\beta}m_3)}, \tag{4.20}$$

where the sum runs over all integers n_1, n_2, m_1, m_2, m_3 whose sum is even, in order for the periodicity to be consistent also with the behavior of fermions in the bulk, as in the QFT discussion around (2.3).

B. Supersymmetry

The solutions described above, as well as their analytic continuations to complex parameters, are supersymmetric if their parameters are related as

$$q=\frac{m}{1+2a}. \tag{4.21}$$

This gives a two-parameter family of solutions in terms of (a,m). In terms of the equivalent set of parameters (a,r_+), the supersymmetry condition becomes

$$q=-a^2+(1+2a)r_+^2\mp ir_+(r_+^2-r_*^2), \tag{4.22}$$

where $r_*^2=2a+a^2$ and the $\mp$ signs correspond to the two branches of a square root. For the upper sign, the expression above equals

$$q=-(a-ir_+)^2(1-ir_+). \tag{4.23}$$

We refer to the corresponding family of SUSY solutions as the first branch. On the other hand, for the lower sign the expression is obtained by sending $i\to -i$, and we call it the second branch. In the following we will focus on the first SUSY branch (4.23), but we will sometimes mention also the results for the second branch.

In the supersymmetric cases, parametrized by (a,r_+), the chemical potentials become

$$\Phi = \frac{3ir_+(1-ir_+)}{2(r_*^2+ir_+)}, \qquad \beta = \frac{2\pi i a(a-ir_+)(r_*^2+ir_+)}{(r_*^2-3iar_+)(r_*^2-r_+^2)},$$
$$\Omega = \frac{(r_*^2+iar_+)(1-ir_+)}{(r_*^2+ir_+)(a-ir_+)}. \tag{4.24}$$

They satisfy

$$\beta(1+2\Omega-2\Phi) = 2\pi i. \tag{4.25}$$

In the Euclidean solutions there is a globally well-defined Killing spinor, which changes sign when going around the Euclidean time cycle, whenever we have $(\pm 2\pi i)$ on the right-hand side. This is consistent with (4.25) and its counterpart on the second branch. Note that except in the special case $\Phi = \frac{1}{2}$, we cannot have Φ and β real and also Ω imaginary in supersymmetric solutions.

In order to relate to the index, following the discussion of Sec. II, we can map the chemical potentials of our gravitational configurations to index parameters using (2.12) (on both branches), namely[20]

$$\sigma_g = \tau_g = \frac{\beta(\Omega-1)}{2\pi i}, \qquad \Delta_g = \frac{\beta(\Phi-\frac{3}{2})}{3\pi i}. \tag{4.26}$$

Such a gravitational configuration contributes to the index with $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_g$. The index parameters on the first branch are given by

$$\sigma_g = \tau_g = \frac{a(1-a)}{r_*^2-3iar_+}, \qquad \Delta_g = -\frac{a(a-ir_+)}{r_*^2-3iar_+}. \tag{4.27}$$

Note that they satisfy

$$2\tau_g - 3\Delta_g = 1 \qquad \text{(on 1st branch)}. \tag{4.28}$$

Thus, one combination of (a, r_+) controls τ_g, Δ_g while another one controls β. On the second SUSY branch, the chemical potentials satisfy $2\tau_g - 3\Delta_g = -1$. The on shell action (4.12) becomes

$$I_{\text{SUGRA}} = -\frac{i\pi^2}{2}\frac{a(a-ir_+)^3}{(1-a)^2(r_*^2-3iar_+)} = \frac{i\pi^2}{2}\frac{\Delta_g^3}{\tau_g^2}. \tag{4.29}$$

The rightmost expression is valid on both branches. Notice that it does not depend on β.

C. Comparison to field theory results

We can rewrite the on shell action (4.29) in terms of the field theory variables. We should reinstate dimensions using the relation (at leading order in $1/N$)

$$\frac{1}{g^3 G_5} = \frac{8c}{\pi} = \frac{2N^2}{\pi}, \tag{4.30}$$

where $a = c$ is the field theory central charge, while G_5 is the five-dimensional Newton constant. Thus the logarithm of the classical contribution of these solutions to the partition function is (in the first branch)

$$\log(Z) = -I_{\text{SUGRA}} = -\frac{1}{g^3 G_5}\frac{i\pi^2}{2}\frac{\Delta_g^3}{\tau_g^2} = -\pi i N^2 \frac{\Delta_g^3}{\tau_g^2}$$
$$= -\frac{\pi i N^2}{27}\frac{(2\tau_g-1)^3}{\tau_g^2}. \tag{4.31}$$

This is exactly the contribution (3.63) of the basic Bethe Ansatz solution $\{1, N, 0\}$ that falls in the first case. Note that, indeed, (4.28) ensures that our parameters always satisfy the field theory conditions for this first case, namely $\Delta_g = [\Delta_g]_{\tau_g}$.

Similarly, the value of Δ_g that we obtain on the second branch of solutions, which is $\Delta_g = (2\tau_g+1)/3$, always satisfies the condition for the second case in our analysis of the basic Bethe Ansatz solution, $\Delta_g = [\Delta_g]'_{\tau_g}$, and also the action of our solution is consistent with this case,

$$\log(Z) = -I_{\text{SUGRA}} = -\pi i N^2 \frac{\Delta_g^3}{\tau_g^2} = -\frac{\pi i N^2}{27}\frac{(2\tau_g+1)^3}{\tau_g^2}, \tag{4.32}$$

which coincides with the contribution (3.63) of the solution $\{1, N, 2\}$ (if we make the identification $\tau_g = \check{\tau} = \tau + 2$).

So, the gravity results agree nicely with the basic Bethe Ansatz solution. Moreover, recall from our discussion above that shifts of Ω and Φ, which correspond to shifts of τ_g and Δ_g by integers, give new solutions which also contribute to the same index. However, we need to still satisfy the constraint (4.28). So, if we start (for instance) from some value of τ_g and Δ_g which satisfies (4.28) on the first branch, we can shift $\tau_g \to \tau_g + 3n$ and $\Delta_g \to \Delta_g + 2n$ and obtain another solution contributing to the same index; it is easy to see that the contribution of this is precisely that of the shifted Bethe Ansatz solution $\{1, N, 3n\}$—which is in the "first case."[21] Moreover, starting from the same values, if we shift $\tau_g \to \tau_g + 3n - 1$, $\Delta_g \to \Delta_g + 2n$ we obtain a SUSY solution on the second branch which also contributes to the same index, and which precisely reproduces the contribution of the shifted Bethe Ansatz solution $\{1, N, 3n-1\}$—which is in the "second case." So, the sum

[20]Recall that $\Phi_1 = \Phi_2 = \Phi_3 = \frac{2}{3}\Phi$, justifying the slightly different definition of Δ_g with respect to (2.12).

[21]Note that n here can be positive or negative. In our analysis of shifted Bethe Ansatz solutions we took $r = 0, \ldots, N-1$ and assumed that r does not scale with N. However, our analysis there works equally well for negative r which does not scale with N, by identifying it with $(N + r)$.

over gravitational solutions contributing to the same index precisely reproduces the sum over two thirds of the Bethe Ansatz solutions with $m = 1$. The leftover solutions are problematic since they lie on Stokes lines, as discussed in Sec. III E, but as in that section, this problem does not arise for generic (unequal) chemical potentials.

However, before we declare success in our matching with the Bethe Ansatz solutions with $m = 1$ (the solutions with $m > 1$ will be discussed later in Sec. VII), we should be more careful. So far in this subsection we only considered shifts of the solutions which retain equality of the angular momenta and of the $U(1)$ charges. However, the most general solution has three different chemical potentials—parametrized by $\Delta_{g,a} = \frac{\beta}{2\pi i}(\Phi_a - 1)$—and two different angular chemical potentials σ_g, τ_g. Each of these can be shifted independently by integers without affecting the boundary conditions, so we should sum over all these additional solutions as well, if they are supersymmetric. The action of the more general solutions is[22]

$$I_{\mathrm{SUGRA}}(\sigma_g, \tau_g, \Delta_{g,1}, \Delta_{g,2}, \Delta_{g,3}) = \pi i N^2 \frac{\Delta_{g,1}\Delta_{g,2}\Delta_{g,3}}{\sigma_g \tau_g}, \tag{4.33}$$

and they are supersymmetric whenever

$$\sigma_g + \tau_g - \Delta_{g,1} - \Delta_{g,2} - \Delta_{g,3} = \pm 1, \qquad (1\mathrm{st}/2\mathrm{nd\,branch}). \tag{4.34}$$

So we have two four-parameter families of shifted solutions that all seem to contribute to the index (even when we happen to evaluate it for $\sigma = \tau$ and $\Delta_1 = \Delta_2 = \Delta_3$). Namely, it appears that in the supersymmetric case we should have contributions from supergravity of the form

$$Z_{S^1\times S^3}(\sigma, \tau, \Delta_1, \Delta_2) \stackrel{?}{=} \sum_{\substack{n_1,n_2,m_1,m_2\in\mathbb{Z} \\ s\in\{+1,-1\}}} e^{-I_{\mathrm{SUGRA}}(\sigma+n_1,\tau+n_2,\Delta_1+m_1,\Delta_2+m_2,\sigma+\tau-\Delta_1-\Delta_2-s+n_1+n_2-m_1-m_2)}. \tag{4.35}$$

However, the only shifts that show up in the Bethe Ansatz computation of the field theory index are the specific ones discussed above, which are related to the $\{1, N, r\}$ family.

One may wonder whether these new shifts correspond to as yet undiscovered BA solutions, or if they might cancel after appropriate resummation. This does not seem to be the case. Indeed, the contribution of some shifted backgrounds to the partition function actually diverges exponentially with the size of the shift, so they had better not contribute for some reason. Specifically, consider starting from the equal chemical potential case, and shift to $\Delta_{g,1} = \Delta_g + n$, $\Delta_{g,2} = \Delta_g + n$, $\Delta_{g,3} = \Delta_g - 2n$, consistently with (4.34). We then have for large $|n|$:

$$\mathbb{R}\mathrm{e}(I_{\mathrm{SUGRA}}) = \pi N^2\, \mathbb{I}\mathrm{m}\left(\frac{2n^3}{\tau_g^2}\right) + \mathcal{O}(n^2). \tag{4.36}$$

So, the contribution $e^{-I_{\mathrm{SUGRA}}}$ from these backgrounds to the partition function would diverge, either for very positive or for very negative n.

In the next sections we will suggest a resolution to the problem: that most of the shifted solutions (which are all complex valued) should not be included in the sum over solutions, because they are unstable towards the condensation of D3-branes, and that when evaluating the gravitational partition function, only stable contributions should be considered. Presumably this criterion can be justified by a careful analysis of the analytic continuation to complex-valued solutions—in particular through the study of Lefschetz thimbles—but we will not attempt to do this here. We will show below that with that criterion, the acceptable shifted solutions (for arbitrary chemical potentials) precisely match the $m = 1$ shifted solutions that contribute to the index.

[22]This is shown for various cases in Appendices C and D, and we assume here that the general form holds.

V. WRAPPED D3-BRANES

Up to now we considered the classical on shell action of our Euclidean solutions. In general, the action receives quantum corrections coming from loops of the gravity fields in these backgrounds. However, we can also have additional nonperturbative corrections coming from wrapped D-branes. It turns out (see Appendix C 3) that wrapped Euclidean D3-branes can be added to our backgrounds while still preserving the same supersymmetry, and without changing the boundary conditions, so that they give nonperturbative corrections to the contributions discussed in the previous section. In this section we compute those corrections for a specific class of wrapped D3-branes and analyze their consequences.

A. Uplift to ten dimensions

In order to analyze configurations with D-branes, we must first find the 10D solution that corresponds to our 5D black holes. Luckily, [90] discusses the embedding of 5D supergravity with $U(1)^3$ gauge symmetry—the so-called STU model—into 10D type IIB supergravity, and the uplift of solutions of the former into the latter. In our context we view the action (4.1) as the $U(1)^3$ action with all three

gauge fields taken to be equal.[23] The uplift of solutions with unequal angular momenta or charges is described in Appendices C and D.

The 10D metric for the equal-charge solutions is given by [90]

$$ds_{10}^2 = ds_5^2 + \sum_{a=1}^{3}\left(d\mu_a^2 + \mu_a^2\left(d\phi_a + \frac{2}{3}A\right)^2\right), \quad (5.1)$$

where ds_5^2 is the 5D solution (4.2), A is the gauge field in (4.2), and we write the S^5 in a way that is natural when it is embedded in $\mathbb{C}^3$ with coordinates $z_a = \mu_a e^{i\phi_a}$, obeying the constraint $\mu_1^2 + \mu_2^2 + \mu_3^2 = 1$, such that for $A = 0$ we have in (5.1) the round metric on S^5. Note that despite the mixing in the last term, the determinant of this metric is the product of the determinant of ds_5^2 times that of the S^5 metric.

The only other 10D field turned on is the self-dual 5-form flux, which is given by

$$F_{(5)} = G_{(5)} + *G_{(5)}, \quad (5.2)$$

with

$$G_{(5)} = -4\epsilon_{(5)} + \frac{1}{3}\sum_{a=1}^{3} d(\mu_a^2) \wedge d\phi_a \wedge *_5 F, \quad (5.3)$$

where $F = dA$, $\epsilon_{(5)}$ is the volume form of ds_5^2, and $*_5$ is the 5D Hodge dual with respect to that metric. From (4.2) we see that

$$F = -\frac{3qr}{\rho^4(1-a^2)}dr \wedge (dt - \nu) - \frac{3q}{2\rho^2(1-a^2)}d\nu. \quad (5.4)$$

In order to evaluate the D3-brane action we need to write $F_{(5)} = dC_{(4)}$, where the potential $C_{(4)}$ is defined up to gauge transformations. In this case, the exact form of $C_{(4)}$ can be chosen to be

$$\begin{aligned} C_{(4)} = {} & \frac{2r\rho^2\cos(2\theta)}{\Xi_a^2}dt \wedge dr \wedge d\phi \wedge d\psi + \frac{1}{3}\sum_{a=1}^{3}\mu_a^2 d\phi_a \wedge \left(*_5 F - \frac{2}{3}A \wedge F\right) \\ & - \frac{1}{2}(\mu_1^2 d(\mu_2^2) - \mu_2^2 d(\mu_1^2)) \wedge \left(d\phi_1 + \frac{2}{3}A\right) \wedge \left(d\phi_2 + \frac{2}{3}A\right) \wedge \left(d\phi_3 + \frac{2}{3}A\right) \\ & + \frac{1}{3}d(\mu_2^2) \wedge d\phi_2 \wedge d\phi_3 \wedge A + \frac{1}{3}d(\mu_1^2) \wedge d\phi_1 \wedge d\phi_3 \wedge A. \end{aligned} \quad (5.5)$$

Locally, we can write

$$*_5 F - \frac{2}{3}A \wedge F = d\alpha_{(2)} \quad (5.6)$$

since the left-hand side is closed thanks to the equation of motion of A, in terms of the following 2-form on 5D spacetime:

$$\alpha_{(2)} = \frac{(3+2\alpha)q}{2\Xi_a\rho^2}dt \wedge \nu - \frac{3q\cos(2\theta)}{4\Xi_a^2}(a dt \wedge (d\phi - d\psi) + d\phi \wedge d\psi). \quad (5.7)$$

The expression in (5.5) is not smooth at the horizon, so one might need to add suitable total-derivative terms in order to fix that, without modifying the integrations over D3-brane world volumes that are discussed below. An alternative gauge-equivalent choice for $C_{(4)}$ is presented in (C48).

Note that in the 10D solution the 5D gauge fields become geometrical, and in the asymptotic behavior of the solutions (5.1), given the boundary condition (4.16), the chemical potentials are realized using the off-diagonal components of the metric, while the angular coordinates ϕ_a still have the standard identifications. Thus, these chemical potentials (unlike the ones for the angular momenta) are realized in our solutions using the "metric realization." If desired, we can think of these coordinates as "tilded coordinates" as in (2.5), and define new coordinates

$$\hat{\phi}_a = \phi_a - \frac{2i}{3}\Phi t_E, \quad (5.8)$$

which have the original metric appearing in their 10D boundary condition but obey twisted identifications under $t_E \to t_E + \beta$.

B. The brane action

Consider now a Euclidean D3-brane in the background (5.1), whose world volume wraps a maximal S^3 inside the

[23] See also (2.11) of [90]. Comparing to (4.1) we see that $A_{\text{there}} = \frac{2}{\sqrt{3}}A_{\text{here}}$, and $A^i_{\text{there}} = \frac{2}{3}A_{\text{here}}$ $(i = 1, 2, 3)$. The scalar fields in the $U(1)^3$ supergravity are all trivial when the charges are equal.

S^5 given (say) by $\mu_1 = 0$, and a maximal S^1 inside the S^3 on the horizon in AdS_5 given (say) by $r = r_+, \theta = \pi/2$ (where the ψ direction shrinks) such that the brane wraps the ϕ direction. We can consider such an embedding either in the Lorentzian or in the Euclidean solution; in the Euclidean solution some combination of the time circle and the ϕ circle shrinks at the horizon, but we can still take our brane to sit at a fixed time and wrap the ϕ direction. This configuration is supersymmetric, as we show in Appendix C 3.

The action of a D3-brane is given in general by

$$S_{\rm D3} = -\frac{1}{g_s(2\pi)^3(\alpha')^2}\int (d^4x\sqrt{-\det(g_{\rm D3})} \mp P[C_{(4)}]), \tag{5.9}$$

where $g_{\rm D3}$ is the induced metric on the D3-brane world volume, $P[C_{(4)}]$ is the pull back of the 4-form $C_{(4)}$ to the world volume, and the $\mp$ sign refers to a brane/antibrane, depending on conventions. Note that in our case the AdS radius (which we previously denoted by $1/g$ and set to one) is given by

$$\frac{1}{g^4} = 4\pi g_s N(\alpha')^2, \tag{5.10}$$

so the prefactor in (5.9) is given by $N/2\pi^2$.

Let us compute each of the two terms in (5.9), for generic (not necessarily supersymmetric) solutions. The second term is quite simple, as only the $\frac{1}{3}d(\mu_2^2)\wedge d\phi_2\wedge d\phi_3\wedge A$ term in (5.5) contributes in this configuration. It factorizes into the integral of $\frac{1}{3}A$ on the S^1 in AdS_5, and the integral of $d(\mu_2^2)\wedge d\phi_2\wedge d\phi_3$ on the S^3 in S^5. The first integral gives

$$\begin{aligned}\frac{1}{3}\int_{S^1} A &= -\frac{q}{2(r_+^2+a^2)(1-a^2)}\int_{S^1}\nu \\ &= -\frac{\pi q a}{(r_+^2+a^2)(1-a^2)},\end{aligned} \tag{5.11}$$

while the second integral just gives $4\pi^2$.

Even though the metric induced on the D3-brane from (5.1) is not diagonal between the S^1 and the S^3 in the D3-brane world volume, its form implies that the determinant of the induced metric on the D3-brane is the product of the determinant of the S^3 metric (coming from the S^5 coordinates) and of $g_{\phi\phi}$ in (4.2) [not including the extra contribution to this from the second term in (5.1)], as we discuss around equation (D24). The integral of the first determinant is just $\mathrm{Vol}(S^3) = 2\pi^2$. On the other hand, the component $g_{\phi\phi}$ on the D3-brane evaluates to

$$g_{\phi\phi} = \left[\frac{(r_+^2+a^2)^2+a^2q}{r_+(r_+^2+a^2)(1-a^2)}\right]^2. \tag{5.12}$$

Therefore, choosing a specific sign for the square root,

$$\int d^4x\sqrt{-\det(g_{\rm D3})} = -4\pi^3 i\frac{(r_+^2+a^2)^2+a^2q}{r_+(r_+^2+a^2)(1-a^2)}. \tag{5.13}$$

The sign is chosen in such a way that in the background of a real and causally well-behaved Lorentzian black hole, for which a, q are real (with $a^2<1$) and r_+ is sufficiently large, the contribution from (5.13) to the path-integral measure $\exp(iS_{\rm D3})$ in bounded in absolute value. Notice that the Wick rotation $t\to -it_{\rm E}$ of the black hole metric has no effect on the Euclidean D3-brane action, because such a brane does not wrap the time direction. The full D3-brane action is

$$\begin{aligned}S_{\rm D3} = -\frac{N}{2\pi^2}\Bigg[&-4\pi^3 i\frac{(r_+^2+a^2)^2+a^2q}{r_+(r_+^2+a^2)(1-a^2)} \\ &\pm 4\pi^3\frac{qa}{(r_+^2+a^2)(1-a^2)}\Bigg],\end{aligned} \tag{5.14}$$

where, as before, the $\pm$ sign refers to a brane/antibrane.

The D3-brane action considerably simplifies when the black hole background is supersymmetric; as we show in Appendix C 3, this is because the branes preserve all supersymmetries of the background. On the first branch where the SUSY condition (4.23) is obeyed, and choosing the upper plus sign in (5.14), we obtain

$$S_{\rm D3} = 2\pi N\frac{a-ir_+}{a-1} = 2\pi N\frac{\Delta_g}{\tau_g}, \tag{5.15}$$

where in the last step we used the expressions (4.27) for the chemical potentials.

We have computed the on shell action of similar branes in the case of two different angular momenta, as reported in Appendix C, and of three different charges, as reported in Appendix D. The result leads us to assume that in the general case of chemical potentials $(\sigma_g, \tau_g, \Delta_{g,a})$, there are supersymmetric D3-branes that wrap either the ϕ or the ψ circle in AdS_5 (and sit at $\theta = \frac{\pi}{2}$ or $\theta = 0$, respectively) and an S^3 inside S^5 for which $\mu_a = 0$ $(a = 1, 2, 3)$, and that their actions are

$$S_{\rm D3}^{\phi} = 2\pi N\frac{\Delta_{g,a}}{\sigma_g}, \qquad S_{\rm D3}^{\psi} = 2\pi N\frac{\Delta_{g,a}}{\tau_g}, \qquad (1\text{st branch}). \tag{5.16}$$

On the second branch of solutions, the parameters of the background obey (4.23) but with $i\to -i$. It turns out that the on shell action (5.14) simplifies when choosing the lower minus sign. This is because the branes that are supersymmetric in the two branches of solutions must have

opposite charge (corresponding to swapping branes with antibranes). This also follows from the analysis of supersymmetry in Appendix C 3. Repeating the same computation as above we then find

$$S_{\mathrm{D3}}^{\phi} = -2\pi N \frac{\Delta_{g,a}}{\sigma_g}, \quad S_{\mathrm{D3}}^{\psi} = -2\pi N \frac{\Delta_{g,a}}{\tau_g}, \quad \text{(2nd branch)}. \tag{5.17}$$

The contribution of the Euclidean D3-brane configurations to the partition function is $\exp(-I_{\mathrm{SUGRA}} + iS_{\mathrm{D3}})$ where I_{SUGRA} is the on shell action of the background black hole solution. Note the relative factor of i that appears in our Euclidean solutions since the branes do not extend in the time direction, and therefore they do not get a factor of i from the Wick rotation to Euclidean time, as noticed earlier. Similarly, we can consider branes wrapping any positive integer linear combinations of these cycles, and we would have in the exponent the sum of the corresponding D3-brane actions.

The analysis above makes sense when the D3-branes decrease the contribution of the solution to the partition function, since otherwise considering an arbitrary number of D3-branes will increase their contribution without a bound. We interpret the latter cases as unstable, perhaps to condensation of these D3-branes that would take us to some other background. Thus, we consider a gravitational background to be stable only if *every* D3-brane satisfies[24]

$$\mathbb{I}\mathrm{m}(S_{\mathrm{D3}}) > 0, \tag{5.18}$$

and we suggest that only these backgrounds should be included in the analysis. Note that the equal-charge backgrounds happen to always satisfy this condition for all their D3-branes, so they are always stable, but generic solutions, including solutions related to the equal-charge case by shifts, do not. We will show in the next section that for these stable solutions the D3-brane contributions precisely match with the nonperturbative corrections to the Bethe Ansatz solutions computed in Sec. III C, and we will analyze the precise implications of the stability condition (5.18).

Note that for real supersymmetric solutions we expect the action to be bounded from below, so that finding D3-branes that decrease the action would be impossible. However, once we continue to complex solutions this is no longer the case, which is why we have to impose the condition (5.18) for every D3-brane. A similar condition on complex configurations was recently suggested in a similar context in [22].

VI. STABLE GRAVITY SOLUTIONS AND THE INDEX

Let us now find the stable gravitational backgrounds that contribute to the partition function with boundary conditions given by chemical potentials $(\sigma = \tau, \Delta_{1,2,3})$, so that we can compare them to the field theory analysis of Sec. III. The gravitational solutions have chemical potentials $(\sigma_g, \tau_g, \Delta_{g,a})$ $(a = 1, 2, 3)$, which could be any integer shift of the ones labeling the index, as long as they satisfy the SUSY constraint (4.34). Recall that the index is parametrized by σ, τ, Δ_1, Δ_2, and that we defined in Sec. III an auxiliary chemical potential Δ_3 such that $\Delta_1 + \Delta_2 + \Delta_3 - \sigma - \tau$ is an integer; from the gravity point of view, $\Delta_{g,3}$ (defined from the third chemical potential) always satisfies this condition (on both branches of supersymmetric solutions) thanks to (4.34), so we can identify also $\Delta_{g,3}$ on the gravity side with the index parameter defined in Sec. III (up to some integer shift).

We shall start by considering gravitational backgrounds in the first branch. Consider the stability condition (5.18) arising from having the union of three branes, one at $\mu_1 = 0$, one at $\mu_2 = 0$, and one at $\mu_3 = 0$, all wrapping the ϕ circle, and from a similar configuration which wraps the ψ circle instead. The action is the sum of the individual brane actions (5.16), so using (4.34) on the first branch, the two conditions are

$$\mathbb{I}\mathrm{m}\left(\frac{\tau_g - 1}{\sigma_g}\right) > 0, \qquad \mathbb{I}\mathrm{m}\left(\frac{\sigma_g - 1}{\tau_g}\right) > 0. \tag{6.1}$$

Now, remember that the gravitational chemical potentials are just integer shifts of the ones in the CFT, so they share the same imaginary part. Using this we can rewrite (6.1) as

$$\mathbb{I}\mathrm{m}(\tau) > \mathbb{I}\mathrm{m}(\tau)\mathbb{R}\mathrm{e}(\tau_g - \sigma_g) > -\mathbb{I}\mathrm{m}(\tau). \tag{6.2}$$

Recalling that $\tau_g - \sigma_g$ is an integer, this can be satisfied only if $\tau_g = \sigma_g$. Thus, the only stable gravitational backgrounds have $\tau_g = \sigma_g$ in the bulk as well.[25]

Next, consider the stability conditions arising from a single brane wrapping the ψ cycle at $\mu_a = 0$, and from the union of two branes wrapping the other two cycles, $\mu_b = 0$ and $\mu_c = 0$ with a, b, c all different. Using (4.34), the two conditions are

[24]Here we are referring to the D3-branes described above, that wrap an S^3 in S^5 and an S^1 in the AdS_5 coordinates. We have found additional SUSY D3-branes that wrap an S^1 in S^5 and the whole S^3 at the horizon in the AdS_5 coordinates, and such branes can induce additional instabilities [see (D35) for their on shell action]. Their interpretation will be discussed elsewhere [58].

[25]For $\tau_g - \sigma_g = \pm 1$ it seems from (6.2) that we are on the boundary of the region of stability. However, recalling that this condition comes from the sum of the conditions for stability of three different D3-branes, for generic values of the chemical potentials at least one of these three D3-branes would lead to an instability.

$$\mathbb{I}\mathrm{m}\left(\frac{\Delta_{g,a}}{\tau_g}\right) > 0, \qquad \mathbb{I}\mathrm{m}\left(\frac{-1-\Delta_{g,a}}{\tau_g}\right) > 0, \quad (6.3)$$

and hence

$$\mathbb{I}\mathrm{m}\left(-\frac{1}{\tau_g}\right) > \mathbb{I}\mathrm{m}\left(\frac{\Delta_{g,a}}{\tau_g}\right) > 0. \quad (6.4)$$

We see that we only have a stable solution when $\Delta_{g,a} = [\Delta_{g,a}]_{\tau_g}$ $(a = 1, 2, 3)$, and not for any other shift of the Δ_a. Recall from the analysis of Sec. III that this condition can only be satisfied if the values of the chemical potentials are in the "first case." Thus, for any specific value of τ_g, if the chemical potentials satisfy the conditions of the "first case" then we have a single stable solution among all possible shifts of the electric chemical potentials, and this solution obeys $\Delta_{g,a} = [\Delta_{g,a}]_{\tau_g}$ such that its action (4.33) precisely reproduces the contribution of the basic Bethe Ansatz solution for $\tau = \tau_g$. On the other hand, if the chemical potentials satisfy the "second case," we do not find any shifted solution within the first branch that is stable for this τ_g.

If we now consider gravitational solutions on the second branch of (4.34), then an analogous stability analysis implies that the only stable bulk solutions have $\tau_g = \sigma_g$ and $\Delta_{g,a} = [\Delta_{g,a}]'_{\tau_g}$. So if the chemical potentials satisfy the "second case" condition we find here a single shifted solution (with this τ_g), and we can check that the action of this solution (4.33) precisely reproduces the contribution of the basic Bethe Ansatz solution for the second case with $\tau = \tau_g$. On the other hand, if the chemical potentials satisfy the "first case" condition, then we do not find any stable solutions within the second branch.

The bottom line is that for each value of τ_g, and of Δ_a (up to integer shifts), we find precisely one stable gravitational solution (on either the first branch or the second branch), and the gravitational action precisely reproduces the value of the basic Bethe Ansatz solution for $\tau = \tau_g$ (which satisfies the "first case" or the "second case" condition, respectively). Now, recalling that τ_g can take any value of the form $\tau_g = \tau + r$ for any integer r, we recognize that the contribution of that τ_g is exactly that of the $\{1, N, r\}$ solution to the Bethe Ansatz equations with chemical potential τ. Thus, we have a one-to-one correspondence between the stable shifted solutions on the gravity side, and the $m = 1$ shifted solutions to the Bethe Ansatz equations, with an exact match of the leading action between the two sides.

Moreover, given the values of the $\Delta_{g,a}$s that we found, the form of the exponentials in the nonperturbative corrections that we found on the gravity side—of the form $e^{iS_{\mathrm{D3}}}$ with S_{D3} given by (5.16) and (5.17)—precisely match with the exponentials that appeared in the corrections to the corresponding Bethe Ansatz solutions in Sec. III C (in each of the two cases). So our matching of the gravity side to the index extends also to these nonperturbative corrections. It would be interesting to match also the coefficients in front of the exponentials in the various nonperturbative corrections between the two sides, but this lies beyond the scope of this paper.

VII. ORBIFOLDS AND $m > 1$ SOLUTIONS

So far we have found a precise match between stable gravitational black hole solutions and the $\{m, n, r\}$ Hong-Liu solutions to the Bethe Ansatz equations with $m = 1$. It is interesting to ask whether we can find gravitational solutions that will agree with the $m > 1$ BA solutions. In this section we show that we can construct orbifold configurations that precisely agree with these solutions (when m remains finite in the large N limit).

A clue to finding these solutions is the simple relation between the leading-order action of the $m > 1$ solutions and that of the basic $\{1, N, 0\}$ solution; they are related by taking $\tau \to \check{\tau} \equiv m\tau + r$, $\Delta_a \to m\Delta_a$, and dividing the action by m.

Consider the Euclidean black hole solutions with equal charges and angular momenta, described in Sec. IV. In 5D, the Euclidean solution with inverse temperature $\tilde{\beta}$ and chemical potentials $\tilde{\Omega}$ and $\tilde{\Phi}$ involves an identification of the coordinates by

$$(t_{\mathrm{E}}, \phi, \psi) \cong (t_{\mathrm{E}} + \tilde{\beta}, \phi - i\tilde{\Omega}\tilde{\beta}, \psi - i\tilde{\Omega}\tilde{\beta}), \quad (7.1)$$

and a gauge field given near the boundary by $A_{\mathrm{bdry}} = i\tilde{\Phi}dt_{\mathrm{E}}$, such that the holonomy around the cycle (7.1) at infinity is $i\tilde{\Phi}\tilde{\beta}$. From the ten-dimensional point of view, as discussed above, in our conventions the identifications of the S^5 coordinates are not shifted, so we have

$$(t_{\mathrm{E}}, \phi, \psi, \phi_1, \phi_2, \phi_3) \cong (t_{\mathrm{E}} + \tilde{\beta}, \phi - i\tilde{\Omega}\tilde{\beta}, \psi - i\tilde{\Omega}\tilde{\beta}, \phi_1, \phi_2, \phi_3), \quad (7.2)$$

with off-diagonal metric components proportional at infinity to $\frac{2}{3}i\tilde{\Phi}$. These can alternatively be described using the hatted coordinates $\hat{\phi}_a$ in (5.8) with an unmodified metric at infinity, but with identifications where $\hat{\phi}_a$ is shifted by $-\frac{2}{3}i\tilde{\Phi}\tilde{\beta}$. The generalization to different charges and/or angular momenta is straightforward, and the resulting identification on the hatted coordinates is

$$(t_{\mathrm{E}}, \phi, \psi, \hat{\phi}_1, \hat{\phi}_2, \hat{\phi}_3) \cong (t_{\mathrm{E}} + \tilde{\beta}, \phi - i\tilde{\Omega}_1\tilde{\beta}, \psi - i\tilde{\Omega}_2\tilde{\beta}, \hat{\phi}_1 - i\tilde{\Phi}_1\tilde{\beta}, \hat{\phi}_2 - i\tilde{\Phi}_2\tilde{\beta}, \hat{\phi}_3 - i\tilde{\Phi}_3\tilde{\beta}). \quad (7.3)$$

On the other hand, in the $(\tilde{\phi}, \tilde{\psi}, \phi_1, \phi_2, \phi_3)$ coordinates—see (2.5)—there is no shift under the identification (7.3)

(such that these coordinates are real even when the chemical potentials are complex), and the chemical potentials are all realized through off-diagonal terms in the behavior of the metric near the boundary.

Now, let us take this 10D background (starting from the case of equal charges and angular momenta, for simplicity) and perform on it a $\mathbb{Z}_m$ orbifold by identifying points related by

$$(t_E,\phi,\psi,\hat{\phi}_1,\hat{\phi}_2,\hat{\phi}_3)\cong\left(t_E+\frac{\tilde{\beta}}{m},\phi-i\tilde{\Omega}\frac{\tilde{\beta}}{m}-\frac{2\pi r}{m},\psi-i\tilde{\Omega}\frac{\tilde{\beta}}{m}-\frac{2\pi r}{m},\right.$$
$$\left.\hat{\phi}_1-\frac{2i}{3}\tilde{\Phi}\frac{\tilde{\beta}}{m}-\frac{2\pi s}{m},\hat{\phi}_2-\frac{2i}{3}\tilde{\Phi}\frac{\tilde{\beta}}{m}-\frac{2\pi s}{m},\hat{\phi}_3-\frac{2i}{3}\tilde{\Phi}\frac{\tilde{\beta}}{m}-\frac{2\pi s}{m}\right),\tag{7.4}$$

for some integers $r=0,\ldots,m-1$ and $s=0,\ldots,2m-1$ (note that we allowed a larger range for s here, since shifting the three angles $\hat{\phi}_a$ by 2π acts as the identity on bosons but gives a minus sign to fermions, so it is a nontrivial operation). Performing this identification m times brings us back to the identification (7.3), so (given that our solutions are all independent of t_E and of the angular coordinates) this is indeed a $\mathbb{Z}_m$ isometry of our original backgrounds, that we can orbifold by in string theory. While in the hatted coordinates it is not obvious that the shifts in (7.4) are consistent with the ranges of the coordinates when the chemical potentials are complex, if we use the real coordinates $(\tilde{\phi},\tilde{\psi},\phi_1,\phi_2,\phi_3)$ we find that only the $\frac{2\pi r}{m}$ and $\frac{2\pi s}{m}$ terms in the shifts remain, consistent with these coordinates being real:

$$(t_E,\tilde{\phi},\tilde{\psi},\phi_1,\phi_2,\phi_3)\cong\left(t_E+\frac{\tilde{\beta}}{m},\tilde{\phi}-\frac{2\pi r}{m},\tilde{\psi}-\frac{2\pi r}{m},\phi_1-\frac{2\pi s}{m},\phi_2-\frac{2\pi s}{m},\phi_3-\frac{2\pi s}{m}\right).\tag{7.5}$$

Thus, (7.4) is a consistent orbifold in string theory (we will discuss its fixed points below).

If we consider the behavior of the orbifolds near the boundary, then we see that the identifications (7.4) that we perform in the orbifold background take exactly the same form as our original identifications on some other black hole solutions (7.2) [with tilded quantities substituted by untilded ones, and in terms of hatted coordinates $\hat{\phi}_a$ as in (7.3)], if we identify

$$\beta=\frac{\tilde{\beta}}{m},\qquad \Omega=\tilde{\Omega}-\frac{2\pi i r}{\tilde{\beta}},\qquad \Phi=\tilde{\Phi}-\frac{3\pi i s}{\tilde{\beta}}.\tag{7.6}$$

Thus, the orbifold background contributes to the sphere partition function with these values of β, Ω, and Φ. Note that the new background has

$$\tau_g=\frac{\beta(\Omega-1)}{2\pi i}=\frac{\tilde{\beta}(\tilde{\Omega}-\frac{2\pi i r}{\tilde{\beta}}-1)}{2\pi i m}=\frac{1}{m}\tilde{\tau}_g-\frac{r}{m},$$
$$\Delta_g=\frac{\beta(\Phi-\frac{3}{2})}{3\pi i}=\frac{\tilde{\beta}(\tilde{\Phi}-\frac{3\pi i s}{\tilde{\beta}}-\frac{3}{2})}{3\pi i m}=\frac{1}{m}\tilde{\Delta}_g-\frac{s}{m},\tag{7.7}$$

which is equivalent to

$$\tilde{\tau}_g=m\tau_g+r\quad\text{and}\quad\tilde{\Delta}_g=m\Delta_g+s.\tag{7.8}$$

General orbifolds of this type will not preserve supersymmetry even if the original background does; the condition for preserving supersymmetry is that the Killing spinor should be antiperiodic under the new identification (7.4) (note that the same Killing spinor can be used as in the original background). This happens precisely when $2\tau_g-3\Delta_g$ is an odd integer, i.e., when

$$2r-3s=m+(2\tilde{\tau}_g-3\tilde{\Delta}_g)\pmod{2m}.\tag{7.9}$$

For any value of r we can find a unique s such that this is satisfied[26] (moreover, recall that the "parent" black hole solution satisfies $2\tilde{\tau}_g-3\tilde{\Delta}_g=\pm1$). When this condition is satisfied, the $\mathbb{Z}_m$ orbifold of the black hole background with parameters $(\tilde{\beta},\tilde{\tau}_g,\tilde{\Delta}_g)$ contributes to the index with these parameters (β,τ_g,Δ_g), since it has the same boundary conditions as the other backgrounds contributing to this index.

The integral of the supergravity action density over the orbifolded background is $1/m$ times the action of the original black hole background, due to the $\mathbb{Z}_m$ identification. Thus, we conclude that in the classical gravity approximation, the contributions of these orbifold backgrounds to the logarithm of the index with parameters (β,τ_g,Δ_g) are precisely $1/m$ times those of a black hole with parameters $(m\beta,m\tau_g+r,m\Delta_g+s)$. This is precisely the same as what we found for the Hong-Liu BA solutions with parameters $\{m,N/m,r\}$ [recall that the leading-order contribution from the BA solutions is invariant under shifting $(m\Delta_g)$ by an integer]. Thus, we have an exact

[26]More precisely, this is true when m is not a multiple of 3. In the more general case of unequal chemical potentials discussed below, this restriction does not arise.

match also for these solutions. Note that the quantization condition for the 5-form flux on the orbifolded background is satisfied only when N/m is an integer, so that exactly the same values of m are allowed in the field theory and on the gravity side.

Consider next the nonperturbative contributions from D3-branes wrapped on the orbifolded background. Recall that our D3-branes sit at the horizon and wrap, say, the ϕ, ϕ_2, ϕ_3 directions, together with an extra coordinate on the S^5. The position of the brane is such that the orbifold acts as a $\mathbb{Z}_m$ shift within the world volume of the D3-brane, and thus its action is $1/m$ times the action of the D3-brane on the original background. On the first branch this was given by $2\pi N\tilde{\Delta}_g/\tilde{\tau}_g$, so we find that on the orbifolded background the D3-brane action is given by $S_{\mathrm{D3}} = 2\pi N(m\Delta_g + s)/m(m\tau_g + r)$. This agrees with the nonperturbative corrections found in Sec. III C if $m\Delta_g + s = [m\Delta_g]_{m\tau_g+r}$.

Of course, in the orbifolded backgrounds we still have the freedom of shifting the various chemical potentials by integers, and we need to perform a similar stability analysis to that of the previous section. And moreover, the generalization of the analysis above to different chemical potentials is straightforward, but we then get separate parameters r_1, r_2 for the shifts of ϕ and ψ in (7.4), and separate parameters s_1 and s_2 for the shifts of ϕ_1 and ϕ_2 there (all in the range $0, \ldots, m-1$), while the shift s_3 of ϕ_3 (in the range $0, \ldots, 2m-1$) is uniquely determined by the supersymmetry condition

$$r_1 + r_2 - s_1 - s_2 - s_3 = m + (2\tilde{\tau}_g - 3\tilde{\Delta}_g) \qquad (\mathrm{mod}\ 2m). \tag{7.10}$$

So we need to generalize the analysis of Sec. VI to allow for all these shifts. The generalization is straightforward and it implies, for instance, that on the first branch the only stable solutions are those obeying $m\Delta_{g,a} + s_a = [m\Delta_{g,a}]_{m\tau_g+r}$ for all $a = 1, 2, 3$, and that they obey $\tau_g = \sigma_g$. As in the $m = 1$ case, this leads to a precise agreement between the stable solutions on the gravity side and the BA solutions, both for the leading-order contribution to the logarithm of the index, and for the form of the nonperturbative corrections discussed in the previous paragraph.

Finally, let us describe the fixed points of these general orbifolds, in which the identifications may be written as

$$(t_{\mathrm{E}}, \tilde{\phi}, \tilde{\psi}, \phi_1, \phi_2, \phi_3) \cong \left(t_{\mathrm{E}} + \frac{\tilde{\beta}}{m}, \tilde{\phi} - \frac{2\pi r_1}{m}, \tilde{\psi} - \frac{2\pi r_2}{m}, \phi_1 - \frac{2\pi s_1}{m}, \phi_2 - \frac{2\pi s_2}{m}, \phi_3 - \frac{2\pi s_3}{m}\right). \tag{7.11}$$

It is clear that there are no fixed points away from the horizon. At the horizon we have one coordinate shrinking to zero size, and the remaining coordinates $\tilde{\phi}$ and $\tilde{\psi}$ form (together with θ) a round S^3, while the S^5 coordinates ϕ_a and μ_a form a round S^5 (recall that our gauge fields in 5D supergravity vanish at the horizon). Generically all the angular coordinates are shifted by the orbifold. However, if (say) r_1 and one of the s_a vanish, then the manifold where $r = r_+$, $\theta = \pi/2$ (such that the $\tilde{\psi}$ circle shrinks) and where $\mu_a = 1$ (such that the two circles of the coordinates ϕ_b with $b \neq a$, that are shifted, shrink as well) is fixed under the orbifold action (7.11). This is a two-dimensional manifold (parametrized by $\tilde{\phi}$ and ϕ_a), and near this fixed manifold the orbifold acts on the transverse space as a supersymmetric $\mathbb{C}^4/\mathbb{Z}_m$ orbifold.

Similarly, if three of the $(r_1, r_2, s_1, s_2, s_3)$ vanish (including at least one of the r_i) we have a four-dimensional space of fixed points (with a transverse $\mathbb{C}^3/\mathbb{Z}_m$ orbifold action), and if four of them vanish we have a six-dimensional space of fixed points (with a transverse $\mathbb{C}^2/\mathbb{Z}_m$ orbifold action).[27] Note that, due to (7.10), it is not possible for all five of these numbers to vanish, consistent with the fact that a $\mathbb{C}/\mathbb{Z}_m$ orbifold cannot be supersymmetric.

When there are fixed points, the orbifolds have light twisted sector states living there, while otherwise all twisted sector states are heavy. Supersymmetry ensures that these twisted sector states do not lead to tachyonic instabilities. In principle the loops of the twisted sector states (whether they are light or heavy) contribute perturbative corrections to the supergravity action, and it would be interesting to verify that their contributions are consistent with the order $\mathcal{O}(1)$ contributions that we found from the Bethe Ansatz approach.

Note that all of our orbifolds involve a shift action on the Euclidean time circle. Thus, they are not related to any $\mathrm{AdS}_5 \times S^5/\mathbb{Z}_m$ backgrounds, and their Lorentzian interpretation is unclear.

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[27] Note that in these cases it is possible that the $\mathbb{Z}_m$ symmetry does not act within the world volume of the wrapped D3-branes. For instance, if $r_1 = s_1 = s_2 = 0$, then it does not act in the world volume of a D3-brane wrapping the ϕ, ϕ_1, and ϕ_2 directions, so its world volume action is not divided by m. However, in such cases the wrapped D3-brane sits at a fixed point of the $\mathbb{Z}_m$ orbifold, and then one can construct also "fractional D3-branes" wrapping the same manifold whose action is smaller by a factor of m, such that the action of the minimal D3-brane still matches what we found from the Bethe Ansatz analysis.

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APPENDIX A: SPECIAL FUNCTIONS

We will use fugacities and chemical potentials related by

$$z = e^{2\pi i u}, \qquad p = e^{2\pi i \sigma}, \qquad q = e^{2\pi i \tau}. \tag{A1}$$

***q*-Pochhammer symbol.**—We use the standard notations

$$(z;q)_n = \prod_{k=0}^{n-1}(1 - zq^k), \qquad (z;q)_\infty = \prod_{k=0}^{\infty}(1 - zq^k) \quad \text{for } |q| < 1. \tag{A2}$$

The expression $(z;q)_\infty$ admits the series expansion and the plethystic representation

$$(z;q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{(q;q)_n} z^n = \exp\left[-\sum_{k=1}^{\infty} \frac{1}{k} \frac{z^k}{1-q^k}\right], \tag{A3}$$

respectively, where the first one converges on the whole domain $|q| < 1$, while the second one converges for $|z|$, $|q| < 1$. Noticing the relation

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} (q;q)_\infty \tag{A4}$$

with the Dedekind eta function, one obtains the modular transformation properties of the q-Pochhammer symbol:

$$(\tilde{q};\tilde{q})_\infty = \sqrt{-i\tau}\, e^{\frac{\pi i}{12}(\tau + 1/\tau)} (q;q)_\infty, \tag{A5}$$

where $\tilde{q} = e^{-2\pi i/\tau}$. The square root is taken with the principal determination, recalling that $\mathbb{I}\mathrm{m}(\tau) > 0$. Finally, we have the asymptotic behaviors

$$(z;q)_\infty \sim (1-z) \quad \text{for } q \to 0, \qquad \log[(z;q)_\infty] \sim -\frac{z}{1-q} \quad \text{for } z \to 0, \tag{A6}$$

where $f \sim g$ means that $\lim f/g = 1$.

Function $\boldsymbol{\theta_0}$.—The elliptic theta function is defined as

$$\theta_0(u;\tau) = (z;q)_\infty (q/z;q)_\infty = \prod_{k=0}^{\infty}(1 - zq^k)(1 - z^{-1}q^{k+1}). \tag{A7}$$

This gives an analytic function on $|q| < 1$ with simple zeros at $z = q^k$ for $k \in \mathbb{Z}$ and no singularities. The infinite product is convergent on the whole domain. We can also give a plethystic definition

$$\theta_0(u;\tau) = \exp\left[-\sum_{k=1}^{\infty} \frac{1}{k} \frac{z^k + (q/z)^k}{1-q^k}\right], \tag{A8}$$

which converges for $|q| < |z| < 1$.

The periodicity relations are

$$\theta_0(u + n + m\tau;\tau) = (-1)^m e^{-2\pi i m u - \pi i m(m-1)\tau} \theta_0(u;\tau) \quad \text{for } n, m \in \mathbb{Z}$$
$$\theta_0(u;\tau) = \theta_0(\tau - u;\tau) = -e^{2\pi i u} \theta_0(-u;\tau). \tag{A9}$$

The modular properties are

$$\theta_0(u;\tau+1) = \theta_0(u;\tau), \qquad \theta_0\left(\frac{u}{\tau}; -\frac{1}{\tau}\right) = -ie^{\frac{\pi i}{\tau}(u^2 + u + \frac{1}{6}) - \pi i u + \frac{\pi i \tau}{6}} \theta_0(u;\tau). \tag{A10}$$

Function $\boldsymbol{\psi}$.—Define, for $\mathbb{I}\mathrm{m}(t) < 0$, the function

$$\psi(t) = \exp\left[t \log(1 - e^{-2\pi i t}) - \frac{1}{2\pi i} \mathrm{Li}_2(e^{-2\pi i t})\right] = \exp\left[-\sum_{\ell=1}^{\infty} \left(\frac{t}{\ell} + \frac{1}{2\pi i \ell^2}\right) e^{-2\pi i t \ell}\right]. \tag{A11}$$

The branch of the logarithm is determined by its series expansion $\log(1-z) = -\sum_{\ell=1}^{\infty} z^\ell/\ell$, whereas $\mathrm{Li}_2(z) = \sum_{\ell=1}^{\infty} z^\ell/\ell^2$ is the dilogarithm. One can show that the branch cut discontinuities of the logarithm and the dilogarithm cancel in the definition of $\psi(t)$, therefore the latter extends to a meromorphic function on the whole complex plane. Some useful properties of $\psi(t)$ are

$$\psi(t)\psi(-t) = e^{-\pi i(t^2-1/6)}, \quad \psi(t+n) = (1-e^{-2\pi i t})^n \psi(t) \quad \text{for } n \in \mathbb{Z}. \tag{A12}$$

In particular, from (A11), $\psi(0) = e^{\pi i/12}$.

Function $\tilde{\Gamma}$.—The elliptic gamma function is defined, in terms of chemical potentials, as

$$\tilde{\Gamma}(u;\sigma,\tau) = \prod_{m,n=0}^{\infty} \frac{1-p^{m+1}q^{n+1}/z}{1-p^m q^n z}. \tag{A13}$$

This definition gives a meromorphic single-valued function on $|p|$, $|q| < 1$, with simple zeros at $z = p^{m+1}q^{n+1}$ and simple poles at $z = p^{-m}q^{-n}$ for m, $n \geq 0$. The infinite product is convergent on the whole domain. We can also give a plethystic definition

$$\tilde{\Gamma}(u;\sigma,\tau) = \exp\left[\sum_{k=1}^{\infty} \frac{1}{k} \frac{z^k - (pq/z)^k}{(1-p^k)(1-q^k)}\right], \tag{A14}$$

which converges for $|pq| < |z| < 1$.

The function satisfies the following periodicity relations:

$$\begin{aligned}
&\tilde{\Gamma}(u;\sigma,\tau) = \tilde{\Gamma}(u;\tau,\sigma),\\
&\tilde{\Gamma}(u+1;\sigma,\tau) = \tilde{\Gamma}(u;\sigma+1,\tau) = \tilde{\Gamma}(u;\sigma,\tau+1) = \tilde{\Gamma}(u;\sigma,\tau),\\
&\tilde{\Gamma}(u+\sigma;\sigma,\tau) = \theta_0(u;\tau)\tilde{\Gamma}(u;\sigma,\tau),\\
&\tilde{\Gamma}(u+\tau;\sigma,\tau) = \theta_0(u;\sigma)\tilde{\Gamma}(u;\sigma,\tau).
\end{aligned} \tag{A15}$$

Moreover

$$\tilde{\Gamma}(u;\sigma,\tau)\tilde{\Gamma}(\sigma+\tau-u;\sigma,\tau) = 1. \tag{A16}$$

The elliptic gamma function has $SL(3,\mathbb{Z})$ modular properties. For σ, τ, σ/τ, $\sigma+\tau \in \mathbb{C}\backslash\mathbb{R}$ there is a "modular formula" [91]:

$$\begin{aligned}
\tilde{\Gamma}(u;\sigma,\tau) &= e^{-\pi i \mathcal{Q}(u;\sigma,\tau)} \frac{\tilde{\Gamma}(\frac{u}{\tau};\frac{\sigma}{\tau},-\frac{1}{\tau})}{\tilde{\Gamma}(\frac{u-\tau}{\sigma};-\frac{1}{\sigma},-\frac{\tau}{\sigma})}\\
&= e^{-\pi i \mathcal{Q}(u;\sigma,\tau)} \frac{\tilde{\Gamma}(\frac{u}{\sigma};-\frac{1}{\sigma},\frac{\tau}{\sigma})}{\tilde{\Gamma}(\frac{u-\sigma}{\tau};-\frac{\sigma}{\tau},-\frac{1}{\tau})},
\end{aligned} \tag{A17}$$

where $\mathcal{Q}$ is the cubic polynomial

$$\begin{aligned}
\mathcal{Q}(u;\sigma,\tau) = &\frac{u^3}{3\sigma\tau} - \frac{\sigma+\tau-1}{2\sigma\tau}u^2\\
&+ \frac{\sigma^2+\tau^2+3\sigma\tau-3\sigma-3\tau+1}{6\sigma\tau}u\\
&+ \frac{(\sigma+\tau-1)(\sigma+\tau-\sigma\tau)}{12\sigma\tau}.
\end{aligned} \tag{A18}$$

In the degenerate case $\sigma = \tau$ the inversion formula above is not valid. For $u \in \mathbb{C}\backslash(\mathbb{Z}+\tau\mathbb{Z})$, however, there is a degenerate relation:

$$\tilde{\Gamma}(u;\tau,\tau) = \frac{e^{-\pi i \mathcal{Q}(u;\tau,\tau)}}{\theta_0(\frac{u}{\tau};-\frac{1}{\tau})} \prod_{k=0}^{\infty} \frac{\psi(\frac{k+1+u}{\tau})}{\psi(\frac{k-u}{\tau})}. \tag{A19}$$

The polynomial $\mathcal{Q}$ reduces to

$$\mathcal{Q}(u;\tau,\tau) = \frac{(2u-2\tau+1)(2u(u+1)-2\tau(2u+1)+\tau^2)}{12\tau^2}. \tag{A20}$$

The function ψ is defined in (A11). Using

$$\mathcal{Q}(u+1;\tau,\tau) - \mathcal{Q}(u;\tau,\tau) = \frac{(u+1)(u+1-2\tau)}{\tau^2} + \frac{5}{6}, \tag{A21}$$

one can check that both sides of (A19) are invariant under $u \to u+1$.

Function $\mathcal{G}$.—This function of u, Δ, τ is defined as

$$\mathcal{G}(u,\Delta;\tau) = \frac{1}{2\pi i}\frac{\partial}{\partial u}\log\left(\frac{\theta_0(\Delta-u;\tau)}{\theta_0(\Delta+u;\tau)}\right). \tag{A22}$$

We can write the series expansion

$$\mathcal{G}(u,\Delta;\tau) = \sum_{\ell=1}^{\infty} \frac{(z^\ell + z^{-\ell})(y^\ell - (q/y)^\ell)}{1-q^\ell}, \tag{A23}$$

which converges for $|q| < |yz| < 1$ and $|q| < |y/z| < 1$, with $z = e^{2\pi i u}$ and $y = e^{2\pi i \Delta}$. Such a domain could be too restrictive; in that case, notice that $\mathcal{G}$ can be written as the sum of two series, each one convergent in one of the two domains, respectively.

We have

$$\mathcal{G}(u,\Delta;\tau) = \mathcal{G}(-u,\Delta;\tau) = -\mathcal{G}(u,\tau-\Delta;\tau). \tag{A24}$$

We also have modular properties, which follow from the ones of θ_0:

$$\begin{aligned}
&\mathcal{G}(u,\Delta;\tau+1) = \mathcal{G}(u,\Delta;\tau),\\
&\mathcal{G}\left(\frac{u}{\tau},\frac{\Delta}{\tau};-\frac{1}{\tau}\right) = \tau - 2\Delta - 1 + \tau\mathcal{G}(u,\Delta;\tau).
\end{aligned} \tag{A25}$$

The periodicity properties are

$$\begin{aligned}
\mathcal{G}(u,\Delta;\tau) &= \mathcal{G}(u+1,\Delta;\tau) = \mathcal{G}(u+\tau,\Delta;\tau)\\
&= \mathcal{G}(u,\Delta+1;\tau),\\
\mathcal{G}(u,\Delta+\tau;\tau) &= 2 + \mathcal{G}(u,\Delta;\tau).
\end{aligned} \tag{A26}$$

In particular, $\mathcal{G}$ is an elliptic function of u, and quasielliptic of Δ.

APPENDIX B: CONTRIBUTION OF HONG-LIU SOLUTIONS

In this Appendix we carry out the computations that lead to the contributions of the HL solutions to the superconformal index, reported in Sec. III A. We will first compute the quantity γ_Δ, and then the Jacobian H.

We proceed to explicitly evaluate the sum in (3.19) that defines the function γ_Δ. As explained in the main text, the strategy is to expand various functions in a common domain of convergence, manipulate the expansions and then obtain exact expressions. By analytic continuation, the latter will be valid everywhere.

1. The case $\Delta \neq 0$

Let us start with the generic case that $\Delta \neq 0$. For $|q|^2 < |y| < 1$, we use the plethystic expansion to evaluate the second summation in the second line of (3.19):

$$n\sum_{j_1\neq j_2}^{m-1}\log\left(\tilde{\Gamma}(v_{j_1j_2}+\Delta;\tau,\tau)\right) = n\sum_{j_1\neq j_2}^{m-1}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\frac{\left(\frac{\xi_{j_1}}{\xi_{j_2}}\right)^\ell y^\ell - \left(\frac{\xi_{j_2}}{\xi_{j_1}}\right)^\ell y^{-\ell}q^{2\ell}}{(1-q^\ell)^2}$$
$$= n\sum_{\ell=1}^{\infty}\frac{1}{\ell}\frac{y^\ell - y^{-\ell}q^{2\ell}}{(1-q^\ell)^2}A_{\ell,m} = N\log\left(\frac{\tilde{\Gamma}(m\Delta;m\tau,m\tau)}{\tilde{\Gamma}(\Delta;\tau,\tau)}\right). \tag{B1}$$

Here we introduced

$$A_{\ell,m} \equiv \sum_{j_1\neq j_2}^{m-1} e^{2\pi i(j_1-j_2)\ell/m} = \begin{cases} m^2-m & \text{for } \ell = 0 \mod m \\ -m & \text{for } \ell \neq 0 \mod m \end{cases}. \tag{B2}$$

We similarly expand the first summation:

$$\sum_{j_1,j_2=0}^{m-1}\sum_{k_1\neq k_2}^{n-1}\log(\tilde{\Gamma}(v_{j_1j_2}+w_{k_1k_2}+\Delta;\tau,\tau)) = \sum_{j_1,j_2=0}^{m-1}\sum_{k_1\neq k_2}^{n-1}\sum_{\ell=1}^{\infty}\frac{\left(\frac{\xi_{j_1}\zeta_{k_1}}{\xi_{j_2}\zeta_{k_2}}\right)^\ell y^\ell - \left(\frac{\xi_{j_2}\zeta_{k_2}}{\xi_{j_1}\zeta_{k_1}}\right)^\ell y^{-\ell}q^{2\ell}}{\ell(1-q^\ell)^2}. \tag{B3}$$

This expansion converges if

$$0 < \mathbb{I}\mathrm{m}(v_{j_1j_2}+w_{k_1k_2}+\Delta) = \frac{k_1-k_2}{n}\mathbb{I}\mathrm{m}(\tau)+\mathbb{I}\mathrm{m}(\Delta) < 2\mathbb{I}\mathrm{m}(\tau) \quad \forall\ k_1\neq k_2. \tag{B4}$$

Taking into account the ranges of $k_{1,2}$, this is the case if

$$\frac{n-1}{n}\mathbb{I}\mathrm{m}(\tau) < \mathbb{I}\mathrm{m}(\Delta) < \frac{n+1}{n}\mathbb{I}\mathrm{m}(\tau). \tag{B5}$$

For any value of n there exists a (small) domain of convergence that we can use to perform our manipulations.[28] We find

$$\text{(B.3)} = \sum_{k_1\neq k_2}^{n-1}\sum_{\ell=1}^{\infty}\frac{\left(\frac{\zeta_{k_1}y}{\zeta_{k_2}}\right)^\ell - \left(\frac{\zeta_{k_2}q^2}{\zeta_{k_1}y}\right)^\ell}{\ell(1-q^\ell)^2}(A_{\ell,m}+m) = m\sum_{k_1\neq k_2}^{n-1}\log(\tilde{\Gamma}(mw_{k_1k_2}+m\Delta;m\tau,m\tau)). \tag{B6}$$

Putting the two terms together we obtain

$$\gamma_\Delta = m\sum_{k_1\neq k_2}^{n-1}\log\left(\tilde{\Gamma}(mw_{k_1k_2}+m\Delta;m\tau,m\tau)\right) + N\log\left(\frac{\tilde{\Gamma}(m\Delta;m\tau,m\tau)}{\tilde{\Gamma}(\Delta;\tau,\tau)}\right), \tag{B7}$$

which, by analyticity, extends to the whole domain of definition of the functions. The sum in the first term on the right-hand side can be evaluated using the "modular formula" (A19). Recall that $\tilde{\Gamma}(u;\tau,\sigma)$ is invariant under integer shifts of τ, σ, and thus we can shift $m\tau \to m\tau + r$. It is convenient to define $\check{\tau} \equiv m\tau + r$. We obtain

[28] Actually, since we are resumming over j_1, j_2 at fixed k_1, k_2, the domain of convergence is even larger.

$$m\sum_{k_1\neq k_2}^{n-1}\log\left[\tilde{\Gamma}\left(\frac{k_1-k_2}{n}\check{\tau}+m\Delta;m\tau,m\tau\right)\right]=-\pi i m\sum_{k_1\neq k_2}^{n-1}\mathcal{Q}\left(\frac{k_1-k_2}{n}\check{\tau}+m\Delta;\check{\tau},\check{\tau}\right)$$
$$-m\sum_{k_1\neq k_2}^{n-1}\log\left[\theta_0\left(\frac{k_1-k_2}{n}+\frac{m\Delta}{\check{\tau}};-\frac{1}{\check{\tau}}\right)\right]+m\sum_{k_1\neq k_2}^{n-1}\sum_{k=0}^{\infty}\log\left(\frac{\psi(\frac{k+1+m\Delta}{\check{\tau}}+\frac{k_1-k_2}{n})}{\psi(\frac{k-m\Delta}{\check{\tau}}+\frac{k_2-k_1}{n})}\right). \tag{B8}$$

Here $\mathcal{Q}$ is the cubic polynomial (A20). Let us evaluate the sums in the second line. For the first sum we use the plethystic expansion (A8) of θ_0, valid for $0<\mathbb{I}\mathrm{m}(m\Delta/\check{\tau})<\mathbb{I}\mathrm{m}(-1/\check{\tau})$. It is convenient to define the variables

$$\tilde{\zeta}_k=e^{2\pi i m w_k/\check{\tau}}=e^{2\pi i k/n},\qquad \tilde{y}=e^{2\pi i m\Delta/\check{\tau}},\qquad \check{q}=e^{-2\pi i/\check{\tau}}. \tag{B9}$$

Then

$$-m\sum_{k_1\neq k_2}^{n-1}\log\left[\theta_0\left(\frac{k_1-k_2}{n}+\frac{m\Delta}{\check{\tau}};-\frac{1}{\check{\tau}}\right)\right]=m\sum_{k_1\neq k_2}^{n-1}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\frac{\left(\frac{\tilde{\zeta}_{k_1}}{\tilde{\zeta}_{k_2}}\right)^{\ell}\tilde{y}^{\ell}+\left(\frac{\tilde{\zeta}_{k_2}}{\tilde{\zeta}_{k_1}}\right)^{\ell}\tilde{y}^{-\ell}\check{q}^{\ell}}{1-\check{q}^{\ell}}$$
$$=m\sum_{\ell=1}^{\infty}\frac{1}{\ell}\frac{\tilde{y}^{\ell}+\tilde{y}^{-\ell}\check{q}^{\ell}}{1-\check{q}^{\ell}}A_{\ell,n}=N\log\left(\frac{\theta_0(\frac{m\Delta}{\check{\tau}};-\frac{1}{\check{\tau}})}{\theta_0(\frac{N\Delta}{\check{\tau}};-\frac{n}{\check{\tau}})}\right). \tag{B10}$$

For the second sum, we use the expansion (A11) valid in the same range:

$$m\sum_{k_1\neq k_2}^{n-1}\sum_{k=0}^{\infty}\log\left(\frac{\psi(\frac{k+1+m\Delta}{\check{\tau}}+\frac{k_1-k_2}{n})}{\psi(\frac{k-m\Delta}{\check{\tau}}+\frac{k_2-k_1}{n})}\right)=m\sum_{k_1\neq k_2}^{n-1}\sum_{k=0}^{\infty}\sum_{\ell=1}^{\infty}\left[\frac{1}{\ell}\left(\frac{k-m\Delta}{\check{\tau}}\tilde{y}^{\ell}\right.\right.$$
$$\left.\left.-\frac{k+1+m\Delta}{\check{\tau}}\left(\frac{\check{q}}{\tilde{y}}\right)^{\ell}\right)+\frac{1}{\ell}\left(\frac{k_2-k_1}{n}+\frac{1}{2\pi i\ell}\right)(\tilde{y}^{\ell}-(\check{q}/\tilde{y})^{\ell})\right]\left(\frac{\tilde{\zeta}_{k_1}}{\tilde{\zeta}_{k_2}}\right)^{\ell}\check{q}^{k\ell}. \tag{B11}$$

Then we conclude that the following sum vanishes:

$$B_{\ell,n}=\sum_{k_1\neq k_2}^{n-1}(k_1-k_2)e^{2\pi i\ell(k_1-k_2)/n}=0. \tag{B12}$$

Therefore

$$\text{(B.11)}=m\sum_{k=0}^{\infty}\sum_{\ell=1}^{\infty}\left[\frac{1}{\ell}\left(\frac{k-m\Delta}{\check{\tau}}\tilde{y}^{\ell}-\frac{k+1+m\Delta}{\check{\tau}}(\check{q}/\tilde{y})^{\ell}\right)+\frac{1}{2\pi i\ell^2}(\tilde{y}^{\ell}-(\check{q}/\tilde{y})^{\ell})\right]\check{q}^{k\ell}A_{\ell,n}$$
$$=\sum_{k=0}^{\infty}\left[m\log\left(\frac{\psi(\frac{n(k+1+m\Delta)}{\check{\tau}})}{\psi(\frac{n(k-m\Delta)}{\check{\tau}})}\right)-N\log\left(\frac{\psi(\frac{k+1+m\Delta}{\check{\tau}})}{\psi(\frac{k-m\Delta}{\check{\tau}})}\right)\right]. \tag{B13}$$

Finally we put all terms together. The expression simplifies using

$$N\log\left(\tilde{\Gamma}(m\Delta;m\tau,m\tau)\right)=-\pi i N\mathcal{Q}(m\Delta;\check{\tau},\check{\tau})$$
$$-N\log\left[\theta_0\left(\frac{m\Delta}{\check{\tau}};-\frac{1}{\check{\tau}}\right)\right]+N\sum_{k=0}^{\infty}\log\left(\frac{\psi(\frac{k+1+m\Delta}{\check{\tau}})}{\psi(\frac{k-m\Delta}{\check{\tau}})}\right). \tag{B14}$$

We obtain

$$\gamma_\Delta = -\pi i m \sum_{k_1,k_2=0}^{n-1} \mathcal{Q}\left(\frac{k_1-k_2}{n}\check{\tau}+m\Delta;\check{\tau},\check{\tau}\right) - N\log\left(\tilde{\Gamma}(\Delta;\tau,\tau)\right)$$
$$- N\log\left[\theta_0\left(\frac{N\Delta}{\check{\tau}};-\frac{n}{\check{\tau}}\right)\right] + m\sum_{k=0}^{\infty}\log\left(\frac{\psi(\frac{k+1+m\Delta}{\check{\tau}/n})}{\psi(\frac{k-m\Delta}{\check{\tau}/n})}\right). \tag{B15}$$

The sum over k_1, k_2 can be performed exactly:

$$-\pi i m \sum_{k_1,k_2=0}^{n-1} \mathcal{Q}\left(\frac{k_1-k_2}{n}\check{\tau}+m\Delta;\check{\tau},\check{\tau}\right) = -\frac{\pi i m}{6}\left(\check{\tau}-m\Delta-\frac{1}{2}\right) + \pi i\frac{N^2}{m}\frac{(\check{\tau}-m\Delta)(\check{\tau}-m\Delta-\frac{1}{2})(\check{\tau}-m\Delta-1)}{3\check{\tau}^2}. \tag{B16}$$

We thus obtain the exact expression in (3.23).

2. The case $\Delta = 0$

The case $\Delta = 0$ requires a separate treatment. Using (A9), (A15), and (A16), we find

$$\gamma_0 = -\sum_{i\neq j}^{N}\log(\theta_0(u_{ij};\tau))|_{(3.16)},$$
$$= -\sum_{j_1,j_2=0}^{m-1}\sum_{k_1\neq k_2}^{n-1}\log(\theta_0(v_{j_1j_2}+w_{k_1k_2};\tau)) - n\sum_{j_1\neq j_2}^{m-1}\log(\theta_0(v_{j_1j_2};\tau)). \tag{B17}$$

The sum in the second term on the second line is computed as follows:

$$-n\sum_{j_1\neq j_2}^{m-1}\log(\theta_0(v_{j_1j_2};\tau)) = -n\sum_{j_1\neq j_2}^{m-1}\left[\log\left(1-\frac{\xi_{j_1}}{\xi_{j_2}}\right)+2\sum_{k=1}^{\infty}\log\left(1-\frac{\xi_{j_1}}{\xi_{j_2}}q^k\right)\right]$$
$$= -N\log(m) + 2n\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty}\frac{1}{\ell}A_{\ell,m}q^{k\ell} = -N\log(m) + 2N\log\left[\frac{(q;q)_\infty}{(q^m;q^m)_\infty}\right]. \tag{B18}$$

We used that $\sum_{j\neq k}^{m}\log(1-e^{2\pi i(j-k)/m}) = m\log(m)$. The sum in the first term can be computed using the plethystic expansion (A8) which converges for $0 < \frac{k_1-k_2}{n}\mathbb{I}\mathrm{m}(\tau) < \mathbb{I}\mathrm{m}(\tau)$. This is not satisfied for all k_1, k_2, therefore we split the sum using (A9) as follows:

$$-\sum_{j_1,j_2=0}^{m-1}\left[\sum_{k_1>k_2}^{n-1}\log(\theta_0(v_{j_1j_2}+w_{k_1k_2};\tau)) + \sum_{k_1<k_2}^{n-1}\log(-e^{2\pi i(v_{j_1j_2}+w_{k_1k_2})}\theta_0(-v_{j_1j_2}-w_{k_1k_2};\tau))\right]$$
$$= -2\sum_{j_1,j_2=0}^{m-1}\sum_{k_1>k_2}^{n-1}\log(\theta_0(v_{j_1j_2}+w_{k_1k_2};\tau)) - m^2\sum_{k_1<k_2}^{n-1}\log(-e^{2\pi i w_{k_1k_2}}),$$
$$= -2\sum_{k_1>k_2}^{n-1}\frac{1}{\ell}\frac{(\frac{\zeta_{k_1}}{\zeta_{k_2}})^\ell + (\frac{\zeta_{k_2}}{\zeta_{k_1}})^\ell q^\ell}{1-q^\ell}(A_{\ell,m}+m) - \sum_{k_1<k_2}^{n-1}\log((-1)^{m^2}e^{2\pi i m^2 w_{k_1k_2}}),$$
$$= -2m\sum_{k_1>k_2}^{n-1}\log(\theta_0(mw_{k_1k_2};m\tau)) - \sum_{k_1<k_2}^{n-1}\log((-1)^{m^2}e^{2\pi i m^2 w_{k_1k_2}}),$$
$$= -m\sum_{k_1\neq k_2}^{n-1}\log(\theta_0(mw_{k_1k_2};m\tau)). \tag{B19}$$

In the last equality we used $(-1)^{m^2-m} = 1$. The sum can be computed using the modular properties (A10):

$$
\begin{aligned}
&- m \sum_{k_1 \neq k_2}^{n-1} \log(\theta_0(m w_{k_1 k_2}; m\tau + r)) \\
&= -\pi i m \sum_{k_1 \neq k_2}^{n-1} \left[\frac{1}{2} - \frac{(k_1 - k_2)^2}{n^2}\check{\tau} - \frac{1}{6\check{\tau}} - \frac{\check{\tau}}{6}\right] - m \sum_{k_1 \neq k_2}^{n-1} \log\left[\theta_0\left(\frac{k_1 - k_2}{n}; -\frac{1}{\check{\tau}}\right)\right], \\
&= \pi i N(n-1)\frac{\check{\tau}(\check{\tau} - \frac{1}{2})(\check{\tau} - 1)}{3\check{\tau}^2} + \pi i m(n-1)\frac{\check{\tau}}{6} - N\log(n) + 2N\log\left[\frac{(\check{q};\check{q})_\infty}{(\check{q}^n;\check{q}^n)_\infty}\right].
\end{aligned} \tag{B20}
$$

The computation of the sum in the second line is as in (B18). Finally, we use the modular properties (A5) of the q-Pochhammer symbol to obtain

$$
2N\log\left[\frac{(q;q)_\infty}{(q^m;q^m)_\infty}\frac{(\check{q};\check{q})_\infty}{(\check{q}^n;\check{q}^n)_\infty}\right] = 2N\log\left[\frac{(q;q)_\infty}{(\check{q}^n;\check{q}^n)_\infty}\right] + N\log(-i\check{\tau}) + \frac{\pi i N}{6}\left(\check{\tau} + \frac{1}{\check{\tau}}\right). \tag{B21}
$$

Recall that $\mathbb{I}\mathrm{m}(\check{\tau}) > 0$, therefore $\log(-i\check{\tau}) = \log(\check{\tau}) - i\pi/2$ with the logarithm in its principal determination. Putting the various pieces together we find (3.25).

3. The Jacobian H

Given the BA operators Q_i written as in (3.29), we compute

$$
\frac{1}{2\pi i}\frac{\partial \log(Q_i)}{\partial u_j} = \sum_{k=1}^{N}\frac{\partial u_{ik}}{\partial u_j}\left[-1 + \sum_{a=1}^{3}\mathcal{G}(u_{ik}, \Delta_a)\right], \tag{B22}
$$

where we defined the function (see also Appendix A)

$$
\mathcal{G}(u, \Delta; \tau) = \frac{1}{2\pi i}\frac{\partial}{\partial u}\log\left(\frac{\theta_0(-u + \Delta; \tau)}{\theta_0(u + \Delta; \tau)}\right) = -\frac{1}{2\pi i}\left[\frac{\theta_0'(u + \Delta; \tau)}{\theta_0(u + \Delta; \tau)} + \frac{\theta_0'(-u + \Delta; \tau)}{\theta_0(-u + \Delta; \tau)}\right] \tag{B23}
$$

with implicit dependence on τ (unless specified). This is an even function of u, namely $\mathcal{G}(-u, \Delta) = \mathcal{G}(u, \Delta)$. Since we treat $u_1, \ldots, u_{N-1}$ as the independent variables while u_N is fixed by the $SU(N)$ constraint, we have $\partial_{u_j} u_{ik} = \delta_{ij} - \delta_{kj} - \delta_{iN} + \delta_{kN}$. Therefore

$$
\frac{1}{2\pi i}\frac{\partial \log(Q_i)}{\partial u_j} = (\delta_{ij} - \delta_{iN})\left(-N + \sum_{k=1}^{N}\sum_{a=1}^{3}\mathcal{G}(u_{ik}, \Delta_a)\right) + \sum_{a=1}^{3}[\mathcal{G}(u_{iN}, \Delta_a) - \mathcal{G}(u_{ij}, \Delta_a)]. \tag{B24}
$$

This leads to the following expression for the Jacobian matrix:

$$
\begin{aligned}
\mathcal{A}_{ij} = &-N(1 + \delta_{ij}) + \sum_{k=1}^{N}\sum_{a=1}^{3}[\delta_{ij}\mathcal{G}(u_{ik}, \Delta_a) + \mathcal{G}(u_{kN}, \Delta_a)] \\
&+ \sum_{a=1}^{3}[\mathcal{G}(u_{iN}, \Delta_a) + \mathcal{G}(u_{jN}, \Delta_a) - \mathcal{G}(u_{ij}, \Delta_a) - \mathcal{G}(0, \Delta_a)],
\end{aligned} \tag{B25}
$$

where $i, j = 1, \ldots, N-1$. Let us compute the following quantity:

$$
\Upsilon \equiv \sum_{l=1}^{N}\mathcal{G}(u_{i_1 l}, \Delta_a; \tau) = \sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\mathcal{G}\left(\frac{j_1 - j}{m} + \frac{k_1 - k}{n}\frac{\check{\tau}}{m}, \Delta_a; \tau\right). \tag{B26}
$$

The sum over j is computed easily, using the series expansion (A23). One has to be careful about the domain of convergence of the series. We sum over j with k fixed; however, there is no domain of Δ_a, τ such that the full series is convergent. One can instead break the series of $\mathcal{G}$ in two, each one convergent in a different domain, and recombine the two pieces at the end. Exploiting

$$\sum_{j=0}^{m-1} e^{2\pi i(j_1-j)\ell/m} = \begin{cases} m & \text{for } \ell = 0 \bmod m \\ 0 & \text{for } \ell \neq 0 \bmod m \end{cases}, \tag{B27}$$

we obtain

$$\Upsilon = m\sum_{k=0}^{n-1} \mathcal{G}\left(\frac{k_1-k}{n}\check{\tau}, m\Delta_a; \check{\tau}\right). \tag{B28}$$

Then we use the modular property (A25) and perform the second sum. We obtain

$$\Upsilon = N\frac{2m\Delta_a + 1 - \check{\tau}}{\check{\tau}} + \frac{N}{\check{\tau}}\mathcal{G}\left(0, \frac{N\Delta_a}{\check{\tau}}; -\frac{n}{\check{\tau}}\right). \tag{B29}$$

Notice in particular that Υ does not depend on i_1 or equivalently on (j_1, k_1). Moreover, Υ is invariant under $m\Delta_a \to m\Delta_a + 1$. Therefore,

$$\begin{aligned}\mathcal{A}_{ij} = N(1+\delta_{ij})\Bigg[-4 + \frac{1}{\check{\tau}}\sum_{a=1}^{3}\Bigg(2m\Delta_a \\ + 1 + \mathcal{G}\left(0, \frac{N\Delta_a}{\check{\tau}}; -\frac{n}{\check{\tau}}\right)\Bigg)\Bigg] \\ + \sum_{a=1}^{3}[\mathcal{G}(u_{iN}, \Delta_a) + \mathcal{G}(u_{jN}, \Delta_a) - \mathcal{G}(u_{ij}, \Delta_a) \\ - \mathcal{G}(0, \Delta_a)].\end{aligned} \tag{B30}$$

Here we recognize the expression in (3.30).

4. Perturbative corrections

We give some evidence that the term $\log[\det(\mathbb{1} + \mathcal{B}^{-1}\mathcal{C}/b_N N)]$ in (3.47) and (3.49) only leads to a contribution of order $\mathcal{O}(1)$ and to nonperturbative corrections in the large N limit, but no $1/N$ perturbative corrections. In order to do that, we expand

$$-\log\left[\det\left(\mathbb{1} + \frac{\mathcal{B}^{-1}\mathcal{C}}{b_N N}\right)\right] = \sum_{\ell=1}^{\infty}\frac{(-1)^\ell}{\ell}\mathrm{Tr}\left[\left(\frac{\mathcal{B}^{-1}\mathcal{C}}{b_N N}\right)^\ell\right] \tag{B31}$$

and compute the first few terms.

First order.—We compute $\mathrm{Tr}(\mathcal{B}^{-1}\mathcal{C})$. The two matrices $\mathcal{B}$ and $\mathcal{C}$ are defined in (3.31) and have size $(N-1)\times(N-1)$. Using the fact that $\mathcal{G}$ is an even function, the trace can be recast as

$$\begin{aligned}\mathrm{Tr}(\mathcal{B}^{-1}\mathcal{C}) &= \sum_{i=1}^{N-1}\mathcal{C}_{ii} - \frac{1}{N}\sum_{i,j=1}^{N-1}\mathcal{C}_{ij} \\ &= \sum_{a=1}^{3}\left[\frac{1}{N}\sum_{i,j=1}^{N}\mathcal{G}(u_{ij}, \Delta_a; \tau) - N\mathcal{G}(0, \Delta_a; \tau)\right].\end{aligned} \tag{B32}$$

The sum inside square brackets was already computed in (B26)–(B29) and it equals Υ, therefore

$$\begin{aligned}\mathrm{Tr}(\mathcal{B}^{-1}\mathcal{C}) &= N\sum_{a=1}^{3}\Bigg[\frac{2[m\Delta_a]_{\check{\tau}} - \check{\tau} + 1}{\check{\tau}} \\ &\quad + \frac{1}{\check{\tau}}\mathcal{G}\left(0, \frac{[m\Delta_a]_{\check{\tau}}}{\check{\tau}/n}; -\frac{n}{\check{\tau}}\right) - \mathcal{G}(0, \Delta_a; \tau)\Bigg], \\ &= N\frac{\check{\tau}\pm 1}{\check{\tau}} + N\sum_{a=1}^{3}\Bigg[\frac{1}{\check{\tau}}\mathcal{G}\left(0, \frac{[m\Delta_a]_{\check{\tau}}}{\check{\tau}/n}; -\frac{n}{\check{\tau}}\right) \\ &\quad - \mathcal{G}(0, \Delta_a; \tau)\Bigg].\end{aligned} \tag{B33}$$

In the first line we used that Υ is invariant under integer shifts of $m\Delta_a$, while the $\pm$ signs in the second line refer to the first and second case of parameter space, respectively. Then, using the large N behavior of $\mathcal{G}$ in (3.38) and

$$b_N([m\Delta]_{\check{\tau}}) = \pm\frac{1}{\check{\tau}} + \mathcal{O}(e^{-N}), \tag{B34}$$

we obtain the large N result

$$\frac{\mathrm{Tr}(\mathcal{B}^{-1}\mathcal{C})}{b_N N} = 1 \pm \check{\tau}\left(1 - \sum_{a=1}^{3}\mathcal{G}(0, \Delta_a; \tau)\right) + \mathcal{O}(e^{-N}) \tag{B35}$$

where, once again, the $\pm$ signs refer to the first and second case, respectively. We see that this quantity only receives nonperturbative corrections at large N.

Second order.—We compute $\mathrm{Tr}(\mathcal{B}^{-1}\mathcal{C}\mathcal{B}^{-1}\mathcal{C})$. Using $\mathcal{C}$ as a symmetric matrix, with some lengthy algebra we obtain

$$\begin{aligned}&\mathrm{Tr}(\mathcal{B}^{-1}\mathcal{C}\mathcal{B}^{-1}\mathcal{C}) \\ &= \sum_{i,j=1}^{N-1}(\mathcal{C}_{ij})^2 - \frac{2}{N}\sum_{i,j,k=1}^{N-1}\mathcal{C}_{ik}\mathcal{C}_{jk} + \frac{1}{N^2}\left(\sum_{i,j=1}^{N-1}\mathcal{C}_{ij}\right)^2, \\ &= \sum_{a,b=1}^{3}\left[\sum_{i,j=1}^{N}\mathcal{G}(u_{ij}, \Delta_a; \tau)\mathcal{G}(u_{ij}, \Delta_b; \tau) - \Upsilon(\Delta_a; \tau)\Upsilon(\Delta_b, \tau)\right].\end{aligned} \tag{B36}$$

This expression is valid for any HL solution $\{m, n, r\}$. However, evaluating the summation in the second line by brute force is quite complicated and here we will content ourselves with the T-transformed solutions, i.e., with the case $m = 1$.

We define the following quantity:

$$\mathcal{U} \equiv \sum_{a,b=1}^{3}\sum_{i,j=1}^{N}\mathcal{G}(u_{ij}, \Delta_a; \tau)\mathcal{G}(u_{ij}, \Delta_b; \tau), \tag{B37}$$

where u_{ij} is as in (3.16) but with $m = 1$. Using the fact that $\mathcal{G}$ is invariant under integer shifts of each of its

arguments (see Appendix A), the modular transformation formula (A25) and the constraint among $[m\Delta_a]_{\check\tau}$, we obtain

$$\begin{aligned}\mathcal{U} &= \sum_{a,b=1}^{3}\sum_{i,j=1}^{N}\mathcal{G}(u_{ij},[\Delta_a]_{\check\tau};\check\tau)\mathcal{G}(u_{ij},[\Delta_b]_{\check\tau};\check\tau),\\ &= N^2\frac{(\check\tau\pm 1)^2}{\check\tau^2}+\frac{2(\check\tau\pm 1)}{\check\tau^2}\sum_{a=1}^{3}\sum_{i,j=1}^{N}\mathcal{G}\left(\frac{u_{ij}}{\check\tau},\frac{[\Delta_a]_{\check\tau}}{\check\tau};-\frac{1}{\check\tau}\right)\\ &\quad+\frac{1}{\check\tau^2}\sum_{a,b=1}^{3}\sum_{i,j=1}^{N}\mathcal{G}\left(\frac{u_{ij}}{\check\tau},\frac{[\Delta_a]_{\check\tau}}{\check\tau};-\frac{1}{\check\tau}\right)\mathcal{G}\left(\frac{u_{ij}}{\check\tau},\frac{[\Delta_b]_{\check\tau}}{\check\tau};-\frac{1}{\check\tau}\right),\end{aligned}\tag{B38}$$

where the $\pm$ signs refer to the first and second case, respectively. Using the series expansion of $\mathcal{G}$ in (A23) and the same method we used to compute Υ after (B26), the summation on the second line gives

$$\sum_{i,j=1}^{N}\mathcal{G}\left(\frac{u_{ij}}{\check\tau},\frac{[\Delta_a]_{\check\tau}}{\check\tau};-\frac{1}{\check\tau}\right)=N^2\mathcal{G}\left(0,\frac{N[\Delta_a]_{\check\tau}}{\check\tau};-\frac{N}{\check\tau}\right).\tag{B39}$$

The summation on the third line involves the object

$$\mathcal{W}_{ab}\equiv\sum_{i,j=1}^{N}\mathcal{G}\left(\frac{u_{ij}}{\check\tau},\frac{[\Delta_a]_{\check\tau}}{\check\tau};-\frac{1}{\check\tau}\right)\mathcal{G}\left(\frac{u_{ij}}{\check\tau},\frac{[\Delta_b]_{\check\tau}}{\check\tau};-\frac{1}{\check\tau}\right)=2\sum_{k,\ell=1}^{\infty}c_a^k c_b^\ell\sum_{i,j=1}^{N}\left[\left(\frac{\tilde z_i}{\tilde z_j}\right)^{k+\ell}+\left(\frac{\tilde z_i}{\tilde z_j}\right)^{k-\ell}\right],\tag{B40}$$

where the coefficients c_a^ℓ are given by

$$c_a^\ell=\frac{\tilde y_a^\ell-(\check q/\tilde y_a)^\ell}{1-\check q^\ell}=\sum_{r=0}^{\infty}[(\check q^r\tilde y_a)^\ell-(\check q^{r+1}/\tilde y_a)^\ell],\tag{B41}$$

and $\tilde z_i=e^{2\pi i u_i/\check\tau}$, $\tilde y_a=e^{2\pi i[\Delta_a]_{\check\tau}/\check\tau}$, $\check q=e^{-2\pi i/\check\tau}$. The sum over i, j can be performed exploiting (B27), and we obtain

$$\mathcal{W}_{ab}=4N^2\sum_{\alpha,\beta=0}^{\infty}c_a^{N(\alpha+1)}c_b^{N(\beta+1)}+2N^2\sum_{\alpha,\beta=0}^{\infty}\sum_{\gamma=1}^{N-1}c_a^{N\alpha+\gamma}(c_b^{N\beta+\gamma}+c_b^{N(\beta+1)-\gamma}).\tag{B42}$$

The first sum can be performed exactly using

$$\sum_{\alpha=1}^{\infty}c_a^{N\alpha}=\frac{1}{2}\mathcal{G}\left(0,\frac{N[\Delta_a]_{\check\tau}}{\check\tau};-\frac{N}{\check\tau}\right).\tag{B43}$$

Substituting back in $\mathcal{W}_{ab}$, in $\mathcal{U}$ and finally in the trace we obtain the expansion

$$\mathrm{Tr}(\mathcal{B}^{-1}\mathcal{C}\mathcal{B}^{-1}\mathcal{C})=\frac{2N^2}{\check\tau^2}\sum_{a,b=1}^{3}\sum_{\alpha,\beta=0}^{\infty}\sum_{\gamma=1}^{N-1}c_a^{N\alpha+\gamma}(c_b^{N\beta+\gamma}+c_b^{N(\beta+1)-\gamma}).\tag{B44}$$

Let us study, in the large N limit, the summations containing the two terms in parentheses separately. The second term involves

$$\begin{aligned}\chi_{ab}^{(2)}&\equiv\sum_{\alpha,\beta=0}^{\infty}\sum_{\gamma=1}^{N-1}c_a^{N\alpha+\gamma}c_b^{N(\beta+1)-\gamma},\\ &=\sum_{\alpha,\beta=0}^{\infty}\sum_{\gamma=1}^{N-1}\sum_{r,s=0}^{\infty}[(\check q^r\tilde y_a)^{N\alpha+\gamma}-(\check q^{r+1}/\tilde y_a)^{N\alpha+\gamma}][(\check q^s\tilde y_b)^{N\beta+N-\gamma}-(\check q^{s+1}/\tilde y_b)^{N\beta+N-\gamma}].\end{aligned}\tag{B45}$$

The fundamental property of $[\Delta_a]_{\check\tau}$ in (3.40) guarantees that $|\check q|<|\tilde y_a|<1$, and similarly $|\check q|<|\tilde y_b|<1$. Therefore the defining sum of $\chi_{ab}^{(2)}$ involves terms whose absolute value is smaller than a number smaller than 1, elevated to the powers $N(\alpha+\beta)+N$. This implies that $\chi_{ab}^{(2)}$ is of order $\mathcal{O}(e^{-N})$, and thus it only produces nonperturbative corrections. Next we consider

$$\chi^{(1)}_{ab} \equiv \sum_{\alpha,\beta=0}^{\infty} \sum_{\gamma=1}^{N-1} c_a^{N\alpha+\gamma} c_b^{N\beta+\gamma}. \tag{B46}$$

By expanding out the coefficients c_a^{ℓ} as in (B41), one obtains four sums. One of them is

$$\begin{aligned} &\sum_{\alpha,\beta=1}^{\infty} \sum_{\gamma=1}^{N-1} \sum_{r,s=0}^{\infty} (\check{q}^r \tilde{y}_a)^{N\alpha+\gamma} (\check{q}^s \tilde{y}_b)^{N\beta+\gamma} \\ &= \sum_{\alpha,\beta,r,s=0}^{\infty} (\check{q}^r \tilde{y}_a)^{N\alpha} (\check{q}^s \tilde{y}_b)^{N\beta} \frac{\check{q}^{r+s} \tilde{y}_a \tilde{y}_b - (\check{q}^{r+s} \tilde{y}_a \tilde{y}_b)^N}{1 - \check{q}^{r+s} \tilde{y}_a \tilde{y}_b} = \sum_{\ell=0}^{\infty} \frac{(\ell+1) \check{q}^{\ell} \tilde{y}_a \tilde{y}_b}{1 - \check{q}^{\ell} \tilde{y}_a \tilde{y}_b} + \mathcal{O}(e^{-N}). \end{aligned} \tag{B47}$$

In the first equality we summed the finite geometric series in γ, while in the second equality we only kept the terms that are finite in the large N limit. The other three sums that constitute $\chi^{(1)}_{ab}$ can be treated similarly. The result, up to nonperturbative corrections, can be nicely expressed in terms of the function

$$\tilde{\gamma}(u;\tau) \equiv \frac{1}{2\pi i} \partial_u \log(\tilde{\Gamma}(u;\tau,\tau)), \tag{B48}$$

which can be expanded as follows:

$$\tilde{\gamma}(u;\tau) = \sum_{k=0}^{\infty} (k+1) \left(\frac{q^k z}{1 - q^k z} + \frac{q^{k+2}/z}{1 - q^{k+2}/z} \right). \tag{B49}$$

One obtains

$$\chi^{(1)}_{ab} = \tilde{\gamma}\left(\frac{[\Delta_a]_{\check{\tau}} + [\Delta_b]_{\check{\tau}}}{\check{\tau}}; -\frac{1}{\check{\tau}} \right) - \tilde{\gamma}\left(\frac{[\Delta_a]_{\check{\tau}} - [\Delta_b]_{\check{\tau}} - 1}{\check{\tau}}; -\frac{1}{\check{\tau}} \right) + \mathcal{O}(e^{-N}). \tag{B50}$$

We can finally substitute into the trace, and obtain the second-order contribution

$$\frac{\mathrm{Tr}(\mathcal{B}^{-1}\mathcal{C}\mathcal{B}^{-1}\mathcal{C})}{b_N^2 N^2} = \frac{1}{2} \sum_{a,b=1}^{3} \left[\tilde{\gamma}\left(\frac{[\Delta_a]_{\check{\tau}} + [\Delta_b]_{\check{\tau}}}{\check{\tau}}; -\frac{1}{\check{\tau}} \right) - \tilde{\gamma}\left(\frac{[\Delta_a]_{\check{\tau}} - [\Delta_b]'_{\check{\tau}}}{\check{\tau}}; -\frac{1}{\check{\tau}} \right) \right] + \mathcal{O}(e^{-N}). \tag{B51}$$

Also at this order we see that there are only nonperturbative corrections at large N.

APPENDIX C: SOLUTIONS WITH TWO ANGULAR MOMENTA

In this Appendix we generalize our discussion in the main body of the paper to black hole solutions with two different angular momenta (and correspondingly two different chemical potentials for them), but still with three equal $U(1)$ charges (and chemical potentials), so that the solutions can be constructed using 5D minimal gauged supergravity. We also discuss the supersymmetry of those solutions, and the uplift to 10D type IIB supergravity. Finally, we study the supersymmetric embeddings of Euclidean D3-branes.

In Appendix E, as an aside, we will present supersymmetric embeddings of Lorentzian D3-branes that generalize the giant graviton and dual giant graviton solutions in empty AdS_5 [59–61] to Lorentzian black hole backgrounds.

1. 5D minimal gauged supergravity

The bosonic Lagrangian of 5D minimal gauged supergravity is (4.1), that we repeat here[29]:

$$\mathcal{L} = (R + 12g^2) * 1 - \frac{2}{3} F \wedge *F + \frac{8}{27} F \wedge F \wedge A, \tag{C1}$$

where A is the graviphoton potential, $F = dA$ is its field strength, and we followed the notation of [5]. As in the main text, we set the dimensionful coupling g to 1.

The authors of [89], generalizing previous work of [88,92,93], constructed a four-parameter family of charged and rotating black hole solutions:

[29]Given a p-form $\omega_{(p)} = \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} dx^{\mu_1} \ldots dx^{\mu_p}$ in d dimensions, we define its Hodge dual in Lorentzian signature as $*\omega_{(p)} = \frac{\sqrt{-g}}{p!(d-p)!} \omega^{\nu_1 \ldots \nu_p} \epsilon_{\nu_1 \ldots \nu_p \mu_1 \ldots \mu_{d-p}} dx^{\mu_1} \ldots dx^{\mu_{d-p}}$ where $\epsilon_{0\ldots(d-1)} = 1$. Then $*^2 = (-1)^{p(d-p)+1}$, as well as $*1 = d\mathrm{vol}_d$ and $\omega_{(p)} \wedge *\omega_{(p)} = \frac{1}{p!} \omega_{\mu_1 \ldots \mu_p} \omega^{\mu_1 \ldots \mu_p} d\mathrm{vol}_d$.

$$
\begin{aligned}
ds^2 &= -\frac{\Delta_\theta[(1+r^2)\rho^2 dt + 2q\nu]dt}{\Xi_a\Xi_b\rho^2} + \frac{2q\nu w}{\rho^2} + \frac{f_t}{\rho^4}\left(\frac{\Delta_\theta dt}{\Xi_a\Xi_b} - w\right)^2 + \frac{\rho^2 dr^2}{\Delta_r} \\
&\quad + \frac{\rho^2 d\theta^2}{\Delta_\theta} + \frac{r^2+a^2}{\Xi_a}\sin^2(\theta)d\phi^2 + \frac{r^2+b^2}{\Xi_b}\cos^2(\theta)d\psi^2, \\
A &= \frac{3q}{2\rho^2}\left(\frac{\Delta_\theta dt}{\Xi_a\Xi_b} - w\right) - \alpha dt,
\end{aligned} \tag{C2}
$$

where

$$
\begin{aligned}
&\nu = b\sin^2(\theta)d\phi + a\cos^2(\theta)d\psi, \qquad w = a\sin^2(\theta)\frac{d\phi}{\Xi_a} + b\cos^2(\theta)\frac{d\psi}{\Xi_b}, \\
&\Xi_a = 1-a^2, \qquad \Xi_b = 1-b^2, \qquad f_t = 2(m+abq)\rho^2 - q^2, \\
&\Delta_\theta = 1 - a^2\cos^2(\theta) - b^2\sin^2(\theta), \qquad \rho^2 = r^2 + a^2\cos^2(\theta) + b^2\sin^2(\theta), \\
&\Delta_r = \frac{(r^2+a^2)(r^2+b^2)(1+r^2) + q^2 + 2abq}{r^2} - 2m.
\end{aligned} \tag{C3}
$$

The field strength can be written as

$$
\begin{aligned}
F &= \frac{3qr}{\rho^4}dr \wedge w - \frac{3q}{\Xi_a\Xi_b\rho^4}\left(r\Delta_\theta dr - \frac{a^2-b^2}{2}(1+r^2)\sin(2\theta)d\theta\right) \wedge dt \\
&\quad - \frac{3q\sin(2\theta)}{2\rho^4}d\theta \wedge \left(a\frac{r^2+a^2}{\Xi_a}d\phi - b\frac{r^2+b^2}{\Xi_b}d\psi\right),
\end{aligned} \tag{C4}
$$

while $*_5 F - \frac{2}{3}A \wedge F = d\alpha_{(2)}$ with

$$
\alpha_{(2)} = \frac{3q}{2\rho^2}\left(\frac{\Delta_\theta dt}{\Xi_a\Xi_b} - w\right) \wedge \left(\nu + \frac{2}{3}\alpha dt\right) - \frac{3q\cos(2\theta)}{4\Xi_a\Xi_b}(dt \wedge (bd\phi - ad\psi) + d\phi \wedge d\psi). \tag{C5}
$$

The coordinates are $(t, r, \theta, \phi, \psi)$, where $\theta \in [0, \pi/2]$ while ϕ, ψ have period 2π. The constant α is arbitrary in Lorentzian space (it can be shifted by gauge transformations). The four parameters are (a, b, q, m), and correspondingly there are four independent conserved charges: the energy E (associated to the Killing vector $\frac{\partial}{\partial t}$), two angular momenta $J_{1,2}$ (associated to the Killing vectors $\frac{\partial}{\partial\phi}$ and $\frac{\partial}{\partial\psi}$, respectively), and the electric charge Q. Setting $a = b$ one recovers the solution in (4.2) which has $J_1 = J_2 \equiv J$.

The outer horizon is the largest positive root of $\Delta_r(r_+) = 0$, and we denote it by r_+. This is a Killing horizon generated by the Killing vector field

$$
V = \frac{\partial}{\partial t} + \Omega_1\frac{\partial}{\partial\phi} + \Omega_2\frac{\partial}{\partial\psi}. \tag{C6}
$$

Here $\Omega_{1,2}$ are the angular velocities at the horizon, measured in a nonrotating frame at infinity. Evaluating the surface gravity one determines the Hawking temperature $T \equiv 1/\beta$. Besides, one defines the electrostatic potential at the horizon, $\Phi = \iota_V A|_{r_+} - \iota_V A|_\infty$, and the entropy S, equal to a quarter of the horizon area. Explicit expressions for the charges and chemical potentials can be found in [5]. Those quantities satisfy the first law of thermodynamics,

$$
dE = TdS + \Omega_1 dJ_1 + \Omega_2 dJ_2 + \Phi dQ. \tag{C7}
$$

It turns out to be convenient to trade the parameter m for r_+, and thus use (a, b, q, r_+) as the independent parameters. Since r_+ is a root of Δ_r, the relation

$$
m = \frac{(r_+^2+a^2)(r_+^2+b^2)(1+r_+^2) + q^2 + 2abq}{2r_+^2} \tag{C8}
$$

can be used to eliminate m in favor of r_+.

The analytic continuation to Euclidean signature, studied in detail in [5,69], is obtained by rotating $t \to -it_E$, $a \to i\tilde{a}$, and $b \to i\tilde{b}$. This yields a real Euclidean metric, though the gauge field becomes imaginary. Regularity of such a metric around $r = r_+$ requires us to compactify the Euclidean time t_E with a period equal to β, and more precisely it requires us to identify

$$(t_E, \phi, \psi) \cong (t_E + \beta, \phi - i\Omega_1\beta, \psi - i\Omega_2\beta) \cong (t_E, \phi + 2\pi, \psi) \cong (t_E, \phi, \psi + 2\pi). \quad \text{(C9)}$$

Notice that the Killing vector (C6) precisely generates rotations of the Euclidean circle that shrinks at r_+. Regularity of the gauge field at r_+, in a gauge which is regular at the horizon, requires $\iota_V A|_{r_+} = 0$, which implies

$$\alpha = \Phi. \quad \text{(C10)}$$

The on shell action can then be computed for regular Euclidean metrics, regularized using background subtraction (which assigns vanishing action to thermal AdS_5). The family of solutions can then be extended to generic complex values of the parameters (a, b, q, r_+), and the on shell action I_{SUGRA} is extended analytically. It satisfies the so-called quantum statistical relation

$$I_{\text{SUGRA}} = \beta(E - TS - \Omega_1 J_1 - \Omega_2 J_2 - \Phi Q). \quad \text{(C11)}$$

a. Supersymmetry

The solutions (C2) become supersymmetric for

$$m = (1 + a + b)q. \quad \text{(C12)}$$

This gives a three-parameter family of complex (and, generically, nonextremal) solutions in terms of (a, b, q). The charges take the values

$$E = \frac{\pi q(3 + ab - (1 + a)b^2 - (1 + b)a^2)}{4(1 - a)(1 - a^2)(1 - b)(1 - b^2)}, \qquad Q = \frac{\pi q}{2(1 - a^2)(1 - b^2)},$$
$$J_1 = \frac{\pi q(2a + b + ab)}{4(1 - a)(1 - a^2)(1 - b^2)}, \qquad J_2 = \frac{\pi q(2b + a + ab)}{4(1 - a^2)(1 - b)(1 - b^2)}, \quad \text{(C13)}$$

and satisfy the linear supersymmetric relation $E = J_1 + J_2 + \frac{3}{2}Q$. In the following it will be convenient to use (a, b, r_+) as the independent parameters, and combining the change of variables (C8) with the SUSY condition (C12) one finds

$$q = -ab + (1 + a + b)r_+^2 \mp i r_+(r_+^2 - r_*^2) \quad \text{with} \quad r_*^2 = a + b + ab. \quad \text{(C14)}$$

Here r_* is the position of the outer horizon for real supersymmetric and extremal solutions, i.e., for the Euclidean rotation of Lorentzian BPS black hole solutions. The $\mp$ signs come from the branch cut of a square root. For the upper sign, the supersymmetry condition becomes

$$q = -(a - ir_+)(b - ir_+)(1 - ir_+) \qquad (\text{1st branch}). \quad \text{(C15)}$$

We refer to the corresponding family of solutions as the "first branch". The "second branch," which follows from the lower sign, is obtained by sending $i \to -i$. In the first branch, the chemical potentials take the values

$$\Phi = \frac{3ir_+(1 - ir_+)}{2(r_*^2 + ir_+)}, \qquad \beta = \frac{2\pi(a - ir_+)(b - ir_+)(r_*^2 + ir_+)}{(r_+^2 - r_*^2)(2(1 + a + b)r_+ + i(r_*^2 - 3r_+^2))},$$
$$\Omega_1 = \frac{(r_*^2 + iar_+)(1 - ir_+)}{(r_*^2 + ir_+)(a - ir_+)}, \qquad \Omega_2 = \frac{(r_*^2 + ibr_+)(1 - ir_+)}{(r_*^2 + ir_+)(b - ir_+)}. \quad \text{(C16)}$$

They satisfy the relation

$$\beta(1 + \Omega_1 + \Omega_2 - 2\Phi) = \pm 2\pi i, \qquad (\text{1st/2nd branch}) \quad \text{(C17)}$$

where the $\pm$ sign depends on the branch. Following [5][30] and the discussion in Sec. II, we define new chemical potentials σ_g, τ_g, Δ_g as

$$\sigma_g = \frac{\beta}{2\pi i}(\Omega_1 - 1), \qquad \tau_g = \frac{\beta}{2\pi i}(\Omega_2 - 1), \qquad \Delta_g = \frac{\beta}{2\pi i}\left(\frac{2}{3}\Phi - 1\right). \quad \text{(C18)}$$

In the first branch they are given by

$$\sigma_g = \frac{(a - 1)(b - ir_+)}{2i(1 + a + b)r_+ - r_*^2 + 3r_+^2}, \quad \Delta_g = \frac{(a - ir_+)(b - ir_+)}{2i(1 + a + b)r_+ - r_*^2 + 3r_+^2}, \quad \tau_g = \frac{(b - 1)(a - ir_+)}{2i(1 + a + b)r_+ - r_*^2 + 3r_+^2}, \quad \text{(C19)}$$

[30]The potentials ω_1, ω_2, φ defined in [5] differ from ours by a simple rescaling.

(while the values in the second branch are obtained sending $i \to -i$ and changing the overall sign). These potentials satisfy

$$\sigma_g + \tau_g - 3\Delta_g = \pm 1 \qquad (1\text{st}/2\text{nd branch}). \tag{C20}$$

In terms of them, the on shell action takes the very simple form

$$I_{\text{SUGRA}} = \frac{i\pi^2}{2g^3 G_5}\frac{\Delta_g^3}{\sigma_g \tau_g} \tag{C21}$$

in both branches. Notice in particular that it does not depend on β. Here, for clarity, we reinstated the coupling g and the 5D Newton constant G_5 that were previously set to 1. This expression generalizes (4.29).

b. Killing spinor

The solutions (C2) with the SUSY condition (C12) admit a Killing spinor ϵ. The Killing vector arising as a bilinear of the Killing spinor is $K = \frac{\partial}{\partial t} + \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \psi}$. The spinor satisfies

$$\mathcal{L}_{\frac{\partial}{\partial t}}\epsilon = \frac{i}{2}(1-2\alpha)\epsilon, \qquad \mathcal{L}_{\frac{\partial}{\partial \phi}}\epsilon = \mathcal{L}_{\frac{\partial}{\partial \psi}}\epsilon = \frac{i}{2}\epsilon, \tag{C22}$$

where $\mathcal{L}$ is the Lie derivative. The second equation implies that $e^{2\pi\mathcal{L}_{\partial/\partial\phi}}\epsilon = e^{2\pi\mathcal{L}_{\partial/\partial\psi}}\epsilon = -\epsilon$, namely, that ϵ is antiperiodic along the two circles parametrized by ϕ and ψ, which is a necessary condition for a spinor to be well defined since the two circles shrink somewhere inside S^3. The first equation, combined with the gauge-field regularity condition (C10) and the SUSY relation (C17) among the chemical potentials, implies $\mathcal{L}_V\epsilon = \mp\frac{\pi}{\beta}\epsilon$, where V is the Killing vector (C6) and $\mp$ correspond to the first/second branch. In turn this guarantees that $e^{-i\beta\mathcal{L}_V}\epsilon = -\epsilon$, and thus that ϵ is antiperiodic along the Euclidean time circle that shrinks at the horizon.

In order to construct the Killing spinor ϵ, it is convenient to use orthotoric coordinates $(t, \xi, \eta, \Phi, \Psi)$ [94]. The coordinate change is given by

$$\begin{aligned} r^2 &= r_*^2 + (a+b)\tilde{m} + \frac{(a+b)^2}{2}\tilde{m} + \frac{a^2-b^2}{2}\tilde{m}\xi, \qquad \theta = \frac{1}{2}\arccos(\eta), \\ \phi &= t - \frac{4(1-a^2)}{(a^2-b^2)\tilde{m}}(\Phi - \Psi), \qquad \psi = t - \frac{4(1-b^2)}{(a^2-b^2)\tilde{m}}(\Phi + \Psi), \end{aligned} \tag{C23}$$

where we defined the new mass parameter[31]

$$\tilde{m} = \frac{m}{(a+b)(1+a)(1+b)(1+a+b)} - 1. \tag{C24}$$

In these coordinates, supersymmetric black hole metrics are described by the simple vielbein

$$\begin{aligned} {}^0E^0 &= f(dt - \omega), \\ E^1 &= \frac{1}{f^{1/2}}\sqrt{\frac{\eta-\xi}{\mathcal{F}(\xi)}}d\xi, \qquad E^2 = \frac{1}{f^{1/2}}\sqrt{\frac{\mathcal{F}(\xi)}{(\eta-\xi)}}(d\Phi + \eta d\Psi), \\ E^3 &= -\frac{1}{f^{1/2}}\sqrt{\frac{\eta-\xi}{\mathcal{G}(\eta)}}d\eta, \qquad E^4 = \frac{1}{f^{1/2}}\sqrt{\frac{\mathcal{G}(\eta)}{(\eta-\xi)}}(d\Phi + \xi d\Psi), \end{aligned} \tag{C25}$$

where $\mathcal{F}(\xi)$ and $\mathcal{G}(\eta)$ are the cubic polynomials

$$\begin{aligned} \mathcal{G}(\eta) &= -\frac{4(1-\eta^2)}{(a^2-b^2)\tilde{m}}[(1-a^2)(1+\eta) + (1-b^2)(1-\eta)], \\ \mathcal{F}(\xi) &= -\mathcal{G}(\xi) - 4\frac{1+\tilde{m}}{\tilde{m}}\left(\frac{2+a+b}{a-b} + \xi\right)^3, \end{aligned} \tag{C26}$$

while

[31]The extremality condition for supersymmetric solutions, namely $r_+ = r_*$, simply reads $\tilde{m} = 0$ [5].

$$f = \frac{24(\eta-\xi)}{\mathcal{F}''(\xi)+\mathcal{G}''(\eta)},$$
$$\omega = -\frac{\mathcal{F}'''+\mathcal{G}'''}{48(\eta-\xi)^2}\left\{\left[\mathcal{F}+(\eta-\xi)\left(\frac{\mathcal{F}'}{2}-\frac{1}{4}\left(\frac{2+a+b}{a-b}+\xi\right)^2\mathcal{F}'''\right)\right](d\Phi+\eta d\Psi)+\mathcal{G}(d\Phi+\xi d\Psi)\right\} + \frac{\mathcal{F}'''\mathcal{G}'''}{288}[(\eta+\xi)d\Phi+\eta\xi d\Psi]+\frac{2}{\tilde{m}}d\Psi. \tag{C27}$$

Explicitly, the function f takes the form

$$f = -\frac{\tilde{m}(a-b)(\eta-\xi)}{(2+a+b)(1+\tilde{m})+(a-b)(\eta+\tilde{m}\xi)}. \tag{C28}$$

One can also verify that, at the location ξ_+ of the outer horizon $r=r_+$, we have

$$\mathcal{F}(\xi_+)=0. \tag{C29}$$

The metric is then $ds^2=-(E^0)^2+(E^1)^2+(E^2)^2+(E^3)^2+(E^4)^2$. The gauge potential is

$$A = \frac{3(1-f)}{2}dt-\alpha dt - \frac{6(1+a)(1+b)(1+\tilde{m})f}{(a-b)^2(a+b)\tilde{m}^2(\eta-\xi)} \times[(a+b-(a-b)\eta)d\Phi+((a+b)\eta-(a-b))d\Psi]. \tag{C30}$$

Using vielbein indices, the nonvanishing components of the field strength are F_{01}, F_{03}, F_{12}, F_{14}, F_{23}, F_{34}, and antisymmetrizations. The metric determinant is $-\det(g)=(\eta-\xi)^2/f^2$.

The Killing spinor equation of 5D minimal gauged supergravity reads

$$\left[\nabla_\mu - iA_\mu - \frac{1}{2}\gamma_\mu - \frac{i}{12}(\gamma_\mu^{\nu\rho}-4\delta_\mu^\nu\gamma^\rho)F_{\nu\rho}\right]\epsilon=0. \tag{C31}$$

We use the orientation $\gamma^{01234}=-i$. Then we impose the relation $\frac{i}{2}(\gamma^{12}-\gamma^{34})\epsilon=\epsilon$, which implies the following projectors:

$$i\epsilon=-\gamma^0\epsilon=-\gamma^{12}\epsilon=\gamma^{34}\epsilon. \tag{C32}$$

One verifies that the following spinor solves (C31):

$$\epsilon=\exp\left\{\frac{i}{2}\left[(3-2\alpha)t-\frac{4(1-a^2)}{(a^2-b^2)\tilde{m}}(\Phi-\Psi)-\frac{4(1-b^2)}{(a^2-b^2)\tilde{m}}(\Phi+\Psi)\right]\right\}\sqrt{f}\epsilon_0, \tag{C33}$$

where ϵ_0 is a constant spinor satisfying (C32). When checking the Killing spinor equation, one encounters a series of square roots. We started from a domain in which all radicands in (C25) are positive, and then analytically continued from there.

c. Killing spinor for equal angular momenta

Black hole solutions with equal angular momenta $J_1=J_2$ correspond to $a=b$ and have been discussed in the main text. However, the orthotoric coordinates in (C23) are singular for $a=b$, and thus the $a=b$ limit must be taken carefully. Following Sec. 4 of [94], we set[32]

$$b=a+\frac{4(1-a^2)}{a\tilde{m}}\lambda \quad \text{with} \quad \tilde{m}=\frac{m}{2a(1+a)^2(1+2a)}-1,$$
$$\Phi=\lambda\tilde{\Phi}, \qquad \Psi=\lambda\tilde{\Psi}, \qquad \xi=-\frac{1}{\lambda}\tilde{\xi}, \tag{C34}$$

and take $\lambda\to 0$. The change of coordinates reduces to

$$r^2=r_*^2+2a(1+a)\tilde{m}+4(1-a^2)\tilde{\xi}, \qquad \theta=\frac{1}{2}\arccos(\eta),$$
$$\phi=t+\frac{\tilde{\Phi}-\tilde{\Psi}}{2}, \qquad \psi=t+\frac{\tilde{\Phi}+\tilde{\Psi}}{2}, \tag{C35}$$

with $r_*^2=2a+a^2$. The functions $\mathcal{F}$ and $\mathcal{G}$ behave as

$$\mathcal{G}(\eta)=\frac{\tilde{\mathcal{G}}(\eta)}{\lambda}+\mathcal{O}(\lambda^0), \qquad \mathcal{F}(\xi)=\frac{\tilde{\mathcal{F}}(\tilde{\xi})}{\lambda^3}+\mathcal{O}(\lambda^{-2}), \tag{C36}$$

with

$$\tilde{\mathcal{G}}(\eta)=1-\eta^2,$$
$$\tilde{\mathcal{F}}(\tilde{\xi})=\left(1-\frac{4\tilde{\xi}}{\tilde{m}}\right)\tilde{\xi}^2+4\frac{1+\tilde{m}}{\tilde{m}}\left(\frac{a\tilde{m}}{2(1-a)}+\tilde{\xi}\right)^3. \tag{C37}$$

The function f and the 1-form ω reduce to

$$f=\frac{2(1-a)\tilde{\xi}}{a(1+\tilde{m})+2(1-a)\tilde{\xi}},$$
$$\omega=\left[\frac{1}{3f}\left(\frac{\tilde{\mathcal{F}}'(\tilde{\xi})}{2\tilde{\xi}}-1\right)-\frac{(1+\tilde{m})a^2}{4(1-a)^2\tilde{\xi}}\right](d\tilde{\Phi}+\eta d\tilde{\Psi}). \tag{C38}$$

In these coordinates, the outer horizon $r=r_+$ is located at

$$\tilde{\xi}=\tilde{\xi}_+=\frac{(a-ir_+)(r_*^2-r_+^2)}{4(1-a)(1+a)^2} \tag{C39}$$

[32]In this case, the extremality condition is $m=2a(1+a)^2\times(1+2a)$, or more easily $\tilde{m}=0$.

on the first branch (while the value on the second branch is obtained by sending $i \to -i$), and one verifies that

$$\tilde{\mathcal{F}}(\tilde{\xi}_+) = 0. \tag{C40}$$

On the other hand, f vanishes at the horizon only in the extremal BPS limit $r_+ \to r_*$.

The vielbein (C25) reduces to

$$\begin{aligned} E^0 &= f(dt-\omega), \qquad E^1 = -\frac{1}{f^{1/2}}\sqrt{\frac{\tilde{\xi}}{\tilde{\mathcal{F}}(\tilde{\xi})}}d\tilde{\xi}, \\ E^2 &= \frac{1}{f^{1/2}}\sqrt{\frac{\tilde{\mathcal{F}}(\tilde{\xi})}{\tilde{\xi}}}(d\tilde{\Phi}+\eta d\tilde{\Psi}), \\ E^3 &= -\frac{1}{f^{1/2}}\sqrt{\frac{\tilde{\xi}}{\tilde{\mathcal{G}}(\eta)}}d\eta, \qquad E^4 = -\frac{1}{f^{1/2}}\sqrt{\tilde{\mathcal{G}}(\eta)\tilde{\xi}}d\tilde{\Psi}. \end{aligned} \tag{C41}$$

In these coordinates the black hole solution reads

$$\begin{aligned} ds^2 &= -f^2(dt-\omega)^2 + \frac{1}{f}\left[\frac{\tilde{\xi}}{\tilde{\mathcal{F}}}d\tilde{\xi}^2 + \frac{\tilde{\mathcal{F}}}{\tilde{\xi}}(d\tilde{\Phi}+\eta d\tilde{\Psi})^2 \right. \\ &\quad \left. + \tilde{\xi}\left(\frac{d\eta^2}{1-\eta^2} + (1-\eta^2)d\tilde{\Psi}^2\right)\right], \\ A &= \left(\frac{3}{2}(1-f)-\alpha\right)dt - \frac{3a^2(1+\tilde{m})f}{8(1-a)^2\tilde{\xi}}(d\tilde{\Phi}+\eta d\tilde{\Psi}). \end{aligned} \tag{C42}$$

In this case, using vielbein indices, the nonvanishing components of the field strength are F_{01}, F_{12}, F_{34} and their antisymmetrizations.

One can check that the following spinor solves the Killing spinor equation (C31):

$$\epsilon = \exp\left\{\frac{i}{2}[(3-2\alpha)t+\tilde{\Phi}]\right\}\sqrt{f}\epsilon_0, \tag{C43}$$

where ϵ_0 is a constant spinor that satisfies the same projections as in (C32).

2. Ten-dimensional type IIB supergravity

In order to discuss D3-brane embeddings, we need to uplift the black hole solutions to 10D type IIB supergravity on $AdS_5 \times S^5$ [90,95]. It turns out that 5D minimal gauged supergravity can be consistently embedded into $AdS_5 \times SE_5$ for any Sasaki-Einstein 5-manifold SE_5 [96]. This general point of view is particularly useful when discussing supersymmetry, so let us review it here.

The bosonic action of 10D type IIB supergravity restricted to the metric G_{MN} and the 5-form flux $F_{(5)}$ (this is a consistent truncation) is

$$S_{\mathrm{IIB}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G}\left[R_{10\mathrm{d}} - \frac{1}{480}F_{M_1\ldots M_5}F^{M_1\ldots M_5}\right], \tag{C44}$$

supplemented by the self-duality condition $F_{(5)} = *F_{(5)}$. We reduce it on a Sasaki-Einstein manifold SE_5, which in general can be locally written as a $U(1)$ fibration over a Kähler Einstein base B. We consider the ansatz:

$$\begin{aligned} ds_{10}^2 &= ds_5^2 + ds^2(B) + (e^9)^2, \quad e^9 = \frac{1}{3}(d\psi_s + \mathcal{A} + 2A) \\ F_{(5)} &= (1+*)G_{(5)}, \quad G_{(5)} = -4\epsilon_{(5)} + \frac{2}{3}J\wedge *_5 F. \end{aligned} \tag{C45}$$

Here B is a Kähler-Einstein 4-manifold, normalized such that $R_{\mathrm{KE}} = 24$ is its scalar curvature, with Kähler form J (so that $\frac{1}{2}J\wedge J = d\mathrm{vol}_B$ is the volume form), and $\mathcal{A}$ is a specific $U(1)$ connection on B with $d\mathcal{A} = 6J$. The resulting Sasaki-Einstein space has $R_{\mathrm{SE}} = 20$. Then ds_5^2 is the 5D spacetime metric, $\epsilon_{(5)}$ is its volume form, $*_5$ is the 5D Hodge dual operator, A is the 5D graviphoton field, and $F = dA$ is its field strength. Using

$$*G_{(5)} = 2J\wedge J\wedge e^9 - \frac{2}{3}F\wedge J\wedge e^9, \tag{C46}$$

the Bianchi identity for $F_{(5)}$ gives the 5D equation of motion:

$$d *_5 F = \frac{2}{3}F\wedge F. \tag{C47}$$

The full expression for $F_{(5)}$, and a possible choice of potential such that $F_{(5)} = dC_{(4)}$, are

$$\begin{aligned} F_{(5)} &= -4\epsilon_{(5)} + 2J\wedge J\wedge e^9 - \frac{2}{3}F\wedge J\wedge e^9 + \frac{2}{3}J\wedge *_5 F, \\ C_{(4)} &= -4\beta_{(4)} + \frac{1}{9}\tilde{\mathcal{A}}\wedge[(3J-F)\wedge e^9 + *_5 F]. \end{aligned} \tag{C48}$$

Here $\beta_{(4)}$ is a 4-form such that $\epsilon_{(5)} = d\beta_{(4)}$, while $\tilde{\mathcal{A}}$ is any connection on the Kähler-Einstein base B such that $d\tilde{\mathcal{A}} = 6J$ (we could for instance choose $\mathcal{A}$, but not necessarily). Einstein's equations give the 5D equation of motion

$$R_{\mu\nu} = -4g_{\mu\nu} + \frac{2}{3}F_{\mu\rho}F_\nu{}^\rho - \frac{1}{9}g_{\mu\nu}F_{\rho\lambda}F^{\rho\lambda}. \tag{C49}$$

In turn, Eqs. (C47) and (C49) follow from the (bosonic part of the) 5D supergravity action (C1).

One can similarly reduce the supersymmetry condition from 10D to 5D [96]. Restricting to the metric and the 5-form flux, the 10D dilatino variation vanishes automatically while the gravitino variation reads

$$\delta_\varepsilon\psi_M = \nabla_M\varepsilon + \frac{i}{16\cdot 5!}F_{P_1\ldots P_5}\Gamma^{P_1\ldots P_5}\Gamma_M\varepsilon, \tag{C50}$$

where ε is the 10D spinor. We decompose the 10D gamma matrices as

$$\Gamma^M = \{\gamma^\mu \otimes \mathbb{1} \otimes \sigma_1, \mathbb{1} \otimes \hat{\gamma}^{\underline{a}} \otimes \sigma_2\}, \tag{C51}$$

where the index $\underline{a} = 5, 6, 7, 8, 9$ runs over the directions of the Sasaki-Einstein 5-manifold, and in particular $\underline{a} = 5, 6, 7, 8$ runs over the directions of the Kähler-Einstein base B, while $\sigma_{1,2}$ are Pauli matrices. We take

$$\gamma^{01234} = -i, \quad \hat{\gamma}^{56789} = 1, \quad \Gamma_{11} \equiv \Gamma^0 \cdots \Gamma^9 = \mathbb{1} \otimes \mathbb{1} \otimes \sigma_3. \tag{C52}$$

The 10D chiral spinor ε of type IIB supergravity satisfies $\Gamma_{11}\varepsilon = -\varepsilon$ (compatibly with our definition of $*$), in other words $\sigma_3\varepsilon = -\varepsilon$. We thus decompose

$$\varepsilon = \epsilon \otimes \chi \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{C53}$$

We assume we are using a vielbein that "diagonalizes" the Kähler form of B, namely, such that $J = e^{56} + e^{78}$. This leads to the relations

$$F_{M_1\ldots M_5}\Gamma^{M_1\ldots M_5}\Gamma_\mu\varepsilon = 40i[24 + 2(\hat{\gamma}^{56} + \hat{\gamma}^{78})F_{\rho\lambda}\gamma^{\rho\lambda}]\gamma_\mu\varepsilon,$$
$$F_{M_1\ldots M_5}\Gamma^{M_1\ldots M_5}\Gamma_{\underline{a}}\varepsilon = 40[24 + 2(\hat{\gamma}^{56} + \hat{\gamma}^{78})F_{\rho\lambda}\gamma^{\rho\lambda}]\hat{\gamma}_{\underline{a}}\varepsilon. \tag{C54}$$

Sasaki-Einstein manifolds admit a Killing spinor χ,

$$\left[\hat{\nabla}_{\underline{a}} + \frac{i}{2}\hat{\gamma}_{\underline{a}}\right]\chi = 0, \tag{C55}$$

where $\hat{\nabla}$ is the covariant derivative on SE_5. With our choice of vielbein, the spinor satisfies the projectors

$$\hat{\gamma}^{56}\chi = \hat{\gamma}^{78}\chi = i\chi, \qquad \hat{\gamma}^9\chi = -\chi, \tag{C56}$$

which imply $\partial\chi/\partial\psi_s = \frac{i}{2}\chi$. Substituting into the 10D gravitino variation (C50) one obtains the gravitino variation (C31) of 5D minimal gauged supergravity.

The case of S^5.—Let us specialize the discussion to the case of the Sasaki-Einstein manifold S^5, which is a globally defined $U(1)$ fibration over $\mathbb{CP}^2$. We use coordinates $(\rho_s, \theta_s, \varphi_s, \zeta_s, \psi_s)$ and take the following vielbein:

$$e^5 = d\rho_s, \qquad e^6 = \frac{1}{4}\sin(2\rho_s)(d\zeta_s - \cos(\theta_s)d\varphi_s), \qquad \mathcal{A} = 3\tan(\rho_s)e^6 - d\zeta_s,$$
$$e^7 = \frac{1}{2}\sin(\rho_s)d\theta_s, \qquad e^8 = \frac{1}{2}\sin(\rho_s)\sin(\theta_s)d\varphi_s, \qquad e^9 = \frac{1}{3}(d\psi_s + \mathcal{A} + 2A). \tag{C57}$$

The ranges of coordinates are $\rho_s \in [0, \frac{\pi}{2}]$, $\theta_s \in [0, \pi]$, $\varphi_s \in [0, 2\pi)$, $\zeta_s \in [0, 4\pi)$, $\psi_s \in [0, 6\pi)$ with the identifications

$$\begin{pmatrix} \psi_s \\ \zeta_s \\ \varphi_s \end{pmatrix} \simeq \begin{pmatrix} \psi_s + 6\pi \\ \zeta_s \\ \varphi_s \end{pmatrix} \simeq \begin{pmatrix} \psi_s - 2\pi \\ \zeta_s + 4\pi \\ \varphi_s \end{pmatrix} \simeq \begin{pmatrix} \psi_s - 2\pi \\ \zeta_s - 2\pi \\ \varphi_s + 2\pi \end{pmatrix}. \tag{C58}$$

The coordinates $(\rho_s, \theta_s, \varphi_s, \zeta_s, \psi_s)$ are related to the coordinates (μ_a, ϕ_a) (with $a = 1, 2, 3$) of Sec. V A by[33]

$$\mu_1 = \sin(\rho_s)\cos(\theta_s/2), \qquad \phi_1 = (2\psi_s + \zeta_s - 3\varphi_s)/6,$$
$$\mu_2 = \sin(\rho_s)\sin(\theta_s/2), \qquad \phi_2 = (2\psi_s + \zeta_s + 3\varphi_s)/6,$$
$$\mu_3 = \cos(\rho_s), \qquad \phi_3 = (\psi_s - \zeta_s)/3. \tag{C60}$$

One can explicitly verify that the spinor

$$\chi = e^{\frac{i}{2}\psi_s}\chi_0 \quad \text{with} \quad i\hat{\gamma}^{56}\chi_0 = i\hat{\gamma}^{78}\chi_0 = \hat{\gamma}^9\chi_0 = -\chi_0, \tag{C61}$$

where χ_0 is a constant spinor, solves the Killing spinor equation (C55).

Notice that $\mathcal{A}$ is singular along the loci $\theta_s = \pi$ (which is $\mu_1 = 0$), $\theta_s = 0$ (which is $\mu_2 = 0$), $\rho_s = \frac{\pi}{2}$ (which is $\mu_3 = 0$), as well as $\rho_s = 0$. Therefore it will not be well suited to describe the 4-form potential through (C48) when discussing the embedding of Euclidean D3-branes and their on shell action. Instead, we will take

$$\tilde{\mathcal{A}} = 3\tan(\rho_s)e^6. \tag{C62}$$

This is well defined for all $\rho_s < \pi/2$, and will lead to smooth limits as we take $\rho_s \to \pi/2$.

[33] We can also introduce complex coordinates on $\mathbb{CP}^2$ given by

$$w_1 = \frac{\mu_1}{\mu_3}e^{i(\phi_1 - \phi_3)} = \tan(\rho_s)e^{i(\zeta_s - \varphi_s)/2}\cos(\theta_s/2),$$
$$w_2 = \frac{\mu_2}{\mu_3}e^{i(\phi_2 - \phi_3)} = \tan(\rho_s)e^{i(\zeta_s + \varphi_s)/2}\sin(\theta_s/2), \tag{C59}$$

then the Kähler potential is $K = \log(1 + |w_1|^2 + |w_2|^2)$ while the Kähler form is $J = \frac{i}{2}\partial\bar{\partial}K$.

Combining the five-dimensional analysis in spacetime with the one in the internal manifold, we conclude that for the 10D uplift of complex supersymmetric (but not necessarily extremal) black hole solutions with two angular momenta, using the frames (C25) and (C57) with the gauge field (C30), the Killing spinor ε takes the form

$$\varepsilon = \exp\left\{\frac{i}{2}\left[(3-2\alpha)t - \frac{4(1-a^2)}{(a^2-b^2)\tilde{m}}(\Phi-\Psi) - \frac{4(1-b^2)}{(a^2-b^2)\tilde{m}}(\Phi+\Psi) + \psi_s\right]\right\}\sqrt{f}\varepsilon_0, \tag{C63}$$

where ε_0 is a constant spinor, moreover

$$\Gamma^{09}\varepsilon = \varepsilon, \qquad \Gamma^{12}\varepsilon = -i\varepsilon, \qquad \Gamma^{34}\varepsilon = \Gamma^{56}\varepsilon = \Gamma^{78}\varepsilon = i\varepsilon, \tag{C64}$$

and obviously the same projections hold for ε_0. In the case of black hole solutions with two equal angular momenta, using the frames (C41) and (C57) with gauge field (C42), the 10D Killing spinor takes the form

$$\varepsilon = \exp\left\{\frac{i}{2}[(3-2\alpha)t + \tilde{\Phi} + \psi_s]\right\}\sqrt{f}\varepsilon_0, \tag{C65}$$

with exactly the same projections as above.

3. D3-brane embeddings

Let us discuss various types of supersymmetric embeddings of D3-branes in the black hole geometries, and their on shell actions. Let $\sigma^{0,1,2,3}$ be world volume coordinates on the D3-brane, and $X^\mu(\sigma)$ the embedding. The supersymmetry (or κ-symmetry) condition, in the absence of world volume flux on the D3-branes, is [97–100]

$$\Theta\varepsilon = \mp i\varepsilon, \tag{C66}$$

where

$$\Theta = \frac{1}{4!}\frac{\epsilon^{\alpha_1\ldots\alpha_4}}{\sqrt{-h}}\frac{\partial X^{\mu_1}}{\partial\sigma^{\alpha_1}}\ldots\frac{\partial X^{\mu_4}}{\partial\sigma^{\alpha_4}}e^{M_1}{}_{\mu_1}\ldots e^{M_4}{}_{\mu_4}\Gamma_{M_1\ldots M_4}. \tag{C67}$$

Here $\alpha = 0,\ldots,3$ are world volume spacetime indices on the brane, $\mu = \{t,\ldots,\psi_s\}$ are spacetime indices in 10D, while $M = 0,\ldots,9$ are vielbein indices in 10D. We take $\epsilon^{0123} = 1$, then $\mp$ in (C66) correspond to brane/antibrane depending on conventions and world volume orientation. Finally

$$h_{\alpha_1\alpha_2} = \frac{\partial X^{\mu_1}}{\partial\sigma^{\alpha_1}}\frac{\partial X^{\mu_2}}{\partial\sigma^{\alpha_2}}G_{\mu_1\mu_2} \tag{C68}$$

is the induced metric on the D3-brane, while $h = \det(h_{\alpha\beta})$.

Notice that the supersymmetry condition (C66)–(C67) is valid for standard Lorentzian embeddings, as well as for Euclidean embeddings upon analytic continuation. On the other hand, once we discuss Euclidean embeddings, the Wick rotation $t \to -it_{\rm E}$ of the background metric does not affect the D3-branes and their supersymmetry because they do not wrap the time coordinate.

a. Equal angular momenta

Consider first the case, discussed in the main text, of complex supersymmetric black hole solutions with two equal angular momenta, namely, with $a = b$. Working with the orthotoric coordinates of Sec. C 1 c in spacetime and the coordinates of (C57) on S^5, we consider the embedding

$$\begin{aligned} &t = \text{const}, \quad \tilde{\xi} = \text{const}, \quad \eta = -1, \quad \tilde{\Phi} = \text{const} - \sigma^0, \quad \tilde{\Psi} = \sigma^0, \\ &\rho_s = \sigma^1, \quad \theta_s = \text{const}, \quad \varphi_s = \text{const}, \quad \zeta_s = \sigma^2, \quad \psi_s = \sigma^3. \end{aligned} \tag{C69}$$

This is a Euclidean D3-brane wrapping a spatial S^1 in AdS_5 (so far at generic radial position $\tilde{\xi}$ and at $\theta = \pi/2$) as well as a maximal $S^3 \subset S^5$ located at $z_1/z_2 = \text{constant}$ in complex coordinates. We compute

$$\sqrt{-h} = i\frac{\sin(2\rho_s)}{6}\sqrt{\frac{\tilde{\mathcal{F}}}{f\tilde{\xi}} - f^2\omega_{\tilde{\Phi}}^2}, \qquad \Theta = \frac{i}{\sqrt{\frac{\tilde{\mathcal{F}}}{f\tilde{\xi}} - f^2\omega_{\tilde{\Phi}}^2}}\left[f\omega_{\tilde{\Phi}}\Gamma^{0569} + \sqrt{\frac{\tilde{\mathcal{F}}}{f\tilde{\xi}}}\Gamma^{2569}\right]. \tag{C70}$$

Here $\omega_{\tilde{\Phi}}$ is the component of the 1-form ω in (C38) along $d\tilde{\Phi}$. We see that, in general, this is not a supersymmetric embedding. However at the horizon, located at $\tilde{\xi} = \tilde{\xi}_+$, we have $\tilde{\mathcal{F}}(\tilde{\xi}_+) = 0$. Choosing the sign of the square root in such a way to match the analytic continuation of the Lorentzian expression (5.13), we find

$$\sqrt{-h} = \pm\frac{\sin(2\rho_s)f\omega_{\tilde{\Phi}}}{6} \quad \text{and} \quad \Theta = \mp\Gamma^{0569} \tag{C71}$$

for an embedding at the horizon, where the upper/lower signs refer to the first/second branch of supersymmetric solutions. Using the projectors (C64) we obtain $\Theta\varepsilon = \mp i\varepsilon$, thus the embedding is supersymmetric when placed at the horizon. We also see that (depending on conventions) one branch of solutions host a supersymmetric brane while the other one a supersymmetric antibrane.

The embeddings we just described wrap a particular family of S^3s inside S^5. However, recall that the uplift of 5D minimal gauged supergravity into 10D type IIB on S^5 breaks $SO(6)_R \cong SU(4)_R \to U(1)_R \times SU(3)$ and is invariant under $SU(3)$. Therefore, there is a more general class of embeddings in which the $S^3 \subset S^5$ is given by the linear complex equation $az_1 + bz_2 + cz_3 = 0$ inside $|z_1|^2 + |z_2|^2 + |z_3|^2 = 1$ [101].

For instance, let us exhibit the embedding at $\mu_3 = 0$ (and at the horizon). This corresponds to

$$t = \text{const}, \qquad \tilde{\xi} = \tilde{\xi}_+, \qquad \eta = -1, \qquad \tilde{\Phi} = \text{const} - \sigma^0, \qquad \tilde{\Psi} = \sigma^0,$$
$$\rho_s = \frac{\pi}{2}, \qquad \theta_s = \sigma^1, \qquad \varphi_s = \sigma^2, \qquad \zeta_s = \sigma^3, \qquad \psi_s = \sigma^3. \tag{C72}$$

With a little bit of algebra one finds

$$\sqrt{-h} = \pm\frac{\sin(\theta_s) f\omega_{\tilde{\Phi}}}{4} \quad \text{and} \quad \Theta = \mp\Gamma^{0789}. \tag{C73}$$

Using the projectors (C64) we obtain $\Theta\varepsilon = \mp i\varepsilon$, thus the embedding is supersymmetric.

On shell action.—Let us determine the on shell action for the Euclidean embedding (C72). Using the change of coordinates (C35) as well as the identifications (4.10) and (C58), we conclude that σ^0 has period 2π, σ^1 has range π, σ^2 has (twisted) period 2π, while σ^3 has period 4π. The D3-brane action (5.9) in our conventions reads

$$S_{\mathrm{D3}} = -\frac{N}{2\pi^2}\int (d^4\sigma\sqrt{-h} \mp P[C_{(4)}]), \tag{C74}$$

where $P[C_{(4)}]$ is the pull back of $C_{(4)}$ to the brane world volume, while the $\mp$ sign refers to branes/antibranes.

Using the horizon location $\tilde{\xi}_+$ on the first branch, given in (C39), one finds the values of the functions

$$f\omega_{\tilde{\Phi}}|_{\xi=\tilde{\xi}_+} = \frac{(a - ir_+)(r_*^2 + ir_+)}{2(1-a^2)(a + ir_+)}, \tag{C75}$$

as well as

$$L \equiv \left.\frac{a^2(1+\tilde{m})f}{2(1-a)^2\tilde{\xi}}\right|_{\xi=\xi_+} = -\frac{a(1 - ir_+)(a - ir_+)}{(1-a^2)(a + ir_+)}, \tag{C76}$$

which equals $-\frac{4}{3}A_{\tilde{\Phi}}$ and appears in the connection (C42). Given the integration ranges, we obtain the integral of the metric part:

$$\int d^4\sigma\sqrt{-h} = 16\pi^3 \int \left.\frac{\sin(\theta_s)}{4} f\omega_{\tilde{\Phi}}\right|_{\tilde{\xi}_+} d\theta_s = 4\pi^3 \frac{(2a + a^2 + ir_+)(a - ir_+)}{(1-a^2)(a + ir_+)}. \tag{C77}$$

Let us now move to the 4-form potential. Using (C48), the only term that contributes to the pull back is $\frac{2}{9}\tilde{A} \wedge J \wedge A$. We compute

$$\int P[C_{(4)}] = -\frac{L}{8}\int \sin(\sigma^1) d\sigma^0 \wedge d\sigma^1 \wedge d\sigma^2 \wedge d\sigma^3 = 4\pi^3 \frac{a(1 - ir_+)(a - ir_+)}{(1-a^2)(a + ir_+)}. \tag{C78}$$

Summing the two contributions using the upper minus sign in (C74), we obtain

$$S_{\mathrm{D3}} = 2\pi N \frac{a - ir_+}{a - 1} = 2\pi N \frac{\Delta_g}{\tau_g} \tag{C79}$$

as in Sec. V B, where in the second equality we used the values (4.27) of the chemical potentials. Recall that this result is obtained on the first branch.

To obtain the on shell action on the second branch, we should send $i \to -i$ in the expression of q and then of $\tilde{\xi}_+$ and the formulas that follow. According to (C73), we should also choose the lower sign in the expression of $\sqrt{-h}$ in order to match the analytic continuation of the Lorentzian expression. Moreover, we should take the lower plus sign in (C74) because antibranes are supersymmetric on the second branch, as follows from the κ-symmetry operator in (C73). Taking all of this into account, we obtain

$$S_{\rm D3} = -2\pi N \frac{a+ir_+}{a-1} = -2\pi N \frac{\Delta_g}{\tau_g} \tag{C80}$$

on the second branch.

b. Generic angular momenta

The discussion in the general case is similar to the previous one and the one in Sec. V B. We consider a Euclidean D3-brane located at the black hole horizon $r = r_+$, wrapping the spatial S^1 in AdS_5 along ϕ at $\theta = \frac{\pi}{2}$, as well as the maximal $S^3 \subset S^5$ located at $\mu_3 = 0$ (which is $z_3 = 0$ in complex coordinates). By the same argument that lead to (5.13), by evaluating the metric component $g_{\phi\phi}$ in generic black hole solutions (C2) one concludes that

$$\int d^4x \sqrt{-\det(g_{\rm D3})} = -4\pi^3 i \frac{(r_+^2+a^2)(r_+^2+b^2)+abq}{r_+(r_+^2+b^2)(1-a^2)}. \tag{C81}$$

The sign of the square root has been chosen in such a way that in the background of real and causally well-behaved Lorentzian black holes, the contribution to $\exp(iS_{\rm D3})$ is bounded in absolute value. We will use this expression to analytically continue the square root to complex metrics.

In order to discuss supersymmetry of the embedding, it is better to use the orthotoric coordinates of Sec. C 1 b in spacetime and the coordinates of (C57) on S^5. We thus consider the embedding

$$t = \text{const}, \quad \xi = \xi_+, \quad \eta = -1, \quad \Phi = \text{const} - \sigma^0, \quad \Psi = \sigma^0,$$
$$\rho_s = \frac{\pi}{2}, \quad \theta_s = \sigma^1, \quad \varphi_s = \sigma^2, \quad \zeta_s = \sigma^3, \quad \psi_s = \sigma^3. \tag{C82}$$

Given the value of q in (C15), the location of the horizon in orthotoric coordinates is

$$\xi_+ = \frac{ir_+(a+b+2) - a - a^2 - b - b^2}{(a-b)(1+a+b-ir_+)} \tag{C83}$$

on the first branch, while the value on the second branch is obtained by sending $i \to -i$. One then derives the following useful identities:

$$\mathcal{F}(\xi_+) = 0, \qquad \mathcal{G}(\pm 1) = 0,$$
$$\omega_\Phi - \omega_\Psi|_{\eta=1} = 0, \qquad \omega_\Phi + \omega_\Psi|_{\eta=-1} = 0. \tag{C84}$$

With them, one computes

$$\sqrt{-h} = \pm \frac{\sin(\theta_s) f(\omega_\Phi - \omega_\Psi)}{8}, \qquad \Theta = \mp \Gamma^{0789}. \tag{C85}$$

In the first formula, the sign of the square root has been chosen in such a way to match[34] the analytic continuation of (C81), leading to the $\pm$ sign that refers to the first/second branch of solutions. The second formula implies $\Theta\varepsilon = \mp i\varepsilon$, showing that the embedding is supersymmetric. It also implies that branes with opposite charge are supersymmetric on the two branches.

Computing the on shell action is easier in the original coordinates of (C2) and for generic (not necessarily supersymmetric) black hole backgrounds. We already computed the Dirac-Born-Infeld part in (C81). The Wess-Zumino part immediately follows from $\frac{1}{3}\int_{S^1} A$. The full D3-brane on shell action is

$$S^\phi_{\rm D3} = -\frac{N}{2\pi^2}\left[-4\pi^3 i \frac{(r_+^2+a^2)(r_+^2+b^2)+abq}{r_+(r_+^2+b^2)(1-a^2)} \pm 4\pi^3 \frac{qa}{(r_+^2+b^2)(1-a^2)}\right], \tag{C86}$$

where the $\pm$ sign refers to branes/antibranes. Substituting the supersymmetric value of q on the two branches, and considering branes (+ sign) on the first branch and antibranes (− sign) on the second branch, we obtain

$$S^\phi_{\rm D3} = 2\pi N \frac{a-ir_+}{a-1} \quad \text{(1st branch)},$$
$$S^\phi_{\rm D3} = -2\pi N \frac{a+ir_+}{a-1} \quad \text{(2nd branch)}. \tag{C87}$$

These formulas reproduce the expressions in (5.16) and (5.17), in the case that $\Delta_{g,a} \equiv \Delta_g$.

Next, consider a Euclidean D3-brane located at the black hole horizon $r = r_+$, wrapping the spatial S^1 in AdS_5 along ψ at $\theta = 0$, as well as the S^3 at $\mu_3 = 0$. By evaluating the metric component $g_{\psi\psi}$ in generic black hole solutions (C2) one concludes that

$$\int d^4x \sqrt{-\det(g_{\rm D3})} = -4\pi^3 i \frac{(r_+^2+a^2)(r_+^2+b^2)+abq}{r_+(r_+^2+a^2)(1-b^2)}. \tag{C88}$$

We will use this expression to perform analytic continuations. In orthotoric coordinates, the embedding reads

[34] Notice that $\sqrt{-\det(g_{\rm D3})}$ and $\sqrt{-h}$ are computed in different coordinates. In order to compare them, one should include the factor $\tilde{m}(b^2-a^2)/8(1-a^2)$ from the Jacobian of the coordinate change from σ^0 to ϕ.

$$t=\text{const}, \quad \xi=\xi_+, \quad \eta=1, \quad \Phi=\text{const}+\sigma^0, \quad \Psi=\sigma^0,$$
$$\rho_s=\frac{\pi}{2}, \quad \theta_s=\sigma^1, \quad \varphi_s=\sigma^2, \quad \zeta_s=\sigma^3, \quad \psi_s=\sigma^3. \tag{C89}$$

One computes[35]

$$\sqrt{-h}=\mp\frac{\sin(\theta_s)f(\omega_\Phi+\omega_\Psi)}{8}, \quad \Theta=\mp\Gamma^{0789}. \tag{C90}$$

This implies $\Theta\varepsilon=\mp i\varepsilon$, showing that the embedding is supersymmetric.

Finally, we compute the D3-brane on shell action on generic black hole backgrounds:

$$S^\psi_{\rm D3}=-\frac{N}{2\pi^2}\left[-4\pi^3 i\frac{(r_+^2+a^2)(r_+^2+b^2)+abq}{r_+(r_+^2+a^2)(1-b^2)} \pm 4\pi^3\frac{qb}{(r_+^2+a^2)(1-b^2)}\right]. \tag{C91}$$

Substituting the value (C15) of q on the two branches, and choosing the upper/lower sign for branes/antibranes on the first/second branch, we obtain

$$S^\psi_{\rm D3}=2\pi N\frac{b-ir_+}{b-1} \quad (\text{1st branch}),$$
$$S^\psi_{\rm D3}=-2\pi N\frac{b+ir_+}{b-1} \quad (\text{2nd branch}), \tag{C92}$$

which reproduce the expressions in (5.16) and (5.17).

c. More supersymmetric D3-branes

It turns out that there exists another class of Euclidean supersymmetric D3-branes, located at the black hole horizon, wrapping the full spatial S^3 in AdS_5 as well as a maximal $S^1\subset S^5$ (say at $\mu_2=\mu_3=0$ for definiteness). Considering the case of generic angular momenta (C2), one finds

$$\int_{\rm D3}d^4x\sqrt{-\det(g_{\rm D3})}=-4\pi^3 i\frac{(r_+^2+a^2)(r_+^2+b^2)+abq}{r_+(1-a^2)(1-b^2)}, \tag{C93}$$

where the sign has been chosen so as to give a contribution bounded in absolute value to the path-integral measure $e^{iS_{\rm D3}}$ on real Lorentzian black hole backgrounds.

In order to compute the contribution from the Wess-Zumino (WZ) term, we use the local expression (C5) for $*_5F-\frac{2}{3}A\wedge F=d\alpha_{(2)}$. Using (C48), the only term in $C_{(4)}$ that contributes to the WZ term is $\frac{1}{9}\tilde{\mathcal{A}}\wedge(*_5F-\frac{2}{3}A\wedge F)$. The integral of the spacetime part is simply

$$\int_{S^3}d\alpha_{(2)}=4\pi^2\frac{3q}{2\Xi_a\Xi_b}. \tag{C94}$$

The integral of $\tilde{\mathcal{A}}$ along S^1 should be computed carefully, taking into account the possible gluing of patches. One obtains $\int\tilde{\mathcal{A}}=6\pi$.

Combining the two contributions, we obtain

$$S_{\rm D3}=-\frac{N}{2\pi^2}\left[-4\pi^3 i\frac{(r_+^2+a^2)(r_+^2+b^2)+abq}{r_+(1-a^2)(1-b^2)} \mp 4\pi^3\frac{q}{(1-a^2)(1-b^2)}\right]. \tag{C95}$$

On the first branch, where $q=-(a-ir_+)(b-ir_+)\times(1-ir_+)$, and choosing the upper minus sign for anti-D3-branes, we obtain

$$S_{\rm D3}=-2\pi N\frac{(a-ir_+)(b-ir_+)}{(a-1)(b-1)}=-2\pi N\frac{\Delta_g^2}{\sigma_g\tau_g} \quad (\text{1st branch}). \tag{C96}$$

On the second branch, choosing the lower plus sign for D3-branes, we obtain

$$S_{\rm D3}=2\pi N\frac{(a+ir_+)(b+ir_+)}{(a-1)(b-1)}=2\pi N\frac{\Delta_g^2}{\sigma_g\tau_g} \quad (\text{2nd branch}). \tag{C97}$$

Let us also check that the embeddings are supersymmetric. In terms of the orthotoric coordinates of Sec. C 1 b, the embedding reads

$$t=\text{const}, \quad \xi=\xi_+, \quad \eta=\sigma^0, \quad \Phi=\sigma^1, \quad \Psi=\sigma^2,$$
$$\rho_s=\frac{\pi}{2}, \quad \theta_s=0, \quad \varphi_s=-\sigma^3, \quad \zeta_s=\sigma^3, \quad \psi_s=\sigma^3. \tag{C98}$$

One computes

$$\sqrt{h}=\pm(\xi_+\omega_\Phi-\omega_\Psi), \quad \Theta=\pm\Gamma^{0349}, \tag{C99}$$

where the sign of the square root has been chosen in such a way to match the analytic continuation of (C93) to the first/second branch of supersymmetric solutions. The second formula implies $\Theta\varepsilon=\pm i\varepsilon$, implying that Euclidean anti-D3-branes are supersymmetric on the first branch while D3-branes are supersymmetric on the second branch.

[35]This time the Jacobian factor of the coordinate change from σ^0 to ψ is $\tilde{m}(a^2-b^2)/8(1-b^2)$.

APPENDIX D: SOLUTIONS WITH THREE ELECTRIC CHARGES

In this Appendix we generalize the solution described in the main text to the case of three different $U(1)$ charges, but with equal angular momenta. There are known nonsupersymmetric asymptotically locally AdS_5 black hole solutions with three different $U(1)$ charges and equal angular momenta [102] (see also [69,103]).[36] They are classical solutions to the $U(1)^3$ 5D gauged supergravity action (which is a consistent truncation of 10D type IIB supergravity on S^5)

$$S_{5D}=\frac{1}{16\pi}\int\left[\left(R+4\sum_{I=1}^{3}(X^I)^{-1}-\frac{1}{2}\partial\vec{\chi}^2\right)*1-\frac{1}{2}\sum_{I=1}^{3}(X^I)^{-2}F^I\wedge *F^I-\frac{1}{6}|\epsilon_{IJK}|A^I\wedge F^J\wedge F^K\right], \tag{D1}$$

where A^I (with $I=1,2,3$) are Abelian gauge fields, $F^I=dA^I$ are their field strengths, $\vec{\chi}=(\chi_1,\chi_2)$ are real scalar fields, $X^1=\exp(-\frac{1}{\sqrt{6}}\chi_1-\frac{1}{\sqrt{2}}\chi_2)$, $X^2=\exp(-\frac{1}{\sqrt{6}}\chi_1+\frac{1}{\sqrt{2}}\chi_2)$, $X^3=\exp(\frac{2}{\sqrt{6}}\chi_1)$ such that $X^1X^2X^3=1$.

The solutions relevant to us are

$$\begin{aligned}
ds_5^2&=(H_1H_2H_3)^{\frac{1}{3}}\left[-\frac{r^2Y}{f_1}dt^2+\frac{r^4}{Y}dr^2+\frac{r^2}{4}(\sigma_1^2+\sigma_2^2)+\frac{f_1}{4r^4H_1H_2H_3}\left(\sigma_3-\frac{2f_2}{f_1}dt\right)^2\right],\\
X^I&=\frac{(H_1H_2H_3)^{\frac{1}{3}}}{H_I},\\
A^I&=\left(\frac{2m}{r^2H_I}s_Ic_I-\alpha_I\right)dt+\frac{ma}{r^2H_I}(c_Is_Js_K-s_Ic_Jc_K)\sigma_3\equiv A^I_t dt+A^I_\psi\sigma_3,
\end{aligned} \tag{D2}$$

where the σ_i are left-invariant 1-forms on a 3-sphere S^3 parametrized by (θ,ϕ,ψ)[37]:

$$\sigma_1+i\sigma_2=e^{-i(\phi+\psi)}(2d\theta+i\sin(2\theta)(d\psi-d\phi)),\qquad \sigma_3=2(\cos^2(\theta)d\psi+\sin^2(\theta)d\phi), \tag{D3}$$

while in the definition of A^I we took I, J, K all differently. The coefficients are radial functions:

$$\begin{aligned}
H_I&=1+\frac{2ms_I^2}{r^2},\\
f_1&=r^6H_1H_2H_3+2ma^2r^2+4m^2a^2[2(c_1c_2c_3-s_1s_2s_3)s_1s_2s_3-s_1^2s_2^2-s_2^2s_3^2-s_3^2s_1^2],\\
f_2&=2ma(c_1c_2c_3-s_1s_2s_3)r^2+4m^2as_1s_2s_3,\\
f_3&=f_1-r^6H_1H_2H_3+2ma^2,
\end{aligned} \tag{D4}$$

and

$$s_I=\sinh(\delta_I),\qquad c_I=\cosh(\delta_I),\qquad I=1,2,3. \tag{D5}$$

The solution therefore depends on five parameters[38]: m, a, δ_1, δ_2, δ_3. Once again, r_+ is the largest root of Y. A useful way to relate it to m is by a change of coordinates to

$$R^2=r^2+2ms_1^2. \tag{D6}$$

Now Y becomes quadratic in m, and the relation between m and R_+ can be easily solved to produce a lengthy expression for m in terms of R_+ that will be omitted here. It requires a branch choice for a square root, and this choice determines whether we will be in the first or second branch.

To the horizon at $r=r_+$ we can associate entropy S, inverse temperature β, angular velocity relative to a nonrotating frame at infinity Ω, and electrostatic potentials Φ^I:

[36]The general black hole solutions with six independent charges—mass, two angular momenta, and three electric charges—have been constructed in [104].

[37]We choose angular coordinates that agree with the body of the paper. They can be related to those of [69] by $\theta_{\text{there}}=2\theta_{\text{here}}$ and $\phi_{\text{there}}=\psi_{\text{here}}-\phi_{\text{here}}$, $\psi_{\text{there}}=\psi_{\text{here}}+\phi_{\text{here}}$.

[38]Despite their similar names, they are not identical to those of the single-charge case of Sec. IV when the charges are equal.

$$S=\frac{\pi^2}{2}\sqrt{f_1(r_+)},\qquad \beta=4\pi r_+\sqrt{f_1(r_+)}\left(\frac{dY}{dr}\Big|_{r=r_+}\right)^{-1},$$
$$\Omega=\frac{f_2(r_+)}{f_1(r_+)},$$
$$\Phi^I=\frac{m}{r_+^2H_I(r_+)}(2s_Ic_I+2a\Omega(c_Is_Js_K-s_Ic_Jc_K)). \tag{D7}$$

The energy E, angular momentum J, and electric charges Q_I are

$$E=E_0+\frac{1}{4}m\pi(3+a^2+2s_1+2s_2+2s_3),$$
$$Q_I=\frac{1}{2}m\pi s_Ic_I,$$
$$J=\frac{1}{2}ma\pi(c_1c_2c_3-s_1s_2s_3), \tag{D8}$$

where $E_0=\frac{3\pi}{32}$ is the energy of empty AdS_5 using holographic renormalization [69].

The solution can be rotated to Euclidean signature where regularity determines the global identification

$$(t_E,\psi,\phi)\cong(t_E+\beta,\psi-i\Omega\beta,\phi-i\Omega\beta), \tag{D9}$$

and the gauge fixing to be $\alpha_I=\Phi_I$, and thus the gauge fields on the boundary are

$$A^I|_{\text{bdry}}=i\Phi_I dt_E. \tag{D10}$$

The boundary metric is again that of $S^1\times S^3$, and Ω appears only through the identification of coordinates. Again, we can shift Ω or Φ_I by $2\pi i/\beta$ to find new solutions with the same boundary conditions.

The solution is supersymmetric if

$$a=e^{-\delta_1-\delta_2-\delta_3}=\prod_{I=1,2,3}(c_I-s_I), \tag{D11}$$

and in that case we can define chemical potentials as the deviation from their BPS values [69]

$$\tau_g=\frac{\beta}{2\pi i}(\Omega-1),\qquad \Delta_{g,a}=\frac{\beta}{2\pi i}(\Phi_a-1). \tag{D12}$$

One finds that they satisfy the constraint

$$2\tau_g-\Delta_{g,1}-\Delta_{g,2}-\Delta_{g,3}=\pm1\qquad(1\text{st}/2\text{nd branch}). \tag{D13}$$

Let us embed this black hole into a solution of 10D type IIB supergravity, and ask what would be the contribution of wrapped D3-branes similar to the ones considered before. The uplift of this solution to 10D using [90] has the metric

$$ds_{10}^2=\tilde\Lambda^{1/2}ds_5^2+\tilde\Lambda^{-1/2}\sum_{I=1}^{3}(X^I)^{-1}(d\mu_I^2+\mu_I^2(d\phi_I+A^I)^2), \tag{D14}$$

where $\tilde\Lambda=\sum_{I=1}^3X^I\mu_I^2$ and $\phi_{1,2,3}$, $\mu_{1,2,3}$ (with $\sum_{I=1}^3\mu_I^2=1$) are a parametrization of S^5 as the phases and magnitudes of three complex numbers $\mu_Ie^{i\phi_I}$ on the unit sphere in $\mathbb{C}^3$. Its vielbeins are

$$e^0=\tilde\Lambda^{1/4}\sqrt{\frac{(H_1H_2H_3)^{1/3}r^2Y}{f_1}}dt,\qquad e^1=\tilde\Lambda^{1/4}\sqrt{\frac{(H_1H_2H_3)^{1/3}r^4}{Y}}dr,$$
$$e^2=\tilde\Lambda^{1/4}\frac{(H_1H_2H_3)^{1/6}r}{2}\sigma_1,\qquad e^3=\tilde\Lambda^{1/4}\frac{(H_1H_2H_3)^{1/6}r}{2}\sigma_2,$$
$$e^4=\frac{\tilde\Lambda^{1/4}\sqrt{f_1}}{2(H_1H_2H_3)^{1/3}r^2}\left(\sigma_3-\frac{2f_2}{f_1}dt\right),\qquad e^5=\frac{\mu_1}{\tilde\Lambda^{1/4}\sqrt{X^1}}(d\phi_1+A^1),$$
$$e^6=\frac{\mu_2}{\tilde\Lambda^{1/4}\sqrt{X^2}}(d\phi_2+A^2),\qquad e^7=\frac{\mu_3}{\tilde\Lambda^{1/4}\sqrt{X^3}}(d\phi_3+A^3), \tag{D15}$$

$$e^8=\tilde\Lambda^{-1/4}\sqrt{\frac{X^3\mu_3^2+X^1\mu_1^2}{\mu_3^2X^1X^3}}\left(d\mu_1+\frac{\mu_1\mu_2X^1}{X^3\mu_3^2+X^1\mu_1^2}d\mu_2\right),$$
$$e^9=\tilde\Lambda^{-1/4}\sqrt{\frac{(X^3\mu_3^2+X^2\mu_2^2)(X^3\mu_3^2+X^1\mu_1^2)-\mu_1^2\mu_2^2X^1X^2}{\mu_3^2X^3X^2(X^3\mu_3^2+X^1\mu_1^2)}}d\mu_2. \tag{D16}$$

When $\mu_1 = 0$, the last two simplify to

$$e^8 = \frac{d\mu_1}{\tilde{\Lambda}^{1/4}\sqrt{X^1}}, \qquad e^9 = \tilde{\Lambda}^{-1/4}\sqrt{\frac{X^3\mu_3^2 + X^2\mu_2^2}{\mu_3^2 X^3 X^2}}d\mu_2. \tag{D17}$$

There is also a self-dual 5-form flux $F_{(5)} = G_{(5)} + *G_{(5)}$ in the ten-dimensional solution, which is given by[39]

$$G_{(5)} = \sum_{I=1}^{3}\left[2X^I(X^I\mu_I^2 - \tilde{\Lambda})\epsilon_{(5)} + \frac{1}{2(X^I)^2}d(\mu_I^2)\wedge((d\phi_I + A^I)\wedge *_5 F^I + X^I *_5 dX^I)\right], \tag{D18}$$

where $\epsilon_{(5)}$ the volume form of the original 5D metric, and hence

$$*G_{(5)} = \sum_{I=1}^{3}\left[2X^I(X^I\mu_I^2 - \tilde{\Lambda}) * \epsilon_{(5)} - \frac{\tilde{\Lambda}^{-2}}{2X^I}dX^I \wedge \tilde{*}_5 d(\mu_I^2) + \frac{\tilde{\Lambda}^{-1/4}}{2(X^I)^2}\tilde{*}_5(d(\mu_I^2)\wedge(d\phi_I + A^I))\wedge F^I\right], \tag{D19}$$

where $\tilde{*}_5$ is the Hodge dual with respect to the metric on S^5. At $\mu_1 = 0$ the third term in $*G_{(5)}$ simplifies to $d(\mu_2^2)\wedge(d\phi_2 + A^2)\wedge(d\phi_3 + A^3)\wedge F^1$, implying

$$C_{(4)} = \frac{1}{2}d(\mu_2^2)\wedge(d\phi_2 + A^2)\wedge(d\phi_3 + A^3)\wedge A^1 + \text{other terms}. \tag{D20}$$

Let us now consider the action of a D3-brane at $r = r_+$, $\theta = \frac{\pi}{2}$, $\mu_1 = 0$, along the ϕ direction. We assume that the term discussed above is the only one contributing to the brane action via the term

$$\int_{\mathrm{D3}} P[C_{(4)}] = \int_{S^3}\mu_2 d\mu_2\wedge d\phi_2\wedge d\phi_3\int_{S^1}A^1_\phi\sigma_3 = 8\pi^3\frac{ma}{r_+^2 H_1(r_+)}(c_1 s_2 s_3 - s_1 c_2 c_3). \tag{D21}$$

The contribution of the tension of the brane is proportional to $\int_{\mathrm{D3}}\sqrt{-\det(g_{\mathrm{D3}})}$. The determinant of the induced metric is

$$\det(g_{\mathrm{D3}}) = g_{\mu_2\mu_2}(g_{\phi\phi}g_{\phi_2\phi_2}g_{\phi_3\phi_3} - g^2_{\phi\phi_2}g_{\phi_3\phi_3} - g^2_{\phi\phi_3}g_{\phi_2\phi_2}). \tag{D22}$$

Using

$$\begin{aligned} g_{\phi\phi} &= \frac{\tilde{\Lambda}^{1/2}f_1}{r^4(H_1H_2H_3)^{2/3}} + \sum_{I=1}^{3}\frac{\mu_I^2(A^I_\phi)^2}{\tilde{\Lambda}^{1/2}X^I}, \qquad g_{\phi\phi_2} = \frac{\mu_2^2A^2_\phi}{\tilde{\Lambda}^{1/2}X^2}, \qquad g_{\phi\phi_3} = \frac{\mu_3^2A^3_\phi}{\tilde{\Lambda}^{1/2}X^3}, \\ g_{\mu_2\mu_2} &= \frac{\mu_2^2X^2 + \mu_3^2X^3}{\tilde{\Lambda}^{1/2}\mu_3^2X^2X^3}, \qquad g_{\phi_2\phi_2} = \frac{\mu_2^2}{\tilde{\Lambda}^{1/2}X^2}, \qquad g_{\phi_3\phi_3} = \frac{\mu_3^2}{\tilde{\Lambda}^{1/2}X^3}, \end{aligned} \tag{D23}$$

we find

$$\sqrt{-\det(g_{\mathrm{D3}})} = \frac{i\mu_2\sqrt{f_1(r_+)}}{r_+^2H_1(r_+)}. \tag{D24}$$

Note that the off-diagonal metric terms from the mixing of ϕ_I and A^I cancel in the determinant, regardless of the details of the 5D solution (and thus do so also in the single-charge solution discussed in the body of the paper).

Let us begin for simplicity from the case where two of the three charges are equal, namely, $Q_1 \neq Q_2 = Q_3$. Correspondingly we take $\delta_2 = \delta_3$. Using the radial coordinate R defined in (D6), the supersymmetry condition (D11), an additional change of parameters $w^2 = 1 + R_+^2(1 - e^{4(\delta_1+\delta_2)})$ in order to express R_+ in terms of w, and some help from *Mathematica*, we find

$$m = -\frac{2e^{2\delta_1+4\delta_2}(1-w)^2(e^{2(\delta_1+\delta_2)} - w)}{(e^{2(\delta_1+\delta_2)}-1)^2(e^{2(\delta_1+\delta_2)}+1)(e^{4(\delta_1+\delta_2)} - e^{4\delta_2}(1-w) + e^{2(\delta_1+\delta_2)}(1-w) - 1)}, \tag{D25}$$

[39]There is a typo in the formula for $G_{(5)}$, Eq. (2.8), in [90]. See footnote 18 in [105].

which reproduces the first branch of (D13). The second branch is obtained by the other possible choice of w, i.e., by $w \to -w$.

In principle, we need to fix the sign of the $\int_{D3} d^4x\sqrt{-\det(g_{D3})}$ like in (5.13). However, this proved to be quite technically challenging. Instead, let us initially consider both possible signs, one with a brane and one with an antibrane, and determine the correct sign by comparing to the 1-charge case.

One can now move forward to compute the action of the brane, and find

$$\begin{aligned} S_{D3} &= -\frac{N}{2\pi^2}\int_{D3}(d^4x \pm \sqrt{-\det(g_{D3})} \pm P[C_{(4)}]), \\ &= \mp 4\pi N\left(\frac{i\sqrt{f_1(r_+)}}{2r_+^2 H_1(r_+)} + \frac{ma}{r_+^2 H_1(r_+)}(c_1 s_2 s_3 - s_1 c_2 c_3)\right), \\ &= \mp 2\pi N \frac{(e^{4\delta_2}+1)(w-1)}{e^{4(\delta_1+\delta_2)} + e^{4\delta_2}(w-1) - e^{2(\delta_1+\delta_2)}(w-1) - 1}, \\ &= \mp 2\pi N \frac{\Phi_1 - 1}{\Omega - 1} = \mp 2\pi N \frac{\Delta_{g,1}}{\tau_g}. \end{aligned} \tag{D26}$$

This agrees with the general expectations described in the main text for a D3-brane and a specific choice of branch for the tension term.

A similar computation can be made for a D3-brane located along $\mu_2 = 0$, instead of $\mu_1 = 0$ (but still keeping the same two charges $Q_2 = Q_3$). The relevant term in $C_{(4)}$ is

$$\begin{aligned} C_{(4)} &= \frac{1}{2} d(\mu_1^2) \wedge (d\phi_1 + A^1) \wedge (d\phi_3 + A^3) \wedge A^2 \\ &\quad + \text{other terms}, \end{aligned} \tag{D27}$$

while the contribution of the induced metric is now (choosing a branch for the square root as before)

$$\sqrt{-\det(g_{D3})} = -\frac{i\mu_1\sqrt{f_1(r_+)}}{r_+^2 H_2(r_+)}. \tag{D28}$$

Using the same change of coordinates and some more help from *Mathematica*, one finds the action of this brane to be

$$\begin{aligned} S_{D3} &= -\frac{N}{2\pi^2}\int_{D3}(d^4x\sqrt{-\det(g_{D3})} - P[C_{(4)}]), \\ &= 4\pi N\left(\frac{i\sqrt{f_1(r_+)}}{2r_+^2 H_2(r_+)} + \frac{ma}{r_+^2 H_2(r_+)}(c_2 s_1 s_3 - s_2 c_1 c_3)\right), \\ &= 2\pi N \frac{w-1}{e^{2(\delta_1+\delta_2)} - 1}, \\ &= 2\pi N \frac{\Phi_2 - 1}{\Omega - 1} = 2\pi N \frac{\Delta_{g,2}}{\tau_g}. \end{aligned} \tag{D29}$$

Both of these calculations agree with the one performed in Sec. V for the case where the three charges are equal.

We have verified numerically that even when the three charges are different, in a supersymmetric setting the action of a brane like the ones discussed above with $\mu_I = 0$ has action $2\pi N\Delta_{g,I}/\tau_g$, as above. We have not been able to derive this result analytically.

One can also look at another kind of wrapped D3-brane—one that wraps an $S^3 \subset AdS_5$ and an $S^1 \subset S^5$, as discussed for equal $U(1)$ charges in Sec. C 3 c. For example, let us look at one located at $\mu_1 = 1$, $r = r_+$, and wrapping θ, ϕ, ψ, ϕ_1. In order to get the second to last term in (D18) and the last term in (D19), $C_{(4)}$ must include

$$C_{(4)} = \frac{1}{2}\mu_1^2 \wedge d\phi_1 \wedge \left(\frac{{}^*{}_5F_1}{X_1^2} - A_3 \wedge F_2\right) + \text{other terms}, \tag{D30}$$

and the action of the brane (choosing a sign for the square root that reproduces the single-charge solution) is

$$\begin{aligned} S_{D3} &= -\frac{N}{2\pi^2}\int_{D3}(d^4x\sqrt{-\det(g_{D3})} - P[C_{(4)}]), \\ &= -4\pi N\left(\frac{i}{2}\sqrt{f_1(r_+)} + mc_1 s_1\right), \end{aligned} \tag{D31}$$

which, for the two charges case $\delta_2 = \delta_3$, results in

$$\begin{aligned} S_{D3} &= -2\pi N \frac{(w-1)^2}{(e^{2(\delta_1+\delta_2)} - 1)^2}, \\ &= -2\pi N \frac{(\Phi_2 - 1)(\Phi_3 - 1)}{(\Omega - 1)^2} = -2\pi N \frac{\Delta_{g,2}\Delta_{g,3}}{\tau_g^2}. \end{aligned} \tag{D32}$$

This is consistent with (C96) on the first branch, when $\Delta_{g,2} = \Delta_{g,3}$.

Similarly for $\mu_3 = 1$ we find

$$C_{(4)} = \frac{1}{2}\mu_3^2 \wedge d\phi_3 \wedge \left(\frac{{}^*{}_5F_3}{X_3^2} - A_2 \wedge F_1\right) + \text{other terms}, \tag{D33}$$

with the action

$$
\begin{aligned}
S_{\mathrm{D3}} &= -\frac{N}{2\pi^2}\int_{\mathrm{D3}}(d^4x\sqrt{-\det(g_{\mathrm{D3}})}-P[C_{(4)}]),\\
&= -4\pi N\left(\frac{i}{2}\sqrt{f_1(r_+)}+mc_3s_3\right),\\
&= -2\pi N\frac{(e^{4\delta_2}+1)(w-1)^2}{(e^{2(\delta_1+\delta_2)}-1)(e^{4(\delta_1+\delta_2)}+e^{4\delta_2}(w-1)-e^{2(\delta_1+\delta_2)}(w-1)-1)},\\
&= -2\pi N\frac{(\Phi_1-1)(\Phi_2-1)}{(\Omega-1)^2} = -2\pi N\frac{\Delta_{g,1}\Delta_{g,2}}{\tau_g^2},
\end{aligned}
\tag{D34}
$$

where again we assumed $\delta_2=\delta_3$ after the second line.

Taking into account the results in this section and in Sec. C 3 c, we make the natural conjecture that, in the general case of three unequal charges and two unequal angular momenta, the on shell action of a supersymmetric Euclidean D3-brane wrapping the S^3 horizon at $r=r_+$ along the AdS_5 coordinates and wrapping the S^1 given by $\mu_a=1$ inside S^5, is

$$
S_{\mathrm{D3}} = -2\pi N\frac{\Delta_{g,b}\Delta_{g,c}}{\sigma_g\tau_g}\quad(\text{1st branch}),\qquad S_{\mathrm{D3}} = 2\pi N\frac{\Delta_{g,b}\Delta_{g,c}}{\sigma_g\tau_g}\quad(\text{2nd branch}),
\tag{D35}
$$

where a, b, c are all different. We checked this claim numerically in various cases.

APPENDIX E: GIANT GRAVITONS ON BLACK HOLE BACKGROUNDS

As an aside, we can use the method of Appendix C 3 to exhibit new giant graviton and dual giant graviton solutions on Lorentzian black hole backgrounds. These are supersymmetric Lorentzian D3-brane embeddings in the background of the BPS (i.e., supersymmetric and extremal) black holes of minimal 5D gauged supergravity,[40] and they generalize the giant and dual giant graviton embeddings in pure $AdS_5\times S^5$ [59–61]. Giant and dual giant graviton embeddings in the near-horizon limit of black hole solutions were found in [106,107].

Giant gravitons.—Using the orthotoric coordinates (C23) in AdS_5, consider the embedding

$$
\begin{aligned}
&t=\sigma^0,\qquad \xi=\text{const},\qquad \eta=\text{const},\qquad \Phi=\text{const},\qquad \Psi=\text{const},\\
&\rho_s=\text{const},\qquad \theta_s=\sigma^1,\qquad \varphi_s=\sigma^2,\qquad \zeta_s=\sigma^3,\qquad \psi_s=\sigma^3+(2\alpha-3)\sigma^0,
\end{aligned}
\tag{E1}
$$

where α is the gauge parameter appearing in the connection in (C2) and (C30), while $\sigma^{0,1,2,3}$ are the world volume coordinates. This is a Lorentzian D3-brane, sitting at an arbitrary position with constant coordinates (outside or at the black hole horizon) in the spatial slices of AdS_5, while wrapping a round $S^3\subset S^5$ of radius $\sin(\rho_s)$ and orbiting at the speed of light along an orthogonal circle inside S^5. Noticing that $f\geq 0$ outside or at the horizon, we compute

$$
\sqrt{-h}=\frac{f\sin(\theta_s)\sin^4(\rho_s)}{8}\quad\text{and}\quad \Theta=-\Gamma^{0789}+\cot(\rho_s)(\Gamma^{6789}-\Gamma^{0678}).
\tag{E2}
$$

Using the projectors (C64) we obtain $\Theta\varepsilon=-i\varepsilon$, showing that the embedding is supersymmetric.

We stress that this result is valid for the full family of supersymmetric complex black hole solutions (C2) with SUSY constraint (C12), although real Lorentzian metrics that are free of pathologies also satisfy the extremality condition $\tilde{m}=0$.

Dual giant gravitons.—We were able to find dual giant graviton solutions only in the case with equal angular momenta, i.e., with $a=b$. Using orthotoric coordinates (C35) in AdS_5, consider the embedding

$$
\begin{aligned}
&t=\sigma^0,\qquad \tilde{\xi}=\text{const},\qquad \eta=\sigma^1,\qquad \tilde{\Phi}=\sigma^2,\qquad \tilde{\Psi}=\sigma^3,\\
&\rho_s=\text{const},\qquad \theta_s=\text{const},\qquad \varphi_s=\text{const},\qquad \zeta_s=\text{const},\qquad \psi_s=(2\alpha-3)\sigma^0.
\end{aligned}
\tag{E3}
$$

This is a Lorentzian D3-brane, wrapping a spatial S^3 in AdS_5 around the black hole at arbitrary constant radius determined by $\tilde{\xi}$, and orbiting at the speed of light along a maximal circle of S^5. For a given choice of sign of the square root, we compute

[40]The supersymmetric embeddings we find are valid more generally in the complexified supersymmetric (not necessarily extremal) black hole backgrounds and are smooth in the extremal limit.

$$\sqrt{-h} = \left(f\omega_{\tilde{\Phi}} - \frac{2}{3}A_{\tilde{\Phi}}\right)\tilde{\xi}, \qquad \Theta = -\Gamma^{0349} + \sqrt{\frac{\tilde{\xi}\,\tilde{\mathcal{F}}}{-hf}}(\Gamma^{0234} - \Gamma^{2349}). \tag{E4}$$

Using the projectors (C64) we obtain $\Theta\varepsilon = -i\varepsilon$, showing that the embedding is supersymmetric. As before, this result is valid for the full family of supersymmetric (not necessarily extremal) complex black hole solutions, and in particular for extremal solutions characterized by $\tilde{m} = 0$. It would be interesting to generalize these supersymmetric embeddings to the case $a \neq b$.

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