Higher spin analogs of linearized topologically massive gravity and linearized new massive gravity

D. Dalmazi[1,*] and A. L. R. dos Santos[2,†]

[1]*UNESP, Campus de Guaratinguetá, DFI CEP 12516-410, Guaratinguetá, São Paulo, Brazil*
[2]*Instituto Tecnológico de Aeronáutica, DCTACEP 12228-900, São José dos Campos, São Paulo, Brazil*

(Received 30 July 2021; accepted 29 September 2021; published 27 October 2021)

We suggest a new spin-4 self-dual model (parity singlet) and a new spin-4 parity doublet in $D = 2 + 1$. They are of higher order in derivatives and are described by a totally symmetric rank-4 tensor without extra auxiliary fields. Despite the higher derivatives they are ghost free. We find gauge invariant field combinations which allow us to show that the canonical structure of the spin-4 (spin-3) models follows the same pattern of its spin-2 (spin-1) counterpart after field redefinitions. For $s = 1, 2, 3, 4$, the spin-s self-dual models of order $2s - 1$ and the doublet models of order $2s$ can be written in terms of three gauge invariants. The cases $s = 3$ and $s = 4$ suggest a restricted conformal higher spin symmetry as a principle for defining linearized topologically massive gravity and linearized "new massive gravity" for arbitrary integer spins. A key role in our approach is played by the fact that the Cotton tensor in $D = 2 + 1$ has only two independent components for any integer spin.

DOI: 10.1103/PhysRevD.104.085023

I. INTRODUCTION

Contrary to the real world in $D = 3 + 1$ where local actions for massless particles of spin-s necessarily describe both helicities $\pm s$, in $D = 2 + 1$ there are local actions for each helicity $+s$ or $-s$; they may be called self-dual models or parity singlets and represent now massive particles. The Maxwell-Chern-Simons (MCS) theory and the linearized topologically massive gravity (TMG) [1] are paradigmatic examples of self-dual models of spin-1 and spin-2 respectively. By means of a soldering procedure [2], see also [3], it is possible to join together opposite helicities into a parity invariant (parity doublet) local action with helicities $+s$ and $-s$. In the spin-1 case we obtain the Maxwell-Proca theory [4,5] while the soldering of spin-2 second-order (in derivatives) self-dual models [6] leads to the massive spin-2 Fierz-Pauli theory [7], see also [8]. Since those massive actions have the same form in arbitrary dimensions we may say that the self-dual models in $D = 2 + 1$ work like building blocks of those massive particles in arbitrary D dimensions.

Another connection with higher dimensions comes from the fact that massive models may be deduced via Kaluza-Klein dimensional reduction of massless particles, see [9]. In particular, the self-dual models in $D = 2 + 1$ may be obtained from massless particles in $D = 3 + 1$ as shown in [10] for $s = 1, 2, 3$. Such a procedure leads to first-order self-dual models which are required in general auxiliary fields in order to produce the so-called Fierz-Pauli conditions. The auxiliary fields may turn into dynamical fields and become obstacles when interactions are considered. It is possible, however, to trade auxiliary fields in higher derivatives and gauge symmetries necessary to eliminate ghosts. Those symmetries may be used as a guiding principle for the introduction of interactions. Here we are especially interested in those higher derivative gauge invariant higher spin models.

For each spin-s there seem to be a "$2s$ rule" in $D = 2 + 1$ such that we have ghost-free self-dual models of jth order in derivatives with $j = 1, 2, \ldots, 2s$. By means of a Noether gauge embedding (NGE) procedure [11] we can systematically climb up from the jth to the $(j + 1)$th order from bottom $(j = 1)$ to top $(j = 2s)$, stepwise eliminating auxiliary fields and adding gauge symmetries. The procedure works nicely for $s = 1, 3/2, 2$, see [5,12,13] respectively, but at $s = 3$ it is only partially successful. In [14] we go from $j = 1$ until $j = 4$, but we have not been able to connect the spin-3 fourth-order model of [14], containing auxiliary fields, with the top sixth-order self-dual model of [15] which has no auxiliary fields.

Since the top model of order $2s$ is known for arbitrary integer [16] and half-integer [17] spin-s we might try as an alternative approach to climb down the ladder of derivatives. This is what we pursue in the present work. We are

*denis.dalmazi@unesp.br
†alessandroribeiros@yahoo.com.br

able to go one step down from the top for spins $s=3$ and $s=4$ without introducing auxiliary fields. We believe that our approach can be generalized for arbitrary integer spins. In the spin-4 case we obtain a new seventh $(2s-1)$-order self-dual model and a new eighth $(2s)$-order doublet model. We show in a gauge invariant way that they are both ghost free. They correspond respectively to the spin-4 analogs of the spin-2 linearized TMG[1] and of the linearized "new massive gravity" (NMG) of [18] respectively.

Although not explicitly Lorentz invariant, we employ here a formalism based on gauge invariants which dispenses the use of gauge conditions. The absence of gauge conditions allows us to show that the canonical structure of the spin-4 (spin-3) case basically coincides with the canonical structure of the lower spin-2 (spin-1) case. Our approach might be useful in investigating other higher derivative models.

II. GENERAL SETUP

Throughout this work[2] we will be using three Lagrangians $\mathcal{L}_k^{(s)}$ of kth order in derivatives with $k=2s, 2s-1, 2s-2$ as basic ingredients for building up spin-s self-dual models[3] $\mathcal{L}_{2s}^{SD}, \mathcal{L}_{2s-1}^{SD}$ and the doublet model $\mathcal{L}_{2s}^{D}$,

$$S_{2s} = b_0 \int d^3x h_{\mu_1\cdots\mu_s} E^{\mu_1}{}_{\rho} C^{\rho\mu_2\cdots\mu_s} \equiv \int d^3x \mathcal{L}_{2s}^{(s)}, \quad (1)$$

$$S_{2s-1} = c_0 \int d^3x h_{\mu_1\cdots\mu_s} C^{\mu_1\cdots\mu_s} \equiv \int d^3x \mathcal{L}_{2s-1}^{(s)}, \quad (2)$$

$$S_{2s-2} = d_0 \int d^3x h_{\mu_1\cdots\mu_s} D^{\mu_1\cdots\mu_s} \equiv \int d^3x \mathcal{L}_{2s-2}^{(s)}, \quad (3)$$

where (b_0, c_0, d_0) are arbitrary overall constants and $h_{\mu_1\cdots\mu_s}$ is our fundamental rank-s field, traceful and symmetric $h_{\mu_1\cdots\mu_s} = h_{(\mu_1\cdots\mu_s)}$. We frequently use

$$E^{\rho\delta} \equiv \epsilon^{\rho\delta\sigma}\partial_\sigma; \qquad \Box\theta_{\rho\sigma} \equiv \Box\eta_{\rho\sigma} - \partial_\rho\partial_\sigma;$$
$$E^{\mu\nu}E^{\alpha\beta} = \Box(\theta^{\mu\beta}\theta^{\nu\alpha} - \theta^{\mu\alpha}\theta^{\nu\beta}). \quad (4)$$

A major role is played by the spin-s Cotton tensor $C_{\mu_1\cdots\mu_s}$. More specifically in $D=2+1$, it appears in [1] and [19] in the spin-2 and spin-3 cases respectively, and for arbitrary integer spin in [20]. It is of order $2s-1$ in derivatives $(C \sim \partial^{2s-1} h)$, fully symmetric, transverse, and traceless,

$$C_{\mu_1\cdots\mu_s} = C_{(\mu_1\cdots\mu_s)}; \quad \partial^\rho C_{\rho\mu_2\cdots\mu_s} = 0; \quad \eta^{\rho\nu} C_{\rho\nu\mu_3\cdots\mu_s} = 0. \quad (5)$$

Later on we give an explicit formula for $C_{\mu_1\cdots\mu_s}$ in the flat space. An extension for the AdS_3 space including half-integer spins is given in [21]. The tensor $D_{\mu_1\cdots\mu_s}$ is of order $2s-2$ in derivatives, fully symmetric too. It is connected with the Cotton tensor via a symmetrized curl,

$$C_{\mu_1\cdots\mu_s} = E_{(\mu_1}{}^{\rho} D_{\rho\mu_2\cdots\mu_s)}. \quad (6)$$

In general there is a multiparametric family of D tensors satisfying (6). We are specially interested in the subset of Lagrangians $\mathcal{L}_{2s-2}^{(s)}$ without particle content.

We first recall the construction of the highest order self-dual model $\mathcal{L}_{2s}^{SD}$, see [15] for spin-3, [16] for arbitrary integer spin, and [17] for arbitrary half-integer. For arbitrary integer spin-s it is given by a linear combination of $\mathcal{L}_{2s}^{(s)}$ and $\mathcal{L}_{2s-1}^{(s)}$. If we choose $c_0 = -mb_0$ we have

$$\mathcal{L}_{2s}^{SD} = b_0[h_{\mu_1\cdots\mu_s} E^{\mu_1}{}_{\rho} C^{\rho\mu_2\cdots\mu_s} - m h_{\mu_1\cdots\mu_s} C^{\mu_1\cdots\mu_s}]. \quad (7)$$

The corresponding equations of motion,

$$E^{(\mu_1}{}_{\rho} C^{\rho\mu_2\cdots\mu_s)} = msC^{\mu_1\cdots\mu_s}, \quad (8)$$

play the role of the Pauli-Lubanski eigenvalue equation in $D=2+1$. If we apply $E^{\gamma}{}_{\mu_1}$ on (8) and use (4), (5), and (8) recursively, we deduce the Klein-Gordon equations:

$$(\Box - m^2) C_{\mu_1\cdots\mu_s} = 0. \quad (9)$$

It can be shown from first principles that the Fierz-Pauli constraints (5) and the dynamic equations (8) and (9) are all we need to have massive particles with helicity $s|m|/m$. However, since we have in general higher-order time derivatives there might be further particles, including ghosts, so the particle content of (7) must be thoroughly investigated. The Lagrangians $\mathcal{L}_{2s}^{(s)}$, $\mathcal{L}_{2s-1}^{(s)}$, and consequently $\mathcal{L}_{2s}^{SD}$ are invariant under a large set of local transformations:

$$\delta h_{\mu_1\cdots\mu_s} = \partial_{(\mu_1}\Lambda_{\mu_2\cdots\mu_s)} + \eta_{(\mu_1\mu_2}\psi_{\mu_3\cdots\mu_s)}, \quad (10)$$

where the gauge parameters $\Lambda_{\mu_1\cdots\mu_{s-1}}$ and $\psi_{\mu_1\cdots\mu_{s-2}}$ are fully symmetric but otherwise arbitrary tensors. Because of those symmetries one can fix convenient gauges and prove that $\mathcal{L}_{2s}^{SD}$ only contains massive particles of helicity $+s$ or $-s$ depending on the sign of m, see [15] and [16] for the spin-3 and spin-4 cases, respectively. The approach we use here allows us to prove the absence of ghosts in the spin $s=3,4$ cases in an off-shell and gauge invariant way as we will see later.

[1]The authors of [17] have also suggested a higher spin "topologically massive" theory of order $2s-1$ in $D=2+1$ but it requires further auxiliary fields, different from ours.

[2]We only work on the flat space and use $\eta_{\mu\nu} = (-,+,+)$. Symmetrizations do not contain numerical factors, e.g., $(\alpha\beta) = \alpha\beta + \beta\alpha$ and $(\alpha\beta\gamma) = \alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta$.

[3]In all sections the lower index in the Lagrangian symbol stands for its order in derivatives, i.e., the highest number of space time derivatives of the rank-s fundamental field $h_{\mu_1\cdots\mu_s}$.

Inspired by $\mathcal{L}_{2s}^{SD}$ we define the lower-order self-dual model $\mathcal{L}_{2s-1}^{SD}$ combining $\mathcal{L}_{2s-1}^{(s)}$ and $\mathcal{L}_{2s-2}^{(s)}$ with $d_0 = -smc_0$,

$$\mathcal{L}_{2s-1}^{SD} = c_0[h_{\mu_1\cdots\mu_s}E^{(\mu_1}{}_{\rho}D^{\rho\mu_2\cdots\mu_s)} - msh_{\mu_1\cdots\mu_s}D^{\mu_1\cdots\mu_s}]. \quad (11)$$

The equations of motion are given by

$$E^{(\mu_1}{}_{\rho}D^{\rho\mu_2\cdots\mu_s)} = msD^{\mu_1\cdots\mu_s}. \quad (12)$$

Since the left-hand side of (12) is the Cotton tensor, by taking the trace and applying a derivative on (12) we deduce, with help of the identities (5), the Fierz-Pauli constraints $\partial^{\mu_1}D_{\mu_1\cdots\mu_s} = 0$ and $\eta^{\mu_1\mu_2}D_{\mu_1\mu_2\cdots\mu_s} = 0$ which are now dynamic equations instead of trivial identities as (5).

The application of the curl $E^{\gamma}{}_{\mu_1}$ on (12) will similarly lead to (9) with $C_{\mu_1\mu_2\cdots\mu_s}$ replaced by $D_{\mu_1\mu_2\cdots\mu_s}$ which confirms that $\mathcal{L}_{2s-1}^{SD}$ contains massive particles of helicity $s|m|/m$. There is, however, no guarantee that no other propagating particles are present. In the four cases $s = 1, 3/2, 2, 3$, see [1], [22], [13], and [23] respectively, there is a master action connecting $\mathcal{L}_{2s-1}^{SD}$ with $\mathcal{L}_{2s}^{SD}$ with a D tensor satisfying (6) and such that $\mathcal{L}_{2s-2}^{(s)}$ has no particle content. For instance, in the $s = 2$ case there are two choices for the D tensor, one corresponds to the linearized Einstein-Hilbert theory and the other one to the Weyl and Transverse diffeomorphisms (WTdiff) model or linearized unimodular gravity, both Lagrangians have no propagating modes in $D = 2 + 1$. The respective self-dual models $\mathcal{L}_{2s-1}^{SD}$ are the linearized topologically massive gravity [1] and linearized unimodular topologically massive gravity [24]. One can also combine $\mathcal{L}_{2s}^{(s)}$ and $\mathcal{L}_{2s-2}^{(s)}$ and build up doublet models $\mathcal{L}_{2s}^{D}$ containing both helicities $+s$ *and* $-s$. They represent higher spin analog of the linearized NMG [18] and of the linearized unimodular NMG [24]. In the next section, as a preparation for Sec. IV where possible choices for $D_{\mu_1\mu_2\cdots\mu_s}$ will be discussed, we give closed formulas for the Cotton tensor and its symmetrized curl on the flat space. They are convenient for our approach based on the use of gauge invariant field combinations.

III. THE COTTON TENSOR AND THE LAGRANGIANS $\mathcal{L}_{2s-1}^{(s)}$ AND $\mathcal{L}_{2s}^{(s)}$

One can think of $\mathcal{L}_{2s-1}^{(s)}$ as the most general spin-s parity odd and Lorentz invariant expression of order $2s-1$ in derivatives invariant under (10). We start with an Ansatz such that invariance under the higher spin analog of linearized diffeomorphisms (diff) $\delta h_{\mu_1\cdots\mu_s} = \partial_{(\mu_1}\Lambda_{\mu_2\cdots\mu_s)}$ is granted, namely,

$$\begin{aligned}\mathcal{L}_{2s-1}^{(s)} &= h_{\mu_1\cdots\mu_s}\Box^{s-1}E^{\mu_1\nu_1}[c_0\theta^{\mu_2\nu_2}\cdots\theta^{\mu_s\nu_s} + c_1\theta^{\mu_2\mu_3}\theta^{\nu_2\nu_3}\theta^{\mu_4\nu_4}\cdots\theta^{\mu_s\nu_s} + \cdots\cdots\cdots]h_{\nu_1\cdots\nu_s}\\ &= h_{\mu_1\cdots\mu_s}\Box^{s-1}E^{\mu_1\nu_1}[c_0\theta^{s-1} + c_1\hat{\theta}^2\theta^{s-3} + c_2\hat{\theta}^4\theta^{s-5} + \cdots\cdots\cdots]^{\mu_2\cdots\mu_s\nu_2\cdots\nu_s}h_{\nu_1\cdots\nu_s},\end{aligned} \quad (13)$$

where c_j with $j = 0, 1, \ldots, [\frac{s-1}{2}]$ are to be determined, and $\hat{\theta}$ stands for the transverse operator $\theta_{\mu_j\mu_{j+1}}$ or $\theta_{\nu_j\nu_{j+1}}$ whose indices are contracted within indices of the same h field. Under generalized Weyl transformations $\delta h_{\nu_1\cdots\nu_s} = \eta_{(\nu_1\nu_2}\psi_{\nu_3\cdots\nu_s)}$ we have the following structure (suppressing indices):

$$\delta\mathcal{L}_{2s-1}^{(s)} = h\Box^{s-1}E(C_1\hat{\theta}\theta^{s-3}\psi + C_2\hat{\theta}^3\theta^{s-5}\psi + \cdots\cdots\cdots), \quad (14)$$

where we have the coefficients

$$C_j = \frac{(s-2j)(s-2j-1)}{2}c_{j-1} + 2j(s-j)c_j; \qquad j = 1, 2, \ldots, \left[\frac{s-1}{2}\right]. \quad (15)$$

Consequently, higher spin reparametrizations and Weyl invariance $C_j = 0$ completely fixes $\mathcal{L}_{2s-1}^{(s)}$ and the Cotton tensor up to an overall constant, i.e.,

$$c_j = \frac{(-1)^j(s-j-1)!}{4^j j!(s-2j-1)!}c_0; \qquad j = 0, 1, 2, \ldots, \left[\frac{s-1}{2}\right]. \quad (16)$$

Comparing (2) with (13) we have a closed formula for the Cotton tensor,

$$C_{\mu_1\cdots\mu_s} = \Box^{s-1}E_{(\mu_1}{}^{\rho}\sum_{j=0}^{[\frac{s-1}{2}]}\tilde{c}_j[\hat{\theta}^j\theta^{s-1-j}h]_{\rho\mu_2\cdots\mu_s)}, \quad (17)$$

where $\tilde{c}_j = c_j$ at $c_0 = 1/s$. For[4] the first four integer spins $\mathcal{L}^{(s)}_{2s-1}$ becomes[5]

$$\mathcal{L}^{(1)}_1 = c_0 A_\mu E^{\mu\nu} A_\nu; \qquad \mathcal{L}^{(2)}_3 = c_0 h_{\mu_1\mu_2} \Box E^{\mu_1\alpha_1}\theta^{\mu_2\alpha_2} h_{\alpha_1\alpha_2}, \tag{18}$$

$$\mathcal{L}^{(3)}_5 = c_0 h_{\mu_1\mu_2\mu_3} \Box^2 E^{\mu_1\alpha_1}\left[\theta^{\mu_2\alpha_2}\theta^{\mu_3\alpha_3} - \frac{1}{4}\theta^{\mu_2\mu_3}\theta^{\alpha_2\alpha_3}\right] h_{\alpha_1\alpha_2\alpha_3}, \tag{19}$$

$$\mathcal{L}^{(4)}_7 = c_0 h_{\mu_1\mu_2\mu_3\mu_4} \Box^3 E^{\mu_1\alpha_1}\left[\theta^{\mu_2\alpha_2}\theta^{\mu_3\alpha_3}\theta^{\mu_4\alpha_4} - \frac{1}{2}\theta^{\mu_2\mu_3}\theta^{\alpha_2\alpha_3}\theta^{\mu_4\alpha_4}\right] h_{\alpha_1\alpha_2\alpha_3\alpha_4}. \tag{20}$$

Notice that all noncontracted indices $\mu_1\cdots\mu_s$ on the right-hand side of (17) come from transverse operators $E_{\mu\nu}$ and $\theta_{\mu\nu}$, so the transverse property of the Cotton tensor is explicit as in the case of the formulas given in [25] in terms of spin projection operators.[6] Although we do not have a general proof we believe that (17) does agree with previous formulas given in [20], see also [27], for arbitrary integer spin.

In order to deduce $\mathcal{L}^{(s)}_{2s}$, instead of taking the symmetrized curl of the Cotton tensor (17) we find more convenient to repeat the same procedure used for $\mathcal{L}^{(s)}_{2s-1}$. We start from a parity even ansatz explicitly invariant under higher spin reparametrizations with arbitrary coefficients b_j, with $j = 0, 1, \ldots, [\frac{s}{2}]$,

$$\mathcal{L}^{(s)}_{2s} = h_{\mu_1\cdots\mu_s} \Box^s [b_0\theta^s + b_1\hat{\theta}^2\theta^{s-2} + b_2\hat{\theta}^4\theta^{s-4} + \cdots\cdots\cdots]^{\mu_1\cdots\mu_s\nu_1\cdots\nu_s} h_{\nu_1\cdots\nu_s}. \tag{21}$$

Notice that any even number of E operators can be traded into θ operators according to (4). Requiring generalized Weyl symmetry we obtain a unique solution up to an overall constant,

$$b_j = \frac{(-1)^j s(s-j-1)!}{4^j j!(s-2j)!} b_0; \qquad j = 0, 1, \ldots, \left[\frac{s}{2}\right]. \tag{22}$$

The first four cases of $\mathcal{L}^{(s)}_{2s}$ are given by

$$\mathcal{L}^{(1)}_2 = b_0 A^\mu \Box \theta_{\mu\nu} A^\nu = -\frac{b_0}{2} F^2_{\mu\nu}, \tag{23}$$

$$\mathcal{L}^{(2)}_4 = b_0 h_{\mu_1\mu_2} \Box^2 \left[\theta^{\mu_1\alpha_1}\theta^{\mu_2\alpha_2} - \frac{1}{2}\theta^{\mu_1\mu_2}\theta^{\alpha_1\alpha_2}\right] h_{\alpha_1\alpha_2} = 4b_0\left(R^2_{\mu\nu} - \frac{3}{8}R^2\right)_{hh}, \tag{24}$$

$$\mathcal{L}^{(3)}_6 = b_0 h_{\mu_1\mu_2\mu_3} \Box^3 \left[\theta^{\mu_1\alpha_1}\theta^{\mu_2\alpha_2}\theta^{\mu_3\alpha_3} - \frac{3}{4}\theta^{\mu_1\mu_2}\theta^{\alpha_1\alpha_2}\theta^{\mu_3\alpha_3}\right] h_{\alpha_1\alpha_2\alpha_3} \tag{25}$$

$$\mathcal{L}^{(4)}_8 = b_0 h_{\mu_1\cdots\mu_4} \Box^4 \left[\theta^{\mu_1\alpha_1}\theta^{\mu_2\alpha_2}\theta^{\mu_3\alpha_3}\theta^{\mu_4\alpha_4} - \theta^{\mu_1\mu_2}\theta^{\alpha_1\alpha_2}\theta^{\mu_3\alpha_3}\theta^{\mu_4\alpha_4} + \frac{\theta^{\mu_1\mu_2}\theta^{\mu_3\mu_4}\theta^{\alpha_1\alpha_2}\theta^{\alpha_3\alpha_4}}{8}\right] h_{\alpha_1\cdots\alpha_4}. \tag{26}$$

In the spin-1 case we have the Maxwell theory while in the spin-2 case we recognize the linearized K term of the NMG theory [18]. In the next section we work out the spin-1 and spin-2 cases in terms of appropriate gauge invariant field combinations as a preparation for the respective spin-3 and spin-4 cases since they turn out to have the same respective canonical structure.

IV. SPIN-1 IN TERMS OF GAUGE INVARIANTS

Before we start we stress that in the present section and throughout this work, the notation i_{2s} and i_{2s-1} stands for local invariants under the gauge transformation (10) which

[4]It is understood that all equalities involving Lagrangians in the present work hold under space-time integrals.

[5]Notice that in the spin-1 case we have replaced h_μ by the usual notation A_μ for the electromagnetic potential in order to avoid confusion with the spin-3 vector trace $h_\mu = \eta^{\rho\gamma} h_{\rho\gamma\mu}$.

[6]For an earlier connection between the Cotton tensor and projection operators in the $s = 2$ case see [26].

are obtained via derivatives of order $2s$ and $2s-1$, respectively, of a rank-s fundamental field. There is in general a subset of the gauge transformations (10) which leaves the lowest-order Lagrangian $\mathcal{L}_{2s-2}^{(s)}$ invariant. Correspondingly we have the barred local invariants $\bar{i}_{2s-1}$ and $\bar{i}_{2s-2}$ of order $2s-1$ and $2s-2$, respectively. Notice that the same symbol may represent different quantities in different spin-s sections, see for instance (28) and (49). However, they both stand for an invariant built out of $2s$ derivatives of a rank-s tensor. Since each section deals only with one value of s, there will be hopefully no confusion.

Now let us start with the simplest spin-1 case where the gauge transformations (10) become the usual $U(1)$ symmetry,

$$\delta A_\mu = \partial_\mu \Lambda. \tag{27}$$

One of the three equations (27) can be used to eliminate the gauge parameter Λ in terms of variations of the gauge field,[7] $\Lambda = \partial_j \delta A_j/\nabla^2$, plugging back in (27) we derive two local gauge invariants $\delta i_{2s} = 0 = \delta i_{2s-1}$ connected with the electric and magnetic fields:

$$i_{2s} = \nabla^2 A_0 - \partial_0(\partial_k A_k) = \vec{\nabla}\cdot\vec{E}; \qquad i_{2s-1} = \hat{\partial}_j A_j = B. \tag{28}$$

We follow the notation of [15] where

$$\hat{\partial}_j = \epsilon_{jk}\partial_k; \qquad \hat{\partial}_i\hat{\partial}_j + \partial_i\partial_j = \nabla^2\delta_{ij};$$
$$\hat{\partial}_i\partial_j - \hat{\partial}_j\partial_i = \nabla^2\epsilon_{ij}; \qquad \hat{\partial}_i\hat{\partial}_i = \partial_j\partial_j = \nabla^2. \tag{29}$$

Introducing the so-called helicity decomposition and redefining the gauge invariants we have

$$A_0 = \rho; \qquad A_j = \partial_j\Gamma + \hat{\partial}_j\gamma, \tag{30}$$

$$(I_{2s}, I_{2s-1}) = (i_{2s}, i_{2s-1})/\nabla^2 = (\rho - \dot{\Gamma}, \gamma). \tag{31}$$

In all cases $s = 1, 2, 3, 4$ we will be able to write down the Lagrangians $\mathcal{L}_{2s}^{(s)}$ and $\mathcal{L}_{2s-1}^{(s)}$ in terms of only two gauge invariants $(I_{2s}^{(s)}, I_{2s-1}^{(s)})$ and $\mathcal{L}_{2s-2}^{(s)}$ in terms of those two and an extra one. In the spin-1 case the Maxwell and the Chern-Simons terms become

$$\mathcal{L}_{2s}^{(s)} = b_0 A^\mu \Box\theta_{\mu\nu}A^\nu = b_0[I_{2s}(-\nabla^2)I_{2s} + I_{2s-1}(-\nabla^2\Box)I_{2s-1}], \tag{32}$$

$$\mathcal{L}_{2s-1}^{(s)} = c_0 A_\mu E^{\mu\nu}A_\nu = -2c_0 I_{2s-1}\nabla^2 I_{2s}. \tag{33}$$

Since the two invariants can be treated as independent fields $(I_{2s}, I_{2s-1}) = (\rho - \dot{\Gamma}, \gamma) \equiv (\bar{\rho}, \gamma)$, it is clear that the Abelian Chern-Simons term (33) has no particle content[8] (topological term). We can combine (32) with (33) in order to produce the topologically massive Chern-Simons theory [1]. For future comparison with the spin-3 case we write it down with the choice $c_0 = -mb_0$,

$$\mathcal{L}_{2s}^{SD} = b_0[A^\mu\Box\theta_{\mu\nu}A^\nu - mA_\mu E^{\mu\nu}A_\nu] \tag{34}$$

$$= b_0\{I_m(-\nabla^2)I_m + I_{2s-1}[-\nabla^2(\Box - m^2)]I_{2s-1}\} \tag{35}$$

where $I_m = I_{2s} - mI_{2s-1} = \rho - \dot{\Gamma} - m\gamma \equiv \tilde{\rho}$ is the non-propagating gauge invariant while the transverse mode $I_{2s-1} = \gamma$ is the propagating one.

In the spin-1 case the Cotton tensor becomes a vector, the dual of the field strength: $C_\mu = E_{\mu\nu}A^\nu$. So, according to (6) we have $D_\mu = A_\mu$. Thus, $\mathcal{L}_{2s-2}^{(s)}$ becomes the usual Proca mass term $A_\mu A^\mu$ which has of course no particle content and no gauge symmetry. So, there are no barred (residual) gauge transformations at all. However, in order to use a unified notation regarding the spin-3 case where nontrivial barred gauge symmetries do in fact exist, we keep calling each of the components of the vector field A_μ a barred gauge invariant and keep using barred notation for some of the invariants. The reader can check that the following expressions, which will have a spin-3 counterpart, hold true:

$$\mathcal{L}_{2s-2}^{(s)} = d_0 A_\mu A^\mu$$
$$= -d_0(I_{2s-1}\nabla^2 I_{2s-1} + \bar{I}_{2s-1}\nabla^2\bar{I}_{2s-1} + \bar{I}_{2s-2}^2) \tag{36}$$

$$= -d_0(I_{2s-1}\nabla^2 I_{2s-1} + I_{2s}^2 + 2I_{2s}\dot{\bar{I}}_{2s-1} + \bar{I}_{2s-1}\Box\bar{I}_{2s-1}) \tag{37}$$

where, recall $(I_{2s}, I_{2s-1}) = (\rho - \dot{\Gamma}, \gamma)$,

$$\bar{I}_{2s-1} \equiv \partial_j A_j/\nabla^2 = \Gamma; \qquad \bar{I}_{2s-2} \equiv I_{2s} + \dot{\bar{I}}_{2s-1} = A_0 = \rho. \tag{38}$$

We can choose $(c_0, d_0) = (mb_0, -m^2 b_0)$ and combine (33) with (37) in order to produce the first-order self-dual model $\mathcal{L}_{2s-1}^{SD}$ of [28]

$$\mathcal{L}_{2s-1}^{SD} = mb_0(A_\mu E^{\mu\nu}A_\nu - mA_\mu A^\mu) \tag{39}$$

[7]Throughout this work, $i, j, k = 1, 2$ and henceforth we use $\partial_0 f$ and $\dot{f}$ equivalently.

[8]Throughout this work we assume vanishing fields at infinity. The Laplacian ∇^2 has only negative eigenvalues such that the frequently appearing operators ∇^2 and $m^2 - \nabla^2$ have an empty kernel.

$$= b_0\left[I_m\nabla^2 I_m + \tilde{I}_{2s}(m^2-\nabla^2)\tilde{I}_{2s} + m^2\bar{I}_{2s-1}\frac{(-\nabla^2)(\Box - m^2)}{m^2-\nabla^2}\bar{I}_{2s-1}\right] \quad (40)$$

where

$$\tilde{I}_{2s} \equiv I_{2s} + \frac{m^2}{m^2-\nabla^2}\dot{\bar{I}}_{2s-1} = \rho + \frac{\nabla^2}{m^2-\nabla^2}\dot{\Gamma}. \quad (41)$$

The three gauge invariants can be treated as independent fields $(I_m, \tilde{I}_{2s}, \bar{I}_{2s-1}) \equiv (-m\tilde{\gamma}, \tilde{\rho}, \Gamma)$. Although (39) is known [29] to be dual too (34), it contains one extra nonpropagating gauge invariant and the propagating mode is now the longitudinal component (Γ) instead of the transverse one (γ).

We finish this section building up the Maxwell-Proca theory (parity doublet). Since the Chern-Simons terms in (34) and (39) have opposite signs, we can simply add them and obtain

$$\mathcal{L}_{2s}^{D} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{m^2}{2}A_\mu \quad (42)$$

$$= I_{2s-1}\frac{[-\nabla^2(\Box - m^2)]}{2}I_{2s-1} + \frac{m^2}{2}\bar{I}_{2s-1}\frac{(-\nabla^2)(\Box - m^2)}{m^2-\nabla^2}\bar{I}_{2s-1} + \tilde{I}_{2s}\frac{(m^2-\nabla^2)}{2}\tilde{I}_{2s}. \quad (43)$$

Now we have both Γ and γ as propagating modes and the gauge invariant $\tilde{I}_{2s}$, connected with the electrostatic potential A_0, is the nonpropagating one. In all higher spin cases $s = 2, 3, 4$ we will be able to write the doublet model in terms of two propagating and one nonpropagating gauge invariant.

V. SPIN-2 IN TERMS OF GAUGE INVARIANTS

In the spin-2 case the gauge transformations become the usual linearized reparametrizations (diff) plus Weyl that we call Wdiff,

$$\delta h_{\mu\nu} = \partial_\mu\Lambda_\nu + \partial_\nu\Lambda_\mu + \eta_{\mu\nu}\psi \equiv \delta_\Lambda h_{\mu\nu} + \delta_\psi h_{\mu\nu}. \quad (44)$$

In (44) we have six equations and four independent gauge parameters; consequently they give rise to $6-4=2$ gauge invariants (I_{2s}, I_{2s-1}). This is all we need to describe $\mathcal{L}_{2s}^{(s)}$ and $\mathcal{L}_{2s-1}^{(s)}$. It is instructive to do it in two steps. First we derive the $6-3=3$ diff invariants. Using the decomposition $\Lambda_0 = A$, $\Lambda_j = \hat{\partial}_j B + \partial_j C$ in $\delta_\Lambda h_{\mu\nu}$ we find

$$A = \frac{\partial_j\delta_\Lambda h_{0j}}{\nabla^2} - \frac{\partial_i\partial_j\delta_\Lambda\dot{h}_{ij}}{2\nabla^4}; \qquad B = \frac{\partial_i\hat{\partial}_j\delta_\Lambda h_{ij}}{\nabla^4}; \qquad C = \frac{\partial_i\partial_j\delta_\Lambda h_{ij}}{2\nabla^4}. \quad (45)$$

Substituting back in $\delta_\Lambda h_{\mu\nu}$ we derive three local diff invariants $\delta_\Lambda\bar{i}_{2s} = 0 = \delta_\Lambda\bar{i}_{2s-1} = \delta_\Lambda\bar{i}_{2s-2}$,

$$\bar{i}_{2s-2} = \hat{\partial}_i\hat{\partial}_j h_{ij}; \qquad \bar{i}_{2s-1} = \nabla^2\hat{\partial}_j h_{0j} - \partial_k\hat{\partial}_j\dot{h}_{kj}, \quad (46)$$

$$\bar{i}_{2s} = \nabla^4 h_{00} - 2\nabla^2\partial_j\dot{h}_{0j} + \partial_i\partial_j\ddot{h}_{ij}, \quad (47)$$

The fact that we have only three independent local diff invariants is in agreement with the Riemannian geometry, since in $D = 2+1$ the Riemann tensor $R_{\mu\nu\alpha\beta}$ is proportional to the Ricci tensor $R_{\mu\nu}$ which has in principle six components but due to the Bianchi identity $\nabla^\mu R_{\mu\nu} = \nabla_\nu R/2$ only three of them are independent. Indeed, one can check that all six components of the linearized tensor $R_{\mu\nu}^L$ can be written in terms of space time derivatives of $(\bar{i}_{2s-2}, \bar{i}_{2s-1}, \bar{i}_{2s})$.

Second, back to the Wdiff symmetry, since $\delta_\psi(\bar{i}_{2s-2}, \bar{i}_{2s-1}, \bar{i}_{2s}) = (\nabla^2\psi, 0, -\nabla^2\Box\psi)$, we have two Wdiff invariants,

$$i_{2s-1} \equiv \bar{i}_{2s-1} = \nabla^2\hat{\partial}_j h_{0j} - \partial_k\hat{\partial}_j\dot{h}_{kj}, \quad (48)$$

$$i_{2s} = \bar{i}_{2s} + \Box\bar{i}_{2s-2} = \nabla^4 h_{00} - 2\nabla^2\partial_j\dot{h}_{0j} + \partial_i\partial_j\ddot{h}_{ij} + \Box\hat{\partial}_i\hat{\partial}_j h_{ij}. \quad (49)$$

Using the helicity decomposition

$$h_{00} = \rho; \qquad h_{0j} = \partial_j\Gamma + \hat{\partial}_j\gamma;$$
$$h_{ij} = \partial_i\partial_j\chi + (\partial_i\hat{\partial}_j + \hat{\partial}_i\partial_j)\theta + \hat{\partial}_i\hat{\partial}_j\varphi \quad (50)$$

and redefining the gauge invariants we have

$$(\bar{I}_{2s-2}, I_{2s-1}, I_{2s}) = \left(\frac{1}{\nabla^4}\right)(\bar{i}_{2s-2}, \bar{i}_{2s-1}, \bar{i}_{2s}) = (\varphi, \gamma - \dot{\theta}, \rho - 2\dot{\Gamma} + \ddot{\chi} + \Box\varphi). \quad (51)$$

The fourth-order linearized K term of the NMG theory [18] and the third-order linearized gravitational Chern-Simons term, see [1], of the TMG can be written in terms of the two Wdiff invariants

$$\mathcal{L}_{2s}^{(s)} = b_0 h_{\mu_1\mu_2}\Box^2\left(\theta^{\mu_1\alpha_1}\theta^{\mu_2\alpha_2} - \frac{\theta^{\mu_1\mu_2}\theta^{\alpha_1\alpha_2}}{2}\right)h_{\alpha_1\alpha_2}$$
$$= \frac{b_0}{2}[I_{2s}\nabla^4 I_{2s} + 4I_{2s-1}\nabla^4\Box I_{2s-1}], \quad (52)$$

$$\mathcal{L}_{2s-1}^{(s)} = c_0 h_{\mu_1\mu_2}\Box E^{\mu_1\alpha_1}\theta^{\mu_2\alpha_2}h_{\alpha_1\alpha_2} = -2c_0 I_{2s-1}\nabla^4 I_{2s}. \quad (53)$$

From (51) we see that the Wdiff invariants may be considered as independent fields $(I_{2s-1}, I_{2s}) \equiv (\tilde{\gamma}, \tilde{\rho})$, thus we have a massless mode in (52) and no particle content in (53). They can be combined together following (7), in order to produce the linearized version of the "new topologically massive gravity" (NTMG) of [13,30]. Choosing $c_0 = -mb_0$ we have

$$\mathcal{L}^{SD}_{2s} = \mathcal{L}^{(s)}_{2s} + \mathcal{L}^{(s)}_{2s-1} = 2b_0\left\{\frac{1}{4}I_{2m}\nabla^4 I_{2m} + I_{2s-1}[\nabla^4(\Box - m^2)]I_{2s-1}\right\} \quad (54)$$

where $I_{2m} = I_{2s} + 2mI_{2s-1} = \rho - \dot{\Gamma} + \ddot{\chi} + \Box\varphi + 2m(\gamma - \dot{\theta})$ and $I_{2s-1} = \gamma - \dot{\theta}$. Note that the change of variables $(I_{2m}, I_{2s-1}) = (\tilde{\rho}, \tilde{\gamma})$ has a trivial Jacobian. So we are able to check that the fourth-order model (54) is free of ghosts and contains only one propagating massive mode $\tilde{\gamma}$ plus the nonpropagating field $\tilde{\rho}$.

Now in order to construct $\mathcal{L}^{(s)}_{2s-2}$ we need to find a D tensor satisfying (6) and such that $\mathcal{L}^{(s)}_{2s-2} = d_0 h_{\mu\nu}D^{\mu\nu}$ has no particle content. In the spin-2 case the Cotton tensor is a symmetric rank-2 tensor, see (18). So with $c_0 = 2$ we need to solve the equation:

$$C_{\mu\nu} = [E_{(\mu}{}^{\alpha}\Box\theta^{\beta}{}_{\nu)}h_{\alpha\beta}] = E_{(\mu}{}^{\rho}D_{\rho\nu)}. \quad (55)$$

The D tensor must be symmetric and of second order in derivatives. After a rather general ansatz $D_{\mu\nu} \sim (\partial^2 h)_{(\mu\nu)}$ we arrive, up to trivial redefinitions $h_{\mu\nu} \to h_{\mu\nu} + \lambda\eta_{\mu\nu}h$, $(\lambda \neq -1/3)$, at a general solution in terms of two arbitrary real parameters (a, b), without loss of generality,

$$D_{\mu\nu} = \Box h_{\mu\nu} - \partial_{(\mu}\partial^{\rho}h_{\rho\nu)} + a\partial_\mu\partial_\nu h + (a\partial^\alpha\partial^\beta h_{\alpha\beta} - b\Box h)\eta_{\mu\nu}. \quad (56)$$

Under the Wdiff gauge transformations (10) we have

$$\delta D_{\mu\nu} = \eta_{\mu\nu}\Box[2(a-b)\partial\cdot\Lambda + (a - 3b + 1)\psi] + \partial_\mu\partial_\nu[2(a-1)\partial\cdot\Lambda + 3a\psi]. \quad (57)$$

The Wdiff invariance of the Cotton tensor $\delta C_{\mu\nu} = E_{(\mu}{}^{\rho}\delta D_{\rho\nu)} = 0$ for arbitrary values of (a, b) follows simply from the tensor structure of $\delta D_{\mu\nu}$. Notice that $\delta D_{\mu\nu} = 0$ for transverse diffeomorphisms (Tdiff): $(\partial\cdot\Lambda, \psi) = (0, 0)$. Thus, $\mathcal{L}^{(s)}_{2s-2} = d_0 h_{\mu\nu}D^{\mu\nu}$ becomes the Tdiff theory in $D = 2 + 1$, see [31],

$$\mathcal{L}^{(s)}_{2s-2}(a, b) = d_0[-\partial_\mu h^{\alpha\beta}\partial^\mu h_{\alpha\beta} + 2\partial^\mu h^{\alpha\beta}\partial_\alpha h_{\mu\beta} - 2a\partial^\mu h\partial^\nu h_{\mu\nu} + b\partial_\mu h\partial^\mu h]. \quad (58)$$

Now we point out an interesting connection with massless spin-2 particles in $D = 3 + 1$. Namely, it is known [32] that Tdiff is the minimal symmetry required for describing helicity ± 2 particles in $D = 3 + 1$ in terms of a symmetric rank-2 tensor $h_{\mu\nu}$. The general solution (57) seems to confirm that this is true also in $D = 2 + 1$, since we can combine $\mathcal{L}^{(s)}_{2s-2}(a, b)$ with the third-order Chern-Simons term (53) and build up a third-order model that generalizes the linearized topologically massive gravity and contains helicity $2|m|/m$ particles. Such a model can be nonlinearly extended to a topologically massive Tdiff gravity since the metric determinant behaves as a scalar field under Tdiff. Although there are descriptions of helicity ± 2 in $D = 2 + 1$ even without gauge symmetry, see [33], those models require auxiliary fields besides the symmetric rank-2 tensor $h_{\mu\nu}$. The FP conditions are enforced via second class constraints instead of local symmetries.

It is important to stress that (58) describes in general a massless scalar field in $D = 2 + 1$. We can only have an empty spectrum if we enlarge the Tdiff symmetry either to unconstrained linearized diffeomorphisms (diff) by fixing[9] $(a, b) = (1, 1)$ or to WTdiff (Weyl plus Tdiff) by choosing $(a, b) = (2/3, 5/9)$. The second case has been investigated in [24] and corresponds to the linearized version of unimodular gravity; its higher spin analog, of second order in derivatives, has been investigated in [34]. A possible generalization of order $2s - 2$ in $D = 2 + 1$ will be studied elsewhere [35] from the point of view of gauge invariants. Here we only work with the linearized Einstein-Hilbert theory $\mathcal{L}^{(s)}_{2s-2} = \mathcal{L}^{(s)}_{2s-2}(1, 1)$. In terms of the diff invariants (51) we have, see also [1],

$$\mathcal{L}^{(s)}_{2s-2} = \mathcal{L}_{\rm LEH} = 2d_0[I_{2s-1}\nabla^4 I_{2s-1} + I_{2s}\nabla^4\bar{I}_{2s-2} - \bar{I}_{2s-2}\nabla^4\Box\bar{I}_{2s-2}]. \quad (59)$$

Since we can redefine $(\bar{I}_{2s-2}, I_{2s-1}, I_{2s}) = (\varphi, \tilde{\gamma}, \tilde{\rho})$, the equations of motion for those fields lead to the triviality of the Einstein-Hilbert (EH) theory in $D = 2 + 1$: $\varphi = 0 = \tilde{\gamma} = \tilde{\rho}$.

Following (11) we can combine the Einstein-Hilbert theory (59) with the third-order Chern-Simons term (53) and build up the linearized version of TMG, choosing $(c_0, d_0) = (-mb_0, -m^2 b_0)$,

$$\mathcal{L}^{SD}_{2s-1} = \mathcal{L}^{(s)}_{2s-1} + \mathcal{L}^{(s)}_{2s-2} = \mathcal{L}_{\rm TMG} = 2m^2 b_0\left[\bar{I}_{2s-2}\nabla^4\Box\bar{I}_{2s-2} - I_{2s-1}\nabla^4 I_{2s-1} - I_{2s}\nabla^4\left(\bar{I}_{2s-2} + \frac{I_{2s-1}}{m}\right)\right]. \quad (60)$$

Since the Lagrangian is linear on I_{2s} we have the functional constraint $I_{2s-1} = -m\bar{I}_{2s-2}$ which leads to $\mathcal{L}^{SD}_{2s-1} = 2m^2 b_0[\bar{I}_{2s-2}\nabla^4(\Box - m^2)\bar{I}_{2s-2}]$ confirming that we have one physical massive mode content without ghosts. Finally we simply add (54) and (60) in order to produce the NMG parity doublet,

[9]Up to trivial shifts $h_{\mu\nu} \to h_{\mu\nu} + \lambda\eta_{\mu\nu}h$ with $\lambda \neq -1/3$.

$$\mathcal{L}_{2s}^{D} = \mathcal{L}_{\rm LNMG} = 2b_0\left[I_{2s-1}\nabla^4(\Box - m^2)I_{2s-1} + m^2\bar{I}_{2s-2}\nabla^4(\Box - m^2)\bar{I}_{2s-2} + \bar{I}_{2s}\frac{\nabla^4}{4}\bar{I}_{2s}\right] \quad (61)$$

where $\bar{I}_{2s} = I_{2s} - 2m^2\bar{I}_{2s-2}$. Since $(\bar{I}_{2s}, I_{2s-1}, \bar{I}_{2s-2})$ are independent degrees of freedom we confirm the doublet ghost free content of the linearized NMG in a gauge invariant way.

VI. SPIN-3

In the rank-3 case the Wdiff transformations (10) become the following 10 equations:

$$\delta h_{\mu\nu\alpha} = \partial_{(\mu}\Lambda_{\nu\alpha)} + \eta_{(\mu\nu}\psi_{\alpha)}, \quad (62)$$

At first sight we have nine gauge parameters on the right-hand side of (62); however, there are only eight independent ones due to the redundancy $\delta(\Lambda_{\nu\alpha}, \psi_\alpha) = (\eta_{\nu\alpha}\phi, -\partial_\alpha\phi)$. An equivalent counting, valid for arbitrary spin $s \geq 3$ as we will see in Sec. VIII, is to consider, without loss of generality, that we can replace arbitrary diff in (62) by traceless diff $\bar{\Lambda}_{\nu\alpha}$ ($\eta^{\nu\alpha}\bar{\Lambda}_{\nu\alpha} = 0$). No redundancy is left in this case.

From (62) we have $10 - 8 = 2$ gauge invariants. In practice we can decompose ψ_μ and $\bar{\Lambda}_{\mu\nu}$ according to formulas similar to (30) and (50) and find out explicit expressions for the eight independent gauge parameters in terms of $\delta h_{\mu\nu\alpha}$, recall that $\bar{\Lambda}_{00} = \bar{\Lambda}_{jj}$. Plugging back in (62), after some work, we have $\delta i_{2s} = 0 = \delta i_{2s-1}$ where the sixth and fifth-order local Wdiff invariants are

$$i_{2s} = \nabla^6 h_{000} - 3\nabla^4\partial_j\partial_0 h_{00j} + 3\nabla^2\partial_j\partial_k\partial_0^2 h_{0jk} - \partial_i\partial_j\partial_k\partial_0^3 h_{ijk} + 3\Box(\nabla^2\hat{\partial}_j\hat{\partial}_k h_{0jk} - \hat{\partial}_j\hat{\partial}_k\partial_l\partial_0 h_{jkl}), \quad (63)$$

$$i_{2s-1} = 3(\partial_i\partial_j\hat{\partial}_k\partial_0^2 h_{ijk} - 2\nabla^2\partial_0\partial_j\hat{\partial}_k h_{0jk} + \nabla^4\hat{\partial}_j h_{00j}) + \Box\hat{\partial}_j\hat{\partial}_k\hat{\partial}_l h_{jkl}. \quad (64)$$

Introducing the helicity decomposition

$$h_{000} = \rho; \qquad h_{00j} = \partial_j\Gamma + \hat{\partial}_j\gamma, \quad (65)$$

$$h_{0jk} = \hat{\partial}_j\hat{\partial}_k\phi_1 + \partial_j\partial_k\phi_2 + \hat{\partial}_{(j}\partial_{k)}\phi_3, \quad (66)$$

$$h_{jkl} = \hat{\partial}_j\hat{\partial}_k\hat{\partial}_l\psi_1 + \hat{\partial}_{(j}\hat{\partial}_k\partial_{l)}\psi_2 + \partial_{(j}\partial_k\hat{\partial}_{l)}\psi_3 + \partial_j\partial_k\partial_l\psi_4, \quad (67)$$

and redefining the invariants we have

$$I_{2s} \equiv i_{2s}/\nabla^6 = \rho - 3\partial_0\Gamma + 3\Box\phi_1 + 3\partial_0^2\phi_2 - \partial_0^3\psi_4 - 3\Box\partial_0\psi_2, \quad (68)$$

$$I_{2s-1} \equiv i_{2s-1}/\nabla^6 = 3\gamma - 6\partial_0\phi_3 + 3\partial_0^2\psi_3 + \Box\psi_1. \quad (69)$$

The next step is to write down $\mathcal{L}_{2s}^{(s)}$ and $\mathcal{L}_{2s-1}^{(s)}$ given in (25) and (19), respectively, in terms of the gauge invariants (68) and (69). This is much more complicated than in the previous $s = 1, 2$ cases where the explicit substitution of the helicity decomposition could be easily carried out. Now we use a short cut. Namely, we suppose that in both cases the searched Lagrangian has the form

$$\mathcal{L} = I_{2s}\hat{A}I_{2s} + I_{2s-1}\hat{B}I_{2s-1} + I_{2s}\hat{C}I_{2s-1}, \quad (70)$$

where $(\hat{A}, \hat{B}, \hat{C})$ are space time differential operators to be found. We restrict the decomposition of (65)–(67) to the smallest number of fields which allows us to find out the unknown differential operators.[10] We have found that the most convenient choice is to keep only ψ_1 and ψ_2. We are left only with spatial components of the fundamental field,

$$h_{ijk} = \hat{\partial}_i\hat{\partial}_j\hat{\partial}_k\psi_1 + \hat{\partial}_{(i}\hat{\partial}_j\partial_{k)}\psi_2. \quad (71)$$

Thus, we have

$$\mathcal{L}_{2s}^{(s)} = b_0 h_{ijk}\left[\theta^{im}\theta^{jn}\theta^{kp} - \frac{3}{4}\theta^{ij}\theta^{mn}\theta^{kp}\right]\Box^3 h_{mnp} \quad (72)$$

$$= b_0\hat{\partial}_i\hat{\partial}_j\hat{\partial}_k\psi_1\left[\delta^{im}\delta^{jn}\delta^{kp} - \frac{3}{4}\delta^{ij}\delta^{mn}\delta^{kp}\right]\Box^3\hat{\partial}_m\hat{\partial}_n\hat{\partial}_p\psi_1 + b_0\hat{\partial}_{(i}\hat{\partial}_j\partial_{k)}\psi_2\left[\delta^{jn}\delta^{kp}(\Box\delta^{im} - 3\partial^i\partial^m) - \frac{3}{4}\delta^{ij}\delta^{mn}(\Box\delta^{kp} - \partial^k\partial^p)\right]\Box^2\hat{\partial}_{(m}\hat{\partial}_n\partial_{p)}\psi_2 \quad (73)$$

$$= b_0\left[\Box\psi_1\left(\frac{-\nabla^6\Box}{4}\right)\Box\psi_1 + (-3\Box\dot{\psi}_3)\left(\frac{-\nabla^6}{4}\right)(-3\Box\dot{\psi}_3)\right] \quad (74)$$

$$= b_0\left[I_{2s-1}\left(\frac{-\nabla^6\Box}{4}\right)I_{2s-1} + I_{2s}\left(\frac{-\nabla^6}{4}\right)I_{2s}\right] \quad (75)$$

where $\Box\theta^{im} = \Box\delta^{im} - \partial^i\partial^m$ and from (68), (69) we have $(I_{2s}, I_{2s-1}) = (-3\Box\dot{\psi}_2, \Box\psi_1)$. Due to the fact that there is always an odd (even) number of dual derivatives $\hat{\partial}$ acting

[10] Alternatively, we believe that it is possible to determine the operators $(\hat{A}, \hat{B}, \hat{C})$, up to an overall constant, by Lorentz invariance, mass dimension, and locality as we have done in some examples.

on ψ_1 (ψ_2) there are no cross terms $\psi_1 \times \psi_2$; they vanish due to $\hat{\partial}\cdot\partial = 0$. For the same reason we have dropped several derivatives in (73). Notice that the two terms inside parentheses in (73) have exactly led to the double time derivatives required to produce the second term of (74) which is a nontrivial check of (70). Similarly, for the fifth-order spin-3 Chern-Simons term, using (71) again, we have

$$\mathcal{L}_{2s-1}^{(s)} = c_0 h_{ijk} E^{im}\left[\theta^{jn}\theta^{kp} - \frac{1}{4}\theta^{jk}\theta^{np}\right]\Box^2 h_{mnp} = c_0 h_{ijk}\left[\theta^{jn}\theta^{kp} - \frac{1}{4}\theta^{jk}\theta^{np}\right]\Box^2 \dot{h}^*_{i(np)}, \tag{76}$$

where we have used $E^{im} = \epsilon^{im}\partial_0$ and defined

$$h^*_{i(np)} \equiv \epsilon^{im} h_{mnp} = -\partial_i\hat{\partial}_n\hat{\partial}_p\psi_1 + \hat{\partial}_i\hat{\partial}_n\hat{\partial}_p\psi_2 - \partial_i\partial_{(n}\hat{\partial}_{p)}\psi_2. \tag{77}$$

Notice that in h_{ijk} we have an odd (even) number of dual derivatives $\hat{\partial}$ acting on ψ_1 (ψ_2) while the opposite applies for $h^*_{i(np)}$; therefore, only cross terms $\psi_1 \times \psi_2$ show up in (76) and we can neglect the last term of (77) due to $\partial\cdot\hat{\partial} = 0$. Consequently,

$$\mathcal{L}_{2s-1}^{(s)} = c_0[\hat{\partial}_i\hat{\partial}_j\hat{\partial}_k\psi_1 + \hat{\partial}_{(i}\hat{\partial}_j\partial_{k)}\psi_2]\left[\delta^{jn}\delta^{kp} - \frac{1}{4}\delta^{jk}\delta^{np}\right] \times \Box^2(-\partial_i\hat{\partial}_n\hat{\partial}_p\psi_1 + \hat{\partial}_i\hat{\partial}_n\hat{\partial}_p\psi_2) = 2c_0\Box\psi_1\nabla^6(-3\Box\dot{\psi}_2) = 2c_0\left(I_{2s-1}\frac{\nabla^6}{4}I_{2s}\right). \tag{78}$$

From (68) and (69) we see that we can define, with a trivial Jacobian, the new fields $(I_{2s}, I_{2s-1}) \equiv (\tilde{\rho}, 3\tilde{\gamma})$ such that I_{2s} and I_{2s-1} are two independent fields as in the previous spin-1 and spin-2 cases. So we can verify by comparing (75) with (32) and (52) as well as (78) with (33) and (53), that the canonical structure of $\mathcal{L}_{2s}^{(s)}$, $\mathcal{L}_{2s-1}^{(s)}$ remains the same up to irrelevant overall numerical factors and powers of $-\nabla^2$ which can be absorbed in redefinitions of the constants (b_0, c_0) and of the invariants (I_{2s}, I_{2s-1}), respectively. There is no obstacle in building up the spin-3 sixth-order self-dual model $\mathcal{L}_{2s}^{SD}$, as originally suggested in [15]. By combining $\mathcal{L}_{2s}^{(s)}$ and $\mathcal{L}_{2s-1}^{(s)}$ with $c_0 = -mb_0$ we have a self-dual model with the same form of (35),

$$\mathcal{L}_{2s}^{SD} = b_0 h_{\mu\nu\rho}\left[\Box\left(\theta^{\mu\alpha}\theta^{\nu\beta}\theta^{\rho\gamma} - \frac{3}{4}\theta^{\mu\nu}\theta^{\alpha\beta}\theta^{\rho\gamma}\right) - mE^{\mu\alpha}\left(\theta^{\nu\beta}\theta^{\rho\gamma} - \frac{1}{4}\theta^{\mu\nu}\theta^{\beta\gamma}\right)\right]\Box^2 h_{\alpha\beta\gamma} \tag{79}$$

$$= \frac{b_0}{4}\{I_m(-\nabla^2)I_m + I_{2s-1}[-\nabla^2(\Box - m^2)]I_{2s-1}\}. \tag{80}$$

From (68) and (69) we see that we can redefine the fields such that $I_{2s-1} \equiv \tilde{\rho}$ and $I_m \equiv I_{2s} + mI_{2s-1} \equiv 3\tilde{\gamma}$. So the particle content of (79) corresponds to only one propagating massive mode.

We move now to the investigation of the fourth-order spin-3 Lagrangian $\mathcal{L}_{2s-2}^{(s)} = b_0 h_{\mu\nu\alpha}D^{\mu\nu\alpha}$, preliminarly studied in [23]. We need to find the symmetric D tensor which solves the equation

$$C_{\mu\nu\alpha} = E_{(\mu}{}^{\rho}D_{\rho\nu\alpha)}, \tag{81}$$

where the spin-3 Cotton tensor can be obtained from (19) or (17). The D tensor must be of fourth order in derivatives ($D_{\mu\nu\alpha} \sim (\hat{\partial}^4 h)_{(\mu\nu\alpha)}$). In the spin-2 case we have started from a general second-order ansatz $D_{\mu\nu} \sim (\hat{\partial}^2 h)_{(\mu\nu)}$ and required (55). Alternatively, we could have obtained (57) by requiring instead that its variation under Wdiff had the tensor structure $\delta D_{\mu\nu} = \partial_\mu\partial_\nu F + \eta_{\mu\nu}\Box G$. This guarantees the Wdiff invariance of the spin-2 Cotton tensor. The Cotton tensor is uniquely determined by its local symmetry and order in derivatives. Since F and G must be linear functions of $\partial\cdot\Lambda$ and ψ the required tensor structure is equivalent to the Tdiff symmetry. The spin-3 and spin-4 cases are completely analogous. In the spin-3 case, the symmetry of the Cotton requires that under (62) we have $\delta D_{\mu\nu\alpha} = \partial_\mu\partial_\nu\partial_\alpha F + \Box\eta_{(\mu\nu}\partial_{\alpha)}G$ where F and G are linear functions of $\Box\Lambda$, $\partial^\mu\partial^\nu\Lambda_{\mu\nu}$ and $\partial^\mu\psi_\mu$. This is equivalent to demand

$$\bar{\delta}\int d^3x h_{\mu\nu\alpha}D^{\mu\nu\alpha} = 0. \tag{82}$$

where the $\bar{\delta}$ gauge transformations correspond to Wdiff with the three scalar restrictions:

$$\eta^{\mu\nu}\Lambda_{\mu\nu} \equiv \Lambda = 0 = \partial^\mu\partial^\nu\Lambda_{\mu\nu} = \partial^\mu\psi_\mu. \tag{83}$$

The general solution to (82) is a two parameter family of Lagrangians,

$$\mathcal{L}_{2s-2}^{(s)}(f,g) = d_0\left[h_{\mu\nu\alpha}\Box^2 h^{\mu\nu\alpha} - \frac{3}{4}h_\mu\Box^2 h^\mu - 3h_{\mu\nu\alpha}\Box\partial^\mu\partial_\beta h^{\beta\nu\alpha} + \frac{3}{2}h_{\mu\nu\alpha}\Box\partial^\mu\partial^\nu h^\alpha + \frac{9}{4}h_{\mu\nu\alpha}\partial^\mu\partial^\nu\partial_\beta\partial_\rho h^{\beta\rho\alpha} + fh_\mu\Box\partial^\mu\partial^\nu h_\nu + gh_{\mu\nu\alpha}\partial^\mu\partial^\nu\partial^\alpha\partial^\beta h_\beta\right] \tag{84}$$

where the parameters (f, g) are so far arbitrary.

The question is which values of the parameters (f, g) render (84) an empty theory? The more symmetry, the less content we have, so we must try to enlarge the $\bar{\delta}$ symmetry as much as possible. In the spin-2 case the Tdiff symmetry associated with the restrictions $(\partial^\mu \Lambda_\mu, \psi) = (0,0)$ could be enlarged either to diff ($\psi = 0$) or to WTdiff ($\partial^\mu \Lambda_\mu = 0$). So the idea is to lift the restrictions (83) as much as possible. The reader can check that there is no solution for (f, g) if we try to keep only one of the three restrictions (83), but in case we keep two of them we have found some solutions. In [35] we will make a general analysis including the more complex spin-4 case. Here we stick to the case where the $\bar{\delta}$ symmetry is enlarged to traceless spin-3 diff plus transverse Weyl transformations (TWdiff),

$$\Lambda = 0 = \partial^\mu \psi_\mu \to (f, g) = \left(\frac{21}{16}, -\frac{9}{4}\right). \quad (85)$$

Correspondingly we define from (84) the fourth-order spin-3 Lagrangian

$$\mathcal{L}_4^{(3)} \equiv \mathcal{L}_{2s-2}^{(s)}\left(\frac{21}{16}, -\frac{9}{4}\right) \quad \text{at } d_0 = -m^2 b_0. \quad (86)$$

The reason we choose (85) is twofold. First, it has already been analyzed, in a fixed gauge, in [23] where it is shown to have an empty spectrum.[11] Second, there will be an analogous case for spin-4 as we will see in the next section.

Following our gauge invariant approach, due to the restrictions $(\Lambda, \partial^\mu \psi_\mu) = (0,0)$ we have seven independent gauge parameters on the right-hand side of the 10 equations (62), thus we have $10 - 7 = 3$ gauge invariants just like the spin-2 Einstein-Hilbert case and the spin-1 Proca mass term. By eliminating the seven independent gauge parameters as functions of $\delta h_{\mu\nu\alpha}$ and plugging back in (62) we obtain two fifth-order invariants and a sixth-order one, $\delta i_{2s-1} = 0 = \delta \bar{i}_{2s-1} = \delta i_{2s}$ where (i_{2s}, i_{2s-1}) are the two invariants under unrestricted transformations (62) given in (63) and (64) while

$$\begin{aligned}\bar{i}_{2s-1} = {} & 3[\nabla^2 \hat{\partial}_i \hat{\partial}_j \partial_k h_{ijk} + 2\hat{\partial}_i \hat{\partial}_j \partial_k \ddot{h}_{ijk} - 3\nabla^2 \hat{\partial}_j \hat{\partial}_k \dot{h}_{0jk}] + 3[\partial_j \partial_k \dot{h}_{0jk} - \nabla^2 \partial_j \dot{h}_{00j}] \\ & + \nabla^2 \partial_i \partial_j \partial_k h_{ijk} - 2\partial_i \partial_j \partial_k \ddot{h}_{ijk} + \nabla^4 \dot{h}_{000}. \end{aligned} \quad (87)$$

After a convenient redefinition, in terms of helicity variables, we have

$$\bar{I}_{2s-1} = \frac{\bar{i}_{2s-1}}{(-2\nabla^6)} = \left[-3\Gamma + \frac{\dot{\rho}}{\nabla^2} - 9\dot{\phi}_1 + 3\dot{\phi}_2 + \nabla^2 \psi_4 - 2\ddot{\psi}_4 + 3(\nabla^2 \psi_2 + 2\ddot{\psi}_2)\right]/(-2). \quad (88)$$

As in (70) we assume that $\mathcal{L}_{2s-2}^{(s)}(21/16, -9/4) = \sum_{K,L} I_K \hat{O}_{KL} I_L$ where the sum run over the three invariants (68), (69), and (88) while $\hat{O}_{KL}$ stands for a symmetric 3×3 matrix differential operator to be found. We have followed a two step procedure. In the first step we keep only $(\psi_1, \psi_4) \neq (0,0)$ in the helicity decomposition (65)–(67) while in the second one we assume $(\psi_1, \psi_2) \neq (0,0)$ such that we respectively have

$$h_{jkl} = \hat{\partial}_j \hat{\partial}_k \hat{\partial}_l \psi_1 + \partial_j \partial_k \partial_l \psi_4 \to (I_{2s}, I_{2s-1}, \bar{I}_{2s-1}) = (-\partial_0^3 \psi_4, \Box \psi_1, \ddot{\psi}_4 - \nabla^2 \psi_4/2), \quad (89)$$

$$h_{ijk} = \hat{\partial}_i \hat{\partial}_j \hat{\partial}_k \psi_1 + \hat{\partial}_{(i} \hat{\partial}_j \partial_{k)} \psi_2 \to (I_{2s}, I_{2s-1}, \bar{I}_{2s-1}) = (-3\Box \dot{\psi}_2, \Box \psi_1, -3(\nabla^2 \psi_2 + 2\ddot{\psi}_2)/2). \quad (90)$$

Direct substitution in $\mathcal{L}_4^{(3)}$ leads respectively to

$$\mathcal{L}_I = -\psi_1 \nabla^6 \Box^2 \psi_1 + \psi_4 \nabla^6 [-\Box^2 + (3/4)\nabla^2 \Box] \psi_4, \quad (91)$$

$$\mathcal{L}_{II} = \Box \psi_1 (-\nabla^6) \Box \psi_1 + 9\psi_2 \Box (\ddot{\psi}_3 - \nabla^2 \psi_2/4) \quad (92)$$

which uniquely determine the fourth-order spin-3 Lagrangian, compared with (36) and (37),

$$\mathcal{L}_{2s-2}^{(s)} = -d_0 (I_{2s-1} \nabla^6 I_{2s-1} + \bar{I}_{2s-1} \nabla^6 \bar{I}_{2s-1} + \bar{I}_{2s-2}^2) \quad (93)$$

$$\begin{aligned} = {} & -d_0 (I_{2s-1} \nabla^6 I_{2s-1} + I_{2s} \nabla^4 I_{2s} + 2I_{2s} \nabla^4 \dot{\bar{I}}_{2s-1} \\ & + \bar{I}_{2s-1} \nabla^4 \Box \bar{I}_{2s-1}) \end{aligned} \quad (94)$$

where we have defined

$$\begin{aligned}\bar{I}_{2s-2} = {} & \nabla^2 (I_{2s} + \dot{\bar{I}}_{2s-1}) = 2\nabla^2 \rho - \ddot{\rho} - 3\nabla^2 \dot{\Gamma} \\ & + 3\nabla^2 (\ddot{\phi}_1 + 2\nabla^2 \phi_1) + 3\nabla^2 \ddot{\phi}_2 - 9\nabla^4 \dot{\psi}_3 - \nabla^4 \dot{\psi}_2. \end{aligned} \quad (95)$$

[11]The spin-3 fourth-order Lagrangian (86) appeared in the literature, see [36], even before [23]. We thank Prof. Karapet Mkrtchyan for bringing that reference to our attention.

Notice from (63) and (87) that all sixth-order terms in the combination $\bar{I}_{2s-2}$ cancel out and we are left with at most four derivatives of the fundamental field $h_{\mu\nu\alpha}$ which justifies the lower index. Now an important technical point must be stressed. In order to establish full analogy with the spin-1 case we should be able to treat $(I_{2s-1}, \bar{I}_{2s-1}, \bar{I}_{2s-2})$ as independent fields. Although $I_{2s-1} \equiv 3\gamma$ decouples from $\bar{I}_{2s-1}$ and $\bar{I}_{2s-2}$, due to the time derivatives on Γ and ρ in (88) and (95) it is not obvious that both $\bar{I}_{2s-1}$ and $\bar{I}_{2s-2}$ can be treated as basic independent fields. In order to prove it we first get rid of time derivatives in (88) redefining Γ via $\bar{I}_{2s-1} \equiv 3\bar{\Gamma}/2$. After such redefinition (95) still has terms of the type $\ddot{\rho}$ which can be eliminated via $\bar{\phi}_1 \equiv \phi_1 - \frac{\rho}{6\nabla^2}$. The final step is to redefine ρ according to $\bar{I}_{2s-2} \equiv 3\nabla^2\bar{\rho}$. The reader can check that the triple change of variables $\bar{\Phi}_J = M_{JK}\Phi_K + G_J$ with $\Phi_J = (\Gamma, \phi_1, \rho)$ and G_J independent of Φ_J is such that all derivatives cancel out in the Jacobian and we have det $M = 1$. Therefore, the fourth-order theory given in (93) or (94), see also (84) with $(f, g) = (21/16, -9/4)$, has no particle content, in agreement with the gauge fixed analysis of [23].

Since the spin-3 Lagrangian (94) has exactly the same form of the Proca mass term (37), similarly for the spin-1 (33) and spin-3 (78) Chern-Simon terms, we can follow the same steps leading to (40), with the choice $(c_0, d_0) = (mb_0, -m^2 b_0)$, and obtain the fifth-order spin-3 self-dual model suggested in [23] in terms of gauge invariants,

$$\mathcal{L}_{2s-1}^{SD} = mb_0 h_{\mu\nu\rho} E^{\mu\alpha}\left(\theta^{\nu\beta}\theta^{\rho\gamma} - \frac{1}{4}\theta^{\mu\nu}\theta^{\beta\gamma}\right)\Box^2 h_{\alpha\beta\gamma} + \mathcal{L}_4^{(3)} \tag{96}$$

$$= \frac{b_0}{4}\left[I_m\nabla^2 I_m + \tilde{I}_{2s}(m^2 - \nabla^2)\tilde{I}_{2s} + m^2\bar{I}_{2s-1}\frac{(-\nabla^2)(\Box - m^2)}{m^2 - \nabla^2}\bar{I}_{2s-1}\right] \tag{97}$$

where $I_m = I_{2s} + mI_{2s-1}$, $\tilde{I}_{2s} \equiv I_{2s} + m^2\dot{\bar{I}}_{2s-1}/(m^2 - \nabla^2)$ and $\mathcal{L}_4^{(3)}$ is given in (86). Such a model is the spin-3 analog of the linearized TMG. Notice however that it is not trivial to show that $(\tilde{I}_{2s}, \bar{I}_{2s-1}, I_m)$ are three independent degrees of freedom. First we notice that γ only appears in I_m, thus the redefinition $I_m = 3m\bar{\gamma}$ does not affect $(\tilde{I}_{2s}, \bar{I}_{2s-1})$. Next we redefine Γ such that $\bar{I}_{2s-1} \equiv -3\bar{\Gamma}$, then we make $\bar{\phi}_1 \equiv \phi_1 - \frac{\rho}{6\nabla^2}$ in order to get rid of time derivatives of ρ in $\tilde{I}_{2s}$, and finally we redefine ρ such that $\tilde{I}_{2s} = 3\nabla^2(m^2 - \nabla^2)\bar{\rho}$. It turns out that the whole Jacobian is trivial.

The doublet model $\mathcal{L}_{2s}^D$, i.e., the spin-3 analog of NMG has been suggested in [37] where it was obtained via soldering of two self-dual models of opposite helicities +3 and −3 as given in (79) or in (96). The same result can be obtained adding (79) and (96) with $c_0 = mb_0$, i.e.,

$$\mathcal{L}_{2s}^D = b_0 h_{\mu\nu\rho}\Box^3\left(\theta^{\mu\alpha}\theta^{\nu\beta}\theta^{\rho\gamma} - \frac{3}{4}\theta^{\mu\nu}\theta^{\alpha\beta}\theta^{\mu\gamma}\right)h_{\alpha\beta\gamma} + \mathcal{L}_4^{(3)} \tag{98}$$

$$= \frac{b_0}{4}\left\{I_{2s-1}[-\nabla^6(\Box - m^2)]I_{2s-1} + m^2\bar{I}_{2s-1}\times\frac{(-\nabla^6)(\Box - m^2)}{m^2 - \nabla^2}\bar{I}_{2s-1} + \tilde{I}_{2s}\nabla^4(m^2 - \nabla^2)\tilde{I}_{2s}\right\}. \tag{99}$$

VII. THE SPIN-4 CASE

In the rank-4 case the Wdiff transformations (10) equations correspond to 15 equations:

$$\delta h_{\mu\nu\alpha\beta} = \partial_{(\mu}\Lambda_{\nu\alpha\beta)} + \eta_{(\mu\nu}\psi_{\alpha\beta)}. \tag{100}$$

By either considering $\Lambda_{\nu\alpha\beta}$ a traceless tensor $\eta^{\nu\alpha}\Lambda_{\nu\alpha\beta} \equiv \Lambda_\beta = 0$ or taking into account the vector redundancy $\delta(\Lambda_{\nu\alpha\beta}, \psi_{\alpha\beta}) = (\eta_{(\nu\alpha}\epsilon_{\beta)}, -\partial_{(\alpha}\epsilon_{\beta)})$ we see that (100) leads to $15 - 13 = 2$ Wdiff local invariants of eighth and seventh order in derivatives, $\delta i_{2s} = 0 = \delta i_{2s-1}$,

$$i_{2s} = \nabla^8 h_{0000} - 4\nabla^6\partial_j\partial_0 h_{00j} + 6\nabla^4\partial_j\partial_k\partial_0^2 h_{00jk} - 4\nabla^2\partial_i\partial_j\partial_k\partial_0^3 h_{0ijk} + \partial_i\partial_j\partial_k\partial_l\partial_0^4 h_{ijkl} + 6\Box[(\nabla^4\hat{\partial}_j\hat{\partial}_k h_{00jk} - 2\hat{\partial}_j\hat{\partial}_k\partial_l\partial_0 h_{0jkl} + \hat{\partial}_j\hat{\partial}_k\partial_l\partial_m\partial_0^2 h_{jklm}] + \Box^2\hat{\partial}_j\hat{\partial}_k\hat{\partial}_l\hat{\partial}_m h_{jklm}, \tag{101}$$

$$i_{2s-1} = \nabla^6\hat{\partial}_j h_{000j} - 3\nabla^4\partial_j\hat{\partial}_k\partial_0 h_{00jk} + 3\nabla^2\partial_i\partial_j\hat{\partial}_k\partial_0^2 h_{0ijk} - \partial_i\partial_j\partial_k\hat{\partial}_l\partial_0^3 h_{ijkl} \times\Box[\nabla^2\hat{\partial}_i\hat{\partial}_j\hat{\partial}_k h_{0ijk} - \partial_i\hat{\partial}_j\hat{\partial}_k\hat{\partial}_l\partial_0 h_{ijkl}]. \tag{102}$$

After the helicity decomposition

$$h_{0000} = \rho; \qquad h_{000j} = \hat{\partial}_j\gamma + \partial_j\Gamma, \tag{103}$$

$$h_{00jk} = \hat{\partial}_j\hat{\partial}_k\phi_1 + \hat{\partial}_{(j}\partial_{k)}\phi_2 + \partial_j\partial_k\phi_3, \tag{104}$$

$$h_{0jkl} = \hat{\partial}_j\hat{\partial}_k\hat{\partial}_l\psi_1 + \hat{\partial}_{(j}\hat{\partial}_k\partial_{l)}\psi_2 + \partial_{(j}\partial_k\hat{\partial}_{l)}\psi_3 + \partial_j\partial_k\partial_l\psi_4+, \tag{105}$$

$$h_{ijkl} = \hat{\partial}_i\hat{\partial}_j\hat{\partial}_k\hat{\partial}_l\beta_1 + \hat{\partial}_{(i}\hat{\partial}_j\hat{\partial}_k\partial_{l)}\beta_2 + \partial_{(i}\partial_j\hat{\partial}_k\hat{\partial}_{l)}\beta_3 + \hat{\partial}_{(i}\partial_j\partial_k\partial_{l)}\beta_4 + \partial_i\partial_j\partial_k\partial_l\beta_5, \tag{106}$$

and redefining the invariants we obtain

$$I_{2s} \equiv i_{2s}/\nabla^8 = \rho - 4\dot{\Gamma} + 6\ddot{\phi}_3 + 6\Box\phi_1 - 4\partial_0^3\psi_4 - 12\Box\partial_0\psi_2 + \partial_0^4\beta_5 + \Box^2\beta_1, \tag{107}$$

$$I_{2s-1} \equiv i_{2s-1}/\nabla^8 = \gamma + 3\ddot{\psi}_3 + \Box\psi_1 - \Box\dot{\beta}_2 - \partial_0^3\beta_4. \tag{108}$$

In order to write down the eighth order Wdiff invariant Lagrangian (26) in terms of gauge invariants we assume that the only nonvanishing fields are β_1 and β_2; therefore,

$$h_{ijkl} = \hat{\partial}_i\hat{\partial}_j\hat{\partial}_k\hat{\partial}_l\beta_1 + \hat{\partial}_{(i}\hat{\partial}_j\hat{\partial}_k\partial_{l)}\beta_2 \to (I_{2s}, I_{2s-1}) = (\Box^2\beta_1, -\Box\dot{\beta}_2), \tag{109}$$

$$\mathcal{L}_{2s}^{(s)} = b_0 h_{ijkl}\left[\theta^{im}\theta^{jn}\theta^{kp}\theta^{lq} - \theta^{ij}\theta^{mn}\theta^{kp}\theta^{lq} + \frac{\theta^{ij}\theta^{mn}\theta^{kl}\theta^{pq}}{8}\right]h_{mnpq} \tag{110}$$

$$= b_0\left[\beta_1\nabla^8\left(1 - 1 + \frac{1}{8}\right)\Box^4\beta_1 + \hat{\partial}_{(i}\hat{\partial}_j\hat{\partial}_k\partial_{l)}\beta_2(\theta^{im}\theta^{jn}\theta^{kp}\theta^{lq} - \theta^{ij}\theta^{mn}\theta^{kp}\theta^{lq})\hat{\partial}_{(m}\hat{\partial}_n\hat{\partial}_p\partial_{q)}\beta_2\right]$$

$$= b_0\left[I_{2s}\frac{\nabla^8}{8}I_{2s} + 2I_{2s-1}\nabla^8\Box I_{2s-1}\right] \tag{111}$$

where we have used $\partial_i\partial_m\Box\theta^{im} = \nabla^2\partial_0^2$. Notice that no cross term $\beta_1 \times \beta_2$ appears due to the odd number of dual derivatives $\hat{\partial}$ which leads to $\hat{\partial}\cdot\partial = 0$. Regarding the seventh-order Chern-Simons term (20) we have

$$\mathcal{L}_{2s-1}^{(s)} = \frac{c_0}{2}h_{ijkl}(2\theta^{jn}\theta^{kp}\theta^{lq} - \theta^{np}\theta^{jk}\theta^{lq})\Box^3\dot{h}^*_{i(npq)} \tag{112}$$

$$= -c_0\hat{\partial}_j\hat{\partial}_k\hat{\partial}_l\beta_1(2\delta^{jn}\delta^{kp}\delta^{lq} - \delta^{np}\delta^{jk}\delta^{lq})\nabla^2\hat{\partial}_n\hat{\partial}_p\hat{\partial}_q\dot{\beta}_2 = -c_0 I_{2s-1}\nabla^8 I_{2s} \tag{113}$$

where we have used $E^{im} = \epsilon^{im}\partial_0$ and

$$\dot{h}^*_{i(npq)} \equiv -\epsilon^{im}h_{mnpq} = -\partial_i\hat{\partial}_n\hat{\partial}_p\hat{\partial}_q\dot{\beta}_1 + \hat{\partial}_i\hat{\partial}_n\hat{\partial}_p\hat{\partial}_q\dot{\beta}_2 - \partial_i[\partial_{(n}\hat{\partial}_p\hat{\partial}_{q)}\dot{\beta}_2]. \tag{114}$$

As in the spin-3 case, only the cross terms $\beta_1 \times \beta_2$ survive in (113) due to $\partial\cdot\hat{\partial} = 0$.

Comparing (52) and (53) with (111) and (113) we see that the canonical structure of spin-2 and spin-4 cases basically coincide. So the linearized NTMG (54) has its spin-4 counterpart, first suggested in [16], with $c_0 = -mb_0$ and we have

$$\mathcal{L}_{2s}^{SD} = b_0\left\{h_{\mu\nu\alpha\beta}\Box^4\left[\theta^{\mu\rho}\theta^{\nu\gamma}\theta^{\alpha\lambda}\theta^{\beta\sigma} - \theta^{\mu\nu}\theta^{\rho\gamma}\theta^{\alpha\lambda}\theta^{\beta\sigma} + \frac{1}{8}\theta^{\mu\nu}\theta^{\rho\gamma}\theta^{\alpha\beta}\theta^{\lambda\sigma}\right]h_{\rho\gamma\lambda\sigma}\right.$$
$$\left. + mh_{\mu\nu\alpha\beta}\Box^3 E^{\mu\gamma}\left(\theta^{\nu\rho}\theta^{\alpha\lambda}\theta^{\beta\sigma} - \frac{1}{2}\theta^{\nu\alpha}\theta^{\rho\lambda}\theta^{\beta\sigma}\right)h_{\gamma\rho\lambda\sigma}\right\} \tag{115}$$

$$= 2b_0\left\{\frac{1}{4}I_{2m}\nabla^4 I_{2m} + I_{2s-1}[\nabla^4(\Box - m^2)]I_{2s-1}\right\} \tag{116}$$

where $\mathcal{L}_{2s}^{(s)}$ is given in (110) while $\mathcal{L}_{2s-1}^{(s)}$ appears in (112); moreover $I_{2m} = I_{2s}/2 + 2mI_{2s-1}$. We can always change variables $(I_{2m}, I_{2s-1}) = (\bar{\rho}/2, \bar{\gamma})$ and treat the two Wdiff invariants as independent degrees of freedom. So we have just one massive mode in $\mathcal{L}_{2s}^{SD}$ as shown in [16] in a fixed gauge.

In order to find the spin-4 analogs of TMG and NMG we first need $\mathcal{L}_{2s-2}^{(s)}$

$$\mathcal{L}_{2s-2}^{(s)} = \int d^3x h_{\mu\nu\alpha\beta}D^{\mu\nu\alpha\beta}(h) \tag{117}$$

where $D^{\mu\nu\alpha\beta}(h) \sim \partial^6 h$ is fully symmetric and satisfies

$$C_{\mu\nu\alpha\beta} = E_{(\mu}{}^{\rho} D_{\rho\nu\alpha\beta)} \tag{118}$$

where $C_{\mu\nu\alpha\beta}$ is the spin-4 Cotton tensor in flat space given in (20). As in the spin-3 case we can alternatively start from a rather general ansatz for $\mathcal{L}^{(s)}_{2s-2}$ with all possible contractions and require $\bar{\delta}\mathcal{L}^{(s)}_{2s-2} = 0$ where the $\bar{\delta}$ transformations correspond to (100) with all possible scalar restrictions on the gauge parameters,

$$\partial^\alpha \eta^{\mu\nu}\Lambda_{\mu\nu\alpha} \equiv \partial\cdot\Lambda = 0 = \partial^\mu\partial^\nu\partial^\alpha\Lambda_{\mu\nu\alpha} = \partial^\mu\partial^\nu\psi_{\mu\nu} = \eta^{\mu\nu}\psi_{\mu\nu}. \tag{119}$$

The general solution to the $\bar{\delta}$ symmetry, or equivalently to (118), is given by a five parameter family of Lagrangians

$$\begin{aligned}\mathcal{L}^{(s)}_{2s-2}[a,b,c,d,e] = d_0[&h_{\mu\nu\alpha\beta}\Box^3 h^{\mu\nu\alpha\beta} - h_{\mu\nu}\Box^3 h^{\mu\nu} + 4\partial_\mu h^{\mu\nu\alpha\beta}\Box^2\partial^\lambda h_{\lambda\nu\alpha\beta} - 2\partial_\mu h^{\mu\nu}\Box^2\partial^\lambda h_{\lambda\nu}\\ &+ 2h_{\mu\nu}\Box^2\partial_\alpha\partial_\beta h^{\mu\nu\alpha\beta} + 5\partial_\mu\partial_\nu h^{\mu\nu\alpha\beta}\Box\partial^\lambda\partial^\sigma h_{\lambda\sigma\alpha\beta} + 4\partial_\mu h^{\mu\nu}\Box\partial^\alpha\partial^\beta\partial^\lambda h_{\lambda\nu\alpha\beta}\\ &+ 2\partial^\alpha\partial^\beta\partial^\lambda h_{\lambda\nu\alpha\beta}\partial_\mu\partial_\rho\partial_\gamma h^{\nu\mu\rho\gamma} + ah\Box^3 h + bh\Box^2\partial_\mu\partial_\nu h^{\mu\nu} + c\partial_\mu\partial_\nu h^{\mu\nu}\Box\partial^\lambda\partial^\sigma h_{\lambda\sigma}\\ &+ dh\Box\partial_\mu\partial_\nu\partial_\lambda\partial_\rho h^{\mu\nu\lambda\rho} + e\partial_\mu\partial_\nu h^{\mu\nu}\partial_\alpha\partial_\beta\partial_\lambda\partial_\rho h^{\alpha\beta\lambda\rho}],\end{aligned} \tag{120}$$

where $h \equiv \eta^{\mu\nu}h_{\mu\nu} \equiv \eta^{\mu\nu}\eta^{\alpha\beta}h_{\mu\nu\alpha\beta}$.

Once again we look for a subset of solutions with an empty spectrum by requiring the maximal possible symmetry. First we have checked that there is no solution invariant under full Wdiff (100) constrained by only one of the restrictions (119). However, we have found at least five sets of two restrictions for which all coefficients (a,b,c,d,e) are fixed; they will be discussed elsewhere [35]. Here we only analyze the case

$$\begin{aligned}\partial\cdot\Lambda = 0 = \partial^\mu\partial^\nu\psi_{\mu\nu} &\to (a,b,c,d,e)\\ &= \left(\frac{3}{25},-\frac{2}{5},-1,\frac{1}{5},2\right).\end{aligned} \tag{121}$$

For convenience we define

$$\mathcal{L}^{(4)}_6 \equiv \mathcal{L}^{(s)}_{2s-2}\left[\frac{3}{25},-\frac{2}{5},-1,\frac{1}{5},2\right] \quad \text{at } d_0 = -m^2 b_0. \tag{122}$$

At first sight (121) does not seem to be a perfect spin-4 analog of (85). However, it turns out that if we start from a general Lagrangian of the form (120) but with all 13 coefficients arbitrary and require invariance under (100) with the restrictions $(\Lambda_\mu, \partial^\mu\partial^\nu\psi_{\mu\nu}) = (0,0)$ we would arrive exactly at (121). Likewise, in the spin-3 case, we have checked that if we start from a fourth-order Lagrangian of the form (84) but with all 7 coefficients arbitrary and require symmetry under (62) with the restrictions $(\Lambda, \partial\cdot\psi) = (0,0)$ we would end up precisely with (85). This means that instead of finding the higher spin analogs of the EH theory by searching for the solutions to (6) which have an empty spectrum, we can use instead a gauge symmetry principle just like the EH theory is completely fixed, up to trivial field redefinitions, by requiring diff symmetry.

Henceforth we take (121) for granted. Note that the transformations (100) restricted by $\partial^\mu\Lambda_\mu = 0 = \partial^\mu\partial^\nu\psi_{\mu\nu}$ still have a vector redundancy of the type discussed after (100) but the vector must be transverse $\partial^\mu\epsilon_\mu = 0$. This means that we have in total $10 + 6 - 2 - 2 = 12$ independent gauge parameters[12] in (100) which leads to $15 - 12 = 3$ gauge invariants just like the previous $s = 1, 2, 3$ cases. Solving (100) for the 12 parameters and plugging back in (100) we obtain three gauge invariants $\bar{\delta}i_{2s} = 0 = \bar{\delta}i_{2s-1} = \bar{\delta}\bar{i}_{2s-2}$. Besides the known invariants of eighth and seventh order given in (101) and (102) we have the sixth order invariant:

$$\begin{aligned}\bar{i}_{2s-2} = &-\Box(\partial_i\partial_j\partial_k\partial_l h_{ijkl} - 2\nabla^2\partial_j\partial_k h_{00jk} + \nabla^4 h_{0000}) + \Box[8\partial_i\partial_j\hat{\partial}_k\hat{\partial}_l h_{ijkl} + 2\nabla^2\hat{\partial}_j\hat{\partial}_k h_{00jk}]\\ &- \Box\hat{\partial}_i\hat{\partial}_j\hat{\partial}_k\hat{\partial}_l h_{ijkl} - 10\nabla^2[\partial_i\partial_j\hat{\partial}_k\hat{\partial}_l h_{ijkl} - 2\partial_j\hat{\partial}_k\hat{\partial}_l h_{0jkl} + \nabla^2\hat{\partial}_j\hat{\partial}_k h_{00jk}].\end{aligned} \tag{123}$$

In terms of helicity variables we have

$$\bar{I}_{2s-2} = \frac{\bar{i}_{2s-2}}{\nabla^6} = 20\nabla^2\dot{\psi}_2 - \frac{\Box}{\nabla^2}\rho + 2\Box\phi_3 - 2(\ddot{\phi}_1 + 4\nabla^2\phi_1) - \nabla^2\Box(\beta_1+\beta_5) - 2\nabla^4\beta_3 - 8\nabla^2\ddot{\beta}_3. \tag{124}$$

[12] Alternatively, the four restrictions $\Lambda_\mu = 0 = \partial^\mu\partial^\nu\psi_{\mu\nu}$ also give $16 - 4 = 12$ gauge parameters, and no redundancy is left in this case.

In order to write down $\mathcal{L}^{(s)}_{2s-2}$ in terms of gauge invariants we suppose that $\mathcal{L}^{(s)}_{2s-2} = \sum_{K,L} I_K \hat{O}_{KL} I_L$ where the sum run over the three invariants (107), (108), and (124). If we first assume that the only nonvanishing fields are (β_1, β_2) and then (β_1, β_5), direct substitution in (122) with the constants given in (121) lead respectively to

$$\mathcal{L}^{(s)}_{2s-2}[\beta_1, \beta_2] = \frac{3}{25}\beta_1 \nabla^8 \Box^3 \beta_1 - 2\beta_2 \nabla^8 \Box^2 \ddot{\beta}_2, \tag{125}$$

$$\mathcal{L}^{(s)}_{2s-2}[\beta_1, \beta_5] = \frac{1}{5}\tilde{\beta}_1 \nabla^{10} \Box (\nabla^2 - 2\Box)\beta_5 + \frac{3}{25}\tilde{\beta}_1 \nabla^8 \Box^3 \tilde{\beta}_1, \tag{126}$$

where $\tilde{\beta}_1 = \beta_1 + \beta_5$. Moreover, if we keep only (γ, Γ), the only nonvanishing components will be h_{000j} and they are such that it is impossible to have a cross term $\gamma \times \Gamma$, consequently $\hat{O}_{78} = 0 = \hat{O}_{87}$. From (125), (126) and $\hat{O}_{78} = 0$ we obtain

$$\mathcal{L}^{(s)}_{2s-2} = d_0 \left[2I_{2s-1} \nabla^8 I_{2s-1} - \frac{1}{5} I_{2s} \nabla^6 \bar{I}_{2s-2} - \frac{2}{25} \bar{I}_{2s-2} \nabla^4 \Box \bar{I}_{2s-2} \right]. \tag{127}$$

Comparing (111), (113), and (127) with the corresponding formulas of the spin-2 case (52), (53), and (59) there is a perfect match after a harmless redefinition $(I_{2s}, \bar{I}_{2s-2}) \to (2I_{2s}, 5\bar{I}_{2s-2})$ in (127). However, we still have to worry whether we can treat $(I_{2s}, I_{2s-1}, \bar{I}_{2s-2})$ given in (107), (108), and (124) as independent degrees of freedom such that the higher time derivatives can be properly hidden via a change of variables in order to avoid ghosts. This is indeed the case as we show now.

First we redefine γ such that $I_{2s-1} = \bar{\gamma}$. This is a trivial one field redefinition that does not affect either I_{2s} or $\bar{I}_{2s-2}$. Next we redefine ρ such that $I_{2s} = \bar{\rho}$ which introduces higher time derivatives of ψ_2 in $\bar{I}_{2s-2}$. They can be cancelled out after $\Gamma = \bar{\Gamma} - 3\Box\psi_2 + (3/2)\dot{\phi}_3$ which allows us to get rid also of time derivatives of ϕ_3 via $\psi_2 = \bar{\psi}_2 - \dot{\phi}_3/(10\nabla^2)$. Finally, we redefine ϕ_3 such that $\bar{I}_{2s-2} = 2\nabla^2 \bar{\phi}_3$. In summary, we have a fivefold change of variables $\bar{\Phi}_I = M_{IJ}\Phi_J + F_J$, where $\Phi_J = (\rho, \Gamma, \psi_2, \phi_3, \gamma)$ and F_J do not depend upon Φ_I, which leads to $(I_{2s}, I_{2s-1}, \bar{I}_{2s-2}) = (\bar{\rho}, \bar{\gamma}, 2\nabla^2\bar{\phi}_3)$. One can check that $\det M = 1$.

Consequently we can define the spin-4 analogs of the spin-2 linearized TMG and linearized NMG. In the first case if we choose $(c_0, d_0) = (-mb_0, -m^2 b_0)$ we have

$$\mathcal{L}^{SD}_{2s-1} \equiv \mathcal{L}^{(s=4)}_{LTMG} = -mb_0 h_{\mu\nu\alpha\beta} \Box^3 E^{\mu\gamma} \times \left(\theta^{\nu\rho}\theta^{\alpha\lambda}\theta^{\beta\sigma} - \frac{1}{2}\theta^{\nu\alpha}\theta^{\rho\lambda}\theta^{\beta\sigma} \right) h_{\gamma\rho\lambda\sigma} + \mathcal{L}^{(4)}_6 \tag{128}$$

$$= 2m^2 b_0 \left[\tilde{\tilde{I}}_{2s-2} \nabla^6 \Box \tilde{\tilde{I}}_{2s-2} - I_{2s-1} \nabla^8 I_{2s-1} - \tilde{I}_{2s} \nabla^8 \left(\tilde{\tilde{I}}_{2s-2} + \frac{I_{2s-1}}{m} \right) \right]. \tag{129}$$

where $\mathcal{L}^{(4)}_6$ is the sixth-order spin-4 Lagrangian in (122). The three gauge invariants $(\tilde{I}_{2s}, I_{2s-1}, \tilde{\tilde{I}}_{2s-2}) = (2I_{2s}, I_{2s-1}, 5\bar{I}_{2s-2})$ can be obtained from (107), (108), and (124). We can repeat the arguments given after (60) and prove that (128) has only one massive propagating mode. It is invariant under the Wdiff transformations (100) constrained by $\partial \cdot \Lambda = 0 = \partial^\mu \partial^\nu \psi_{\mu\nu}$.

In the second case of the spin-4 linearized NMG model we add the Lagrangians (115) and (128) such that the higher spin Chern-Simons term cancel out and we have

$$\mathcal{L}^{D}_{2s} \equiv \mathcal{L}^{(s=4)}_{\rm LNMG} = b_0 h_{\mu\nu\alpha\beta} \Box^4 \left[\theta^{\mu\rho}\theta^{\nu\gamma}\theta^{\alpha\lambda}\theta^{\beta\sigma} - \theta^{\mu\nu}\theta^{\rho\gamma}\theta^{\alpha\lambda}\theta^{\beta\sigma} + \frac{1}{8}\theta^{\mu\nu}\theta^{\rho\gamma}\theta^{\alpha\beta}\theta^{\lambda\sigma} \right] h_{\rho\gamma\lambda\sigma} + \mathcal{L}^{(4)}_6 \tag{130}$$

$$= 2b_0 [I_{2s-1} \nabla^6 (\Box - m^2) I_{2s-1} + m^2 \bar{I}_{2s-2} \nabla^4 (\Box - m^2) \bar{I}_{2s-2} + \bar{I}_{2s} \nabla^8 \bar{I}_{2s}] \tag{131}$$

where $\bar{I}_{2s} = \tilde{I}_{2s} - 2m^2 \tilde{\tilde{I}}_{2s-2}$. The spin-4 NMG theory is also invariant under (100) restricted by the scalar conditions $\partial \cdot \Lambda = 0 = \partial^\mu \partial^\nu \psi_{\mu\nu}$.

VIII. GAUGE INVARIANTS AND THE COTTON AND *D* TENSOR

In all cases $s = 1, 2, 3, 4$ worked out here we have been able to find two invariants under (10), (i_{2s}, i_{2s-1}), which play an instrumental role. The transformations (10) can be rewritten without loss of generality as

$$\delta h_{\mu_1 \cdots \mu_s} = \partial_{(\mu_1} \bar{\Lambda}_{\mu_2 \cdots \mu_s)} + \eta_{(\mu_1 \mu_2} \psi_{\mu_3 \cdots \mu_s)}, \tag{132}$$

where $\eta^{\mu_2\mu_3} \bar{\Lambda}_{\mu_2 \cdots \mu_s} = 0$. Since all three tensors in (132) are fully symmetric, their number of components in 3D is given by

$$N_{\delta h} = \frac{3.4. \cdots (3+s-1)}{s!} = \frac{(s+1)(s+2)}{2},$$
$$N_{\bar{\Lambda}} = \frac{3.4. \cdots (3+s-2)}{(s-1)!} - \frac{3.4. \cdots (3+s-4)}{(s-3)!} = 2s-1,$$
$$N_{\psi} = \frac{3.4. \cdots (3+s-3)}{(s-2)!} = \frac{s(s-1)}{2}.$$

Therefore we always have only two invariants under (132) for arbitrary integer spin-s,

$$N_I = \frac{(s+1)(s+2)}{2} - \left[2s - 1 + \frac{s(s-1)}{2}\right] = 2. \quad (133)$$

On the other hand, the Cotton tensor $C_{\mu_1\cdots\mu_s}$ is also fully symmetric, transverse, and traceless, see (5). Therefore, the same counting applies to $C_{\mu_1\cdots\mu_s}$ which must have only two independent components invariant under (132). This raises the question about the connection between the invariants and the Cotton tensor. For all cases presented here we have found $\mathcal{L}_{2s-1}^{(s)} \sim 2c_0 i_{2s-1}\nabla^{-2s} i_{2s} = c_0 h_{\mu_1\cdots\mu_s} C^{\mu_1\cdots\mu_s}$. Since $h_{00\cdots 0}$ and $\hat{\partial}_j h_{00\cdots j}$ appear linearly in i_{2s} and in i_{2s-1}, respectively, from the functional derivatives $\hat{\partial}_j \delta S_{2s-1}/\delta h_{00\cdots j}$ and $\delta S_{2s-1}/\delta h_{00\cdots 0}$, we learn that $(i_{2s}, i_{2s-1}) \sim (\hat{\partial}_j C_{00\cdots j}, C_{0\cdots 0})$. We have confirmed in all cases $s = 1, 2, 3, 4$ that this is indeed the case. Thus, we do not need to solve for the gauge parameters in (132) in terms of $\delta h_{\mu_1\cdots\mu_s}$ in order to obtain (i_{2s}, i_{2s-1}).

Whenever we consider the Lagrangian $\mathcal{L}_{2s-2}^{(s)}$ the number of symmetries is decreased by one unit and consequently we need one more gauge invariant which corresponds to $\bar{I}_{2s-1}$ (odd spin) or to $\bar{I}_{2s-2}$ (even spin). In the even spin cases $s = 2, 4$ we see that I_{2s} appears linearly in $\mathcal{L}_{2s-2}^{(s)}$, see (59) and (127). Since $\dot{\Gamma}$ is only present in I_{2s}, see (51) and (128), it turns out that $\partial_j D_{00\cdots j} \sim \dot{\bar{I}}_{2s-2}$, so we have a shortcut to obtain also $\bar{I}_{2s-2}$. In the odd spin case we have not been able to find any shortcut for $\bar{I}_{2s-1}$ which might avoid long calculations involving the elimination of the gauge parameters in the gauge transformations (10).

IX. CONCLUSION

Here we have suggested spin-4 analogs of the linearized TMG and linearized NMG, see (128) and (130), respectively. Although those models are of seventh and eighth order in derivatives, respectively, we have shown that they are ghost-free and moreover they have exactly the same canonical structure of their spin-2 counterpart when written in terms of appropriate gauge invariants, see (129), (131) and compare with (60) and (61). The canonical structure of the linearized spin-4 NTMG (115), suggested in [16], also coincides with its spin-2 counterpart, compare (54) with (116). We have also shown that the spin-3 linearized TMG, NTMG, and NMG have the same canonical structure of the spin-1 first-order self-dual model of [28], Maxwell-Chern-Simons and Maxwell-Proca models, respectively.

An important ingredient for higher spin linearized TMG and NMG is the D tensor in the action $\mathcal{L}_{2s-2}^{(s)}$. It is the tensor whose symmetrized curl is the Cotton tensor (6). This condition guarantees that the self-dual model of order $2s-1$ built by combining $\mathcal{L}_{2s-1}^{(s)}$ and $\mathcal{L}_{2s-2}^{(s)}$, see (11), contains particles with helicity $s|m|/m$. However, there might be further particles including ghosts. In general, the condition (6) leads to a multiparametric family of Lagrangians $\mathcal{L}_{2s-2}^{(s)}$. In the rank-2 case $\mathcal{L}_{2s-2}^{(s)}$ becomes the two parameter family of Tdiff models [31] which in $D = 2+1$ contains only a scalar field in the spectrum which might have a wrong overall sign depending on the parameters of the model. Consequently the third-order self-dual model defined in (11) might contain a scalar ghost besides the helicity $2|m|/m$ particle. In order to avoid such extra modes we have learned from previous works [1,13,23,24,29] that $\mathcal{L}_{2s-2}^{(s)}$ must have an empty spectrum. In the spin-2 case this leads to only two possibilities. Namely, either Tdiff is extended to diff (Einstein-Hilbert) or to WTdiff (linearized unimodular gravity). The corresponding self-dual models become the TMG of [1] and the unimodular TMG of [24] respectively.

For higher spins $s \geq 3$ we are still investigating [35] possible candidates for $\mathcal{L}_{2s-2}^{(s)}$ satisfying (6) and without particle content. There is, however, at least one natural higher spin version of the linearized Einstein-Hilbert (LEH) theory in $D = 2+1$ in the $s = 3$ and $s = 4$ cases. In the $s = 3$ case it is the fourth-order Lagrangian given in (86); see [23]. It is invariant under (62) restricted by $\Lambda = 0 = \partial^\mu \cdot \psi_\mu$. The $s = 4$ case corresponds to the sixth-order Lagrangian in (122) which is invariant under (100) with the restrictions $\partial_\mu \Lambda^\mu = 0 = \partial^\mu \partial^\nu \psi_{\mu\nu}$. Just like the LEH theory in $D = 2+1$, we have shown here, in a gauge invariant way, that both (86) and (122) have no particle content.

The Lagrangians $\mathcal{L}_{\rm LEH}$, $\mathcal{L}_4^{(3)}$, and $\mathcal{L}_6^{(4)}$ share another interesting feature. The LEH theory is uniquely determined by its order in derivatives and invariance under linearized diffeomorphisms. Likewise (86) and (122) are uniquely determined by their order in derivatives and invariance under restricted Λ and Weyl transformations. Namely, if we start with a Lagrangian of the form (84) but with all seven terms with arbitrary coefficients and require invariance under (62) restricted by $\Lambda = 0 = \partial^\mu \psi_\mu$ we end up with $\mathcal{L}_4^{(3)}$. Similarly, beginning with a Lagrangian of the form (120) with 13 arbitrary constants and demanding invariance under (100) restricted by $\Lambda_\mu = 0 = \partial^\mu \partial^\nu \psi_{\mu\nu}$ we arrive precisely at $\mathcal{L}_6^{(4)}$. Notice that there is no need of requiring (6). From this point of view $\mathcal{L}_{2s-2}^{(s)}$ is on the same footing of $\mathcal{L}_{2s}^{(s)}$ and $\mathcal{L}_{2s-1}^{(s)}$—they are all completely determined by their order in derivatives and a local symmetry, namely a restricted conformal higher spin symmetry. Moreover, $(\mathcal{L}_{\rm LEH}, \mathcal{L}_4^{(3)}, \mathcal{L}_6^{(4)})$ all have one less symmetry than Wdiff (10). It is tempting to generalize the above symmetry restrictions for arbitrary integer spins in order to infer an arbitrary spin-s canonical structure for all singlets of order $2s$ and $2s-1$ and doublets of order $2s$.

We point out that the method we have used here for investigating the particle content of higher derivative theories dispenses the use of gauge conditions which clarifies the underlying canonical structure. Furthermore, it holds off-shell which is especially useful for the doublet models $\mathcal{L}_{2s}^{D}$ with both helicities $\pm s$ where the relative sign of the two massive modes is crucial for absence of ghosts. We mention that the extension of the gauge invariant formulation to curved backgrounds is very promising, see [38]. Regarding the generalization of our self-dual models of order $2s-1$ to curved backgrounds see the recent work [39].

Instead of stepping down the ladder of higher spin self-dual models mentioned in the Introduction as we have done here going from the eighth-order spin-4 self-dual model to the seventh-order one, one might try to go up the ladder starting from the first-order spin-4 self-dual model of [40]. However, a recent attempt, see [41], along such direction gets stuck apparently at the fourth-order model which is the same order at which the spin-3 case stops [14]. One might try[13] to go up the ladder by either using the master action or the Noether gauge embedding procedure starting from the first-order action of [42] for arbitrary spin self-dual models.

Finally, we noted that the most general D-tensor solution to (6) in the spin-2 case leads to Tdiff (transverse diffeomorphisms) theories. It is known that Tdiff is the minimal symmetry for massless spin-2 particles in $D = 3+1$ which can be related to massive particles of helicity ± 2 in $D = 2+1$ via dimensional reduction. Since for $s = 3$ and $s = 4$ the D-tensor definition (6) can be traded into a symmetry principle under general transformations (10) restricted by scalar constraints, see (83) and (119), it may be worth investigating the role of those symmetries in $D = 3+1$.

[13] We thank an anonymous referee for an inspring question about that point.

ACKNOWLEDGMENTS

The work of D. D. is partially supported by CNPq (Grant No. 306380/2017-0). A. L. R. dos S. has been supported by a CNPq-PDJ (Grant No. 160784/2019-0).

[1] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982).
[2] M. Stone, Illinois preprint, Report No. ILL-(TH)-89-23, 1989; Phys. Rev. Lett. **63**, 731 (1989); Nucl. Phys. **B327**, 399 (1989).
[3] C. Wotzasek, arXiv:hep-th/9806005.
[4] R. Banerjee and S. Kumar, Phys. Rev. D **60**, 085005 (1999).
[5] D. Dalmazi, A. de Souza Dutra, and E. M. C. Abreu, Phys. Rev. D **74**, 025015 (2006); **79**, 109902(E) (2009).
[6] S. Deser and J. McCarthy, Phys. Lett. B **246**, 441 (1990).
[7] M. Fierz, Helv. Phys. Acta **12**, 3 (1939); M. Fierz and W. Pauli, Proc. R. Soc. A **173**, 211 (1939).
[8] D. Dalmazi and E. L. Mendonça, Phys. Rev. D **80**, 025017 (2009).
[9] S. D. Rindani and M. Sivakumar, Phys. Rev. D **32**, 3238 (1985).
[10] D. Dalmazi, arXiv:2102.05483.
[11] M. A. Anacleto, A. Ilha, J. R. S. Nascimento, R. F. Ribeiro, and C. Wotzasek, Phys. Lett. B **504**, 268 (2001).
[12] E. L. Mendonça, D. S. Lima, and A. L. R. dos Santos, Phys. Lett. B **783**, 387 (2018).
[13] D. Dalmazi and E. L. Mendonça, J. High Energy Phys. 09 (2009) 011.
[14] E. L. Mendonça and D. Dalmazi, Phys. Rev. D **91**, 065037 (2015).
[15] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, Ann. Phys. (Amsterdam) **325**, 1118 (2010).
[16] E. A. Bergshoeff, M. Kovacevic, J. Rosseel, P. K. Townsend, and Y. Yin, Classical Quant. Grav. **28**, 245007 (2011).
[17] S. M. Kuzenko and M. Ponds, J. High Energy Phys. 10 (2018) 160.
[18] E. Bergshoeff, O. Hohm, and P. K. Townsend, Phys. Rev. Lett. **102**, 201301 (2009).
[19] T. Damour and S. Deser, Ann. Inst. Henri Poincaré. Phys.Theor. **47**, 277 (1987).
[20] C. N. Pope and P. K. Townsend, Phys. Lett. B **225**, 245 (1989).
[21] S. M. Kuzenko and M. Ponds, J. High Energy Phys. 05 (2021) 275.
[22] E. L. Mendonca, H. L. de Oliveira, and P. H. F. Nogueira, Eur. Phys. J. Plus **135**, 800 (2020).
[23] D. Dalmazi, A. L. R dos Santos, and R. R Lino dos Santos, Phys. Rev. D **98**, 105002 (2018).
[24] D. Dalmazi, A. L. R. dos Santos, S. Ghosh, and E. L. Mendonça, Eur. Phys. J. C **77**, 620 (2017).
[25] E. I. Buchbinder, S. M. Kuzenko, J. La Fontaine, and M. Ponds, Phys. Lett. B **790**, 389 (2019).
[26] D. Dalmazi, Phys. Rev. D **80**, 085008 (2009).
[27] M. Henneaux, S. Hrtner, and A. Leonard, J. High Energy Phys. 01 (2016) 073.
[28] P. K. Townsend, K. Pilch, and P. van Nieuwenhuizen, Phys. Lett **136B**, 38 (1984).
[29] S. Deser and R. Jackiw, Phys. Lett. **139B**, 371 (1984).
[30] R. Andringa, E. A. Bergshoeff, M. de Roo, O. Hohm, E. Sezgin, and P. K. Townsend, Classical Quant. Grav. **27**, 025010 (2010).
[31] E. Alvarez, D. Blas, J. Garriga, and E. Verdaguer, Nucl. Phys. **B756**, 148 (2006); D. Blas, Ph.D. thesis, University of Barcelona, 2008.
[32] J. J. van der Bij, H. van Dam, and Y. J. Ng, Physica (Amsterdam) **116A**, 307 (1982).
[33] C. Aragone and A. Khoudeir, Phys. Lett. B **173**, 141 (1986).

[34] E. D. Skvortsov and M. A. Vasiliev, Phys. Lett. B **664**, 301 (2008).
[35] D. Dalmazi and A. L. R. dos Santos (to be published).
[36] E. Joung and K. Mkrtchyan, J. High Energy Phys. 11 (2012) 153.
[37] D. Dalmazi, A. L. R. dos Santos, E. L. Mendona, and R. Schimidt Bittencourt, Phys. Rev. D **100**, 065011 (2019).
[38] M. Jaccard, M. Maggiore, and E. Mitsou, Phys. Rev. D **87**, 044017 (2013).
[39] D. Hutchings, S. M. Kuzenko, and M. Ponds, arXiv:2107.12201.
[40] C. Aragone and A. Khoudeir, arXiv:hep-th/9307004v1.
[41] E. L. Mendonça and H. L. Oliveira, arXiv:2108.09323.
[42] I. V. Tyutin and M. A. Vasiliev, Teor. Mat. Fiz. **113N1**, 45 (1997) [Theor. Math. Phys. **113**, 1244 (1997)].