

ϕ^4 kink mass at two loops

Jarah Evslin 

*Institute of Modern Physics, NanChangLu 509, Lanzhou 730000, China
and University of the Chinese Academy of Sciences, YuQuanLu 19A, Beijing 100049, China*

 (Received 5 June 2021; accepted 16 September 2021; published 12 October 2021)

The two-loop correction to the mass of the ϕ^4 kink is $0.0126\lambda/m$ in terms of the coupling λ and the meson mass m evaluated at the minimum of the potential. This is calculated using a recently proposed alternative to collective coordinates. Both the kink energy and the vacuum energy are IR divergent at this order. To cancel the divergence, the two energy densities are subtracted before integrating over space, or equivalently a finite counterterm is added to the Hamiltonian density to cancel the vacuum energy density. All spatial integrals are performed analytically. However in the last step of our calculation, integrals over virtual momenta are performed numerically.

DOI: [10.1103/PhysRevD.104.085013](https://doi.org/10.1103/PhysRevD.104.085013)

I. INTRODUCTION

The ϕ^4 double well model in $1 + 1$ dimensions has two vacua. In each, a real scalar field has a mass of $m = 2\beta$. A classical kink solution of mass

$$Q_0 = \frac{8\beta^3}{3\lambda} \quad (1.1)$$

interpolates between the two vacua. The one-loop mass, defined as the difference between the kink state energy and the vacuum energy, was calculated in [1] to be

$$Q_1 = \left(-\frac{3}{\pi} + \frac{1}{2\sqrt{3}} \right) \beta. \quad (1.2)$$

The kink state energy and vacuum energy were calculated using the kink Hamiltonian and vacuum Hamiltonian respectively. In principle the theory is defined by the vacuum Hamiltonian, but the use of the kink Hamiltonian to calculate the kink state energy is justified by the fact that the two Hamiltonians have the same spectrum.

The problem is that the two Hamiltonians need to be regularized, and regularization changes their spectra. After regularization, the energies depend on the regulators and, as explained in [2], even for the same value of the regulator, the kink state energy calculated using the kink Hamiltonian may no longer be equal to that calculated using the defining vacuum Hamiltonian. It is of course the eigenvalue of the

defining Hamiltonian which gives the mass, and so if these two eigenvalues are not equal, then the calculation that uses the kink Hamiltonian will obtain the wrong answer. In Ref. [2] it was shown that for some regulators this mismatch persists even when the regulator is taken to infinity. Worse yet, while there have been many proposed principles in the literature [3,4], there is no established criterion for determining which regulators lead to this mismatch.¹

This ambiguity can be avoided when the kink mass is fixed by another principle, such as integrability or supersymmetry, if the regularization preserves this property. However the ϕ^4 theory has neither property. The ambiguity can also be avoided at one loop, where the problem reduces to finding the density of states of a free theory [5], and so it was eventually shown that the one-loop result (1.2) is nonetheless correct.

What about two loops? The goal of the present paper will be to calculate the two-loop mass of the ϕ^4 kink.

We will use the kink sector perturbation theory introduced in Ref. [6] in the context of this model and generalized to other models in [7]. This framework provides a simpler alternative to the more traditional collective coordinate approaches [8,9], as one considers a linear decomposition of the fields, treating the kink zero mode perturbatively. This avoids nonlinear canonical transformations, as the decomposed fields already obey canonical commutation relations. On the other hand, it only produces a reasonably convergent expansion when the kink position

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

¹Of course one can obtain nonperturbative results on the lattice or using Hamiltonian truncation or Borel resummation, without recourse to a kink Hamiltonian, but that will not be our approach here. We will see in Sec. VI that our method yields superior precision at weak coupling.

is close to some reference point, and so it has so far only been applied to study the properties of a lone, stationary kink.

This method begins with the observation that the kink Hamiltonian H' can be defined via a similarity transformation of the original Hamiltonian H

$$H' = \mathcal{D}_f^{-1} H \mathcal{D}_f \quad (1.3)$$

where \mathcal{D}_f is a unitary operator. Two similar operators necessarily have the same spectrum. The key innovation in this procedure is that one first regularizes H and then defines the regularized H' via (1.3). Now there is a single regulator, and the spectrum of H' is equal to that of the defining H at each value of the regulator, so that the energy calculated using H' agrees with the correct energy, which would be obtained using H . Thus the problem is solved. In Ref. [10] this method was used to find the two-loop energy of a general kink in its ground state.

Unfortunately, in the case of the ϕ^4 theory, and any theory whose potential has a nonvanishing third derivative at its minima, this energy suffers from an infrared (IR) divergence. This divergence is to be expected, because the vacuum at this order has a constant energy density of [11–13]

$$\rho^{(\text{vac})}(x) = \left(\frac{1}{24} - \frac{\psi^{(1)}(1/3)}{16\pi^2} \right) \lambda \sim -0.0222644\lambda \quad (1.4)$$

and so an infinite energy. Here $\psi^{(1)}(z) = \partial_z^2 \ln(\Gamma(z))$ is the polygamma function. The mass, which is the difference between these two infinite quantities, is expected to be finite.

How can these two infinite quantities be reliably subtracted? Usually one removes IR divergences by calculating an experimentally accessible quantity. One might be tempted to ask how much energy one needs to create a kink. Unfortunately this is impossible in the quantum theory [14] as the kink changes the boundary conditions, creating a lone soliton would require an infinite action. One could also remove IR divergences by putting a system in a box, but it is known that boundary conditions can lead to mass shifts which persist even in the limit that the box size is taken to infinity.

One is always free to set the exact vacuum energy to zero by including a counterterm in the defining Hamiltonian density which exactly cancels the vacuum energy density [15]. As this counterterm is a c -number, Eq. (1.3) implies that it will also appear in the kink Hamiltonian density. Our procedure is equivalent to modifying the calculation of [10] by including this counterterm, defined by the condition that the vacuum energy vanishes.

While equivalent, we find that it is more convenient to subtract the vacuum energy density from the kink energy density rather than to subtract it directly from the kink Hamiltonian density. This requires an *ad hoc* convention for the definition of the kink energy density, but after

integration over space the result is independent of this convention. The complication is that the kink is not an eigenstate of the Hamiltonian density operator, nor is the vacuum. One might try to define the energy density as the expectation value of the Hamiltonian density operator, but alas the Hamiltonian eigenstates have constant momenta and so are not normalizable. This could in principle be resolved by introducing wave packets, but a finite width wave packet would increase the energy and in the limit that the width goes to infinity, the divergence would return. In principle one could nonetheless subtract the vacuum energy inside of the support of the wave packet as the wave packet size goes to infinity, but this procedure would be ambiguous as a finite shift in the definition of the support of the wave packet would lead to a finite shift in the limit, and so in the calculated mass.

We instead adopt the following definition. If $|0\rangle$ is the kink ground state after the similarity transform (1.3), then

$$H'|0\rangle = Q|0\rangle. \quad (1.5)$$

If H' is the integral of a Hamiltonian density $\mathcal{H}'(x)$, then we can expand $\mathcal{H}'(x)|0\rangle$ in a basis of the Hilbert space, with $|0\rangle$ an element of the basis. Integrating over x this must reduce to (1.5) and so the coefficients of all basis elements except for $|0\rangle$ will integrate to zero while the coefficient of $|0\rangle$ integrates to Q . Since the coefficient of $|0\rangle$ in the expansion of $\mathcal{H}'(x)|0\rangle$ integrates to Q , we define it to be the energy density $\rho(x)$. While this definition of $\rho(x)$ may depend on the choice of basis, as $\rho(x)$ may mix with the coefficients of some other basis elements under a basis transformation, this ambiguity should disappear after the integration over x as all other coefficients integrate to zero.

In the case of the Sine-Gordon model, at this order the correction to the vacuum energy vanishes and so this issue does not arise. Also the two-loop kink mass is known using integrability [16,17]. However, one could have shifted the defining Hamiltonian density by a constant, leading to a finite vacuum energy density and an IR divergence in the kink energy. In that case, our prescription would be equivalent to simply shifting the Hamiltonian back by this constant, which of course is the correct way to remove this spurious divergence.

We begin in Sec. II with a review of the calculation of the two-loop kink ground state energy for a general kink. In Sec. III we restrict our attention to the ϕ^4 double well and evaluate the finite terms, reducing the problem to a finite-dimensional integral of elementary functions. In Sec. IV we treat the IR divergent terms, subtracting the corresponding vacuum energy as described above. As a result, the entire energy is computed in the form of a three-dimensional integral over elementary functions. In Sec. V this integral is evaluated numerically. Finally, the result is compared with the literature in Sec. VI. Some of the most important notation is summarized in Table I.

TABLE I. Summary of Notation.

Operator	Description
$\phi(x), \pi(x)$	The real scalar field and its conjugate momentum
$b_k^\dagger, b_k, B_k^\dagger, B_k$	Creation and annihilation operators in normal mode basis
ϕ_0	Zero mode of $\phi(x)$ in normal mode basis
$::_a, ::_b$	Normal ordering with respect to a or b operators respectively
Hamiltonian	Description
H	The original Hamiltonian
H'	H with $\phi(x)$ shifted by kink solution $f(x)$
H_n	The ϕ^n term in H'
Symbol	Description
β	Half of the scalar mass
λ	Coupling constant
$f(x)$	The classical kink solution
\mathcal{D}_f	Operator that translates $\phi(x)$ by the classical kink solution
$g_B(x)$	The kink linearized translation mode
$g_k(x)$	Continuum normal mode or shape mode
$g_S(x)$	Shape mode
γ_i^{mn}	Coefficient of $\phi_0^m B^{\dagger n} 0\rangle_0$ in order i ground state $ 0\rangle_i$
V_{ijk}	Derivative of the potential contracted with various functions
\mathcal{C}_{ijk}	Coefficient which appears in all continuous V_{ijk}
α_{ijk}	Coefficient which appears in contribution of V_{ijk} to energy
$\sigma_{ijk}(x)$	Quantity which integrates to V_{ijk}
$\Phi_{ijk}^{IJ}(x)$	Matrix characterizing x -dependence of σ_{ijk}
Δ_{ij}	Integral of $g_i(x)g_j(x)$
$\rho(x)$	Energy density arising from three virtual continuum modes
$\rho^{(\text{vac})}(x)$	Vacuum energy density
$\mathcal{I}(x)$	Contraction factor from Wick's theorem
k_i	The analog of momentum for normal modes
ω_k	The frequency corresponding to k
Q_n	n -loop correction to kink energy
State	Description
$ K\rangle, \Omega\rangle$	Kink and vacuum sector ground states
$ 0\rangle$	Translation of $ K\rangle$ by \mathcal{D}_f^{-1}
$ 0\rangle_i$	Translation of $ K\rangle$ by \mathcal{D}_f^{-1} at order i

II. REVIEW

Consider a $(1 + 1)$ -dimensional theory of a real scalar field ϕ and its conjugate momentum π , defined by the Hamiltonian

$$\begin{aligned}
 H &= \int dx \mathcal{H}(x) \\
 \mathcal{H}(x) &= \frac{1}{2} : \pi(x) \pi(x) :_a + \frac{1}{2} : \partial_x \phi(x) \partial_x \phi(x) :_a \\
 &\quad + \frac{1}{\lambda} : V[\sqrt{\lambda} \phi(x)] :_a
 \end{aligned} \tag{2.1}$$

where $::_a$ is normal-ordering of the usual operators that create and annihilate plane waves and we always work in the Schrodinger picture. We remind the reader that normal ordering eliminates all UV divergences in canonical scalar theories in $1 + 1$ dimensions with arbitrary nonderivative, local interactions. Our procedure may be repeated with other renormalization schemes, leading to similar results. For example, as shown in Ref. [12], normal ordering with normal ordering mass² m , as performed below, is equivalent to dimensional regularization with a modified minimal subtraction, at mass scale $\mu = m$, using a renormalized mass squared of

$$m_{\text{d.r.}}^2 = m^2 \tag{2.2}$$

and a counterterm

$$\delta \mathcal{H}(x) = -\frac{3\lambda}{8\pi} \left(\frac{2}{\epsilon} - \gamma_E + \log(4\pi) \right) \phi^2(x) + O(\lambda^2). \tag{2.3}$$

The vacuum energy may be set to zero by including additional constant counterterms, given explicitly in Ref. [12]. Note that an order $O(\lambda^2)$ contribution to (2.2) would contribute to the two-loop kink mass, but this contribution vanishes [12]. On the other hand, the corresponding contribution to the counterterm $\delta \mathcal{H}(x)$ only contributes to the kink mass starting at three loops. A convenient lattice renormalization scheme also leads to a shifted m^2 and can be found in Eq. (14) of Ref. [18].

The classical equations of motion admit the kink solution

$$\phi(x, t) = f(x). \tag{2.4}$$

To perturbatively treat oscillations about the kink, one would like to expand about the kink solution. This may be done using the passive transformation of the fields $\phi \rightarrow \phi' = \phi - f$ or else the active transformation of the functionals acting on the fields

$$F[\phi] \rightarrow F'[\phi] = F[\phi']. \tag{2.5}$$

We opt for the second approach, realized as follows. Defining the displacement operator \mathcal{D}_f

²Our m is the mass at the bottom of the potential, which is equal to \tilde{m} in Ref. [12], while our λ is 4 times larger.

$$\mathcal{D}_f = \exp\left(-i \int dx f(x) \pi(x)\right) \quad (2.6)$$

we define the kink Hamiltonian and kink momentum as

$$H' = \mathcal{D}_f^\dagger H \mathcal{D}_f, \quad P' = \mathcal{D}_f^\dagger P \mathcal{D}_f. \quad (2.7)$$

The theory is rendered UV finite by normal ordering, and so (2.5) and (2.7) are equivalent. However, our procedure may be implemented with a general regulator and in that case (2.7) should be taken as our definition of kink sector operators, as the similarity transform guarantees that kink sector operators will have the same spectra as the original operators.

A quick calculation [6] shows

$$\begin{aligned} H' &= \mathcal{D}_f^\dagger H \mathcal{D}_f = Q_0 + \sum_{n=2}^{\infty} H_n, \\ H_{n(>2)} &= \frac{1}{n!} \int dx V^{(n)}[\sqrt{\lambda} f(x)] : \phi^n(x) :_a \\ H_2 &= \frac{1}{2} \int dx [: \pi^2(x) :_a + : (\partial_x \phi(x))^2 :_a \\ &\quad + V''[g f(x)] : \phi^2(x) :_a]. \end{aligned} \quad (2.8)$$

Let $|K\rangle$ be the kink ground state, and Q its energy

$$H|K\rangle = Q|K\rangle, \quad P|K\rangle = 0. \quad (2.9)$$

Then we may define

$$|0\rangle = \mathcal{D}_f^\dagger |K\rangle \quad (2.10)$$

which is an eigenstate,³ of the kink Hamiltonian and momentum

$$H'|0\rangle = Q|0\rangle, \quad P'|0\rangle = 0. \quad (2.11)$$

Define a semiclassical expansion of this state and its eigenvalue

$$|0\rangle = \sum_{i=0} |0\rangle_i, \quad Q = \sum_{j=0} Q_j \quad (2.12)$$

where $|0\rangle_i$ is the i th order ground state and Q_j is the j -loop correction to its mass. At j loops the state is determined up to $i = 2j - 2$.

³In our perturbative approach to the kink Hamiltonian, $|0\rangle$ is the perturbative ground state. For example it can be expanded in a power series in the coupling and the leading order term in the expansion is annihilated by the normal mode annihilation operators. However, nonperturbatively there are lower energy eigenstates, such as \mathcal{D}_f^\dagger acting on the vacuum of the defining Hamiltonian or more generally on low lying states in the vacuum sector.

In this note we will be interested in the two-loop correction to the energy of the kink ground state [10]

$$\begin{aligned} Q_2 &= \sum_{j=1}^5 Q_2^{(j)}, \quad Q_2^{(1)} = \frac{V_{\mathcal{I}\mathcal{I}}}{8}, \quad Q_2^{(2)} = -\frac{1}{8} \int^+ \frac{dk |V_{\mathcal{I}k}|^2}{2\pi \omega_k^2} \\ Q_2^{(3)} &= -\frac{1}{48} \int^+ \frac{d^3k}{(2\pi)^3} \frac{|V_{k_1 k_2 k_3}|^2}{\omega_{k_1} \omega_{k_2} \omega_{k_3} (\omega_{k_1} + \omega_{k_2} + \omega_{k_3})} \\ Q_2^{(4)} &= \frac{1}{16Q_0} \int^+ \frac{d^2k}{(2\pi)^2} \frac{|(\omega_{k_1} - \omega_{k_2}) \Delta_{k_1 k_2}|^2}{\omega_{k_1} \omega_{k_2}} \\ &= \frac{1}{16} \int^+ \frac{d^2k}{(2\pi)^2} \frac{|V_{Bk_1 k_2}|^2}{\omega_{k_1} \omega_{k_2} (k_1 + k_2)^2} \\ Q_2^{(5)} &= -\frac{1}{8Q_0^2} \int dx |f''(x)|^2 = -\frac{1}{8Q_0} \int^+ \frac{dk}{2\pi} |\Delta_{kB}|^2 \\ &= -\frac{1}{8} \int^+ \frac{dk |V_{BBk}|^2}{2\pi \omega_k^4}. \end{aligned} \quad (2.13)$$

Here we have introduced the matrix

$$\Delta_{ij} = \int dx g_i(x) g_j'(x) \quad (2.14)$$

and the symbol

$$V_{\mathcal{I}^m \mathcal{I} \alpha_1 \dots \alpha_n} = \int dx V^{(2m+n)}[\sqrt{\lambda} f(x)] \mathcal{I}^m(x) g_{\alpha_1}(x) \dots g_{\alpha_n}(x) \quad (2.15)$$

where $V^{(n)}$ is the n th derivative of $\lambda^{n/2-1} V$ with respect to its argument and $g_k(x)$ are continuous and discrete normal modes of frequency ω_k , which solve the linearized equations of motion for H' . In particular $g_B(x)$ is the zero mode with $\omega_B = 0$. Here m is an arbitrary nonnegative integer. Feynman diagrams can be defined for our kink sector perturbation theory [19], shown in Fig. 1. In these diagrams, the symbol V is the vertex factor for a vertex with m loops going from the vertex to itself and n lines with a single endpoint at the vertex.

The normalization and phases of the normal modes are chosen so that

$$\begin{aligned} \int dx g_{k_1}(x) g_{k_2}^*(x) &= 2\pi \delta(k_1 - k_2), \\ \int dx |g_S(x)|^2 &= \int dx |g_B(x)|^2 = 1 \\ g_k(-x) &= g_k^*(x) = g_{-k}(x), \quad g_S(-x) = g_S^*(x) \end{aligned} \quad (2.16)$$

where $g_S(x)$ is any discrete normal mode and k_i are real.

We have also introduced the notation that $\int^+ dk/2\pi$ is an integral over all real values of k corresponding to the

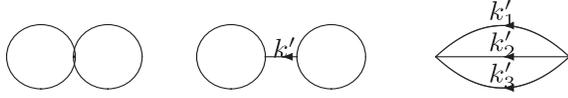


FIG. 1. Q_2 in pictures, as described in Ref. [19]. Each vertex represents an interaction in H' , with operator ordering running to the left and a factor of \mathcal{I} for each loop at a single vertex. The three diagrams correspond to $Q_2^{(1)}$, $Q_2^{(2)}$ and $Q_2^{(3)}$ respectively while $Q_2^{(4)}$ and $Q_2^{(5)}$ arise from replacing a normal mode with one or two zero modes in the last diagram.

continuous normal modes as well as a discrete sum over the imaginary values of k corresponding to discrete normal modes, like the shape mode of the ϕ^4 kink. The zero mode is not included in this sum.

The function $\mathcal{I}(x)$ arises from each loop at a single vertex, or equivalently by transforming the normal ordering $::_a$ in terms of operators which create plane waves, used in the definition of the Hamiltonian, to a normal ordering in terms of operators which create normal modes $::_b$. It solves the equation [20]

$$\partial_x \mathcal{I}(x) = \int^+ \frac{dk}{2\pi} \frac{1}{2\omega_k} \partial_x |g_k(x)|^2 \quad (2.17)$$

and tends asymptotically to 0.

We have used the identities

$$V_{BBk} = -\frac{\omega_k^2}{\sqrt{Q_0}} \Delta_{kB}, \quad V_{Bk_1k_2} = \frac{\omega_{k_2}^2 - \omega_{k_1}^2}{\sqrt{Q_0}} \Delta_{k_1k_2} \quad (2.18)$$

to write (2.13) in several equivalent forms. The expressions on the right are more complicated, but following [19] their diagrammatic interpretation, shown in Fig. 1, is more clear. There it is explained that $Q_2^{(1)}$ corresponds to two loops at a

point, $Q_2^{(2)}$ to two loops connected by an internal line, $Q_2^{(3)}$ corresponds to two points connected by 3 internal lines and the next two diagrams are obtained from the third by replacing, respectively, one or two normal mode internal lines by one or two zero-mode internal lines. These diagrams are each UV-finite, as loops at a point lead to factors of the finite function $\mathcal{I}(x)$. In the more standard diagrammatic approach of Refs. [21,22], which use a UV cutoff, each of these diagrams corresponds to a UV-finite sum of individually divergent diagrams including diagrams with counterterms. However, in our case no UV cutoff or counterterms are needed as normal ordering in (2.1) has already removed all UV divergences.

III. IR-FINITE CONTRIBUTIONS

A. The ϕ^4 double well

Now let us specialize to the ϕ^4 double well theory, corresponding to the potential

$$V[\sqrt{\lambda}\phi(x)] = \frac{\lambda\phi^2}{4} (\sqrt{\lambda}\phi(x) - \beta\sqrt{8})^2 \quad (3.1)$$

and stationary classical kink solution

$$f(x) = \beta\sqrt{\frac{2}{\lambda}}(1 + \tanh(\beta x)). \quad (3.2)$$

At the vacua $\phi = 0$ and $\phi = \beta\sqrt{8/\lambda}$, we define m^2 to be the meson mass squared and so

$$m = 2\beta. \quad (3.3)$$

The continuum normal modes, shape mode and zero mode are

$$g_k(x) = \frac{e^{-ikx}}{\omega_k \sqrt{k^2 + \beta^2}} [k^2 - 2\beta^2 + 3\beta^2 \text{sech}^2(\beta x) - 3i\beta k \tanh(\beta x)]$$

$$g_S(x) = -i\sqrt{\frac{3\beta}{2}} \tanh(\beta x) \text{sech}(\beta x), \quad g_B(x) = \frac{\sqrt{3\beta}}{2} \text{sech}^2(\beta x) \quad (3.4)$$

and have frequencies

$$\omega_k = \sqrt{4\beta^2 + k^2}, \quad \omega_S = \beta\sqrt{3}, \quad \omega_B = 0. \quad (3.5)$$

One easily finds the derivatives

$$V^{(3)}[\sqrt{\lambda}f(x)] = 6\sqrt{2\lambda}\beta \tanh(\beta x), \quad V^{(4)}[\sqrt{\lambda}f(x)] = 6\lambda \quad (3.6)$$

and, solving (2.17), the loop function

$$\mathcal{I}(x) = \frac{1}{4\sqrt{3}} \text{sech}^2(\beta x) \tanh^2(\beta x) - \frac{3}{8\pi} \text{sech}^4(\beta x). \quad (3.7)$$

B. Some useful integrals

In what follows, all derivatives over x will be performed analytically using

$$\int dx e^{-ikx} \operatorname{sech}^{2n}(\beta x) = \begin{cases} 2\pi\delta(k) & \text{if } n = 0 \\ \frac{\pi}{(2n-1)!k} \left[\prod_{j=0}^{n-1} \left(\frac{k^2}{\beta^2} + (2j)^2 \right) \right] \operatorname{csch}\left(\frac{\pi k}{2\beta}\right) & \text{if } n > 0 \end{cases}$$

$$\int dx e^{-ikx} \operatorname{sech}^{2n+1}(\beta x) = \frac{\pi}{(2n)! \beta} \left[\prod_{j=0}^{n-1} \left(\frac{k^2}{\beta^2} + (2j+1)^2 \right) \right] \operatorname{sech}\left(\frac{\pi k}{2\beta}\right)$$

$$\int dx e^{-ikx} \operatorname{sech}^{2n}(\beta x) \tanh(\beta x) = -i \frac{\pi}{(2n)! \beta} \left[\prod_{j=0}^{n-1} \left(\frac{k^2}{\beta^2} + (2j)^2 \right) \right] \operatorname{csch}\left(\frac{\pi k}{2\beta}\right)$$

$$\int dx e^{-ikx} \operatorname{sech}^{2n+1}(\beta x) \tanh(\beta x) = -i \frac{\pi k}{(2n+1)! \beta^2} \left[\prod_{j=0}^{n-1} \left(\frac{k^2}{\beta^2} + (2j+1)^2 \right) \right] \operatorname{sech}\left(\frac{\pi k}{2\beta}\right). \quad (3.8)$$

C. The last term

The energy of the kink ground state is expressed in (2.13) as $\sum_{i=1}^5 Q_2^{(i)}$. The simplest term to evaluate is the fifth. One need only use

$$f''(x) = -\frac{2}{\sqrt{\lambda}} \beta^3 \operatorname{sech}^2(\beta x) \tanh(\beta x) \quad (3.9)$$

to obtain

$$\int dx |f''(x)|^2 = \frac{4}{\lambda} \beta^6 \int dx (\operatorname{sech}^4(\beta x) - \operatorname{sech}^6(\beta x)) = \frac{16\beta^5}{15\lambda}. \quad (3.10)$$

Alternately one may evaluate $Q_2^{(5)}$ using

$$\Delta_{SB} = i\pi \frac{3\beta}{8\sqrt{2}}, \quad \Delta_{kB} = i\pi \frac{\sqrt{3}}{8} \frac{k^2 \omega_k}{\beta^{3/2} \sqrt{\beta^2 + k^2}} \operatorname{csch}\left(\frac{\pi k}{2\beta}\right). \quad (3.11)$$

The result is the same and will be summarized in Sec V.

D. The penultimate term

The contribution $Q_2^{(4)}$ can be found by inserting

$$\Delta_{kS} = -i\pi \frac{\sqrt{3}}{4\sqrt{2}} \frac{(3\beta^2 + k^2) \sqrt{\beta^2 + k^2}}{\beta^{3/2} \omega_k} \operatorname{sech}\left(\frac{\pi k}{2\beta}\right)$$

$$\Delta_{k_1 k_2} = i\pi (k_1 - k_2) \delta(k_1 + k_2) + i\pi \frac{3}{4} \left(\frac{\omega_{k_1}}{\omega_{k_2}} - \frac{\omega_{k_2}}{\omega_{k_1}} \right) \times \frac{4\beta^2 + k_1^2 + k_2^2}{\sqrt{\beta^2 + k_1^2} \sqrt{\beta^2 + k_2^2}} \operatorname{csch}\left(\frac{\pi(k_1 + k_2)}{2\beta}\right) \quad (3.12)$$

into (2.13) and integrating over continuum modes k and adding the contribution from the shape mode S . The contribution from the Dirac delta function vanishes, as it is multiplied by zero in (2.13).

E. Contractions with the potential

The vertex factors are defined in (2.15). Those involving a zero-mode are

$$V_{BBB} = V_{SSB} = 0, \quad V_{SBB} = -i\pi \frac{9\sqrt{3}\lambda}{32} \beta^{3/2}$$

$$V_{kBB} = -i\pi \frac{3\sqrt{\lambda}}{16\sqrt{2}} \frac{k^2 \omega_k^3}{\beta^3 \sqrt{\beta^2 + k^2}} \operatorname{csch}\left(\frac{\pi k}{2\beta}\right)$$

$$V_{kSB} = i\pi \frac{3\sqrt{\lambda}}{16} \frac{(3\beta^2 + k^2)(k^2 + \beta^2)^{3/2}}{\beta^3 \omega_k} \operatorname{sech}\left(\frac{\pi k}{2\beta}\right)$$

$$V_{k_1 k_2 B} = -i\pi \frac{3\sqrt{3}\lambda}{8\sqrt{2}} \frac{(k_1^2 - k_2^2)^2 (4\beta^2 + k_1^2 + k_2^2)}{\beta^{3/2} k_1 k_2 \sqrt{\beta^2 + k_1^2} \sqrt{\beta^2 + k_2^2}} \times \operatorname{csch}\left(\frac{\pi(k_1 + k_2)}{2\beta}\right). \quad (3.13)$$

These are consistent with the identities (2.18).

Those involving the loop factor $\mathcal{I}(x)$, given in (3.7), are

$$V_{\mathcal{I}\mathcal{I}} = \frac{\lambda}{70\beta} \left(1 - \frac{4\sqrt{3}}{\pi} + \frac{54}{\pi^2} \right)$$

$$V_{\mathcal{I}k} = i \frac{\sqrt{\lambda}}{32\sqrt{6}} \frac{k^2 \omega_k}{\beta^4 \sqrt{\beta^2 + k^2}} \times [2\pi(-2\beta^2 + k^2) + 3\sqrt{3}\omega_k^2] \operatorname{csch}\left(\frac{\pi k}{2\beta}\right)$$

$$V_{\mathcal{I}S} = i \frac{3\sqrt{\lambda}}{64} \sqrt{\beta} (3\sqrt{3} - 2\pi). \quad (3.14)$$

These yield $Q_2^{(1)}$ and $Q_2^{(2)}$. So the rest of this note will be concerned with $Q_2^{(3)}$.

The only divergence arises from the term with three continuum normal modes. The remaining finite terms are

$$\begin{aligned} V_{SSS} &= i\pi \frac{9\sqrt{3\lambda}}{16} \beta^{3/2}, & V_{kSS} &= i\pi \frac{3\sqrt{\lambda} k^2 \omega_k (2\beta^2 - k^2)}{8\sqrt{2} \beta^3 \sqrt{\beta^2 + k^2}} \operatorname{csch}\left(\frac{\pi k}{2\beta}\right) \\ V_{k_1 k_2 S} &= -i\pi \frac{3\sqrt{3\lambda} (17\beta^4 - (k_1^2 - k_2^2)^2)(\beta^2 + k_1^2 + k_2^2) + 8\beta^2 k_1^2 k_2^2}{8 \beta^{3/2} k_1 k_2 \sqrt{\beta^2 + k_1^2} \sqrt{\beta^2 + k_2^2}} \operatorname{sech}\left(\frac{\pi(k_1 + k_2)}{2\beta}\right). \end{aligned} \quad (3.15)$$

F. The divergent term

Although we use (2.13) to evaluate the energy of the kink ground state, one could also use it to evaluate the vacuum ground state. One would simply set $f(x)$ to be the corresponding expectation value of $\phi(x)$ and the $g_k(x)$ would be the corresponding linearized solutions, which are just plane waves. Only the third term $Q_2^{(3)}$ would be nonzero and it would be divergent. The problem is that $V_{k_1 k_2 k_3}$ diverges when $\sum_i k_i = 0$. In that case it would be proportional to a delta function reflecting momentum conservation in the third figure in Fig. 1.

The infrared behavior of the kink sector is identical, as at long distances the only effect of the kink is a phase-shift which is not relevant to this divergence. The continuum normal modes tend asymptotically to plane waves, albeit shifted, and the potential tends to exactly the same constant as in the vacuum case. Therefore, for small $\sum_i k_i$, the symbol $V_{k_1 k_2 k_3}$ approaches the vacuum value. This is good news, as it means that the kink energy and the vacuum energy have the same divergence, so their difference, the kink mass, is finite.

The long distance divergence of $V_{k_1 k_2 k_3}$ arises from the fact that at small $\sum_i k_i$ it tends to the integral of a constant. Thus, to regularize this divergence, we must study the integrand, which we will call $\sigma(x)$

$$\begin{aligned} V_{k_1 k_2 k_3} &= \int dx \sigma_{k_1 k_2 k_3}(x) = \sum_{I=0}^3 \sum_{J=0}^1 V_{k_1 k_2 k_3}^{IJ}, & V_{k_1 k_2 k_3}^{IJ} &= \int dx \sigma_{k_1 k_2 k_3}^{IJ}(x) \\ \sigma_{k_1 k_2 k_3}(x) &= V^{(3)}[\sqrt{\lambda} f(x)] g_{k_1}(x) g_{k_2}(x) g_{k_3}(x) = \sum_{I=0}^3 \sum_{J=0}^1 \sigma_{k_1 k_2 k_3}^{IJ}(x). \end{aligned} \quad (3.16)$$

The indices I and J are respectively the number of powers of $\operatorname{sech}^2(\beta x)$ and $\tanh(\beta x)$ in each term. We separate out the k -dependence, the x -dependence and a universal k -dependent coefficient by introducing yet more notation

$$\begin{aligned} \sigma_{k_1 k_2 k_3}^{IJ}(x) &= C_{k_1 k_2 k_3} \Phi_{k_1 k_2 k_3}^{IJ} e^{-ix(k_1 + k_2 + k_3)} \operatorname{sech}^{2I}(\beta x) \tanh^J(\beta x) \\ C_{k_1 k_2 k_3} &= 6\sqrt{2\lambda} \frac{\beta}{\omega_{k_1} \omega_{k_2} \omega_{k_3} \sqrt{\beta^2 + k_1^2} \sqrt{\beta^2 + k_2^2} \sqrt{\beta^2 + k_3^2}}. \end{aligned} \quad (3.17)$$

For future reference we will use a hat to define functions restricted to $\sum k_i = 0$

$$\hat{\Phi}_{k_1 k_2}^{IJ} = \Phi_{k_1, k_2, -k_1 - k_2}^{IJ}, \quad \hat{C}_{k_1 k_2} = C_{k_1, k_2, -k_1 - k_2}. \quad (3.18)$$

The x integrals may be performed analytically, yielding

$$\begin{aligned} V_{k_1 k_2 k_3}^{00} &= C_{k_1 k_2 k_3} \Phi_{k_1 k_2 k_3}^{00} 2\pi \delta(k_1 + k_2 + k_3) = \hat{C}_{k_1 k_2} \hat{\Phi}_{k_1 k_2}^{00} 2\pi \delta(k_1 + k_2 + k_3) \\ V_{k_1 k_2 k_3}^{10} &= C_{k_1 k_2 k_3} \Phi_{k_1 k_2 k_3}^{10} \frac{\pi \sum_i^3 k_i}{\beta^2} \operatorname{csch}\left(\frac{\pi \sum_i^3 k_i}{2\beta}\right) \\ V_{k_1 k_2 k_3}^{20} &= C_{k_1 k_2 k_3} \Phi_{k_1 k_2 k_3}^{20} \frac{\pi \sum_i^3 k_i}{6\beta^2} \left(\frac{(\sum_i^3 k_i)^2}{\beta^2} + 4\right) \operatorname{csch}\left(\frac{\pi \sum_i^3 k_i}{2\beta}\right) \\ V_{k_1 k_2 k_3}^{30} &= C_{k_1 k_2 k_3} \Phi_{k_1 k_2 k_3}^{30} \frac{\pi \sum_i^3 k_i}{120\beta^2} \left(\frac{(\sum_i^3 k_i)^2}{\beta^2} + 4\right) \left(\frac{(\sum_i^3 k_i)^2}{\beta^2} + 16\right) \operatorname{csch}\left(\frac{\pi \sum_i^3 k_i}{2\beta}\right) \end{aligned} \quad (3.19)$$

and

$$\begin{aligned}
 V_{k_1 k_2 k_3}^{01} &= -iC_{k_1 k_2 k_3} \Phi_{k_1 k_2 k_3}^{01} \frac{\pi}{\beta} \operatorname{csch}\left(\frac{\pi \sum_i^3 k_i}{2\beta}\right) \\
 V_{k_1 k_2 k_3}^{11} &= -iC_{k_1 k_2 k_3} \Phi_{k_1 k_2 k_3}^{11} \frac{\pi(\sum_i^3 k_i)^2}{2\beta^3} \operatorname{csch}\left(\frac{\pi \sum_i^3 k_i}{2\beta}\right) \\
 V_{k_1 k_2 k_3}^{21} &= -iC_{k_1 k_2 k_3} \Phi_{k_1 k_2 k_3}^{21} \frac{\pi(\sum_i^3 k_i)^2}{24\beta^3} \left(\frac{(\sum_i^3 k_i)^2}{\beta^2} + 4\right) \operatorname{csch}\left(\frac{\pi \sum_i^3 k_i}{2\beta}\right) \\
 V_{k_1 k_2 k_3}^{31} &= -iC_{k_1 k_2 k_3} \Phi_{k_1 k_2 k_3}^{31} \frac{\pi(\sum_i^3 k_i)^2}{720\beta^3} \left(\frac{(\sum_i^3 k_i)^2}{\beta^2} + 4\right) \left(\frac{(\sum_i^3 k_i)^2}{\beta^2} + 16\right) \operatorname{csch}\left(\frac{\pi \sum_i^3 k_i}{2\beta}\right). \tag{3.20}
 \end{aligned}$$

However if we simply substitute these results into (2.13), the terms containing $V^{0J}V^{0J'}$ would be divergent, even after integration over momenta. One delta function or simple pole could be used to do the k_3 integral, but the other delta function or simple pole would leave an infinite result. This is of course to be expected, because the kink ground state really does have a infinite energy. We will need to subtract the vacuum energy before evaluating some x integrals.

The terms with $I > 0$ are not divergent. So let us decompose V into three imaginary components

$$V_{k_1 k_2 k_3} = V_{k_1 k_2 k_3}^{00} + V_{k_1 k_2 k_3}^{01} + V_{k_1 k_2 k_3}^F. \tag{3.21}$$

The first contains a δ function divergence, the second a simple pole and the third is finite

$$\begin{aligned}
 V_{k_1 k_2 k_3}^F &= C_{k_1 k_2 k_3} \frac{\pi \sum_i^3 k_i}{\beta^2} \operatorname{csch}\left(\frac{\pi \sum_i^3 k_i}{2\beta}\right) \sum_{I=1}^3 \frac{1}{(2I-1)!} \left(\Phi_{k_1 k_2 k_3}^{I0} - \frac{\sum_i^3 k_i}{2I\beta} i\Phi_{k_1 k_2 k_3}^{I1} \right) \prod_{j=1}^{I-1} \left(\frac{(\sum_i^3 k_i)^2}{\beta^2} + (2j)^2 \right) \\
 \hat{V}_{k_1 k_2}^F &= V_{k_1, k_2, -k_1-k_2}^F = \hat{C}_{k_1 k_2} \frac{2}{\beta} \hat{\Phi}_{k_1 k_2}^{10}. \tag{3.22}
 \end{aligned}$$

Defining the symmetrized products

$$\begin{aligned}
 S_1^n &= k_1^n + k_2^n + k_3^n, & S_2^n &= (k_1 k_2)^n + (k_1 k_3)^n + (k_2 k_3)^n, & S_3^n &= (k_1 k_2 k_3)^n \\
 S_2^{mn} &= k_1^m k_2^n + k_1^m k_3^n + k_2^m k_3^n + k_1^n k_2^m + k_1^n k_3^m + k_2^n k_3^m \tag{3.23}
 \end{aligned}$$

one may use (3.4), (3.16), and (3.17) to calculate the coefficients of the triple product of the continuous normal modes

$$\begin{aligned}
 \Phi_{k_1 k_2 k_3}^{00} &= 3i\beta[-4\beta^4 S_1^1 + \beta^2(2S_2^{21} + 9S_3^1) - S_3^1 S_2^1] \\
 \Phi_{k_1 k_2 k_3}^{10} &= 3i\beta[16\beta^4 S_1^1 + \beta^2(-5S_2^{21} - 18S_3^1) + S_3^1 S_2^1] \\
 \Phi_{k_1 k_2 k_3}^{20} &= 9i\beta^3[-7\beta^2 S_1^1 + S_2^{21} + 3S_3^1], & \Phi_{k_1 k_2 k_3}^{30} &= 27i\beta^5 S_1^1 \\
 \Phi_{k_1 k_2 k_3}^{01} &= -8\beta^6 + \beta^4(18S_2^1 + 4S_3^1) + \beta^2(-2S_2^2 - 9S_3^1 S_1^1) + S_3^2 \\
 \Phi_{k_1 k_2 k_3}^{11} &= 3\beta^2[12\beta^4 + \beta^2(-15S_2^1 - 4S_3^1) + (S_2^2 + 3S_3^1 S_1^1)] \\
 \Phi_{k_1 k_2 k_3}^{21} &= 9\beta^4[-6\beta^2 + (3S_2^1 + S_3^1)], & \Phi_{k_1 k_2 k_3}^{31} &= 27\beta^6. \tag{3.24}
 \end{aligned}$$

Similarly, at $\sum_i k_i = 0$ one may define the symmetrized products

$$\hat{S}_2 = \frac{k_1^2 + k_2^2 + (k_1 + k_2)^2}{2}, \quad \hat{S}_3 = k_1 k_2 (k_1 + k_2) \tag{3.25}$$

and write the reduced coefficients

$$\begin{aligned}
 \hat{\Phi}_{k_1 k_2}^{00} &= -3i\beta\hat{S}_3(3\beta^2 + \hat{S}_2), & \hat{\Phi}_{k_1 k_2}^{10} &= 3i\beta\hat{S}_3(3\beta^2 + \hat{S}_2), \\
 \hat{\Phi}_{k_1 k_2}^{20} &= \hat{\Phi}_{k_1 k_2}^{30} = 0 \\
 \hat{\Phi}_{k_1 k_2}^{01} &= \hat{S}_3^2 - 2\beta^2\hat{S}_2^2 - 10\beta^4\hat{S}_2 - 8\beta^6, \\
 \hat{\Phi}_{k_1 k_2}^{11} &= 3\beta^2(4\beta^2 + \hat{S}_2)(3\beta^2 + \hat{S}_2) \\
 \hat{\Phi}_{k_1 k_2}^{21} &= -9\beta^4(6\beta^2 + \hat{S}_2), & \hat{\Phi}_{k_1 k_2}^{31} &= 27\beta^6. \tag{3.26}
 \end{aligned}$$

Now we have completed our decomposition of the kink ground state energy Q_2 . In the next section we will subtract the vacuum energy and then reassemble Q_2 to arrive at the kink mass.

IV. CANCELING IR DIVERGENCES

A. Defining the energy density

Our strategy for canceling the IR divergence in the two-loop kink energy Q_2 is to subtract the vacuum energy density from the kink energy density before performing the spatial integration. Our first task is thus to define the kink energy density.

Recall that our state $|0\rangle$ is an eigenstate of the kink Hamiltonian. The corresponding eigenvalue equation is

$$(H' - Q)|0\rangle = 0, \quad Q = \sum_i Q_i. \quad (4.1)$$

Expanding this equation in powers of the coupling,

$$\sum_{j=0}^i (H_{i+2-j} - Q_{\frac{i-j}{2}+1})|0\rangle_j = 0. \quad (4.2)$$

The first two orders do not involve the two-loop correction Q_2

$$Q_1|0\rangle_0 = H_2|0\rangle_0, \quad 0 = H_3|0\rangle_0 + H_2|0\rangle_1. \quad (4.3)$$

It appears at the third order

$$Q_2|0\rangle_0 = H_4|0\rangle_0 + H_3|0\rangle_1 + (H_2 - Q_1)|0\rangle_2. \quad (4.4)$$

We also expand the kink Hamiltonian density order by order

$$H_i = \int dx \mathcal{H}_i(x), \quad \mathcal{H}_{i(>2)}(x) = \frac{1}{i!} V^{(i)}[\sqrt{\lambda}f(x)] : \phi^i(x) :_a. \quad (4.5)$$

In the Schrodinger picture, the fields can be expanded in any basis of functions. We will expand them in terms of normalized kink normal modes [6]

$$\begin{aligned} \phi(x) &= \phi_C(x) + \phi_S(x) + \phi_B(x) \\ \phi_C(x) &= \int \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega_k}} (b_k^\dagger + b_{-k}) g_k(x) \\ \phi_S(x) &= \frac{1}{\sqrt{2\omega_S}} (b_S^\dagger - b_S) g_S(x) \\ \phi_B(x) &= \phi_0 g_B(x). \end{aligned} \quad (4.6)$$

The Hamiltonian density allows us to define the functions $\rho(x)$, as the expansion of $\mathcal{H}(x)|0\rangle$ in a Fock basis

$$\begin{aligned} &\sum_{j=0}^i \mathcal{H}_{i+2-j}(x)|0\rangle_j \\ &= \sum_{mn} \int^+ \frac{d^n k}{(2\pi)^n} \rho_i^{mn}(k_1 \cdots k_n; x) \phi_0^m B_{k_1}^\dagger \cdots B_{k_n}^\dagger |0\rangle_0 \end{aligned} \quad (4.7)$$

where $B_k^\dagger = b_k^\dagger / \sqrt{2\omega_k}$. Integrating the $i = 2$ equation over x and using (4.4) one obtains

$$\begin{aligned} &Q_2|0\rangle_0 + Q_1|0\rangle_2 \\ &= \sum_{mn} \int^+ \frac{d^n k}{(2\pi)^n} \left(\int dx \rho_2^{mn}(k_1 \cdots k_n; x) \right) \phi_0^m B_{k_1}^\dagger \cdots B_{k_n}^\dagger |0\rangle_0. \end{aligned} \quad (4.8)$$

Choose $|0\rangle_{i>0}$ to be orthogonal to $|0\rangle_0$, as one is always free to do in old-fashioned perturbation theory,⁴ then project onto $|0\rangle_0$ to obtain

$$Q_2 = \int dx \rho_2^{00}(x). \quad (4.9)$$

This ρ_2^{00} will be our definition of the kink energy density. As the zero-mode part of the Fock basis is not orthogonal, this choice of projection was somewhat arbitrary. It depended on our choice of basis for the space of functions of ϕ_0 . If another basis of functions for the ϕ_0 were used in (4.7), such as a set of polynomials, the kink energy density $\rho_2^{00}(x)$ could be shifted by some linear combination of the $\rho_2^{m0}(x)$. However, as $|0\rangle$ is an eigenstate of H' , these each integrate to zero and so the resulting kink mass would be unchanged.

B. Evaluating the energy density

Let us expand the kink ground state $|0\rangle$ in this same Fock basis

⁴At i th order, the state $|0\rangle$ is the sum $\sum_{j=0}^i |0\rangle_j$. Imagine that all of these $|0\rangle_j$ have been found such that (4.2) is solved. If $|0\rangle_i$ is not orthogonal to $|0\rangle_0$, then the projection of $|0\rangle_i$ in the $|0\rangle_0$ direction can be subtracted from $|0\rangle_i$ and added to $|0\rangle_0$, which rescales $|0\rangle_0$. After this, the inner product will vanish. This procedure does not change the sum and so the sum is still an eigenvector of $H - Q$ up to order i . Since the rescaling of $|0\rangle_0$ is order i in the coupling, it can be removed if desired by rescaling the sum without affecting any other $|0\rangle_j$, since the corresponding correction would be of order $i + j$.

$$|0\rangle_i = \sum_{m,n=0}^{\infty} |0\rangle_i^{mn},$$

$$|0\rangle_i^{mn} = Q_0^{-i/2} \int^+ \frac{d^n k}{(2\pi)^n} \gamma_i^{mn}(k_1 \dots k_n) \phi_0^m B_{k_1}^\dagger \dots B_{k_n}^\dagger |0\rangle_0. \quad (4.10)$$

Here γ are defined to be the coefficients of this expansion.

The divergence in Q_2 arises from $Q_2^{(3)}$ which in turn was derived in Ref. [10] from the mixing of the free ground state $|0\rangle_0 = |0\rangle_0^{00}$ with a kink state with three virtual normal modes $|0\rangle_1^{03}$, given by

$$\gamma_1^{03} = -\frac{\sqrt{Q_0}}{6 \sum_i^3 \omega_{k_i}} V_{k_1 k_2 k_3}. \quad (4.11)$$

The contribution of this state to Q_2 arises from the $H_3|0\rangle_1$ term in $H'|0\rangle$, as seen in (4.4). More precisely, it arises from the first term on the right-hand side of

$$H_3 = \frac{1}{6} \int dx V^{(3)}[\sqrt{\lambda} f(x)] : \phi^3(x) :_a$$

$$= \frac{1}{6} \int dx V^{(3)}[\sqrt{\lambda} f(x)] : \phi^3(x) :_b$$

$$+ \frac{1}{2} \int dx V^{(3)}[\sqrt{\lambda} f(x)] \phi(x) \mathcal{I}(x) \quad (4.12)$$

where we have changed normal ordering in terms of plane wave creation operators $::_a$ to normal ordering in terms of normal mode creation operators $::_b$ using the Wick's theorem of Ref. [20].

The corresponding two-loop energy contribution comes from ϕ^3 acting on $|0\rangle_1^{03}$. This contains terms of the form (4.11) where 0, 1, 2, or 3 of the k are continuum modes and the other are shape modes. For each continuum mode, the contribution to the vacuum energy arises from ϕ_C which contains $b_k g_{-k}(x)$. As $g_{-k}(x) = g_k^*(x)$, this yields a factor of $g_k^*(x)$. For each shape mode, the contribution arises from ϕ_S which contains $-b_S g_S(x)$, contributing a factor of $-g_S(x)$. However due to our convention $g_k(-x) = g_k^*(x)$, and the antisymmetry of $g_S(x)$, $g_S(x)$ is imaginary and so this contribution may be written $g_S^*(x)$. Thus the contributions from the three factors of $\phi(x)$ are $g_{k_1}^*(x) g_{k_2}^*(x) g_{k_3}^*(x)$, multiplied by the $V^{(3)}$ in H_3 , yielding $V_{k_1 k_2 k_3}^*$, irrespectively of which k are continuum and discrete modes. The three factors of $1/\sqrt{2}$ from the decomposition of ϕ_c in (4.6) combine with the three in the B^\dagger in the $|0\rangle_1^{03}$ term of (4.10) and the $1/6$ in (4.12), yielding a $1/48$. In all one finds

$$H_3|0\rangle_1^{03} \supset Q_2^{(3)}|0\rangle_0,$$

$$Q_2^{(3)} = - \int^+ \frac{d^3 k}{(2\pi)^3} \alpha_{k_1 k_2 k_3} |V_{k_1 k_2 k_3}|^2 |0\rangle_0 \quad (4.13)$$

where

$$\alpha_{k_1 k_2 k_3} = \frac{1}{48 \omega_{k_1} \omega_{k_2} \omega_{k_3} (\omega_{k_1} + \omega_{k_2} + \omega_{k_3})}. \quad (4.14)$$

The divergence arises from $k_1 + k_2 + k_3 \sim 0$ which only occurs in the real, continuous part of the integral

$$Q_2^{(3)} \supset Q_2^{(33)} = - \int \frac{d^3 k}{(2\pi)^3} \alpha_{k_1 k_2 k_3} |V_{k_1 k_2 k_3}|^2 |0\rangle_0 \quad (4.15)$$

where we defined $Q_2^{(3I)}$ to be the contribution to $Q_2^{(3)}$ in which I of the three normal modes are continuous.

Now we wish to tame this IR divergence by first writing it as an integral of an energy density, defined as in (4.7). Using

$$\mathcal{H}_3(x) = \frac{1}{6} V^{(3)}[\sqrt{\lambda} f(x)] : \phi^3(x) :_b$$

$$+ \frac{1}{2} V^{(3)}[\sqrt{\lambda} f(x)] \phi(x) \mathcal{I}(x) \quad (4.16)$$

one finds that the corresponding term in (4.7) is

$$\mathcal{H}_3(x)|0\rangle_1^{03} \supset \rho(x)|0\rangle_0, \quad \rho(x)$$

$$= - \int \frac{d^3 k}{(2\pi)^3} \alpha_{k_1 k_2 k_3} V_{k_1 k_2 k_3} \sigma_{-k_1, -k_2, -k_3}(x). \quad (4.17)$$

This is our definition of the energy density corresponding to IR-divergent $Q_2^{(33)}$. Indeed, as desired, formally it satisfies

$$Q_2^{(33)} = \int dx \rho(x) \quad (4.18)$$

although this is infinite. The expression (4.17) for the two-loop energy density corresponding to $Q_2^{(33)}$ is a main result of this work. The derivation can easily be modified to obtain the energy density corresponding to any term, but since the other terms are already IR-finite we will not need to do this.

We will now proceed to decompose $\rho(x)$ into pieces with various x -dependences

$$\rho(x) = \sum_{I, I'=0}^3 \sum_{J, J'=0}^1 \rho_{IJ I' J'}(x),$$

$$\rho_{IJ I' J'}(x) = - \int \frac{d^3 k}{(2\pi)^3} \alpha_{k_1 k_2 k_3} V_{k_1 k_2 k_3}^{IJ} \sigma_{-k_1, -k_2, -k_3}^{I' J'}(x). \quad (4.19)$$

The divergences lie in the $I = I' = 0$ terms.

We will separate the finite and infinite terms

$$\rho(x) = \rho^{(\text{div})}(x) + \rho^{(\text{fin})}(x), \quad \rho^{(\text{div})}(x) = \sum_{J, J'=0}^1 \rho_{0J 0 J'}(x). \quad (4.20)$$

The finite part may be integrated over x

$$\begin{aligned}
 Q_2^{(33\text{fin})} &= \int dx \rho^{(\text{fin})}(x) = q_{0F} + q_{1F} + q_{FF} \\
 q_{0F} &= -2 \int \frac{d^3k}{(2\pi)^3} \alpha_{k_1 k_2 k_3} V_{k_1 k_2 k_3}^{00} V_{-k_1 -k_2 -k_3}^F \\
 &= -\frac{4}{\beta} \int \frac{d^2k}{(2\pi)^2} \hat{\alpha}_{k_1 k_2} \hat{C}_{k_1 k_2}^2 \hat{\Phi}_{k_1 k_2}^{00} \hat{\Phi}_{-k_1 -k_2}^{10} \\
 q_{1F} &= -2 \int \frac{d^3k}{(2\pi)^3} \alpha_{k_1 k_2 k_3} V_{k_1 k_2 k_3}^{01} V_{-k_1 -k_2 -k_3}^F \\
 q_{FF} &= - \int \frac{d^3k}{(2\pi)^3} \alpha_{k_1 k_2 k_3} V_{k_1 k_2 k_3}^F V_{-k_1 -k_2 -k_3}^F
 \end{aligned} \tag{4.21}$$

where we have defined

$$\hat{\alpha}_{k_1 k_2} = \alpha_{k_1, k_2, -k_1 -k_2}. \tag{4.22}$$

In all $Q_2^{(33)}$ consists of seven contributions to the kink ground state energy: q_{0F} , q_{1F} , q_{FF} and the divergent integrals of the four $\rho_{0J0J'}(x)$.

In the Introduction we claimed that this approach is equivalent to adding an IR counterterm to $\mathcal{H}_4(x)$, equal and opposite to the vacuum energy density $\rho^{(\text{vac})}(x)$. Let us now justify that claim. In that case, the kink Hamiltonian would be an x integral over the total kink Hamiltonian density, which includes $\mathcal{H}_3(x) + \mathcal{H}_4(x)$. Then the eigenvalue equation (4.4) for Q_2 would include

$$\begin{aligned}
 H'|0\rangle &= \int dx \mathcal{H}'(x)|0\rangle \supset \int dx (\mathcal{H}_4(x)|0\rangle_0 + \mathcal{H}_3(x)|0\rangle_1) \\
 &\supset \int dx (-\rho^{(\text{vac})}(x)|0\rangle_0 + \mathcal{H}_3(x)|0\rangle_1^{03}).
 \end{aligned} \tag{4.23}$$

One sees that these two terms should indeed be added before performing the x integration. First we will manipulate $\rho^{(\text{vac})}(x)$ to cast this subtraction in a form in which we may analytically integrate over x .

C. The vacuum energy density

Using the third derivative of the potential

$$V^{(3)}[\phi_0] = 6\sqrt{2\lambda\beta} \tag{4.24}$$

and old-fashioned perturbation theory, one finds the one-loop vacuum energy density [12,13]

$$\begin{aligned}
 \rho^{(\text{vac})}(x) &= -72\lambda\beta^2 \int \frac{d^3p}{(2\pi)^3} \alpha_{p_1 p_2 p_3} 2\pi\delta(p_1 + p_2 + p_3) \\
 &= -72\lambda\beta^2 \int \frac{d^2k}{(2\pi)^2} \hat{\alpha}_{k_1 k_2}.
 \end{aligned} \tag{4.25}$$

This is independent of x , but we will manipulate it to introduce a spurious x -dependence shortly.

The identity

$$|\Phi_{k_1 k_2 k_3}^{00}|^2 + |\Phi_{k_1 k_2 k_3}^{01}|^2 = \frac{72\lambda\beta^2}{C_{k_1 k_2 k_3}^2} \tag{4.26}$$

allows us to replace the $72\lambda\beta^2$ in (4.25) with a sum of two more complicated terms. We use this to define the decomposition

$$\begin{aligned}
 \rho^{(\text{vac})}(x) &= \rho_0^{(\text{vac})}(x) + \rho_1^{(\text{vac})}(x), \\
 \rho_J^{(\text{vac})}(x) &= - \int \frac{d^2k}{(2\pi)^2} \hat{\alpha}_{k_1 k_2} \hat{C}_{k_1 k_2}^2 |\hat{\Phi}_{k_1 k_2}^{0J}|^2
 \end{aligned} \tag{4.27}$$

where J runs over 0 and 1. Intuitively these are the decomposition in terms of the contributions from the even and odd parts of $\sigma(x)$, whose integrals respectively lead to a delta function and a simple pole.

Multiplying by the identity

$$1 = -2i\text{sign}(x) \int \frac{dk_3}{2\pi} \frac{e^{ix \sum_i^3 k_i}}{\sum_i^3 k_i} \tag{4.28}$$

one finds

$$\rho_1^{(\text{vac})}(x) = 2i\text{sign}(x) \int \frac{d^3k}{(2\pi)^3} \frac{e^{ix \sum_i^3 k_i}}{\sum_i^3 k_i} \hat{\alpha}_{k_1 k_2} \hat{C}_{k_1 k_2}^2 |\hat{\Phi}_{k_1 k_2}^{01}|^2. \tag{4.29}$$

We will not need the analogous formula for $\rho_0^{(\text{vac})}$.

The advantage of these more complicated forms is that the vacuum energy density now has the same structure, as a matrix in k , as the kink energy density. Thus we can subtract the densities at each k separately. We will see that, once the vacuum energy density has been subtracted, the energy density becomes integrable over x for each n -tuple of k_i . This means that we may perform the x integration before the k integration, as both are anyway finite but we are only able to perform the x integration analytically.

Nonetheless this trick is somewhat expensive numerically, as it yields our only three-dimensional integration over k_i which does not decrease exponentially in $\sum_i^3 k_i$. To resolve this problem, we make the following observation. Momentum p is conserved at each interaction. k is not the momentum, and so the sum of k , which we call a does not need to vanish. However, at high k , the normal mode solutions tend to plane waves with momentum $p = k$. So at high k , a is roughly equal to the sum of p which is zero. Stated differently, amplitudes tend to be suppressed when a is large. Our trick will be to introduce an arbitrary cutoff $|a|$ in the three-dimensional integral and then to break up the

integral into contributions above and below the cutoff. Luckily the contribution above the cutoff can be reduced to a two-dimensional integral when the cutoff is nonzero. The IR divergence arises from the contribution beneath the cutoff.

We use the sine integral function

$$\text{Si}(a) = \int_0^a dt \frac{\sin(t)}{t}. \quad (4.30)$$

Using

$$i \left(\frac{\text{sign}(x)}{2} - \frac{\text{Si}(ax)}{\pi} \right) = \int_{-\infty}^{-k_1-k_2-a} \frac{dk_3}{2\pi} \frac{e^{ix(\sum_i^3 k_i)}}{\sum_i^3 k_i} + \int_{-k_1-k_2+a}^{\infty} \frac{dk_3}{2\pi} \frac{e^{ix(\sum_i^3 k_i)}}{\sum_i^3 k_i} \quad (4.31)$$

one can replace (4.28) with

$$1 = -2i\text{sign}(x) \int_{-k_1-k_2-a}^{-k_1-k_2+a} \frac{dk_3}{2\pi} \frac{e^{ix \sum_i^3 k_i}}{\sum_i^3 k_i} + \left(1 - \frac{2\text{sign}(x)\text{Si}(ax)}{\pi} \right). \quad (4.32)$$

This allows one to reduce the k_3 range of integration

$$\rho_1^{(\text{vac})}(x) = 2i\text{sign}(x) \int \frac{d^2k}{(2\pi)^2} \int_{-k_1-k_2-a}^{-k_1-k_2+a} \frac{dk_3}{2\pi} \frac{e^{ix \sum_i^3 k_i}}{\sum_i^3 k_i} \hat{\alpha}_{k_1 k_2} \hat{C}_{k_1 k_2}^2 |\hat{\Phi}_{k_1 k_2}^{01}|^2 + R(x, a) \quad (4.33)$$

where a is any positive number and the remainder is

$$R(x, a) = - \left(1 - \frac{2\text{sign}(x)\text{Si}(ax)}{\pi} \right) \int \frac{d^2k}{(2\pi)^2} \hat{\alpha}_{k_1 k_2} \hat{C}_{k_1 k_2}^2 |\hat{\Phi}_{k_1 k_2}^{0J}|^2. \quad (4.34)$$

The remainder is easily integrated over x

$$\int dx R(x, a) = - \frac{4}{\pi a} \int \frac{d^2k}{(2\pi)^2} \hat{\alpha}_{k_1 k_2} \hat{C}_{k_1 k_2}^2 |\hat{\Phi}_{k_1 k_2}^{01}|^2. \quad (4.35)$$

Notice that in the limit in which $a \rightarrow 0$, this integral is infinite, as it is equal to $\int dx \rho_1^{(\text{vac})}(x)$.

D. Canceling IR divergences

We are now ready to subtract the vacuum energy density from the kink energy density components defined in (4.20). We will do this one term at a time.

I. $\rho_{0000}(x)$

The first contribution to the kink energy density, resulting from two even $\sigma(x)$ terms, is

$$\begin{aligned} \rho_{0000}(x) &= - \int \frac{d^3k}{(2\pi)^3} \alpha_{k_1 k_2 k_3} V_{k_1 k_2 k_3}^{00} \sigma_{-k_1, -k_2, -k_3}^{00}(x) \\ &= - \int \frac{d^3k}{(2\pi)^3} \alpha_{k_1 k_2 k_3} \hat{C}_{k_1 k_2} \hat{\Phi}_{k_1 k_2}^{00} 2\pi \delta(k_1 + k_2 + k_3) \mathcal{C}_{k_1 k_2 k_3} \Phi_{-k_1 -k_2 -k_3}^{00} e^{ix(k_1 + k_2 + k_3)} \\ &= - \int \frac{d^2k}{(2\pi)^2} \hat{\alpha}_{k_1 k_2} \hat{C}_{k_1 k_2}^2 |\hat{\Phi}_{k_1 k_2}^{00}|^2 = \rho_0^{(\text{vac})}(x). \end{aligned} \quad (4.36)$$

Subtracting the corresponding contribution to the vacuum energy (4.27) one obtains the corresponding contribution to the two-loop kink ground state mass

$$\rho_{00}^{(\text{dif})}(x) = \rho_{0000}(x) - \rho_0^{(\text{vac})}(x) = 0. \quad (4.37)$$

2. $\rho_{0101}(x)$

The next contribution arises from the two odd $\sigma(x)$ terms

$$\begin{aligned} \rho_{0101}(x) &= - \int \frac{d^3k}{(2\pi)^3} \alpha_{k_1 k_2 k_3} V_{k_1 k_2 k_3}^{01} \sigma_{-k_1, -k_2, -k_3}^{01}(x) \\ &= 2i \int \frac{d^3k}{(2\pi)^3} \alpha_{k_1 k_2 k_3} \mathcal{C}_{k_1 k_2 k_3}^2 |\Phi_{k_1 k_2 k_3}^{01}|^2 \frac{\pi}{2\beta} \operatorname{csch}\left(\frac{\pi \sum_i^3 k_i}{2\beta}\right) e^{ix(k_1+k_2+k_3)} \tanh(\beta x). \end{aligned} \quad (4.38)$$

Subtracting the vacuum energy density (4.29) one finds

$$\begin{aligned} \rho_{0101}(x) - \rho_1^{(\text{vac})}(x) &= 2i \int \frac{d^3k}{(2\pi)^3} e^{ix(k_1+k_2+k_3)} \\ &\times \left[\alpha_{k_1 k_2 k_3} \mathcal{C}_{k_1 k_2 k_3}^2 |\Phi_{k_1 k_2 k_3}^{01}|^2 \frac{\pi}{2\beta} \operatorname{csch}\left(\frac{\pi \sum_i^3 k_i}{2\beta}\right) \tanh(\beta x) - \frac{1}{\sum_i^3 k_i} \hat{\alpha}_{k_1 k_2} \hat{\mathcal{C}}_{k_1 k_2}^2 |\hat{\Phi}_{k_1 k_2}^{01}|^2 \operatorname{sign}(x) \right]. \end{aligned} \quad (4.39)$$

In the limit $|x| \rightarrow \infty$ the term in square brackets tends to a finite value for all k , as the residues of the simple poles are equal and opposite. Therefore we may integrate over x to obtain the corresponding contribution to the kink mass

$$\begin{aligned} q_{11} &= \int dx (\rho_{0101}(x) - \rho_1^{(\text{vac})}(x)) \\ &= 4 \int \frac{d^3k}{(2\pi)^3} \left[-\alpha_{k_1 k_2 k_3} \mathcal{C}_{k_1 k_2 k_3}^2 |\Phi_{k_1 k_2 k_3}^{01}|^2 \left(\frac{\pi}{2\beta}\right)^2 \operatorname{csch}^2\left(\frac{\pi \sum_i^3 k_i}{2\beta}\right) + \frac{1}{(\sum_i^3 k_i)^2} \hat{\alpha}_{k_1 k_2} \hat{\mathcal{C}}_{k_1 k_2}^2 |\hat{\Phi}_{k_1 k_2}^{01}|^2 \right]. \end{aligned} \quad (4.40)$$

The term in brackets has only a first order pole at $k_1 + k_2 + k_3 = 0$ as the residues of the second order poles cancel. We define this integral to be the principal value, which is finite and real. Any other prescription would lead to a finite contribution at arbitrarily small k , which is unphysical as long wavelength modes have measure zero support on the kink.

3. $\rho_{0001}(x) + \rho_{0100}(x)$

Now we have subtracted the entire vacuum energy, but we still have two divergent terms left in the kink energy. These are the cross terms arising from one even and one odd $\sigma(x)$. As a consistency check on our calculation, their sum must be finite.

The two terms are

$$\begin{aligned} \rho_{0100}(x) &= - \int \frac{d^3k}{(2\pi)^3} \alpha_{k_1 k_2 k_3} V_{k_1 k_2 k_3}^{01} \sigma_{-k_1, -k_2, -k_3}^{00}(x) \\ &= 2i \int \frac{d^3k}{(2\pi)^3} \alpha_{k_1 k_2 k_3} \mathcal{C}_{k_1 k_2 k_3}^2 \Phi_{k_1 k_2 k_3}^{01} \Phi_{-k_1 -k_2 -k_3}^{00} \frac{\pi}{2\beta} \operatorname{csch}\left(\frac{\pi \sum_i^3 k_i}{2\beta}\right) e^{ix \sum_i^3 k_i} \end{aligned} \quad (4.41)$$

and

$$\begin{aligned} \rho_{0001}(x) &= - \int \frac{d^3k}{(2\pi)^3} \alpha_{k_1 k_2 k_3} V_{k_1 k_2 k_3}^{00} \sigma_{-k_1, -k_2, -k_3}^{01}(x) \\ &= - \int \frac{d^3k}{(2\pi)^3} \alpha_{k_1 k_2 k_3} \hat{\mathcal{C}}_{k_1 k_2} \hat{\Phi}_{k_1 k_2}^{00} 2\pi \delta(k_1 + k_2 + k_3) \mathcal{C}_{k_1 k_2 k_3} \Phi_{-k_1 -k_2 -k_3}^{01} \tanh(\beta x) \\ &= - \int \frac{d^2k}{(2\pi)^2} \hat{\alpha}_{k_1 k_2} \hat{\mathcal{C}}_{k_1 k_2}^2 \hat{\Phi}_{k_1 k_2}^{00} \hat{\Phi}_{-k_1 -k_2}^{01} \tanh(\beta x) \\ &= - \int \frac{d^3k}{(2\pi)^3} \hat{\alpha}_{k_1 k_2} \hat{\mathcal{C}}_{k_1 k_2}^2 \hat{\Phi}_{k_1 k_2}^{00} \hat{\Phi}_{-k_1 -k_2}^{01} \left[-i \frac{\pi}{\beta} e^{ix \sum_i^3 k_i} \operatorname{csch}\left(\frac{\pi \sum_i^3 k_i}{2\beta}\right) \right] \\ &= -2i \int \frac{d^3k}{(2\pi)^3} \hat{\alpha}_{k_1 k_2} \hat{\mathcal{C}}_{k_1 k_2}^2 \hat{\Phi}_{-k_1 -k_2}^{00} \hat{\Phi}_{k_1 k_2}^{01} \frac{\pi}{2\beta} \operatorname{csch}\left(\frac{\pi \sum_i^3 k_i}{2\beta}\right) e^{ix \sum_i^3 k_i}. \end{aligned} \quad (4.42)$$

Their sum is

$$\begin{aligned} \rho_{0100}(x) + \rho_{0001}(x) &= 2i \int \frac{d^3k}{(2\pi)^3} \frac{\pi \sum_i^3 k_i}{2\beta} \operatorname{csch}\left(\frac{\pi \sum_i^3 k_i}{2\beta}\right) e^{ix \sum_i^3 k_i} T_{k_1 k_2 k_3} \\ T_{k_1 k_2 k_3} &= \frac{\alpha_{k_1 k_2 k_3} \mathcal{C}_{k_1 k_2 k_3}^2 \Phi_{k_1 k_2 k_3}^{01} \Phi_{-k_1 -k_2 -k_3}^{00} - \hat{\alpha}_{k_1 k_2} \hat{\mathcal{C}}_{k_1 k_2}^2 \hat{\Phi}_{k_1 k_2}^{01} \hat{\Phi}_{-k_1 -k_2}^{00}}{\sum_i^3 k_i}. \end{aligned} \quad (4.43)$$

The numerator vanishes linearly as $\sum_i^3 k_i$ tends to zero, and so

$$\hat{T}_{k_1 k_2} = T_{k_1, k_2, -k_1 -k_2} = \left. \frac{\partial}{\partial k_3} (\alpha_{k_1 k_2 k_3} \mathcal{C}_{k_1 k_2 k_3}^2 \Phi_{k_1 k_2 k_3}^{01} \Phi_{-k_1 -k_2 -k_3}^{00}) \right|_{k_3 = -k_1 -k_2} \quad (4.44)$$

is finite, as is the integrand. Now we may integrate over x

$$\begin{aligned} q_{10} &= \int dx (\rho_{0100}(x) + \rho_{0001}(x)) \\ &= 2i \int \frac{d^3k}{(2\pi)^3} \frac{\pi \sum_i^3 k_i}{2\beta} \operatorname{csch}\left(\frac{\pi \sum_i^3 k_i}{2\beta}\right) T_{k_1 k_2 k_3} 2\pi \delta\left(\sum_i^3 k_i\right) \\ &= 2i \int \frac{d^2k}{(2\pi)^2} \hat{T}_{k_1 k_2}. \end{aligned} \quad (4.45)$$

In all, we have found five contributions to the IR finite $Q_2^{(33)} = \int dx \rho(x)$, obtained by subtracting $\rho^{(\text{vac})}(x)$ from the integrand at each x

$$\int dx (\rho(x) - \rho^{(\text{vac})}(x)) = q_{0F} + q_{1F} + q_{FF} + q_{11} + q_{10} \quad (4.46)$$

each of which is finite.

V. NUMERICAL INTEGRATION

Now that all x integration has been done analytically, we will perform the k integration in (2.13) numerically. Uncertainties will be reported in parentheses for those integrals that dominate the error budget.

Two summands require no integration over the momenta k and so are easily determined analytically using (3.14) and (3.2) respectively

$$Q_2^{(1)} = \frac{1}{560} \left(1 - \frac{4\sqrt{3}}{\pi} + \frac{54}{\pi^2}\right) \frac{\lambda}{\beta}, \quad Q_2^{(5)} = -\frac{1}{8} \frac{9\lambda^2}{64\beta^6} \frac{16\beta^5}{15\lambda} = -\frac{3}{80} \frac{\lambda}{\beta}. \quad (5.1)$$

Numerically, these are

$$Q_2^{(1)} \sim 0.00761791 \frac{\lambda}{\beta}, \quad Q_2^{(5)} = -0.0375 \frac{\lambda}{\beta}. \quad (5.2)$$

Next, again using (3.14), we find the second contribution

$$\begin{aligned} Q_2^{(2)} &= -\frac{1}{8} \int \frac{dk}{2\pi} \frac{|V_{\mathcal{I}k}|^2}{\omega_k^2} - \frac{|V_{\mathcal{I}S}|^2}{8\omega_S^2} \\ &= -\frac{1}{8} \frac{\lambda}{6144\beta^8} \int \frac{dk}{2\pi} \frac{k^4}{\beta^2 + k^2} [2\pi(-2\beta^2 + k^2) + 3\sqrt{3}\omega_k^2]^2 \operatorname{csch}^2\left(\frac{\pi k}{2\beta}\right) \\ &= -\frac{1}{8} \frac{3}{4096} [-2\pi + 3\sqrt{3}]^2 \frac{\lambda}{\beta} \sim (-0.000961713 - 0.000108182) \frac{\lambda}{\beta} \\ &\sim -0.0010699 \frac{\lambda}{\beta}. \end{aligned} \quad (5.3)$$

The fourth term can be found using (3.12)

$$\begin{aligned}
 Q_2^{(4)} &= \frac{1}{16Q_0} \int \frac{d^2k}{(2\pi)^2} \frac{|(\omega_{k_1} - \omega_{k_2})\Delta_{k_1k_2}|^2}{\omega_{k_1}\omega_{k_2}} + \frac{1}{8Q_0} \int \frac{d^1k}{(2\pi)^1} \frac{|(\omega_{k_1} - \omega_S)\Delta_{k_1S}|^2}{\omega_{k_1}\omega_S} \\
 &= \frac{27\pi^2\lambda}{2048\beta^3} \int \frac{d^2k}{(2\pi)^2} \frac{(\omega_{k_1} - \omega_{k_2})^2(k_1^2 - k_2^2)^2}{k_1^3k_2^3} \frac{(4\beta^2 + k_1^2 + k_2^2)^2}{(\beta^2 + k_1^2)(\beta^2 + k_2^2)} \operatorname{csch}^2\left(\frac{\pi(k_1 + k_2)}{2\beta}\right) \\
 &\quad + \frac{3\sqrt{3}\pi^2\lambda}{2048\beta^3} \int \frac{dk}{2\pi} \frac{(\omega_k - \beta\sqrt{3})^2(3\beta^2 + k^2)^2(\beta^2 + k^2)}{\beta\omega_k\beta^3\omega_k^2} \operatorname{sech}^2\left(\frac{\pi k}{2\beta}\right) \\
 &\sim (0.001481577 + 0.002358405) \frac{\lambda}{\beta} \sim 0.00383998 \frac{\lambda}{\beta}. \tag{5.4}
 \end{aligned}$$

It remains to evaluate the third term $Q_2^{(3)}$. Recall that this term results from two vertices with three normal modes each. The normal modes may be the shape mode S or continuum modes. Let us first decompose the result into summands involving various numbers of continuum modes

$$\begin{aligned}
 Q_2^{(3)} &= \sum_{k=0}^3 Q_2^{(3k)}, \quad Q_2^{(30)} = -\frac{1}{48} \frac{|V_{SSS}|^2}{3\omega_S^4} \\
 Q_2^{(31)} &= -\frac{1}{16} \int \frac{d^1k}{(2\pi)^1} \frac{|V_{SSk_1}|^2}{\omega_{k_1}\omega_S^2(\omega_{k_1} + 2\omega_S)} \\
 Q_2^{(32)} &= -\frac{1}{16} \int \frac{d^2k}{(2\pi)^2} \frac{|V_{Sk_1k_2}|^2}{\omega_{k_1}k_2\omega_S(\omega_{k_1} + k_2 + \omega_S)} \\
 Q_2^{(33)} &= q_{0F} + q_{1F} + q_{FF} + q_{11} + q_{10}. \tag{5.5}
 \end{aligned}$$

Only the term with three continuum modes is divergent. The others can be found using (3.13)

$$\begin{aligned}
 Q_2^{(30)} &= -\frac{3\pi^2}{4096} \frac{\lambda}{\beta} \sim -0.00722871 \frac{\lambda}{\beta} \\
 Q_2^{(31)} &= -\frac{3\pi^2\lambda}{2048\beta^8} \int \frac{dk}{2\pi} \frac{k^4\omega_k(2\beta^2 - k^2)^2}{(\omega_k + 2\sqrt{3}\beta)(\beta^2 + k^2)} \operatorname{csch}^2\left(\frac{\pi k}{2\beta}\right) \sim -0.0008745152 \frac{\lambda}{\beta} \\
 Q_2^{(32)} &= -\frac{9\sqrt{3}\pi^2\lambda}{1024\beta^4} \int \frac{d^2k}{(2\pi)^2} \frac{[(17\beta^4 - (k_1^2 - k_2^2)^2)(\beta^2 + k_1^2 + k_2^2) + 8\beta^2k_1^2k_2^2]^2 \operatorname{sech}^2\left(\frac{\pi(k_1+k_2)}{2\beta}\right)}{k_1^3k_2^3(\omega_{k_1} + k_2 + \sqrt{3}\beta)(\beta^2 + k_1^2)(\beta^2 + k_2^2)} \sim -0.0311512(1) \frac{\lambda}{\beta}. \tag{5.6}
 \end{aligned}$$

Finally we are ready for the five terms (4.46) involving three continuum modes. These are the only terms which have analogs in the vacuum sector, defined by replacing each g_k with the corresponding plane wave and the kink Hamiltonian with the defining Hamiltonian, so that all Δ vanish and V is proportional to a Dirac delta function. Thus these are the only terms for which we subtract the vacuum contribution. As an abuse of notation, we continue to call this contribution $Q_2^{(33)}$ after this infinite subtraction.

A single divergent term in V yields a delta function or simple pole which can be used to do the k_3 integral. However $Q_2^{(33)}$ is quadratic in V and so it is the second

divergence in V which causes an infinity in $Q_2^{(33)}$. This means that the three terms (4.21) which involve zero or one divergent terms V^{00} or V^{01} in the two V factors, and finite terms V^F from the others, are manifestly finite and require no vacuum subtraction. Let us begin with these three terms.

The first contribution arises from the cross term between one interaction V^{00} proportional to Dirac delta function in momentum space, corresponding to the position-independent part of the triple product of three normal modes, with the finite terms V^F in the triple product. As the delta function can be used to do the k_3 integral, this contribution is given by a two-dimensional integral

$$q_{0F} = 54\beta^3\lambda \int \frac{d^2k}{(2\pi)^2} \frac{k_1^2 k_2^2 (k_1 + k_2)^2 (3\beta^2 + k_1^2 + k_1 k_2 + k_2^2)^2}{\omega_{k_1}^3 \omega_{k_2}^3 \omega_{k_1+k_2}^3 (\omega_{k_1} + \omega_{k_2} + \omega_{k_1+k_2}) (\beta^2 + k_1^2) (\beta^2 + k_2^2) (\beta^2 + (k_1 + k_2)^2)} \sim 0.0375390(1) \frac{\lambda}{\beta}. \quad (5.7)$$

For the three-dimension terms that follow, we will often encounter the combination

$$\alpha_{k_1 k_2 k_3} \mathcal{C}_{k_1 k_2 k_3}^2 = \frac{3\lambda\beta^2}{2\omega_{k_1}^3 \omega_{k_2}^3 \omega_{k_3}^3 (\omega_{k_1} + \omega_{k_2} + \omega_{k_3}) (\beta^2 + k_1^2) (\beta^2 + k_2^2) (\beta^2 + k_3^2)}. \quad (5.8)$$

The next finite term is the cross term between the antisymmetric V^{01} and the finite V^F

$$q_{1F} = \frac{3\pi^2\lambda}{\beta} \int \frac{d^3k}{(2\pi)^3} \frac{\Phi_{k_1 k_2 k_3}^{01} (\sum_i^3 k_i) \text{csch}^2(\frac{\pi \sum_i^3 k_i}{2\beta})}{\omega_{k_1}^3 \omega_{k_2}^3 \omega_{k_3}^3 (\omega_{k_1} + \omega_{k_2} + \omega_{k_3}) (\beta^2 + k_1^2) (\beta^2 + k_2^2) (\beta^2 + k_3^2)} \quad (5.9)$$

$$\times \sum_{I=1}^3 \frac{1}{(2I-1)!} \left(-i\Phi_{k_1 k_2 k_3}^{I0} - \frac{\sum_i^3 k_i}{2I\beta} \Phi_{k_1 k_2 k_3}^{I1} \right) \prod_{j=1}^{I-1} \left(\frac{(\sum_i^3 k_i)^2}{\beta^2} + (2j)^2 \right) \sim 0.006070(1) \frac{\lambda}{\beta}. \quad (5.10)$$

Here the integral is evaluated according to the principal value prescription, as it must be real and no finite contribution from spatial infinity, corresponding to $\sum_i^3 k_i = 0$, is expected.

The last finite term corresponds to the product of the two finite terms V^F

$$q_{FF} = -\frac{3\pi^2\lambda}{2\beta^2} \int \frac{d^3k}{(2\pi)^3} \frac{(\sum_i^3 k_i)^2 \text{csch}^2(\frac{\pi \sum_i^3 k_i}{2\beta})}{\omega_{k_1}^3 \omega_{k_2}^3 \omega_{k_3}^3 (\omega_{k_1} + \omega_{k_2} + \omega_{k_3}) (\beta^2 + k_1^2) (\beta^2 + k_2^2) (\beta^2 + k_3^2)} \times \left[\sum_{I=1}^3 \frac{1}{(2I-1)!} \left(i\Phi_{k_1 k_2 k_3}^{I0} + \frac{\sum_i^3 k_i}{2I\beta} \Phi_{k_1 k_2 k_3}^{I1} \right) \prod_{j=1}^{I-1} \left(\frac{(\sum_i^3 k_i)^2}{\beta^2} + (2j)^2 \right) \right]^2 \sim -0.0163976(4) \frac{\lambda}{\beta}. \quad (5.11)$$

Recall that, only in the case of q_{11} , the integral decreases as $1/k$ in all three directions and so the integral is truly three-dimensional. In all other three-dimensional integrals, and in the kink sector contribution to q_{11} , the k_3 integrand converges exponentially as the result of a csch function. However, as explained above, the k_3 integration of the vacuum sector contribution can be performed analytically

above any given cutoff a , leaving the two-dimensional integral given in Eq. (4.35). We therefore choose this cutoff to be at least $\sum_i^3 k_i = 12$, so that the kink energy term, as it is exponentially suppressed in $\sum_i^3 k_i$, is easily calculated to within our error budget. Above this threshold the choice of cutoff has negligible effect on our result

$$q_{11} = 6\lambda\beta^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_{k_1}^3 \omega_{k_2}^3 (\beta^2 + k_1^2) (\beta^2 + k_2^2)} \left[-\frac{|\Phi_{k_1 k_2 k_3}^{01}|^2 (\frac{\pi}{2\beta})^2 \text{csch}^2(\frac{\pi \sum_i^3 k_i}{2\beta})}{\omega_{k_3}^3 (\sum_{j=1}^3 \omega_{k_j}) (\beta^2 + k_3^2)} + \frac{1}{(\sum_i^3 k_i)^2 \omega_{k_1+k_2}^3 (\omega_{k_1} + \omega_{k_2} + \omega_{k_1+k_2}) (\beta^2 + (k_1 + k_2)^2)} \right] \sim 6\lambda\beta^2 \int \frac{d^2k}{(2\pi)^2} \int_{-k_1-k_2-a}^{-k_1-k_2+a} dk_3 \frac{1}{2\pi \omega_{k_1}^3 \omega_{k_2}^3 (\beta^2 + k_1^2) (\beta^2 + k_2^2)} \times \left[-\frac{|\Phi_{k_1 k_2 k_3}^{01}|^2 (\frac{\pi}{2\beta})^2 \text{csch}^2(\frac{\pi \sum_i^3 k_i}{2\beta})}{\omega_{k_3}^3 (\sum_{j=1}^3 \omega_{k_j}) (\beta^2 + k_3^2)} + \frac{1}{(\sum_i^3 k_i)^2 \omega_{k_1+k_2}^3 (\omega_{k_1} + \omega_{k_2} + \omega_{k_1+k_2}) (\beta^2 + (k_1 + k_2)^2)} \right] + \frac{4}{\pi a} \int \frac{d^2k}{(2\pi)^2} \hat{\alpha}_{k_1 k_2} \hat{\mathcal{C}}_{k_1 k_2}^2 |\hat{\Phi}_{k_1 k_2}^{01}|^2 \sim 0.033043(2) \frac{\lambda}{\beta}. \quad (5.12)$$

Finally the cross term between the two divergent V yields

$$q_{10} = -3\lambda\beta^2 \int \frac{d^2k}{(2\pi)^2} \frac{\partial}{\partial k_3} \left(\frac{i\Phi_{k_1 k_2 k_3}^{00} \Phi_{k_1 k_2 k_3}^{01}}{\omega_{k_1}^3 \omega_{k_2}^3 \omega_{k_3}^3 (\omega_{k_1} + \omega_{k_2} + \omega_{k_3}) (\beta^2 + k_1^2) (\beta^2 + k_2^2) (\beta^2 + k_3^2)} \right) \Big|_{k_3=-k_1-k_2} \sim 0.0124290(1) \frac{\lambda}{\beta}. \quad (5.13)$$

VI. CONCLUDING REMARKS

Adding all 12 contributions one finds that the two-loop correction to the ϕ^4 kink mass is

$$Q_2 \sim 0.006316(2) \frac{\lambda}{\beta} = 0.012633(5) \frac{\lambda}{m}. \quad (6.1)$$

This is positive, in agreement with small λ lattice results [23], instantaneous frame Fock space truncation results [11,24] and Borel resummation [12]. It also agrees with the preferred value of the mass correction in light front [25] and conformal space [26] truncations. On the other hand, the lattice study in Ref. [27] found a kink mass beneath the one-loop result. Fig. 8 of Ref. [11] appears to show a mass correction of 0.018 ± 0.008 at $\lambda = 0.4$ and $m = 1$, if the dot size is to be interpreted as the error bar, which may be compared with our result of $Q_2 \sim 0.005$. Our result also lies just beyond the lower error bar of Fig. 11 of Ref. [12]. In both cases, this light tension is smaller than some of the individual mass contributions, such as $Q_2^{(5)}$ which, at $\lambda = 0.4$ and $m = 1$, contributes -0.03 to Q_2 .

In principle, the methods cited above include nonperturbative and in the case of Ref. [12] also higher order perturbative information, and so their results are more reliable as the coupling grows. In practice, the uncertainties in all of these studies are of the same order as the difference between the calculated kink mass and the one loop result (1.2). In contrast, the uncertainty on our Q_2 is several thousand times smaller than its central value. In that sense, our method offers far more precise results, although as the coupling increases the accuracy will be compromised by higher order and also nonperturbative corrections, with the latter arising from virtual kink-antikink pair creation.

While Q_2 is positive, the coefficient is not large enough to invalidate the observation of Ref. [11] that a naive extrapolation of the mass formula, together with the semiclassical calculation of the bound state masses in Ref. [26], suggests that at large enough coupling, all bound states of kinks and so potentially all topologically trivial particles are no longer in the spectrum. Similarly, the coupling at which the meson mass is twice the kink mass, and so the meson may decay [24,28], is hardly affected. Of course there is no reason to trust our perturbative analysis beyond weak coupling.

Chang duality [29] implies that the vacuum Hamiltonian, at each value of λ/β^2 beneath about 8, is equal to the

vacuum Hamiltonian at a large value of λ/β^2 , as the shift in the classical Hamiltonian is exactly compensated by the change in normal ordering as the mass changes. In other words, each quantum Hamiltonian arises from two distinct classical Hamiltonians with $\beta^2 > 0$. We do not expect our semiclassical expansion to be reliable at such large couplings, although the positive Q_2 found here is consistent with the possibility that the kink belongs to a Chang-symmetric family of Hamiltonian eigenstates which exists at all couplings and is continuous in the coupling. This possibility is also weakly supported by the fact that the kink mass appears to become flat at the right of Fig. 11 of Ref. [12]. Stronger evidence arises from the $\beta^2 < 0$ Chang dual, whose Hamiltonian we recall is the same operator. Reference [12] found that its mass gap is well defined throughout this region and coincides with the kink mass when the $\beta^2 > 0$ theory is at coupling below the critical coupling. In an intermediate regime of couplings including the self-dual point, one expects that the \mathbb{Z}_2 symmetry, whose breaking is responsible for the classical kink solution, is restored. Thus any continuation of the kink state in that region would be quite interesting.

Moreover, this duality implies that our perturbative kink states, found at weak coupling, are also eigenstates of a strongly coupled Hamiltonian to the same precision. In particular, they become exact eigenstates in the infinite λ/β^2 limit. In this limit they do not become the semiclassical solitons of the strongly coupled theory, which one expects to be deformed beyond recognition by quantum corrections. For example, the \mathcal{D}_f in the construction uses the classical kink solution $f(x)$ of the original classical Hamiltonian, which is not a solution of the classical equations of motion of the dual Hamiltonian. These may therefore provide an example of a quantum soliton unrelated to any classical solution of the same system. As this a property that any monopoles appearing in Yang-Mills would also possess, they could prove to be a useful testing ground.

ACKNOWLEDGMENTS

J. E. is supported by the CAS Key Research Program of Frontier Sciences Grant No. QYZDY-SSW-SLH006 and the NSFC MianShang grants No. 11875296 and No. 11675223. J.E. also thanks the Recruitment Program of High-end Foreign Experts for support.

[1] R. F. Dashen, B. Hasslacher, and A. Neveu, Nonperturbative methods and extended hadron models in field theory 2. Two-dimensional models and extended hadrons, *Phys. Rev. D* **10**, 4130 (1974).

[2] A. Rebhan and P. van Nieuwenhuizen, No saturation of the quantum Bogomolnyi bound by two-dimensional supersymmetric solitons, *Nucl. Phys.* **B508**, 449 (1997).

- [3] A. Litvintsev and P. van Nieuwenhuizen, Once more on the BPS bound for the SUSY kink, [arXiv:hep-th/0010051](#).
- [4] A. S. Goldhaber, A. Litvintsev, and P. van Nieuwenhuizen, Local Casimir energy for solitons, *Phys. Rev. D* **67**, 105021 (2003).
- [5] N. Graham and R. L. Jaffe, Unambiguous one loop quantum energies of $(1+1)$ -dimensional bosonic field configurations, *Phys. Lett. B* **435**, 145 (1998).
- [6] J. Evslin, Manifestly finite derivation of the quantum kink mass, *J. High Energy Phys.* 11 (2019) 161.
- [7] J. Evslin, Well-defined quantum soliton masses without supersymmetry, *Phys. Rev. D* **101**, 065005 (2020).
- [8] N. H. Christ and T. D. Lee, Quantum expansion of soliton solutions, *Phys. Rev. D* **12**, 1606 (1975).
- [9] J. L. Gervais, A. Jevicki, and B. Sakita, Perturbation expansion around extended particle states in quantum field theory, *Phys. Rev. D* **12**, 1038 (1975).
- [10] J. Evslin and H. Guo, Two-loop scalar kinks, *Phys. Rev. D* **103**, 125011 (2021).
- [11] S. Rychkov and L. G. Vitale, Hamiltonian truncation study of the ϕ^4 theory in two dimensions. II. The \mathbb{Z}_2 -broken phase and the Chang duality, *Phys. Rev. D* **93**, 065014 (2016).
- [12] M. Serone, G. Spada, and G. Villadoro, $\lambda\phi_2^4$ theory—Part II. the broken phase beyond NNNN(NNNN)LO, *J. High Energy Phys.* 05 (2019) 047.
- [13] H. Liu, Y. Zhou, and J. Evslin, Perturbative ground states of the 1+1d double-well, *Eur. Phys. J. C* **80**, 357 (2020).
- [14] S. Mandal, Low energy effective quantum field theoretical description of excitations about kink soliton configuration, [arXiv:1903.07711](#).
- [15] S. Coleman, Notes from Sidney Coleman's physics 253a: Quantum field theory, [arXiv:1110.5013](#).
- [16] R. F. Dashen, B. Hasslacher, and A. Neveu, The particle spectrum in model field theories from semiclassical functional integral techniques, *Phys. Rev. D* **11**, 3424 (1975).
- [17] A. Luther, Eigenvalue spectrum of interacting massive fermions in one-dimension, *Phys. Rev. B* **14**, 2153 (1976).
- [18] W. Loinaz and R. S. Willey, Monte Carlo simulation calculation of critical coupling constant for continuum ϕ^4 in two-dimensions, *Phys. Rev. D* **58**, 076003 (1998).
- [19] J. Evslin and H. Guo, Excited kinks as quantum states, *Phys. Rev. D* **104**, 085013 (2021).
- [20] J. Evslin, Normal ordering normal modes, *Eur. Phys. J. C* **81**, 92 (2021).
- [21] H. J. de Vega, Two-loop quantum corrections to the soliton mass in two-dimensional scalar field theories, *Nucl. Phys.* **B115**, 411 (1976).
- [22] J. Verwaest, Higher order correction to the sine-Gordon soliton mass, *Nucl. Phys.* **B123**, 100 (1977).
- [23] A. Ardekani and A. G. Williams, Lattice study of the kink soliton and the zero mode problem for ϕ^4 in two-dimensions, *Aust. J. Phys.* **52**, 929 (1999).
- [24] Z. Bajnok and M. Lajer, Truncated Hilbert space approach to the 2d ϕ^4 theory, *J. High Energy Phys.* 10 (2016) 050.
- [25] D. Chakrabarti, A. Harindranath, L. Martinovic, and J. P. Vary, Kinks in discrete light cone quantization, *Phys. Lett. B* **582**, 196 (2004).
- [26] A. Coser, M. Beria, G. P. Brandino, R. M. Konik, and G. Mussardo, Truncated conformal space approach for 2D Landau-Ginzburg theories, *J. Stat. Mech.* (2014) P12010.
- [27] A. Rajantie and D. J. Weir, Quantum kink and its excitations, *J. High Energy Phys.* 04 (2009) 068.
- [28] G. Mussardo, Neutral bound states in kink-like theories, *Nucl. Phys.* **B779**, 101 (2007).
- [29] S. J. Chang, The existence of a second order phase transition in the two-dimensional ϕ^4 field theory, *Phys. Rev. D* **13**, 2778 (1976); **16**, 1979(E) (1977).