Some remarks on the Hamiltonian for unimodular gravity

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Several formulations of the first order approach to unimodular gravity are presented. There is always a particular one such that it is *classically* equivalent to the second order formulation; this we call *educated*. It is often at variance with the *naive* approach, in which the Lagrangian is taken as given exactly by the same expression as in the second order formulation; only the number and character of the independent variables changes. Namely, typically some of the momenta are now considered as coordinates. The ensuing Hamiltonians are thereby discussed, and their physical differences pointed out.

I. INTRODUCTION

It is well known that every second order equation of motion (EoM) can be written in first order language (FO) just by introducing new dependent variables. This is usually believed to hold true when there are infinite variables, like in field theory. Nevertheless, recently some curious behavior have been pointed out when the gravitational field is considered. Namely, it would seem that there is a crucial difference between Lagrangians linear in curvature (Einstein-Hilbert) and Lagrangians involving higher orders in curvature. In the latter case, FO is *not* equivalent to the usual second order approach (SO), and the connection field encapsulates many different spin components [1].

This is indeed a fact when the same Lagrangian that is usually worked out in SO is considered as FO; this just means that the metric and the connection are treated as fully independent fields. This will dub herewith as the *naive approach* or *naive FO*. The reason is that is it always possible to build up a slightly more complicated FO Lagrangian, such that its EoM are completely equivalent as those obtained in the usual SO approach. This we shall dub *educated FO*. We shall give many examples in the body of the paper.

A simple example, a naive FO for Einstein-Hilbert Lagrangian, would be

$$S = \int \sqrt{|g|} d^n x g^{\mu\nu} (\partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\nu\lambda} + \Gamma^\sigma_{\sigma\lambda} \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\lambda}), \quad (1)$$

and a educated FO,

$$S = \int \sqrt{|g|} d^n x \left\{ g^{\mu\nu} (\partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\mu \Gamma^\lambda_{\nu\lambda} + \Gamma^\sigma_{\sigma\lambda} \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\mu\sigma} \Gamma^\sigma_{\nu\lambda}) \right.$$
$$\left. + \lambda^{\mu\nu}_\tau \left(\frac{1}{2} g^{\tau\sigma} (-\partial_\sigma g_{\mu\nu} + \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu}) - \Gamma^\tau_{\mu\nu} \right) \right\}, \quad (2)$$

where $\lambda^{\mu\nu}_\tau$ is a Lagrange multiplier.[1]

We have also recently studied a modification of general relativity, unimodular gravity ([2] and references therein) in which the set of admissible metrics is restricted to those with unit determinant. The symmetries of the theory are thereby reduced from the set of all diffeomorphisms, $Diff(M)$, to those that preserve the unimodular condition, namely the transverse ones $TDiff(M)$. Those transverse vector fields generate the subgroup of *volume preserving diffeomorphisms*. We shall always employ the notation,

$$\gamma_{\mu\nu}, \quad (3)$$

to denote a metric, such that

$$\gamma \equiv \det \gamma_{\mu\nu} = -1. \quad (4)$$

Sometimes it is useful to generate an unimodular metric out of an arbitrary one by means of a Weyl rescaling,

$$\gamma_{\mu\nu} \equiv g^{-1/n} g_{\mu\nu}. \quad (5)$$

In this paper, the expressions *linear* or *quadratic* are employed always as referring to Riemann's curvature tensor.

Our aim in this paper is to elaborate on those ambiguities, introducing educated FO and computing, in

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[1]Throughout this work, we follow the Landau-Lifshitz spacelike conventions, in particular, the metric is $\eta_{\mu\nu} = (+, -, -, -)$ and $R^\mu_{\ \nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \cdots$; we omit the factor $-\frac{1}{2\kappa^2}$ in the Einstein-Hilbert action.

particular, the corresponding Hamiltonians. The point is that in any diffeomorphism invariant theory, the total Hamiltonian is a constraint that must be put equal to zero. It is often the case that at least for asymptotically flat space-times, the physical energy is embodied in a boundary term, precisely the sort of thing that changes in every "equivalent" formulation of the physical theory.

A general observation [3] is the following. The second order variation of any Lagrangian depending on the metric and the connection field is symbolically,

$$\delta S = \int \frac{\delta S}{\delta \Gamma} \frac{\delta \Gamma}{\delta g} + \frac{\delta S}{\delta g}, \tag{6}$$

whereas the first order one read

$$\frac{\delta S}{\delta \Gamma} = 0$$
$$\frac{\delta S}{\delta g} = 0. \tag{7}$$

This clearly shows that FO implies SO; the opposite is untrue.

The determination of a Hamiltonian for the gravitational field is an old problem [4–6]. As has been already pointed out, the bulk Hamiltonian vanishes (again, this is actually a generic property of all diffeomorphism invariant theories), and this is the origin of the constraints to be imposed in any canonical quantization of the gravitational field (confer, for example, [7] and references therein).

We will try to be quite specific on the subtle differences between general relativity and unimodular gravity in this respect.

Let us stress again that there is also a boundary term which is quite important because it fully determines the numerical value of the energy associated to asymptotically at gravitational fields, (ADM) [5]. We also would like to give a detailed computation in this case, which could depend on the precise formulation of the theory. Although we will devote an initial section to introduce the problem in an explicit *physicist* notation using components, the main part of the paper will be written in a more covariant formalism using frame fields and differential forms. This is almost mandatory once higher order (in curvature) Lagrangians are considered, in which case the component notation becomes exceedingly cumbersome.

II. THEORIES LINEAR IN CURVATURE. THE EINSTEIN-HILBERT LAGRANGIAN

A. The Einstein-Hilbert Lagrangian

The Einstein-Hilbert Lagrangian in FO formalism is

$$\mathcal{L}_{\text{EH}} = \sqrt{|g|} g^{\mu\nu} (\partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\lambda\mu} + \Gamma^\lambda_{\tau\lambda} \Gamma^\tau_{\mu\nu} - \Gamma^\lambda_{\tau\mu} \Gamma^\tau_{\lambda\nu}). \tag{8}$$

A related action principle (educated version) that would be equivalent to Einstein-Hilbert's would read

$$\mathcal{L}_{\text{EH}} = -\Gamma^\lambda_{\mu\nu} \partial_\lambda (\sqrt{|g|} g^{\mu\nu}) + \Gamma^\lambda_{\lambda\mu} \partial_\nu (\sqrt{|g|} g^{\mu\nu})$$
$$+ \sqrt{|g|} g^{\mu\nu} (\Gamma^\lambda_{\tau\lambda} \Gamma^\tau_{\mu\nu} - \Gamma^\lambda_{\tau\mu} \Gamma^\tau_{\lambda\nu}). \tag{9}$$

The variation respect to the metric reads

$$-\Gamma^\lambda_{\mu\nu} \partial_\lambda \left(\sqrt{|g|} \left(\frac{1}{2} g^{\mu\nu} h - h^{\mu\nu} \right) \right) + \Gamma^\lambda_{\lambda\mu} \partial_\nu \left(\sqrt{|g|} \left(\frac{1}{2} g^{\mu\nu} h - h^{\mu\nu} \right) \right)$$
$$- \sqrt{|g|} \left((\Gamma^\lambda_{\tau\lambda} \Gamma^\tau_{\mu\nu} - \Gamma^\lambda_{\tau\mu} \Gamma^\tau_{\lambda\nu}) - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} (\Gamma^\lambda_{\tau\lambda} \Gamma^\tau_{\alpha\beta} - \Gamma^\lambda_{\tau\alpha} \Gamma^\tau_{\lambda\beta}) \right) h^{\mu\nu} = 0; \tag{10}$$

under integration by parts, we recover the Einstein field equation,

$$\frac{1}{2} R h - R_{\mu\nu} h^{\mu\nu} = -G_{\mu\nu} h^{\mu\nu} = 0, \tag{11}$$

and the variation respect to the connection of the Lagrangian reads

$$\left[-\partial_a (\sqrt{|g|} g^{bc}) + \delta^b_a \partial_d (\sqrt{|g|} g^{cd}) - \frac{1}{2} \sqrt{|g|} g^{\mu\nu} (\delta^c_\mu \Gamma^b_{a\nu} + \delta^b_\mu \Gamma^c_{a\nu} + \delta^b_\nu \Gamma^c_{a\mu} + \delta^c_\nu \Gamma^b_{a\mu} - \delta^c_a \Gamma^b_{\mu\nu} - \delta^b_a \Gamma^c_{\mu\nu} - \delta^b_\mu \delta^c_\nu \Gamma^\lambda_{a\lambda} - \delta^c_\mu \delta^b_\nu \Gamma^\lambda_{a\lambda}) \right] A^a_{bc} = 0, \tag{12}$$

which using

$$\partial_\lambda (\sqrt{|g|} g^{\mu\nu}) = \sqrt{|g|} (g^{\mu\nu} \Gamma^\tau_{\tau\lambda} - g^{\tau\nu} \Gamma^\mu_{\tau\lambda} - g^{\mu\tau} \Gamma^\nu_{\tau\lambda}) \tag{13}$$

reduces to zero. This fact shows [8] that the linear Lagrangian does not need an educated form, in the sense that the naive FO is already equivalent to SO.

Let us examine now what is the situation in the unimodular setting. Consider a linear unimodular metric in FO, $\gamma_{\mu\nu}$ and some associated torsionless connection, $\omega^{\mu}_{\nu\lambda}$. We are interested in the action,

$$S \equiv \int d^n x \gamma^{\nu\sigma}(\partial_\mu \omega^\mu_{\nu\sigma} - \partial_\sigma \omega^\mu_{\nu\mu} + \omega^\mu_{\lambda\mu}\omega^\lambda_{\nu\sigma} - \omega^\mu_{\lambda\sigma}\omega^\lambda_{\nu\mu}), \quad (14)$$

where the covariant derivative acts on the covariant indices only. Please note that the Levi-Civita connection associated to the unimodular metric satisfies

$$\omega_\sigma \equiv \omega^\lambda_{\lambda\sigma} = 0 \quad (15)$$

and

$$\omega^\sigma \equiv \gamma^{\alpha\beta}\omega^\sigma_{\alpha\beta} = \partial_\lambda \omega^{\sigma\lambda}. \quad (16)$$

In terms of an arbitrary nonunitary metric [2], $g_{\mu\nu}$

$$\gamma_{\mu\nu} \equiv g^{-\frac{1}{n}}g_{\mu\nu}, \quad (17)$$

this formulation introduces a redundant Weyl gauge symmetry,

$$g_{\mu\nu} \rightarrow \Omega^2(x)g_{\mu\nu}. \quad (18)$$

The Levi-Civita connection associated to $\gamma_{\mu\nu}$ is

$$\omega^\mu_{\nu\rho} = \Gamma^\mu_{\nu\rho} + \frac{1}{2n}g^{\mu\lambda}\left(\frac{\partial_\lambda g}{g}g_{\nu\rho} - \frac{\partial_\nu g}{g}g_{\lambda\rho} - \frac{\partial_\rho g}{g}g_{\nu\lambda}\right), \quad (19)$$

where $\Gamma^\mu_{\nu\rho}$ is the Levi-Civita connection associated to the general metric $g_{\mu\nu}$, that is, Christoffel's symbols.

Next, we present our notation for the ADM formalism [5,7,9]. The metric tensor $g_{\mu\nu}$ of spacetime induces a metric $h_{\mu\nu}$ on the spatial hypersurface $\sum_t$,

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \quad (20)$$

where n_μ is the unit normal. The extrinsic curvature tensor of the spatial hypersurface $\sum_t$ is defined as

$$K_{\mu\nu} = \nabla_\mu n_\nu + n_\mu a_\nu = \nabla_\mu n_\nu + n_\mu n^\lambda \nabla_\lambda n_\nu. \quad (21)$$

The extrinsic curvature can be written as the Lie derivative of the induced metric $h_{\mu\nu}$ on $\sum_t$ along the unit normal n to $\sum_t$,

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n h_{\mu\nu}, \quad (22)$$

and the decomposition of the scalar curvature R of spacetime can be written as

$$R = {}^{(3)}R + K_{\mu\nu}K^{\mu\nu} - K^2 + 2\nabla_\mu(n^\mu K - a^\mu). \quad (23)$$

In the given ADM coordinate base, the components of the metric of spacetime read

$$g_{00} = N^2 \quad g_{0i} = N_i \quad g_{ij} = h_{ij}, \quad (24)$$

the extrinsic curvature tensor (22) is written as

$$K_{ij} = \frac{1}{2}\mathcal{L}_n h_{ij} = \frac{1}{2N}(\partial_t h_{ij} - D_i N_j - D_j N_i). \quad (25)$$

(1) *Einstein Hilbert in second order*

What happens in second order UG [10] is that the lapse is not an independent dynamical variable, because

$$N^2|h| = 1, \quad (26)$$

where

$$h \equiv \det g_{ij}^{(n-1)} \equiv \det h_{ij}. \quad (27)$$

This means that it is not compulsory to impose the Hamiltonian constraint,

$$\mathcal{H} = 0 \quad (28)$$

$$\{\mathcal{H}(x), \mathcal{H}(x')\} = (\mathcal{H}^i(x) + \mathcal{H}^i(x'))\partial_i\delta(x - x')$$
$$\{\mathcal{H}_i(x), \mathcal{H}(x')\} \sim \mathcal{H}(x)\partial_i\delta(x - x')$$
$$\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} \sim \mathcal{H}_i(x')\partial_j\delta(x - x')$$
$$\qquad + \mathcal{H}_j(x)\partial_i\delta(x - x'), \quad (29)$$

but only the weaker condition,

$$\mathcal{H} = \lambda, \quad (30)$$

where λ is determined by the physical boundary conditions. This is the usual unimodular setting [1] in Hamiltonian language.

(2) *Einstein Hilbert in naive first order*

It has already been pointed out that it has been proved in [8] that even the naive FO Einstein-Hilbert is equivalent to the usual SO Einstein's equations. As for the Hamiltonian, there are several possibilities. We could, for example, start with the Lagrangian of GR in naive FO ADM form (and neglecting boundary terms),

$$\mathcal{L}_{\text{EH}} = N\sqrt{h}[{}^{(3)}R + K_{ij}K^{ij} - K^2], \quad (31)$$

where the variables are the spatial metric h_{ij} and the extrinsic curvature, K_{ij}. With these assumptions all

momenta vanish, and the Hamiltonian just coincides with the potential.

(3) *Einstein Hilbert in educated first order*

Let us instead start with the Lagrangian of GR in educated FO ADM form [7,9] (and neglecting boundary terms),

$$\mathcal{L}_{\text{EH}} = N\sqrt{h}\left[{}^{(3)}R + \frac{1}{N}K^{ij}(\partial_t h_{ij} - D_i N_j - D_j N_i)\right.$$
$$\left. - K_{ij}K^{ij} - 2KT + K^2\right], \qquad (32)$$

where D_i is the induced covariant derivative in the three manifold Σ_t, and we have defined $T \equiv \frac{1}{2N}h^{ij}(\partial_t h_{ij} - D_i N_j - D_j N_i)$. The EoM for the field K^{ij} implies

$$\frac{1}{N}(\partial_t h_{ij} - D_i N_j - D_j N_i) - 2K_{ij} - 2h_{ij}T + 2h_{ij}K = 0, \qquad (33)$$

then

$$K_{ij} = \frac{1}{2N}(\partial_t h_{ij} - D_i N_j - D_j N_i), \qquad (34)$$

with $T = K$; note if we reintroduce the expression of K_{ij} in (32), we recover the standard SO Lagrangian in ADM variables, and the conjugate momenta p^{ij} are

$$N_i \rightarrow p^i = 0$$
$$K_{ij} \rightarrow p^{ij} = 0$$
$$h_{ij} \rightarrow p^{ij} = \sqrt{h}K_{ij}, \qquad (35)$$

therefore, the Hamiltonian,

$$H_{\text{EH}} = -N\sqrt{h}{}^{(3)}R + p^{ij}(D_i N_j + D_j N_i)$$
$$+ \frac{N}{\sqrt{h}}(p_{ij}p^{ij} + p^2). \qquad (36)$$

(4) *Unimodular Einstein Hilbert in second order*

Again here the unimodular constraint,

$$N\sqrt{h} = 1 \qquad (37)$$

implies that the unimodular Lagrangian reads

$$\mathcal{L}_{\text{UG}} = {}^{(3)}R + \frac{1}{N}K^{ij}(\partial_t h_{ij} - D_i N_j - D_j N_i)$$
$$- K_{ij}K^{ij} - 2KT + K^2, \qquad (38)$$

and the conjugate momenta p^{ij} are

$$N_i \rightarrow p^i = 0$$
$$K_{ij} \rightarrow p^{ij} = 0$$
$$h_{ij} \rightarrow p^{ij} = \frac{1}{N}K_{ij}, \qquad (39)$$

and the Hamiltonian,

$$H_{\text{UG}} = {}^{(3)}R + p^{ij}(D_i N_j + D_j N_i) + N^2(p_{ij}p^{ij} + p^2). \qquad (40)$$

(5) *Unimodular educated first order*

The unimodular version of (9) yields

$$\mathcal{L}_{UE} = -\Gamma^\lambda_{\mu\nu}\partial_\lambda(\gamma^{\mu\nu}) + \Gamma^\lambda_{\lambda\mu}\partial_\nu(\gamma^{\mu\nu})$$
$$- \gamma^{\mu\nu}(\Gamma^\lambda_{\tau\mu}\Gamma^\tau_{\lambda\nu} - \Gamma^\lambda_{\tau\lambda}\Gamma^\tau_{\mu\nu}), \qquad (41)$$

the conjugate momenta p^{ij} are

$$\Gamma^\lambda_{\mu\nu} \rightarrow p^{\mu\nu}_\lambda = 0$$
$$\gamma_{00} \rightarrow p^{00} = -\Gamma^0_{00} + \Gamma^\lambda_{\lambda 0}$$
$$\gamma_{0i} \rightarrow p^{0i} = -\Gamma^0_{0i} + \Gamma^\lambda_{\lambda i}$$
$$\gamma_{ij} \rightarrow p^{ij} = -\Gamma^0_{ij}, \qquad (42)$$

and the Hamiltonian reads

$$H_{UE} = \Gamma^i_{\mu\nu}\partial_i\gamma^{\mu\nu} - \Gamma^\lambda_{\lambda\mu}\partial_i\gamma^{i\mu} + \gamma^{\mu\nu}(\Gamma^\lambda_{\tau\mu}\Gamma^\tau_{\lambda\nu} - \Gamma^\lambda_{\tau\lambda}\Gamma^\tau_{\mu\nu}). \qquad (43)$$

B. Schrödinger's Lagrangian

The Einstein-Hilbert Lagrangian, can be written as

$$\mathcal{L}_{\text{EH}} = \sqrt{|g|}R = \partial_\lambda(\sqrt{|g|}g^{\mu\nu}\Gamma^\lambda_{\mu\nu}) - \partial_\nu(\sqrt{|g|}g^{\mu\nu}\Gamma^\lambda_{\lambda\mu}) + \mathcal{L}_S, \qquad (44)$$

where

$$\partial_\lambda(\sqrt{|g|}g^{\mu\nu}) = \sqrt{|g|}(g^{\mu\nu}\Gamma^\tau_{\tau\lambda} - g^{\tau\nu}\Gamma^\mu_{\tau\lambda} - g^{\mu\tau}\Gamma^\nu_{\tau\lambda}), \qquad (45)$$

then, up to a total derivative,

$$\mathcal{L}_S = \sqrt{|g|}g^{\mu\nu}\mathfrak{R}_{\mu\nu} = \sqrt{|g|}g^{\mu\nu}(\Gamma^\lambda_{\tau\mu}\Gamma^\tau_{\lambda\nu} - \Gamma^\lambda_{\tau\lambda}\Gamma^\tau_{\mu\nu}), \qquad (46)$$

which is just the $\Gamma\Gamma$ Schrödinger's [11] Lagrangian. It is then plain that the Einstein-Hilbert and Schrödinger's Lagrangian differ by a total derivative, so that they yield the same equations of motion when considered in second order.

The energy-momentum tensor of Schrödinger's Lagrangian is

$$T^\mu_\nu = \frac{\partial \mathcal{L}_S}{\partial(\partial_\mu g_{\alpha\beta})}\partial_\nu g_{\alpha\beta} - \mathcal{L}_S \delta^\mu_\nu$$

$$= \frac{\sqrt{g}}{2}[2\Gamma^\mu_{\alpha\beta} - \Gamma^\mu_{\rho\sigma}g^{\rho\sigma}g_{\alpha\beta} - \delta^\mu_\alpha\Gamma^\lambda_{\beta\lambda} - \delta^\mu_\beta\Gamma^\lambda_{\alpha\lambda}]g^{\alpha\beta}{}_{,\nu}$$

$$- \delta^\mu_\nu\sqrt{|g|}g^{\alpha\beta}(\Gamma^\lambda_{\tau\alpha}\Gamma^\tau_{\lambda\beta} - \Gamma^\lambda_{\tau\lambda}\Gamma^\tau_{\alpha\beta}). \tag{47}$$

It is curious that it corresponds to the so-called Einstein energy pseudo tensor [6], which in first order formalism, reduces to

$$T^\mu_\nu = -\mathcal{L}_S \delta^\mu_\nu. \tag{48}$$

What about Schrödinger's Lagrangian considered as a first order one? The dependence on the variables $\Gamma^\alpha_{\beta\gamma}$ and $g_{\mu\nu}$ is algebraic, so that the Lagrangian is equivalent to the Hamiltonian,

$$H \equiv V = -L. \tag{49}$$

(i) Let us now include physical sources for the graviton, $T^{\mu\nu}$, and for the connection field, $j^{\beta\gamma}_\alpha$ in Schrödinger's Lagrangian,

$$\mathcal{L}_{\text{Smatter}} \equiv \sqrt{g}(g^{\mu\nu}\mathfrak{L}_{\mu\nu} + g^{\mu\nu}T_{\mu\nu} + \Gamma^\alpha_{\beta\gamma}j^{\beta\gamma}_\alpha). \tag{50}$$

The variation with respect to the metric[2] of (50) yields

$$\delta\mathcal{L}_S = \sqrt{|g|}\left(-\mathfrak{L}^{\mu\nu} + \frac{1}{2}\mathfrak{L}g^{\mu\nu} - T^{\mu\nu} + \frac{1}{2}Tg^{\mu\nu}\right.$$
$$\left. + \frac{1}{2}g^{\mu\nu}\Gamma^\alpha_{\beta\gamma}j^{\beta\gamma}_\alpha\right)\delta g_{\mu\nu}. \tag{53}$$

The trace of the EoM yields

$$(n-2)(\mathfrak{L} + T) + n\Gamma^\alpha_{\beta\gamma}j^{\beta\gamma}_\alpha = 0; \tag{54}$$

in the absence of sources, this implies

$$\mathfrak{L} = 0. \tag{55}$$

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[2]There is a small subtlety here. The variation of the scalar $T \equiv g^{\mu\nu}T_{\mu\nu} = g_{\mu\nu}T^{\mu\nu}$ is

$$\delta g^{\mu\nu}T_{\mu\nu} \neq \delta g_{\mu\nu}T^{\mu\nu}. \tag{51}$$

The explanation is that it is *not* equivalent to assume $\delta T^{\mu\nu} = 0$ than to assume $\delta T_{\mu\nu} = 0$. Indeed,

$$\delta T^{\mu\nu} = \delta g_{\alpha\lambda}T^\lambda_\beta + \delta g_{\beta\lambda}T^\lambda_\alpha + g_{\alpha\mu}g_{\beta\nu}\delta T^{\mu\nu}. \tag{52}$$

Here, we are assuming $\delta T_{\mu\nu} = 0$.

Neglecting for the time being the connection source, the EoM for the graviton field reads

$$\mathfrak{L}_{\mu\nu} + T_{\mu\nu} = 0. \tag{56}$$

The variation of (50) with respect to the connection yields

$$\delta\mathcal{L}_S = \left\{\frac{1}{2}g^{\mu\nu}[\delta^c_\mu\Gamma^b_{a\nu} + \delta^b_\mu\Gamma^c_{a\nu} + \delta^b_\nu\Gamma^c_{a\mu} + \delta^c_\nu\Gamma^b_{a\mu}\right.$$
$$\left. - \delta^c_a\Gamma^b_{\mu\nu} - \delta^b_a\Gamma^c_{\mu\nu} - \delta^b_\mu\delta^c_\nu\Gamma^\lambda_{a\lambda} - \delta^c_\mu\delta^b_\nu\Gamma^\lambda_{a\lambda}] + j^{bc}_a\right\}A^a_{bc}, \tag{57}$$

where $A^a_{bc} = \delta\Gamma^a_{bc}$, then

$$2\Gamma^{b|c}_a + 2\Gamma^{c|b}_a - \delta^c_a g^{\mu\nu}\Gamma^b_{\mu\nu} - \delta^b_a g^{\mu\nu}\Gamma^c_{\mu\nu}$$
$$- 2g^{bc}\Gamma^\lambda_{a\lambda} + 2j^{bc}_a = 0 \tag{58}$$

trace (58), with δ^a_c

$$g^{\mu\nu}[(1-n)\Gamma^c_{\mu\nu} + 2j^{c\lambda}_\lambda] = 0. \tag{59}$$

Now we trace (58), with g_{cb}

$$(4-2n)\Gamma^\lambda_{a\lambda} - 2g_{ac}g^{\mu\nu}\Gamma^c_{\mu\nu} + 2g_{bc}j^{bc}_a = 0. \tag{60}$$

(ii) Let us work out the linear approximation,

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$
$$\Gamma^\alpha_{\beta\gamma} = 0 + A^\alpha_{\beta\gamma}, \tag{61}$$

where we just have seen that

$$\mathfrak{L}^{(L)} = \frac{1}{2}hT - h^{\mu\nu}T_{\mu\nu} + \frac{1}{2}hA^\lambda_{\mu\nu}j^{\mu\nu}_\lambda. \tag{62}$$

At any rate, it is plain that without sources,

$$\mathfrak{L}^{(L)}_{\alpha\beta} = 0, \tag{63}$$

it does then seem impossible to recover Newton's equation in the appropiate linear limit in FO. We have just seen that in SO, we recover Einstein's equations exactly. The reason for this apparent contradiction is that the difference between Schrödinger and Einstein-Hilbert Lagrangians is a total derivative only when considered in SO, but it is not when considered in FO.

This then illustrates a dramatic instance of a nonequivalence of FO and SO approaches in a

theory of gravity linear in curvature. The standard lore [8] was that FO and SO were equivalent for theories linear un curvature (such as the standard Einstein-Hilbert Lagrangian), and nonequivalence appears only in theories involving higher powers of Riemann's tensor.

In conclusion, whereas the $\Gamma - \Gamma$ Lagrangian correctly reproduce Einstein's equations when considered in SO, it predicts flat space in vacuum as the only solution in FO. No gravitational waves exist in this formulation.

III. THEORIES QUADRATIC IN CURVATURE

Quadratic theories can be written in the general form,

$$S = \frac{1}{2} \int d^n x \sqrt{|g|} \{ R^\mu{}_{\nu\rho\sigma}[\Gamma] P_{\mu\mu'}{}^{\nu\nu'\rho\rho'\sigma\sigma'}[g] R^{\mu'}{}_{\nu'\rho'\sigma'}[\Gamma] \}, \quad (64)$$

where the tensor $P_{\mu\mu'}{}^{\nu\nu'\rho\rho'\sigma\sigma'}[g]$ depends only on the metric $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$. Let us work out the basic example, where

$$P_{\mu\mu'}{}^{\nu\nu'\rho\rho'\sigma\sigma'}[g] = \delta^{\mu'}_\mu \delta^\nu_{\mu'} g^{\rho\rho'} g^{\sigma\sigma'}. \quad (65)$$

Other contractions of Riemann's tensor can be worked out along similar rules. An educated first order version of the action principle is given by promoting $R^\mu{}_{\nu\rho\sigma}$ to an independent variable together with the connection Γ and the metric tensor g,

$$S = \frac{1}{2} \int d^n x \sqrt{|g|} \{ R^\mu{}_{\nu\rho\sigma} g^{\rho\alpha} g^{\sigma\beta} R^\nu{}_{\mu\alpha\beta} $$
$$+ 2 R_\mu{}^{\nu\rho\sigma} (\partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\lambda\rho} \Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\lambda\sigma} \Gamma^\lambda_{\nu\rho}) \}. \quad (66)$$

In FO, it is not necessary to introduce auxiliary fields as in [12]. We shall assume that the field $R_\mu{}^{\nu\rho\sigma}$ has the symmetries of Riemann's tensor.

In fact, the Lagrangian EoM ensure that it is given on shell by the Riemann tensor corresponding to the dynamical connection Γ,

$$R^\mu{}_{\nu\rho\sigma}[\Gamma] \equiv \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\lambda\rho} \Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\lambda\sigma} \Gamma^\lambda_{\nu\rho}. \quad (67)$$

The canonical momenta are given by

$$g_{\mu\nu} \to p^{\mu\nu} \sim 0$$
$$R^\mu{}_{\nu\rho\sigma} \to p_\mu{}^{\nu\rho\sigma} \sim 0$$
$$\Gamma^\mu_{\nu i} \to p_\mu{}^{\nu i} = \sqrt{|g|} (R_\mu{}^{\nu 0 i} + R_\mu{}^{i 0 \nu}). \quad (68)$$

This object is symmetric in (νi), and as usual, greek indices run from $(0 \ldots n-1)$, and latin indices from the middle of the alphabet run from $1 \ldots n-1$. When contracted with some other tensor with those symmetries, it is not

necessary to make those explicit. Let us define the auxiliary variable,

$$q_\mu{}^{\nu i} \equiv \sqrt{|g|} (R_\mu{}^{\nu 0 i} - R_\mu{}^{i 0 \nu}). \quad (69)$$

In conclusion,

$$R_\mu{}^{\nu 0 i} = \frac{1}{2 \sqrt{|g|}} (q_\mu{}^{\nu i} - p_\mu{}^{\nu i}), \quad (70)$$

now we need to substitute certain components of Riemann's tensor by the corresponding momenta. The computation becomes heavy, and we refrain from reproducing it here; we shall give a simplified treatment using differential forms momentarily.

IV. COVARIANT APPROACH IN TERMS OF THE FRAME FIELD AND THE SPIN CONNECTION

When discussing a Hamiltonian formalism, it is unavoidable to introduce a noncovariant distinction between space and time. This can be done however in such a way that as many symmetries as possible are respected. The time direction will be characterized by a vector field,

$$Z \equiv Z^\mu \partial_\mu, \quad (71)$$

then acting on any exact form,

$$i_Z d\alpha = \pounds_Z \alpha - d i_Z \alpha \equiv \dot{\alpha} - d i_Z \alpha \quad (72)$$

($\pounds_Z \alpha$ is the generalization of the concept of *time derivative*). It is always possible to locally choose an adapted coordinates, such that $Z = \partial_t$ with $i_Z dt = 1$. It is then natural [6] to define the time and space projections on an arbitrary form α as

$$\hat{\alpha} \equiv i_Z \alpha$$
$$\underline{\alpha} \equiv \alpha - dt \wedge \hat{\alpha}_t. \quad (73)$$

The induced projections on the exterior differential read

$$d\alpha \equiv \underline{d\alpha} + dt \wedge \widehat{d\alpha} = \underline{d\alpha} + dt \wedge (\dot{\alpha} - d\hat{\alpha}). \quad (74)$$

It will also prove convenient to decompose the differential operator $d = dx^\mu \wedge \partial_\mu = dt \wedge \partial_t + dx^k \wedge \partial_k$, and we define

$$d \equiv dt \wedge \hat{d} + \underline{d}, \quad (75)$$

with $\hat{d} \equiv \pounds_Z$.

After this small introduction, let us write down the variation of the first order Lagrangian. Our purpose is to get an expression for the associated Hamiltonian by particularizing later for an explicit form of the variation,

$$\delta \mathcal{L}_{\text{FO}} = \sum \left[d(\delta \phi^k \wedge p_k) + \delta \phi^k \wedge \frac{\delta \mathcal{L}_{\text{FO}}}{\delta \phi^k} + \frac{\delta \mathcal{L}_{\text{FO}}}{\delta p_k} \wedge \delta p_k \right]. \tag{76}$$

This formula does not assume anything about the variation. Consider the particular case of a time translation, that is, a Lie derivative along the vector field Z,

$$\delta \phi^k = \pounds_Z \phi^k. \tag{77}$$

It follows in general that

$$di_Z \mathcal{L}_{\text{FO}} = \pounds_Z \mathcal{L}_{\text{FO}} = \sum \left[d(\pounds_Z \phi^k \wedge p_k) \right.$$
$$\left. + \pounds_Z \phi^k \wedge \frac{\delta \mathcal{L}_{\text{FO}}}{\delta \phi^k} + \frac{\delta \mathcal{L}_{\text{FO}}}{\delta p_k} \wedge \pounds_Z p_k \right]. \tag{78}$$

This means that there is a first integral,

$$H(Z) \equiv \sum \pounds_Z \phi^k \wedge p_k - i_Z \mathcal{L}_{\text{FO}}, \tag{79}$$

that satisfies the identity,

$$-dH(Z) = \sum \left[\pounds_Z \phi^k \wedge \frac{\delta \mathcal{L}_{\text{FO}}}{\delta \phi^k} + \frac{\delta \mathcal{L}_{\text{FO}}}{\delta p_k} \wedge \pounds_Z p_k \right]. \tag{80}$$

It is a conserved current on shell, which its explicit expression is

$$H(Z) = \sum [d(i_Z \phi^k \wedge p_k) + i_Z \phi^k \wedge dp_k$$
$$+ d\phi^k \wedge i_Z p_k + i_Z \Lambda] \tag{81}$$

can be written like a displacement vector plus a total differential,

$$H(Z) \equiv Z^\mu H_\mu + dB(Z), \tag{82}$$

where

$$B(Z) = \sum i_Z \phi^k \wedge p_k \tag{83}$$

compare the differential of this expression $dH = dZ^\mu \wedge H_\mu + Z^\mu dH_\mu$ with (80). We learn that

$$Z^\mu H_\mu = \sum \left[-i_Z \phi^k \wedge \frac{\delta \mathcal{L}_{\text{FO}}}{\delta \phi^k} + \frac{\delta \mathcal{L}_{\text{FO}}}{\delta p_k} \wedge i_Z p_k \right], \tag{84}$$

so that H_μ itself vanishes on shell, and all contribution to the energy comes from the boundary term, $B(Z)$.

A. Einstein-Hilbert theory

In terms of the frame one forms,[3]

$$e^a \equiv e^a_\mu dx^\mu, \tag{85}$$

and the curvature two form,

$$R_{ab} \equiv \frac{1}{2} R_{ab\mu\nu} dx^\mu \wedge dx^\nu, \tag{86}$$

the Einstein-Hilbert action can be written [1] as the integral over spacetime,

$$S = \int e^a \wedge e^b \wedge *R_{ab} = \int R_{ab} \wedge *(e^a \wedge e^b), \tag{87}$$

where the curvature is expressed in terms on the connection one forms,

$$\omega_{ab} \equiv \omega_{ab\mu} dx^\mu. \tag{88}$$

Note $\omega_{ab\mu} = -\omega_{ba\mu}$, then

$$R_{ab} \equiv d\omega_{ab} + \omega_{ac} \wedge \omega^c{}_b. \tag{89}$$

In second order formalism, the connection one forms are determined by the torsionless condition,

$$de^a + \omega^a{}_b \wedge e^b = 0, \tag{90}$$

but in this paper, we would like to stick to the first order formalism, in which ω is an independent field.

Let us explain in detail how this comes about.

$$S = \frac{1}{2} \int \epsilon_{abcd} e^a \wedge e^b \wedge R^{cd}$$
$$= \frac{1}{4} \int \epsilon_{abcd} e^a_\mu e^b_\nu R^{cd}{}_{\rho\sigma} dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \tag{91}$$

but

$$dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma = d^n x \sqrt{|g|} \epsilon^{\mu\nu\rho\sigma} \tag{92}$$

$$S = \frac{1}{4} \int d^n x \sqrt{|g|} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu\nu cd} R^{cd}{}_{\rho\sigma}$$
$$= \frac{1}{2} \int d^n x \sqrt{|g|} \delta^{\rho\sigma}_{cd} R^{cd}{}_{\rho\sigma} = \int d^n x \sqrt{|g|} R. \tag{93}$$

---

[3]Flat or Lorentz indices are raised or lowered with the flat metric η_{ab}, whereas Einstein or curved indices do that with the metric $g_{\mu\nu}$.

The conjugate momenta are given by

$$e^a \to p_a \equiv \frac{\partial \mathcal{L}}{\partial de^a} = 0$$

$$\omega_{ab} \to p^{ab} \equiv \frac{\partial \mathcal{L}}{\partial d\omega_{ab}} = *(e^a \wedge e^b), \quad (94)$$

then the Legendre transform is performed though the construct [6],

$$\Lambda \equiv \sum d\phi^i \wedge p_i - \mathcal{L}$$
$$= d\omega_{ab} \wedge p^{ab} - (d\omega_{ab} + \omega_a{}^e \wedge \omega_{eb}) \wedge *(e^a \wedge e^b)$$
$$= -\omega_a{}^e \wedge \omega_{eb} \wedge p^{ab}. \quad (95)$$

This defines a first order Lagrangian, namely,

$$\mathcal{L}_{FO} \equiv \sum d\phi^i \wedge p_i - \Lambda$$
$$= d\omega^{ab} \wedge p_{ab} + \omega^{ae} \wedge \omega_e{}^b \wedge p_{ab} = R^{ab} \wedge p_{ab}, \quad (96)$$

which EoM read

$$\frac{\delta \mathcal{L}_{FO}}{\delta e^a} = 0$$

$$\frac{\delta \mathcal{L}_{FO}}{\delta \omega^{ab}} = -dp_{ab} + \omega^c{}_a \wedge p_{cb} - \omega^c{}_b \wedge p_{ac}$$

$$\frac{\delta \mathcal{L}_{FO}}{\delta p_{ab}} = R^{ab}. \quad (97)$$

In our case,

$$H(Z) \equiv \sum \pounds(Z)\phi^i \wedge p_i - i_Z L_{FO}$$
$$= \pounds(Z)\omega^{ab} \wedge p_{ab} - i_Z(p_{ab} \wedge R^{ab}), \quad (98)$$

the displacement piece of the Hamiltonian, is then given in this language by the vector valued three form,

$$H_\mu = -\omega^{ab}{}_\mu \wedge (-dp_{ab} + \omega^c{}_a \wedge p_{cb} - \omega^c{}_b \wedge p_{ac})$$
$$+ R^{ab} \wedge p_{ab\mu}. \quad (99)$$

It follows that the one forms $p_{ab\mu}$ are given in terms of the frame by

$$p_{ab\mu} = \frac{1}{2}\epsilon_{abcd}(e^c_\mu e^d - e^d_\mu e^c); \quad (100)$$

this then the form that the Hamiltonian constraint and momentum constraints take in this formalism. The boundary term, (83), is given by the two form,

$$B = i_Z \omega^{ab} \wedge p_{ab} = \frac{1}{2}\epsilon^{ab}{}_{cd}\omega_{ab\mu}Z^\mu \wedge e^c \wedge e^d. \quad (101)$$

Particularizing for Schwarzschild's metric,

$$ds^2 = f^2(r)dt^2 - \frac{dr^2}{f^2(r)} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \quad (102)$$

where $f(r) = \sqrt{1 - \frac{r_s}{r}}$, with $r_s = 2GM$, then the frame field read

$$e^0 = f(r)dt$$

$$e^1 = \frac{dr}{f(r)}$$

$$e^2 = rd\theta$$

$$e^3 = r\sin\theta d\phi \quad (103)$$

in such a way that the nontrivial connection one forms are given by

$$\omega^0_1 = f'(r)e^0$$

$$\omega^2_1 = \frac{f(r)}{r}e^2$$

$$\omega^3_1 = \frac{f(r)}{r}e^3$$

$$\omega^3_2 = \frac{\cot\theta}{r}e^3. \quad (104)$$

The integral of the boundary over the two sphere S^2_∞ at infinity,

$$t = \text{constant}$$

$$r = R \uparrow \infty \quad (105)$$

reads

$$\int_{S^2_\infty} B = r_s\pi dt(Z) = \frac{\kappa^2 M_\odot}{4}dt(Z). \quad (106)$$

B. Unimodular gravity

In [1], it has been proved that if the simplest FO Lagrangian,

$$S_{UG} = \int R_{ab} \wedge *(\hat{e}^a \wedge \hat{e}^b) \quad (107)$$

is chosen where

$$\hat{e}^a \equiv e^{-1/n}e_a, \quad (108)$$

then there is on shell a nonvanishing torsion, which however vanishes in the Weyl gauge $e = 1$. The reason is that this Lagrangian is Weyl invariant with inert spin connection, but the torsionless condition is not.

One way out is to impose a nontrivial Weyl transformation of the spin connection in such a way that the

torsionless condition is Weyl invariant. The resulting Lagrangian has the drawback in that it depends not only on $\hat{e}_a$ but also on e.

The simplest alternative would probably be to consider again the Lagrangian (107) but with the Weyl dependent connection. This is what will be implicitly done here, although we shall not be interested in the Weyl symmetry in this paper. Then,

$$S_{\mathrm{UG}} = \int R_{ab} \wedge *(e^{-2/n} e^a \wedge e^b), \qquad (109)$$

in this case, the conjugate momenta is

$$p^{ab} \equiv *(e^{-2/n} e^a \wedge e^b). \qquad (110)$$

Therefore, the Legendre transform result

$$\Lambda = d\omega^{ab} \wedge p_{ab} - (d\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b) \wedge *(e^{-2/n} e^a \wedge e^b)$$
$$= -\omega^{ac} \wedge \omega_c{}^b \wedge p_{ab}, \qquad (111)$$

so that

$$\mathcal{L}_{\mathrm{FO}} = d\omega^{ab} \wedge p_{ab} + \omega^{ac} \wedge \omega_c{}^b \wedge p_{ab}, \qquad (112)$$

which is exactly the same as before. The displacement piece of the Hamiltonian is then given in this language by the vector-valued three form,

$$H_\mu = -\omega^{ab}{}_\mu \wedge (-dp_{ab} + \omega^c{}_a \wedge p_{cb} - \omega^c{}_b \wedge p_{ac})$$
$$+ R^{ab} \wedge p_{ab\mu}. \qquad (113)$$

It follows that the one forms $p_{ab\mu}$ are given in terms of the frame by

$$p_{ab\mu} = \frac{1}{2} \epsilon_{abcd} (\hat{e}^c_\mu \hat{e}^d - \hat{e}^d_\mu \hat{e}^c). \qquad (114)$$

This then the form that the Hamiltonian constraint and momentum constraints take in this formalism. The explicit expression obviously coincide with the general-relativistic ones in the Weyl gauge e = 1, and when the vector Z is chosen in the ADM manner like

$$Z = n \frac{\partial}{\partial x^0} + N^i \frac{\partial}{\partial x^i}, \qquad (115)$$

we recover the unimodular constraint $N\sqrt{h} = 1$.

The boundary term, (83), in the Hamiltonian is now

$$B_{\mathrm{UG}} = i_Z \omega_{ab} \wedge p^{ab} = \frac{1}{2} \epsilon^{ab}{}_{cd} \hat{\omega}_{ab\mu} Z^\mu \wedge \hat{e}^c \wedge \hat{e}^d, \qquad (116)$$

particularizing to Schwarzschild's metric,

$$e = r^2 \sin\theta. \qquad (117)$$

Using the frame field, (103), we obtain the connection,

$$\hat{\omega}_{01\mu} Z^\mu = \left[-\frac{1}{2r} f(r) + f'(r) \right] f(r) dt(Z) \qquad (118)$$

$$B_{\mathrm{UG}} = \frac{1}{2} \frac{1}{r\sin^{1/2}\theta} \left[-\frac{1}{2r} f(r) + f'(r) \right]$$
$$\times f(r) dt(Z) r^2 \sin\theta d\theta \wedge d\phi, \qquad (119)$$

but

$$f'(r) = \frac{1}{2f(r)} \frac{r_s}{r^2} \qquad (120)$$

over the sphere at infinity,

$$\int_{S^2_\infty} B_{\mathrm{UG}} = -\frac{1}{4} dt(Z) \int_{S^2_\infty} \sin^{1/2}\theta d\theta \wedge d\phi$$
$$= -\sqrt{2\pi} \left[\Gamma\left(\frac{3}{4}\right) \right]^2 dt(Z). \qquad (121)$$

In is instructive in this context to consider the unimodular frame [13], due to Schwarzschild[4] himself,

$$ds^2 = f_0 dt^2 - f_1 dx_1^2 - f_2 \left(\frac{dx_2^2}{1-x_2^2} + (1-x_2^2) dx_3^2 \right), \qquad (122)$$

where $x_1 = r^3/3$, $x_2 = -\cos\theta$, $x_3 = \phi$ and

$$f_0(x_1) = 1 - \frac{2GM}{(3x_1 + b)^{1/3}}$$

$$f_1(x_1) = \frac{(3x_1 + b)^{-4/3}}{1 - \frac{2GM}{(3x_1+b)^{1/3}}}$$

$$f_2(x_1) = (3x_1 + b)^{2/3}, \qquad (123)$$

where b is a constant of integration and $f_0 f_1 f_2^2 = 1$, then the frame field reads

$$e^0 = \sqrt{f_0} dt$$
$$e^1 = \sqrt{f_1} dx_1$$
$$e^2 = \frac{\sqrt{f_2}}{\sqrt{1-x_2^2}} dx_2$$
$$e^3 = \sqrt{(1-x_2^2) f_2} dx_3. \qquad (124)$$

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[4]Please beware of an annoying erratum in Eq. (8) of the preprint version of [13].

The boundary term, (83), in the Hamiltonian is now

$$B_{\text{UGSch}} = \frac{1}{2}\epsilon^{ab}{}_{cd}\hat{\omega}_{ab\mu}Z^{\mu} \wedge \hat{e}^c \wedge \hat{e}^d. \qquad (125)$$

We can derive

$$de^0 = -\frac{GM}{f_2}dt \wedge e^1, \qquad (126)$$

but $de^0 + \omega^0{}_1 \wedge e^1 = 0$, we obtain the connection one form,

$$\hat{\omega}_{01\mu}Z^{\mu} = \frac{GM}{f_2}dt(Z), \qquad (127)$$

and

$$B_{\text{UGSch}} = \frac{GM}{2}dt(Z)dx_2 \wedge dx_3, \qquad (128)$$

over the sphere at infinity,

$$\int_{S^2_{\infty}} B_{\text{UG}} = \frac{GM}{2}dt(Z)\int_{S^2_{\infty}} \sin\theta d\theta \wedge d\phi = \pi r_s dt(Z). \qquad (129)$$

This result reproduces (106).

C. Schrödinger's Lagrangian

The Einstein-Hlbert action can be written

$$S = \int R_{ab} \wedge *(e^a \wedge e^b)$$
$$= \frac{1}{2}\int \epsilon^{abcd}(d\omega_{ab} + \omega_{af} \wedge \omega^f{}_b) \wedge (e_c \wedge e_d). \qquad (130)$$

Let us write $d(e_c \wedge e_d) = de_c \wedge e_d - e_c \wedge de_d$ and using the torsionless condition, (90), obtain

$$S = \frac{1}{2}\int d^n x \epsilon^{abcd}\{d[\omega_{ab} \wedge (e_c \wedge e_d)]$$
$$+ 2\omega_{ab} \wedge \omega_{cf} \wedge e^f \wedge e_d + \omega_{af} \wedge \omega^f{}_b \wedge e_c \wedge e_d\} \qquad (131)$$

in terms of the frame components on the connection field,

$$S = \frac{1}{2}\int d^n x \epsilon^{abcd}\{2\omega_{abu}\omega_{cfv}e^u \wedge e^v \wedge e^f \wedge e_d$$
$$+ \omega_{afu}\omega^f_{bv}e^u \wedge e^v \wedge e_c \wedge e_d\}, \qquad (132)$$

where we neglect the total derivative, and

$$e^u \wedge e^v \wedge e^a \wedge e^b = d^n x e \epsilon^{uvab}. \qquad (133)$$

We get

$$S = \frac{1}{2}\int d^n x e \epsilon^{abcd}\{2\omega_{ab}{}^u\omega_c{}^{fv}\epsilon_{uvfd} + \omega_{af}{}^u\omega^{f\cdot v}{}_b\epsilon_{uvcd}\} \qquad (134)$$

but

$$\epsilon^{abcd}\epsilon_{uvfd} = \delta^{abc}_{uvf}$$
$$\epsilon^{abcd}\epsilon_{uvcd} = 2\delta^{ab}_{uv} \qquad (135)$$

and finally,

$$S = 3\int d^n x e[\omega_{ca}{}^a\omega_b{}^{cb} + \omega_{abc}\omega^{cba}]. \qquad (136)$$

This is Schrödinger's Lagrangian in terms of forms. In FO, there are no nonvanishing momenta, so that the Hamiltonian is just

$$H(Z) = -i_Z\mathcal{L}. \qquad (137)$$

Thereby, there is no boundary term.

Incidentally, the value of the Schrödinger's Lagrangian for Schwarzschild's solution is

$$\mathcal{L} = \frac{6r - 4r_s + 2r\cot^2\theta}{r^3}. \qquad (138)$$

D. Quadratic theories

In terms of the one form,

$$\Gamma^{\mu}_{\nu} \equiv \Gamma^{\mu}_{\nu\lambda}dx^{\lambda}, \qquad (139)$$

and the two form,

$$R^{\mu}{}_{\nu} \equiv \frac{1}{2}R^{\mu}{}_{\nu\rho\sigma}dx^{\rho} \wedge dx^{\sigma}, \qquad (140)$$

the preceding Lagrangian reads

$$S = \int R^{\mu}{}_{\nu} \wedge *R^{\nu}{}_{\mu} + 2R^{\mu}{}_{\nu} \wedge *(d\Gamma^{\nu}_{\mu} + \Gamma^{\nu}_{\rho} \wedge \Gamma^{\rho}_{\mu}), \qquad (141)$$

and the EoM read

$$\frac{\delta S}{\delta R^{\mu}{}_{\nu}} = R^{\nu}{}_{\mu} - (d\Gamma^{\nu}_{\mu} + \Gamma^{\nu}_{\rho} \wedge \Gamma^{\rho}_{\mu}) = 0$$

$$\frac{\delta S}{\delta \Gamma^{\mu}{}_{\nu}} = *2dR^{\nu}{}_{\mu} + R^{\nu}{}_{\sigma} \wedge *\Gamma^{\sigma}_{\mu} - R^{\lambda}{}_{\mu} \wedge *\Gamma^{\nu}_{\lambda} = 0, \qquad (142)$$

because $d* = *\delta$. The corresponding momenta read

$$p \equiv \frac{\partial L}{\partial dR^\mu{}_\nu} = 0$$

$$p^\nu{}_\mu \equiv \frac{\partial L}{\partial d\Gamma^\mu{}_\nu} = 2 * R^\nu{}_\mu; \qquad (143)$$

therefore, the Legendre transform result,

$$\Lambda = d\Gamma^\mu{}_\nu \wedge p^\nu{}_\mu - \frac{1}{4} p^\nu{}_\mu \wedge *p^\mu{}_\nu - p^\mu{}_\nu \wedge (d\Gamma^\nu{}_\mu + \Gamma^\nu{}_\rho \wedge \Gamma^\rho{}_\mu)$$

$$= -\frac{1}{4} p^\nu{}_\mu \wedge *p^\mu{}_\nu - p^\mu{}_\nu \wedge (\Gamma^\nu{}_\rho \wedge \Gamma^\rho{}_\mu) \qquad (144)$$

and the FO Lagrangian in this language reads

$$\mathcal{L}_{\mathrm{FO}} = \frac{1}{4} p^\nu{}_\mu \wedge *p^\mu{}_\nu + p^\mu{}_\nu \wedge (d\Gamma^\nu{}_\mu + \Gamma^\nu{}_\rho \wedge \Gamma^\rho{}_\mu). \qquad (145)$$

Therefore, the FO EoM read

$$\frac{\delta \mathcal{L}_{\mathrm{FO}}}{\delta \Gamma^\mu{}_\nu} = dp^\nu{}_\mu + p^\nu{}_\sigma \wedge \Gamma^\sigma{}_\mu - p^\lambda{}_\mu \wedge \Gamma^\nu{}_\lambda$$

$$\frac{\partial \mathcal{L}_{\mathrm{FO}}}{\partial p^\mu{}_\nu} = *\frac{1}{4} p^\nu{}_\mu + (d\Gamma^\nu{}_\mu + \Gamma^\nu{}_\rho \wedge \Gamma^\rho{}_\mu) \qquad (146)$$

using the spacetime decomposition, (84),

$$Z^\mu H_\mu = -i_Z \Gamma^\mu{}_\nu \wedge (dp^\nu{}_\mu + p^\nu{}_\sigma \wedge \Gamma^\sigma{}_\mu - p^\lambda{}_\mu \wedge \Gamma^\nu{}_\lambda)$$

$$+ \left(*\frac{1}{4} p^\nu{}_\mu + (d\Gamma^\nu{}_\mu + \Gamma^\nu{}_\rho \wedge \Gamma^\rho{}_\mu) \right) \wedge i_Z p^\mu{}_\nu. \qquad (147)$$

Then,

$$H_\lambda = -\Gamma^\mu{}_{\nu\lambda} \wedge (dp^\nu{}_\mu + p^\nu{}_\sigma \wedge \Gamma^\sigma{}_\mu - p^\lambda{}_\mu \wedge \Gamma^\nu{}_\lambda)$$

$$+ 2 \left(*\frac{1}{4} p^\nu{}_\mu + (d\Gamma^\nu{}_\mu + \Gamma^\nu{}_\rho \wedge \Gamma^\rho{}_\mu) \right) \wedge \epsilon^{\alpha\beta}{}_{\lambda\sigma} R^\mu{}_{\nu\alpha\beta} dx^\sigma$$

$$(148)$$

because

$$i_Z p^\mu{}_\nu = i_Z \epsilon^{\alpha\beta}{}_{\rho\sigma} R^\mu{}_{\nu\alpha\beta} dx^\rho \wedge dx^\sigma = 2\epsilon^{\alpha\beta}{}_{\rho\sigma} R^\mu{}_{\nu\alpha\beta} Z^\rho dx^\sigma. \qquad (149)$$

As is usual in diffeomorphism invariant theories, H_λ itself vanishes on shell, and all contribution to the energy comes from the boundary term, (83),

$$B = i_Z \Gamma^\mu{}_\nu \wedge p^\nu{}_\mu = 2\Gamma^\mu{}_{\nu\lambda} Z^\lambda \wedge *R^\nu{}_\mu$$

$$= \frac{1}{2e} \Gamma^\mu{}_{\nu\lambda} Z^\lambda \wedge \epsilon^\nu{}_\mu{}^{\alpha\beta} R_{\alpha\beta\rho\sigma} dx^\rho \wedge dx^\sigma. \qquad (150)$$

Now we need integrate over the sphere,

$$B = \frac{1}{e} \Gamma^\mu{}_{\nu\lambda} Z^\lambda \wedge \epsilon^\nu{}_\mu{}^{\alpha\beta} R_{\alpha\beta 23} dx^2 \wedge dx^3, \qquad (151)$$

particularizing to Schwarzschild metric $R_{2323} = [1 - f^2(r)] \sin^2\theta$. Then,

$$B = \frac{2}{e} \Gamma^\mu{}_{\nu\lambda} Z^\lambda \wedge \epsilon^\nu{}_\mu{}^{23} [1 - f^2(r)] \sin^2\theta d\theta \wedge d\phi, \qquad (152)$$

again for Schwarzschild,

$$B = \frac{2}{e} [\Gamma^0{}_{10} - \Gamma^1{}_{00}] dt(Z) [1 - f^2(r)] r^2 \sin^3\theta d\theta \wedge d\phi, \qquad (153)$$

but $\Gamma^0{}_{10} = \frac{f'(r)}{f(r)}$ and $\Gamma^1{}_{00} = f^4(r)\Gamma^0{}_{10}$

$$B = \frac{2f'(r)}{f(r)} [1 - f^4(r)] dt(Z) [1 - f^2(r)] \sin^2\theta d\theta \wedge d\phi, \qquad (154)$$

with $f(r) = \sqrt{1 - \frac{r_s}{r}}$ over the sphere at infinity, with the usual assignment,

$$dt(Z) = 1, \qquad (155)$$

it yields vanishing Hamiltonian energy. As is well-known, there are other formulations [14] that assign a finite energy to those configurations.

V. CONCLUSIONS

We have computed the Hamiltonian corresponding to different first order versions of unimodular gravity.

It must be stressed that the *naive* approach, in which the Lagrangian is taken as given exactly by the same expression as in the second order approach, with the proviso that the role of the independent variables is changed, namely the connection field and the metric field are now to be treated as independent, we stress, is not always equivalent to the more usual second order one.

Theories linear in curvature have been studied, both the one that corresponds to the standard Einstein-Hilbert Lagrangian as well as the one related to Schrödinger's version quadratic in the connection field. This last version is particularly interesting insofar as it can be viewed as giving a rationale for (Einstein's) energy-momentum pseudotensor. While it is in fact true that both versions differ by a total derivative, it is not less true that one of the most interesting aspects of the Hamiltonian in general covariant framework is precisely the boundary term, also a total derivative, and precisely this boundary term usually depends on those total derivatives.

Theories quadratic in curvature have also been considered. In this case, the proliferation of indices quickly becomes overwhelming. At any rate, the usual formalism

is unsatisfactory here insofar as it yields vanishing energy in this case.

It has been found often convenient in this paper to use the language of frame fields and differential forms. This is the more true when dealing with theories with Lagrangians quadratic in curvature, although the formalism saves much space even in simpler contexts.

We are working in a frame formulation of the ideas in [14] in order to give a satisfactory definition of energy in the quadratic case. We hope to be able to report on it in due time.

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