Gravitational multipole renormalization

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We study the effect of scattering gravitational radiation off the static background curvature, up to second order in Newton constant, known in the literature as tail and tail-of-tail processes, for generic electric and magnetic multipoles. Starting from the multipole expansion of composite compact objects, and as expected due to the known electric quadrupole case, both long- and short-distance (UV) divergences are encountered. The former disappear from properly defined observables, the latter are renormalized, and their associated logarithms give rise to a classical renormalization group flow. UV divergences alert for incompleteness of the multipolar description of the composite source and are expected not to be present in a UV-complete theory, as explicitly derived in the literature for the case of conservative dynamics. Logarithmic terms from tail-of-tail processes associated to generic magnetic multipoles are computed in this work for the first time.

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I. INTRODUCTION

The recent detections of gravitational waves emitted by compact binary coalescences [1–3], observed by the LIGO [4] and Virgo [5] large interferometric detectors, made the compelling case for improving the knowledge of binary system dynamics, as its features are imprinted in the details of the detected waveforms.

The starting point of this work is the multipolar action, describing the coupling of a compact source to an external gravitational field in general relativity. When the multipoles describe a composite source with internal velocity v and size r, like in the case of compact binary coalescence, the multipolar expansion parameter is v, and in this case the gravitational radiation emitted by a time-varying multipole has angular frequency $\omega \sim v/r$.

Building on the multipole expansion, we study a specific class of post-Minkowskian (PM) corrections up to second order in the Newton constant G_N. At $O(G_N)$ beyond leading-order emission, one encounters leading nonlinear *hereditary* effects, i.e., terms depending on the history of the source rather than on an instantaneous state at retarded time. Historically, these have been divided into *memory* and *tail* effects [6], the former arising from scattering of radiation onto radiation [7], the latter from scattering of radiation onto the static background curvature sourced by the total mass E of the system [8]. The denominations are related to the nature of the phenomenological effects they have on the waveform: The tail part of the waveform arrives later than the "wave front," being delayed by the scattering, and then smoothly fades off with time; the memory part is a persistent zero-frequency effect which is still present well after the wave front has passed.

While hereditary in the waveform, radiation-radiation scattering leads to a vanishing effect in the emitted flux [9] and to an instantaneous (i.e., nonhereditary) contribution to the conservative energy [10]; tail effects, on the other hand, give a hereditary contribution to the waveform [8] and to the conservative energy [11] (later confirmed in Ref. [12]) while giving an instantaneous contribution to the flux emission from circular orbits [13]. The scattering of radiation off the angular-momentum-dependent static background curvature leads to instantaneous terms both in the waveform [14] and in the conservative energy shift [10] and no contribution to the flux.

In particular, only the (mass) tail-corrected emission process involves a large-distance, or infrared (IR), divergence,

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as thoroughly explained in Ref. [13], which, however, disappears from suitably defined observables. In the waveform, the IR tail divergences are relatively imaginary with respect to the leading order, and they exponentiate to a pure phase, so disappearing from the flux. While, in principle, still showing up in the waveform, analogous to the well-known infinite phase shift induced by the Coulomb potential in scattering amplitudes [15], one has to consider that actual detections do not measure the instantaneous absolute value of the phase but phase differences between different times, and the infinity cancels out of any observable quantity [16]. Note, however, that finite contributions of the tail effect for different multipoles are different, and their nonzero difference *is* physical, while the IR divergent part is common to all multipoles [17] and cancels out in the difference.

Note that observability of the finite shift in the waveform phase generated by the tail effect has already been investigated long ago in Refs. [18,19], and, unfortunately, the possibility of it being measured is scarce, as such an effect appears as $G_N E\omega \sim v^3$ correction to the leading-order phase which goes as v^{-5}, hence a fourth-order post-Newtonian (PN) effect [19], where $v^2 \sim G_N E/r$ is the expansion parameter of the PN approximation. Current knowledge of PN-expanded waveforms stops at 3.5PN order; see Ref. [9] for a review and Ref. [20] for the most recent tests on real data. Note that finite contributions of the tail affect the waveform phase at the same order as a shift Δt in the arrival time of the signal, which enters the phase with a term $\sim 2\pi f \Delta t \sim v^3(\Delta t/G_N E)$.

The main focus of the present work is the analysis of (mass) tail-of-tail effects at waveform level or, equivalently, in the language of field theory, in one-point amplitudes. They come with both IR and UV divergences; the former are consistent with the exponentiation to a phase of the simple tail IR divergences, and the latter have associated logarithmic terms that give rise to renormalization group equations, which can be integrated to compute all-orders leading logarithmic corrections, as already done for the logarithms from the electric quadrupole case [21].

In particular, we generalize the computation of logarithmic terms in tail-of-tail processes, already known in the electric case from the results obtained in Ref. [22] for the mass quadrupole and in Ref. [8] for all the electric multipoles, to magnetic multipoles at all orders. While sharing the same topology, diagrams of increasing multipole order become more intricate because of the presence of an increasing number of momenta. In PN scaling, moving from a multipole to the following one adds a power of v to the coupling; hence, tail diagrams involving the electric (magnetic) 2^n-multipole affect one-point amplitudes starting at $1/2 + n/2$ $(1 + n/2)$ PN order. Multipoles corrected by gravitational self-interactions are also called in the literature *radiative* multipoles [9], to differentiate from *source* multipoles, which instead designate the source terms in the fundamental multipolar expansions.

Note, however, that, when multipoles of composite objects like binary systems are expressed in terms of individual binary constituents, they can naturally be expanded in v^2, i.e., in a PN series, whose terms are determined by a *matching* procedure, which for the mass quadrupole has been completed in an effective field theory (EFT) framework up to second PN order [23] and to fourth PN order in the multipolar-post-Minkowskian approach [24].

By analogy with the conservative dynamics case treated in detail in Ref. [25], we expect that the UV divergence in the tail-of-tail process will be canceled by analogous divergences in the expression of the PN-corrected source multipoles, to leave a finite, consistent result. After all, the multipole expansion is bound to fail at a short enough distance, i.e., when the actual internal structure of the composite system becomes important.

The paper is structured as follows: In Sec. II, we give an overview of the method, treating in detail the known case of tail process, building on which we obtain new results for the tail-of-tail process in Sec. III. Section IV concludes the present work with a discussion of the results.

II. METHOD

A. Generalities

We will proceed from and expand along the lines of Ref. [13], which applies to the radiative gravitational sector the EFT approach developed in Ref. [26], known as nonrelativistic general relativity.

At a large distance from the source, its interaction with gravity can be encoded in terms of multipoles as in the following effective Lagrangian, whose form is uniquely dictated by the symmetries and scaling of the theory[1]:

$$\begin{aligned}\mathcal{S}_{\rm mult} &= \int {\rm d}t\left(\frac{1}{2}Eh_{00} - \frac{1}{2}\epsilon^{ijk}L_i h_{0j,k} - \frac{1}{2}I^{ij}\mathcal{E}_{ij} - \frac{1}{6}I^{ijk}\mathcal{E}_{ij,k} + \frac{2}{3}J^{ij}\mathcal{B}_{ij} + \cdots\right)\\ &= \int {\rm d}t\left[\frac{1}{2}Eh_{00} - \frac{1}{2}\epsilon^{ijk}L_i h_{0j,k} - \sum_{r\geq 0}\left(c_r^{(I)} I^{iji_1\ldots i_r}\partial_{i_1}\ldots\partial_{i_r}\mathcal{E}_{ij} - c_r^{(J)} J^{iji_1\ldots i_r}\partial_{i_1}\ldots\partial_{i_r}\mathcal{B}_{ij}\right)\right],\end{aligned} \tag{1}$$

[1] We use the mostly plus metric signature and the speed of light $c = 1$ throughout the paper. Latin indices run over $\{1, 2, 3\}$ and are raised and lowered by Kronecker deltas.

with [27]

$$c_r^{(I)} = \frac{1}{(r+2)!}, \qquad c_r^{(J)} = \frac{2(r+2)}{(r+3)!}, \tag{2}$$

where E and L_i are, respectively, energy and angular momentum, $I^{iji_1\ldots i_r}$ ($J^{iji_1\ldots i_r}$) are generic electric (magnetic) source 2^n-poles for $n \geq 2$, $n = r+2$ (i.e., from quadrupole on), and $\mathcal{E}_{ij}$ and $\mathcal{B}_{ij}$ denote, respectively, the electric and magnetic part of the Riemann tensor.[2]

In case one is interested in applications to compact binary systems, the source multipoles appearing in Eq. (1) can be explicitly related to individual constituents' parameters by means of a matching procedure, as done up to 2PN for the mass quadrupole I^{ij} within the EFT approach in Ref. [23] and to higher orders within the multipolar Minkowskian formalism; see [17,24,28–30], and references therein.

In the present work, we are mainly interested in the universal properties (i.e., not depending on the short-scale features of the source) of the gravitational waveform, so our focus will not be on the matching procedure but rather on the study of emission amplitudes, as expressed in terms of the generic multipoles $I^{iji_1\ldots i_r}$ and $J^{iji_1\ldots i_r}$, with particular emphasis on the divergences appearing in dimensional regularization and on the associated logarithmic terms. We are also not studying here conservative effects associated to emission and reabsorption of radiative modes, for which we refer to Refs. [10,31].

We work in the harmonic gauge, as in Refs. [9,32], which is equivalent to using the following form for the pure (bulk) gravity action:

$$S_{\rm bulk} = 2\Lambda^2 \int {\rm d}^{d+1}x\sqrt{-g}\left[R(g) - \frac{1}{2}\Gamma_\mu\Gamma^\mu\right], \tag{3}$$

where $R(g)$ is the Ricci scalar, $\Gamma^\mu \equiv g^{\rho\sigma}\Gamma^\mu_{\rho\sigma}$, $\Gamma^\mu_{\rho\sigma}$ being the standard Christoffel coefficients, and $\Lambda^{-2} \equiv 32\pi G_N \mu^{3-d}$. Note that for the number of purely spatial dimensions $d \neq 3$ an inverse length μ appears, as it is necessary to relate Λ, which has dimensions $({\rm mass}/{\rm length}^{d-2})^{1/2}$, to the ordinary $3+1$-dimensional Newton constant G_N.

We find it useful to decompose the metric via a Kaluza-Klein parameterization [33]:

$$g_{\mu\nu} = e^{2\phi/\Lambda}\begin{pmatrix} -1 & A_j/\Lambda \\ A_i/\Lambda & e^{-c_d\phi/\Lambda}\gamma_{ij} - A_iA_j/\Lambda^2 \end{pmatrix}, \tag{4}$$

[2]Terms proportional to the center of mass position and velocity in the multipole expansions have been neglected. We denote by an overdot the time derivative and by ϵ_{ijk} the three-dimensional Levi-Civita tensor. If the d-dimensional Levi-Civita tensor is used instead, one has $\epsilon_{ijk}\epsilon_{ilm} = (d-2)(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})$ and the extra $d-2$ factor must be compensated by an inverse rescaling of the magnetic multipoles $J^{iji_1\ldots i_r}$.

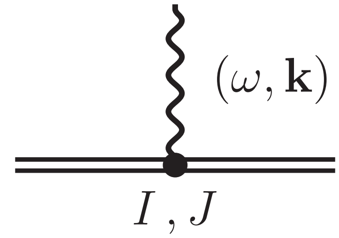

FIG. 1. Feynman diagram representing the leading-order emission amplitude.

with $\gamma_{ij} \equiv \delta_{ij} + \sigma_{ij}/\Lambda$, and $c_d \equiv 2\frac{(d-1)}{(d-2)}$. In this decomposition, one can write at linear order

$$\Lambda\mathcal{E}_{ij} \simeq -\frac{1}{2}(\ddot{\sigma}_{ij} - \dot{A}_{i,j} - \dot{A}_{j,i}) + \phi_{,ij} + \frac{\delta_{ij}}{d-2}\ddot{\phi} + O(h^2),$$
$$\Lambda\mathcal{B}_{ij} \simeq \frac{1}{4}\epsilon_{ikl}[\dot{\sigma}_{jk,l} - \dot{\sigma}_{jl,k} + A_{l,jk} - A_{k,jl} + \frac{2}{d-2}(\dot{\phi}_{,k}\delta_{jl} - \dot{\phi}_{,l}\delta_{jl})] + O(h^2), \tag{5}$$

where h denotes the generic metric perturbation around Minkowski spacetime.

The radiative, transverse-traceless part of the metric perturbation corresponds to the transverse-traceless part of σ_{ij} (also denoted σ_{ij} for simplicity), and the leading-order amplitude for emission of gravitational mode with on-shell 4-momentum $(\omega, \mathbf{k})$, with $\omega^2 = \mathbf{k}^2$, by a generic electric (I) or magnetic (J) multipole can be written as[3]

$$i\mathcal{A}_0(\omega,\mathbf{k}) = \sum_r \frac{(-i)^{r+1}}{2\Lambda}\sigma^*_{ij}(\omega,\mathbf{k})k_{i_1}\ldots k_{i_r} \times [c_r{}^{(I)}\omega^2 I^{iji_1\ldots i_r}(\omega) + c_r^{(J)}\omega\epsilon_{ikl}k_l J^{jki_1\ldots i_r}(\omega)], \tag{6}$$

with its corresponding Feynman diagrams in Fig. 1.

By applying standard tools for Feynman diagram computations, one can derive $O(G_N)$ and $O(G_N^2)$ corrections to the emission amplitude in Eq. (6), which will be shown in the next sections. The explicit expression for propagators and interaction vertices can be read from Ref. [34] and will not be reported here, with the only modification that for emission processes retarded Green's functions have to be used, which can be represented as

$$G_R(\omega,\mathbf{k}) = \lim_{{\rm a}\to 0^+}\frac{1}{(\omega + i{\rm a})^2 - \mathbf{k}^2}, \tag{7}$$

[3]Our choice for the metric signature implies that uppercase spatial indices are equivalent to lowercase ones. Taking advantage of this fact, we will allow a little abuse of notation in indices position to make equations more appeasing to the eye.

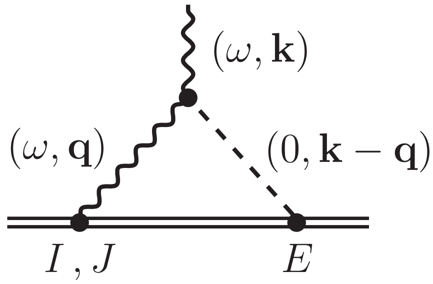

FIG. 2. Feynman diagram representing the tail emission amplitude.

and in all propagators in the rest of this paper we will denote by a an arbitrary small, positive quantity. The gravitational field can be obtained (at leading order) in Fourier space by multiplying the (leading-order) amplitude (6) by the retarded Green's function (7), as it is causally determined by the source. Boundary conditions are specified by the pole displacement in the inverse space representation of the Green's function; hence, their effect shows up only for the region of momenta having $|\mathbf{k}| = \omega$.[4]

B. Tails

The computation of the tail amplitude involving the energy and the electric quadrupole was first derived in Ref. [8], and it has been rederived in Ref. [13] with effective field theory methods; here, we report the results involving generic electric and magnetic multipoles, as represented in Fig. 2 as a warm-up for subsequent calculations. Note that the gravitational mode attached to the conserved energy E has a vanishing time component.

Adopting the notation $\int_{\mathbf{q}} \equiv \int \frac{d^d q}{(2\pi)^d}$, in the electric case one has ($\omega^2 = \mathbf{k}^2$)

$$\begin{aligned} i\mathcal{A}^{(e)}_{r-tail}(\omega,\mathbf{k}) &= (-i)^{r+1}\left(\frac{Ec_r^{(I)}}{4\Lambda^3}\right) I^{iji_1\dots i_r}(\omega)\int_{\mathbf{q}}\frac{1}{[\mathbf{q}^2-(\omega+i\mathrm{a})^2]}\frac{1}{(\mathbf{k}-\mathbf{q})^2}\times q_{i_1}\dots q_{i_r} \\ &\quad\times\left[\omega^4\delta_{ai}\delta_{bj}+2\omega^2 q_i(k-q)_a\delta_{bj}+\frac{2}{c_d}q_iq_j(k-q)_a(k-q)_b\right]\sigma^*_{ab}(\omega,\mathbf{k}) \\ &\simeq i\mathcal{A}^{(e)}_{r0}(\omega,\mathbf{k})(iG_N E\omega)\left[-\frac{(\omega+i\mathrm{a})^2}{\tilde{\mu}^2}\right]^{\epsilon_{\mathrm{IR}}/2}\left[\frac{2}{\epsilon_{\mathrm{IR}}}-2\kappa_{r+2}+O(\epsilon_{\mathrm{IR}})\right], \end{aligned} \tag{8}$$

where $\mathcal{A}^{(e)}_{r0}$ is the electric part of the 2^{2+r}-multipole in Eq. (6), $\epsilon \equiv d-3$, $\tilde{\mu}^2 \equiv \pi\mu^2 e^{-\gamma}$, with γ the Euler constant,

$$\kappa_{r+2} \equiv \frac{2r^2+13r+22}{(r+2)(r+3)(r+4)} + H_r, \tag{9}$$

and H_r is the rth *harmonic number* defined by $H_r \equiv \sum_{i=1}^r 1/i$. The second line in Eq. (8) is determined by the bulk interactions of the tail diagram, which depends on $\sigma^2\phi$, $\sigma A\phi$, and $\sigma\phi^2$ interactions contained in the Einstein-Hilbert action. Expanding also the factor $[-(\omega+i\mathrm{a})^2/\tilde{\mu}^2]^{\epsilon/2}$ in Eq. (8) for $\epsilon \to 0$, recalling the cut in the negative real semiaxis of the ω complex plane, one finally gets[5]

$$i\mathcal{A}^{(e)}_{r-tail}(\omega,\mathbf{k}) \simeq i\mathcal{A}^{(e)}_{r0}(\omega,\mathbf{k})(iG_N E\omega)\left[\frac{2}{\epsilon_{\mathrm{IR}}}-2\kappa_{r+2}-i\pi\,\mathrm{sgn}(\omega)+\log\left(\frac{\omega^2}{\tilde{\mu}^2}\right)\right]. \tag{10}$$

An analogous calculation for the magnetic multipole gives

$$\begin{aligned} i\mathcal{A}^{(m)}_{r-tail}(\omega,\mathbf{k}) &= (-i)^{r+1}\left(\frac{Ec_r^{(J)}}{4\Lambda^3}\right)\omega\epsilon_{ikl}J^{jki_1\dots i_r}(\omega)\int_{\mathbf{q}}\frac{1}{[\mathbf{q}^2-(\omega+i\mathrm{a})^2]}\frac{1}{(\mathbf{k}-\mathbf{q})^2}\times q_{i_1}\dots q_{i_r} \\ &\quad\times q_l[\omega^2\delta_{aj}+q_j(k-q)_a]\sigma^*_{ai}(\omega,\mathbf{k}) \\ &\simeq i\mathcal{A}^{(m)}_{r0}(\omega,\mathbf{k})(iG_N E\omega)\left[\frac{2}{\epsilon_{\mathrm{IR}}}-2\pi_{r+2}-i\pi\,\mathrm{sgn}(\omega)+\log\left(\frac{\omega^2}{\tilde{\mu}^2}\right)\right], \end{aligned} \tag{11}$$

[4]Note that in Ref. [13] Feynman Green's functions have been adopted instead. As pointed out in Ref. [12], such a prescription does not generally allow one to obtain the correct imaginary part of the amplitude (see also footnote 5).

[5]Note the presence of the $\mathrm{sgn}(\omega)$ term in Eq. (10), which is necessary to ensure that the tail corrections satisfy the reality property $\mathcal{A}^*(\omega) = \mathcal{A}(-\omega)$, to ensure a real waveform in direct space. Had one used Feynman Green's function, one would have had $(\omega^2 + i\mathrm{a})$ replacing $(\omega + i\mathrm{a})^2$ in Eq. (8), then obtaining $-i\pi$ instead of $-i\pi\,\mathrm{sgn}(\omega)$ in Eq. (10).

with

$$\pi_{r+2} \equiv \frac{r+1}{(r+2)(r+3)} + H_{r+1}. \tag{12}$$

The integrals have been computed using the formulas reported in the Appendix, and the divergences encountered here are of the IR type, hence the index "IR" to ϵ in Eqs. (8), (10), and (11). They are the leading order of an unobservable divergent phase term common to all multipoles; the finite terms proportional to κ_{r+2}, π_{r+2} (first computed in Ref. [35]) are also exponentiated to a phase [16], which is, however, multipole dependent and so, in principle, observable. Note that the contribution of the $-i\pi \mathrm{sgn}(\omega)$ term in the square brackets is real relative to $\mathcal{A}_0$; hence, it is the only contribution from the tail process to the emission flux at $G_N E\omega \sim v^3$ order.

The amplitudes (8) and (11) are proportional to waveforms; hence, they can be inverse-Fourier transformed to give the waveforms in the time domain, with the result that the logarithmic terms in ω are responsible for nonlocal terms in direct space (i.e., in time) first individuated in Ref. [8]. Note that the IR divergence arises from the loop integral displayed in Eq. (8), as it is clearly shown by changing the integration variable to $\mathbf{q}' \equiv \mathbf{q} - \mathbf{k}$:

$$\mathcal{A}_{\mathrm{tail}}|_{\mathrm{IR-div}}(\omega) \propto \int_{\mathbf{q}'} \frac{1}{(2\mathbf{k}\cdot\mathbf{q}' + \mathbf{q}'^2)\mathbf{q}'^2}, \tag{13}$$

and it is present only for terms whose numerator, which is set to unity for clarity in Eq. (13), is nonvanishing for $\mathbf{q}' \to 0$. An analog process can be considered by replacing the energy E insertion of the tail diagram with the angular momentum L, which, however, comes with one gradient, i.e., one power of $\mathbf{q}'$ [see Eq. (1)], thus having no divergence and producing a local result both in Fourier and in direct space, as can be explicitly checked in Ref. [29]; for this reason, it has been dubbed "failed" angular momentum tail in Ref. [10].

Another qualitatively different process, the *memory*, can be considered at $O(G_N)$ order. It can be obtained by replacing the conserved quantity source insertion of the tail diagram (E or L) with a time-dependent multipole I' or J', giving rise to an amplitude of the type

$$\mathcal{A}_{\mathrm{memory}}(\omega) \propto \int \frac{d\omega'}{2\pi} \int_{\mathbf{q}} \frac{I(\omega-\omega')I'(\omega')}{[\mathbf{q}^2 - (\omega-\omega'+i\mathrm{a})^2][(\mathbf{k}-\mathbf{q})^2 - (\omega'+i\mathrm{a})^2]}, \tag{14}$$

which is not divergent but gives rise to a product of (Fourier transformed) dynamical multipoles, which in direct space involve a convolution in time [29]. In particular, the contribution from $I(\omega-\omega')I'(\omega')$ for $\omega \to 0$ gives rise to a nonvanishing zero-frequency effect, the memory effect [7].

III. RESULTS FOR THE TAIL-OF-TAIL PROCESS

We derive in this section the divergent and logarithmic parts of the more challenging tail-of-tail contributions, at second order in $G_N E\omega$ beyond leading order (equivalent to relative 3PN for binary systems), which is where UV divergences make their first appearance.

The tail-of-tail contribution to the radiative multipole has been derived in detail in Ref. [36] and in Ref. [13] for the electric quadrupole case only (terms $E^2 \times I_{ij}$) within EFT methods, which we generalize in this section to the $E^2 \times (I, J)$ case, for electric and magnetic multipoles of any order.

The tail-of-tail process receives contributions from three different diagrams given in Fig. 3.

The diagrams in Figs. 3(a) and 3(b) can be computed using standard integration techniques, bringing pure UV divergences for any multipole, as described in Ref. [13] for the quadrupole case, as can be shown as follows. After the first loop integration over $\mathbf{p}$, which can be performed via the first equation in (A1), and after dropping the tensor structure for clarity, one is left with an integral similar to the tail one of Eq. (13):

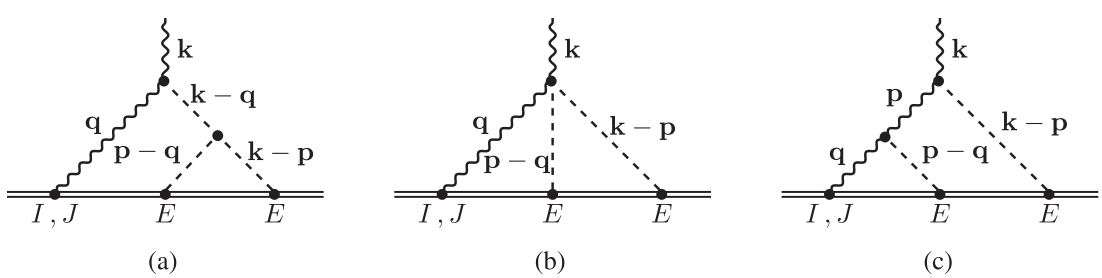

FIG. 3. Feynman diagrams describing the tail-of-tail process. We label explicitly in the figure only the space components of the momenta, the time component being ω, with $\omega^2 = \mathbf{k}^2$ for wavy lines and vanishing for dashed straight lines.

$$\mathcal{A}_{a,b-\mathrm{tail}^2}|_{\mathrm{div}} \sim \int_{\mathbf{q}} \frac{1}{[\mathbf{q}^2-(\omega+i\mathrm{a})^2][(\mathbf{k}-\mathbf{q})^2]^{m-d/2}} = \int_{\mathbf{q}'} \frac{1}{[2\mathbf{k}\cdot\mathbf{q}'+\mathbf{q}'^2][\mathbf{q}'^2]^{m-d/2}}, \tag{15}$$

which, however, has the crucial difference from Eq. (13) of having m a positive integer, giving a half-integer exponent for the $\mathbf{q}'^2$ term, hence leading to a pure UV divergence, when combined with the $\mathbf{q}'^2$ part of the $(2\mathbf{k}\cdot\mathbf{q}'+\mathbf{q}'^2)$ propagator, and no IR divergence. As noted in Ref. [13], such diagrams correspond to the scattering of the emitted radiation with the $1/r^2$ relativistic correction to the static potential.

The diagrams in Figs. 3(a) and 3(b) give for the electric and magnetic case (see the Appendix for details)

$$i\mathcal{A}^{(e,m)}_{a,b-\mathrm{tail}^2}(\omega,\mathbf{k}) \simeq i\mathcal{A}^{(e,m)}_{r0}(\omega,\mathbf{k})(G_N E\omega)^2\left[-\frac{(\omega+i\mathrm{a})^2}{\tilde{\mu}^2}\right]^{\epsilon_{\mathrm{UV}}}\left[\frac{\alpha^{(e,m)}_{a,b}(r)}{\epsilon_{\mathrm{UV}}}+O(\epsilon^0)\right], \tag{16}$$

$$\alpha^{(e)}_a(r) \equiv \frac{2r^3+3r^2-r+1}{(2r-1)(2r+1)(2r+3)(2r+5)}, \tag{17}$$

$$\alpha^{(e)}_b(r) \equiv -2\frac{(16r^3+56r^2+24r-31)}{(2r-1)(2r+1)(2r+3)(2r+5)}, \tag{18}$$

$$\alpha^{(m)}_a(r) \equiv \frac{2r^3+11r^2+21r+17}{(2r+1)(2r+3)(2r+5)(2r+7)}, \tag{19}$$

$$\alpha^{(m)}_b(r) \equiv -2\frac{(16r^3+104r^2+187r-74)}{(2r+1)(2r+3)(2r+5)(2r+7)}. \tag{20}$$

For the more intricate diagram in Fig. 3(c), which can be decomposed in terms of the same master integrals (A1), we report its amplitude before integration, split in terms of the gravitational polarization propagating in the internal wavy lines of the diagram in Fig. 3(c) ($\omega^2=\mathbf{k}^2$). For the electric case, one has

$$\begin{aligned} i\mathcal{A}^{(e)}_c(\omega,\mathbf{k}) = (-i)^{r+1}&\left(-\frac{E^2c^{(I)}_r}{4\Lambda^5}\right)\omega^2 I^{iji_1\ldots i_r}(\omega)\\ &\times\int_{\mathbf{p},\mathbf{q}}\frac{q_{i_1}\ldots q_{i_r}}{[\mathbf{q}^2-(\omega+i\mathrm{a})^2][\mathbf{p}^2-(\omega+i\mathrm{a})^2](\mathbf{p}-\mathbf{q})^2(\mathbf{p}-\mathbf{k})^2}\\ &\times\sigma^*_{ab}(\omega,\mathbf{k})\left\{-\frac{1}{2}\omega^4\delta_{ia}\delta_{jb}\qquad\{\sigma^2\}\right.\\ &\quad+\omega^2[q_aq_j-2p_aq_j+p_ap_j]\delta_{ib}\qquad\{A\sigma\}\\ &\quad-\frac{1}{c_d}q_iq_jp_ap_b\qquad\{\phi^2\}\\ &\quad+\frac{1}{c_d}q_i[q_jq_a-2q_jp_a+p_jp_a]p_b\qquad\{\phi A\}\\ &\quad+\frac{1}{c_d}[(q-p)_ip_jp_ap_b-q_iq_j(q-p)_aq_b]\qquad\{\phi\sigma\}\\ &\quad\left.+q_i[q_jp_b-q_bp_j+(\mathbf{p}\cdot\mathbf{q})\delta_{bj}]p_a\qquad\{A^2\}\right\} \end{aligned} \tag{21}$$

$$\simeq i\mathcal{A}^{(e)}_{r0}(\omega,\mathbf{k})(G_NE\omega)^2\left[-\frac{(\omega+i\mathrm{a})^2}{\tilde{\mu}^2}\right]^{\epsilon}\left[-\frac{2}{\epsilon^2_{\mathrm{IR}}}+\frac{\alpha^{(e)}_c(r)}{\epsilon}\right],$$

$$\alpha^{(e)}_c(r) \equiv 2\left[(r+1)\frac{128r^6+1728r^5+8968r^4+21490r^3+20607r^2-1228r-8628}{(r+2)(r+3)(r+4)(2r-1)(2r+1)(2r+3)(2r+5)(2r+7)}+2H_r\right], \tag{22}$$

and for the magnetic case:

$$
\begin{aligned}
i\mathcal{A}_c^{(m)}(\omega,\mathbf{k}) = (-i)^{r+1}\left(\frac{E^2 c_r^{(J)}}{8\Lambda^5}\right)\omega\epsilon_{ikl}J^{jki_1\ldots i_r}(\omega)\\
\times\int_{\mathbf{p},\mathbf{q}}\frac{q_l q_{i_1}\ldots q_{i_r}}{[\mathbf{q}^2-(\omega+i\mathrm{a})^2][\mathbf{p}^2-(\omega+i\mathrm{a})^2](\mathbf{p}-\mathbf{q})^2(\mathbf{p}-\mathbf{k})^2}\\
\times\sigma^*_{ab}(\omega,\mathbf{k})\Big\{+\omega^4\delta_{ia}\delta_{jb} \qquad \{\sigma^2\}\\
+\omega^2[(q-p)_j p_a\delta_{ib}-p_i p_a\delta_{jb}-q_j(q-p)_a\delta_{ib}] \qquad \{A\sigma\}\\
-\frac{1}{c_d}[p_i(q-p)_j+p_j(q-p)_i]p_a p_b \qquad \{\phi\sigma\}\\
-\frac{1}{c_d}q_j p_i p_a p_b \qquad \{A\phi\}\\
-q_j p_a[(\mathbf{p}\cdot\mathbf{q})\delta_{bi}-p_i q_b] \qquad \{A^2\}\Big\}\\
\simeq i\mathcal{A}_{r0}^{(m)}(\omega,\mathbf{k})(G_N E\omega)^2\left[-\frac{(\omega+i\mathrm{a})^2}{\tilde{\mu}^2}\right]^{\epsilon}\left[-\frac{2}{\epsilon_{\mathrm{IR}}^2}+\frac{\alpha_c^{(m)}(r)}{\epsilon}\right],\\
\alpha_c^{(m)}(r)\equiv 4\left[\frac{32r^6+448r^5+2396r^4+6268r^3+8433r^2+5430r+1269}{(r+2)(r+3)(2r+1)(2r+3)(2r+5)(2r+7)(2r+9)}+H_{r+1}\right].
\end{aligned}
\tag{23}
$$

While leaving the details of the computation to the Appendix, we highlight that, contrarily to the single pole that contains both UV and IR divergences, the double pole (due uniquely to the $\{\sigma^2\}$ contribution) is purely IR and universal, as expected from the exponentiation of the simple tail IR divergence. Indeed, expanding the divergent phase at order $(G_N E\omega)^2$, one obtains schematically

$$
\begin{aligned}
e^{iG_N E\omega(\frac{2}{\epsilon_{\mathrm{IR}}}-2\rho^{(e,m)})} \simeq 1 + iG_N E\omega\left(\frac{2}{\epsilon_{\mathrm{IR}}}-2\rho^{(e,m)}\right)\\
-(G_N E\omega)^2\left(\frac{2}{\epsilon_{\mathrm{IR}}^2}-\frac{4\rho^{(e,m)}}{\epsilon_{\mathrm{IR}}}+O(\epsilon_{\mathrm{IR}}^0)\right)\\
+O((G_N E\omega)^3);
\end{aligned}
\tag{24}
$$

i.e., the knowledge of the $O(\epsilon_{\mathrm{IR}}^0)$ tail term, in Eq. (24) indicated generically with $\rho^{(e,m)}$ in the term linear in $G_N E\omega$, allows one to isolate the simple pole IR divergence of the tail-of-tail process (quadratic piece in $G_N E\omega$), which, in turn, can be subtracted from Eqs. (21) and (23) to finally identify the UV one.

IV. SUMMARY AND DISCUSSION

The general structure of the emission amplitude, including post-Minkowskian multipolar corrections, is

$$
\begin{aligned}
i\mathcal{A}(\omega,\mathbf{k}) = e^{i\frac{\phi_{\mathrm{IR}}(\omega)}{\epsilon_{\mathrm{IR}}}}\sum_r\frac{(-i)^{r+1}}{2\Lambda}\sigma^*_{ij}(\omega,\mathbf{k})k_{i_1}\ldots k_{i_r}\\
\times[c_r{}^{(I)}\omega^2 I_{\mathrm{rad}}^{iji_1\ldots i_r}(\omega)+c_r^{(J)}\omega\epsilon_{ikl}k_l J_{\mathrm{rad}}^{jki_1\ldots i_r}(\omega)],
\end{aligned}
\tag{25}
$$

where $(I,J)_{rad}^{jki_1\cdots i_r}$ are the so-called *radiative multipoles* and

$$
\phi_{\mathrm{IR}}(\omega)\equiv 2G_N E\omega\left(\frac{\omega^2}{\tilde{\mu}^2}\right)^{\epsilon_{\mathrm{IR}}/2}
\tag{26}
$$

is the coefficient of the IR pole, which is, however, unobservable, because it represents a global phase shift common to every multipolar contribution of the emission amplitude. Likewise unobservable is the logarithmic term generated in $\phi_{\mathrm{IR}}/\epsilon_{\mathrm{IR}}$ at ϵ_{IR}^0 order.

Differently from IR divergences, UV ones make their first appearance at second PM order and have an important physical interpretation, as they signal the breakdown of the point particle approximation for the composite object and must be regularized. Applying standard regularization and renormalization procedures, one can obtain physical results from our UV-divergent amplitude. Note that, while such procedures have been first developed and are routinely used in *quantum* field theory, they can be also applied here to our completely classical setting, as they depend on the *field theory* nature of the problem.

The divergence can be absorbed in the definition of the (divergent) *bare* source multipoles $(I,J)_B^{iji_1\ldots i_r}$, related to the renormalized, finite source multipoles $(I,J)_R^{iji_1\ldots i_r}$ by a divergent factor:

$$
I_B^{iji_1\ldots i_r}(\omega)=\left[1-\frac{\beta^{(e)}(r)}{2\epsilon_{\mathrm{UV}}}(G_N E\omega)^2\right]I_R^{iji_1\ldots i_r}(\omega,\mu)
\tag{27}
$$

and analogously for the magnetic multipoles. From the calculation of the previous section, we found

$$\beta^{(e)}(r) \equiv 2(\alpha_a^{(e)} + \alpha_b^{(e)} + \alpha_c^{(e)} - 4\kappa_{r+2}) = -2\frac{15r^4 + 150r^3 + 568r^2 + 965r + 642}{(r+2)(r+3)(2r+3)(2r+5)(2r+7)}, \tag{28}$$

$$\begin{aligned}\beta^{(m)}(r) &\equiv 2(\alpha_a^{(m)} + \alpha_b^{(m)} + \alpha_c^{(m)} - 4\pi_{r+2}) \\ &= -2\frac{60r^6 + 900r^5 + 5535r^4 + 17306r^3 + 28228r^2 + 22101r + 5778}{(r+2)(r+3)(2r+1)(2r+3)(2r+5)(2r+7)(2r+9)},\end{aligned} \tag{29}$$

where the electric coefficients $\beta^{(e)}(r)$ have been first determined in Ref. [8] and we have computed in this work for the first time the expression for the magnetic ones $\beta^{(m)}(r)$.

Substituting for (I, J) in the amplitudes of the previous section the bare source multipoles $(I, J)_B$ expression (27), one finds finite expressions for the amplitudes in terms of the renormalized multipoles. Hence, up to the second post-Minkowskian order, *radiative multipoles* entering the physical amplitude (25) can be related to renormalized source multipoles via

$$\begin{aligned}I_{\rm rad}^{iji_1\ldots i_r}(\omega) &\simeq I_R^{iji_1\ldots i_r}(\omega,\mu)e^{-2iG_N E\omega\kappa_{r+2}} \\ &\times \left[1 + \pi G_N|\omega|E + \frac{\beta^{(e)}(r)}{2}(G_N E\omega)^2\left(\log\frac{\omega^2}{\tilde{\mu}^2} + \mathcal{O}(\epsilon^0)\right)\right]\end{aligned} \tag{30}$$

and analogously for the magnetic case. In this renormalization procedure, which relies on large-scale physics and does not depend on the specific UV structure of the system, the finite $\mathcal{O}(\epsilon^0)$ contribution is left undetermined and must be fixed by comparison with observations or a fine-grained description of the source.

The leading-order (real) tail correction $\pi G_N E|\omega|$ is multipole independent and is generated by the imaginary part of the $\epsilon_{\rm IR}^{-1}(-(\omega + i\text{a})^2)^{\epsilon_{\rm IR}}$ term, which is finite for $\epsilon_{\rm IR} \to 0$, as derived in Sec. II B. At the same post-Minkowskian order of the leading tail, there are further finite contributions, not displayed in Eq. (30), coming from the angular momentum (failed) tail and the memory effect, which for compact binaries are suppressed with respect to the leading order in the post-Newtonian expansion by a factor of v^2. The expression of such terms in the time domain can be found in Ref. [17] for the first multipoles ($r = 0, 1$). As to the finite phases proportional to κ_{r+2} and π_{r+2}, they are, in principle, observable as discussed in the introduction, because they are not universal.

Note that, as the physical emission amplitude is directly related to the radiative multipoles $(I, J)_{\rm rad}$ and cannot depend on the arbitrary renormalization scale μ, the renormalized multipoles must acquire at 2PM order a μ dependence to compensate the explicit dependence on μ of the expression (30), hence the argument μ added to $(I, J)_R$ already in Eq. (27).

This leads to the renormalization group equation

$$\frac{{\rm d}I_R^{iji_1\ldots i_r}(\omega,\mu)}{{\rm d}\log\mu} = \beta^{(e)}(r)(G_N E\omega)^2 I_R^{iji_1\ldots i_r}(\omega,\mu), \tag{31}$$

which is solved by [13]

$$I_R^{iji_1\ldots i_r}(\omega,\mu) = \left(\frac{\mu}{\mu_0}\right)^{\beta^{(e)}(r)(G_N E\omega)^2} I_R^{iji_1\ldots i_r}(\omega,\mu_0) \tag{32}$$

and analogously for the magnetic multipoles $J_R^{iji_1\ldots i_r}(\omega,\mu)$.

The above equations make manifest the role of $\beta^{(e,m)}(r)$ as beta functions controlling the running of the radiative multipoles. The renormalization group equation of the electric quadrupole [21] has been used to resum an infinite series of leading logarithmic terms in the gauge-invariant expression for energy and angular momentum of compact binaries. While the phenomenological impact for gravitational waveforms is expected to be modest (we remind that the lowest-order UV logarithms enter the waveform is 3PN), with the beta functions known at all multipole orders it is possible to compute the leading logarithmic terms in the energy, which are of the type $(M^2\log)^n \times (d^{n+2}I_L/dt^{n+2})^2$, at subleading PN orders. This allows the possibility of additional, highly nontrivial checks with the PN-expanded version of extreme mass ratio results, in analogy to what is done in Ref. [21] at leading PN order, where terms given in Ref. [37] for $n \le 7$ (contributing to the energy of circular orbit up to 22PN order) could be explicitly checked.

Knowledge of all the beta functions allows for an extension of this approach. In particular, before the present work, only the first magnetic coefficient $\beta^{(m)}(0)$ coefficient was known and found to be equal to the electric one; according to our finding, this equality is accidental and does not hold for other multipoles.

In the case of compact binaries, alternatively to the universal renormalization procedure, one can exploit the explicit knowledge of the system at small scales, as has been done in the 4PN study of the conservative sector [25], to cancel the UV divergence from the multipolar dynamics (also called the far zone) with an IR divergence coming from the PN-expanded dynamics of individual binary components interacting via the exchange of longitudinal gravitational modes (the near zone). In this case, the cancellation should come from the explicit determination of the *source* multipoles in terms of the binary constituents' variables at 3PN order, as preliminary confirmed by Ref. [38], and the previously undetermined $\mathcal{O}(\epsilon^0)$ term appearing in Eq. (30) is expected to be unambiguously predicted in terms of the UV details of the system.

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APPENDIX: USEFUL INTEGRALS

All integrals involved in tail diagram computations, as well as in amplitudes (a) and (b) of the tail-of-tail process, can be derived (eventually after iteration) from the following standard one-loop scalar master integrals:

$$J_{ab}(\mathbf{q}) \equiv \int_{\mathbf{p}} \frac{1}{\mathbf{p}^{2a}(\mathbf{p}-\mathbf{q})^{2b}} = (\mathbf{q}^2)^{d/2-a-b} \frac{\Gamma(a+b-d/2)\Gamma(d/2-a)\Gamma(d/2-b)}{(4\pi)^{d/2}\Gamma(a)\Gamma(b)\Gamma(d-a-b)},$$
$$I_a(\omega) \equiv \int_{\mathbf{q}} \frac{1}{[(\mathbf{k}-\mathbf{q})^2]^a[\mathbf{q}^2-(\omega+i\mathrm{a})^2]} = [-(\omega+i\mathrm{a})^2]^{d/2-a-1} \frac{\Gamma(a+1-d/2)\Gamma(d-2a-1)}{(4\pi)^{d/2}\Gamma(d-a-1)}, \tag{A1}$$

where in the I_a equation it is understood that $\mathbf{k}^2 = \omega^2$. The eventual presence of tensorial structures at the numerator is accounted by the usual scalarization procedure plus some combinatorics. For instance, borrowing notation from Ref. [39],

$$\int_{\mathbf{q}} \frac{q_{i_1}\cdots q_{i_n}}{[(\mathbf{k}-\mathbf{q})^2]^a[\mathbf{q}^2-(\omega+i\mathrm{a})^2]^b} = \sum_{m=0}^{[n/2]} S_{a,b}(n,m),$$
$$S_{a,b}(n,m) \equiv \frac{[-(\omega+i\mathrm{a})^2]^{d/2-a-b+m}}{2^m(4\pi)^{d/2}} \frac{\Gamma(a+b-d/2-m)\Gamma(a+n-2m)\Gamma(d+2m-2a-b)}{\Gamma(a)\Gamma(b)\Gamma(d+n-a-b)} \times \{[\delta]^m[k]^{n-2m}\}_{i_1\ldots i_n}, \tag{A2}$$

where $\{[\delta]^m[k]^{n-2m}\}_{i_1\ldots i_n}$ is symmetric in its n indices, it involves m Kronecker deltas and $n-2m$ occurrences of k vectors, and $[n/2]$ is the integer part of $n/2$. To write the amplitude of the diagrams in Figs. 3(a) and 3(b), we preliminarily define

$$\tilde{\delta}_{abcd} \equiv \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} - \frac{2}{d-2}\delta_{ab}\delta_{cd}, \tag{A3}$$

$$D^{(1)}_{abcd} = \delta_{ab}\delta_{cd} - \frac{1}{2}\delta_{ac}\delta_{bd}, \tag{A4}$$

$$D^{(2)}_{abcdef} = \frac{1}{4}\delta_{ab}\delta_{ce}\delta_{df} + \frac{1}{2}\delta_{cd}\delta_{ae}\delta_{bf} - \delta_{ac}\delta_{be}\delta_{df}, \tag{A5}$$

$$D^{(3)}_{abcdefmn} = -\frac{1}{4}\delta_{ab}\delta_{mn}\delta_{ce}\delta_{df} - \frac{1}{2}\delta_{mn}\delta_{cd}\delta_{ae}\delta_{bf} + \delta_{mn}\delta_{ac}\delta_{be}\delta_{df} + \frac{1}{2}\delta_{am}\delta_{bn}\delta_{ce}\delta_{df} + D^{(1)}_{arbs} \times (\delta_{rc}\delta_{se}\delta_{mf}\delta_{nd} - \delta_{rc}\delta_{se}\delta_{md}\delta_{nf} + \delta_{rm}\delta_{se}\delta_{cd}\delta_{nf} - \delta_{rc}\delta_{sm}\delta_{ef}\delta_{nd}), \tag{A6}$$

$$D^{(4)}_{abcdef} = 2\delta_{ae}\delta_{bc}\delta_{df} - \delta_{ad}\delta_{be}\delta_{cf} - 2\delta_{af}\delta_{be}\delta_{cd} + \delta_{af}\delta_{bc}\delta_{de}. \tag{A7}$$

In the diagram in Fig. 3(a), the propagator labeled by q can carry a σ or an A polarization (the others are fixed, as only ϕ couples to the conserved energy E), the two separate contributions being

$$
\begin{aligned}
i\mathcal{A}^{(e)}_{a,\sigma}(\omega,\mathbf{k}) = {}& \frac{(-i)^{r+1}E^2c_r^{(I)}}{16c_d\Lambda^5}\omega^2 I^{iji_1\ldots i_r}(\omega)\int_{\mathbf{q}}\frac{q_{i_1}\ldots q_{i_r}}{(\mathbf{k}-\mathbf{q})^2[\mathbf{q}^2-(\omega+i\mathrm{a})^2]}\int_{\mathbf{p}}\frac{1}{(\mathbf{p}-\mathbf{q})^2(\mathbf{k}-\mathbf{p})^2}\\
&\times (p-q)_\beta(p-k)_\delta D^{(1)}_{\alpha\beta\gamma\delta}\\
&\times\{D^{(2)}_{abcdef}\omega^2\tilde{\delta}_{ab\alpha\gamma}[\delta_{ic}\delta_{jd}\sigma^*_{ef}(\omega,\mathbf{k})+\delta_{ie}\delta_{jf}\sigma^*_{cd}(\omega,\mathbf{k})]\\
&+D^{(3)}_{abcdefmn}[-\delta_{ia}\delta_{jb}((q-k)_m k_n\tilde{\delta}_{cd\alpha\gamma}\sigma^*_{ef}(\omega,\mathbf{k})+k_m(q-k)_n\tilde{\delta}_{ef\alpha\gamma}\sigma^*_{cd}(\omega,\mathbf{k}))\\
&+\delta_{ic}\delta_{jd}q_m(\tilde{\delta}_{ab\alpha\gamma}k_n\sigma^*_{ef}(\omega,\mathbf{k})+\tilde{\delta}_{ef\alpha\gamma}(q-k)_n\sigma^*_{ab}(\omega,\mathbf{k}))\\
&+\delta_{ie}\delta_{jf}q_n(\tilde{\delta}_{ab\alpha\gamma}k_m\sigma^*_{cd}(\omega,\mathbf{k})+\tilde{\delta}_{cd\alpha\gamma}(q-k)_m\sigma^*_{ab}(\omega,\mathbf{k}))]\}
\end{aligned}
\tag{A8}
$$

and

$$
\begin{aligned}
i\mathcal{A}^{(e)}_{a,A}(\omega,\mathbf{k}) = {}& \frac{-(-i)^{r+1}E^2c_r^{(I)}}{32c_d\Lambda^5}\omega^2 I^{iji_1\ldots i_r}(\omega)\int_{\mathbf{q}}\frac{q_{i_1}\ldots q_{i_r}}{(\mathbf{k}-\mathbf{q})^2[\mathbf{q}^2-(\omega+i\mathrm{a})^2]}\int_{\mathbf{p}}\frac{1}{(\mathbf{p}-\mathbf{q})^2(\mathbf{k}-\mathbf{p})^2}\\
&\times D^{(1)}_{\alpha\beta\gamma\delta}D^{(4)}_{abcdei}q_j[(p-q)_\beta(p-k)_\delta+(p-q)_\delta(p-k)_\beta](q-k)_c\tilde{\delta}_{\alpha\gamma de}\sigma^*_{ab}(\omega,\mathbf{k}).
\end{aligned}
\tag{A9}
$$

Similarly, for the magnetic case

$$
\begin{aligned}
i\mathcal{A}^{(m)}_{a,\sigma}(\omega,\mathbf{k}) = {}& \frac{(-i)^{r+1}E^2c_r^{(J)}}{64c_d\Lambda^5}\omega\epsilon_{ikl} J^{jki_1\ldots i_r}(\omega)\int_{\mathbf{q}}\frac{q_{i_1}\ldots q_{i_r}}{(\mathbf{k}-\mathbf{q})^2[\mathbf{q}^2-(\omega+i\mathrm{a})^2]}\int_{\mathbf{p}}\frac{1}{(\mathbf{p}-\mathbf{q})^2(\mathbf{k}-\mathbf{p})^2}\\
&\times D^{(1)}_{\alpha\beta\gamma\delta}q_l[(p-q)_\beta(p-k)_\delta+(p-q)_\delta(p-k)_\beta]\\
&\times\{D^{(2)}_{abcdef}\omega^2\tilde{\delta}_{ab\alpha\gamma}[\tilde{\delta}_{ijcd}\sigma^*_{ef}(\omega,\mathbf{k})+\tilde{\delta}_{ijef}\sigma^*_{cd}(\omega,\mathbf{k})]\\
&+D^{(3)}_{abcdefmn}[-\tilde{\delta}_{ijab}((q-k)_m k_n\tilde{\delta}_{cd\alpha\gamma}\sigma^*_{ef}(\omega,\mathbf{k})+k_m(q-k)_n\tilde{\delta}_{ef\alpha\gamma}\sigma^*_{cd}(\omega,\mathbf{k}))\\
&+\tilde{\delta}_{ijcd}q_m(\tilde{\delta}_{ab\alpha\gamma}k_n\sigma^*_{ef}(\omega,\mathbf{k})+\tilde{\delta}_{ef\alpha\gamma}(q-k)_n\sigma^*_{ab}(\omega,\mathbf{k}))\\
&+\tilde{\delta}_{ijef}q_n(\tilde{\delta}_{ab\alpha\gamma}k_m\sigma^*_{cd}(\omega,\mathbf{k})+\tilde{\delta}_{cd\alpha\gamma}(q-k)_m\sigma^*_{ab}(\omega,\mathbf{k}))]\}
\end{aligned}
\tag{A10}
$$

and

$$
\begin{aligned}
i\mathcal{A}^{(m)}_{a,A}(\omega,\mathbf{k}) = {}& \frac{-(-i)^{r+1}E^2c_r^{(J)}}{64c_d\Lambda^5}\omega\epsilon_{ikl} J^{jki_1\ldots i_r}(\omega)\int_{\mathbf{q}}\frac{q_{i_1}\ldots q_{i_r}}{(\mathbf{k}-\mathbf{q})^2[\mathbf{q}^2-(\omega+i\mathrm{a})^2]}\int_{\mathbf{p}}\frac{1}{(\mathbf{p}-\mathbf{q})^2(\mathbf{k}-\mathbf{p})^2}\\
&\times D^{(1)}_{\alpha\beta\gamma\delta}D^{(4)}_{abcdei}q_l q_j[(p-q)_\beta(p-k)_\delta+(p-q)_\delta(p-k)_\beta](q-k)_c\tilde{\delta}_{\alpha\gamma de}\sigma^*_{ab}(\omega,\mathbf{k}).
\end{aligned}
\tag{A11}
$$

The calculation of the diagram in Fig. 3(b) is similar and gives

$$
\begin{aligned}
i\mathcal{A}^{(e)}_{b}(\omega,\mathbf{k}) = {}& (-i)^{r+1}\left(\frac{E^2c_r^{(I)}}{16\Lambda^5}\right) I^{iji_1\ldots i_r}(\omega)\int_{\mathbf{q}}\frac{q_{i_1}\ldots q_{i_r}}{[\mathbf{q}^2-(\omega+i\mathrm{a})^2]}\int_{\mathbf{p}}\frac{1}{(\mathbf{p}-\mathbf{q})^2(\mathbf{k}-\mathbf{p})^2}\\
&\times\left\{\delta_{ib}\omega^4+\delta_{ib}\frac{\omega^2}{c_d}(\mathbf{p}-\mathbf{q})\cdot(\mathbf{p}-\mathbf{k})-\frac{4}{c_d}\omega^2(p-q)_i(p-k)_b\right\}\sigma^*_{bj}(\omega,\mathbf{k}),
\end{aligned}
\tag{A12}
$$

$$
\begin{aligned}
i\mathcal{A}^{(m)}_{b}(\omega,\mathbf{k}) = {}& (-i)^{r+1}\left(\frac{E^2c_r^{(J)}}{16\Lambda^5}\right)\omega\epsilon_{ikl} J^{jki_1\ldots i_r}(\omega)\int_{\mathbf{q}}\frac{q_{i_1}\ldots q_{i_r}}{[\mathbf{q}^2-(\omega+i\mathrm{a})^2]}\int_{\mathbf{p}}\frac{1}{(\mathbf{p}-\mathbf{q})^2(\mathbf{k}-\mathbf{p})^2}\\
&\times q_l\left\{\omega^2\delta_{ai}\delta_{bj}+\frac{1}{c_d}(\mathbf{p}-\mathbf{q})\cdot(\mathbf{p}-\mathbf{k})\delta_{ai}\delta_{bj}-\frac{2}{c_d}(p-q)_a[(p-k)_i\delta_{bj}+(p-k)_j\delta_{bi}]\right\}\sigma^*_{ab}(\omega,\mathbf{k}).
\end{aligned}
\tag{A13}
$$

Tail-of-tail amplitude (c) is more complicated, as it involves the following family of two-loop integrals (always $\omega^2 = \mathbf{k}^2$):

$$I_{\rm in}[a_1,a_2,a_3,a_4,a_5] \equiv \int_{\mathbf{p},\mathbf{q}} \frac{1}{[\mathbf{q}^2-(\omega+i\mathrm{a})^2]^{a_1}[\mathbf{p}^2-(\omega+i\mathrm{a})^2]^{a_2}(\mathbf{p}-\mathbf{q})^{2a_3}(\mathbf{p}-\mathbf{k})^{2a_4}(\mathbf{q}-\mathbf{k})^{2a_5}}. \tag{A14}$$

The general expression is long and complicated; here, we focus only on the part which is singular in the $d \to 3$ limit, which is the relevant one in the renormalization procedure. Using the standard technique of integration by parts implemented by the software Reduze [40], one can express the main scalar integral as

$$\begin{aligned} I_{\rm in}[1,1,1,1,0] = {} & \frac{1}{\omega^2}\frac{3d-8}{4(d-3)}\int_{\mathbf{p},\mathbf{q}}\frac{1}{(\mathbf{p}^2-(\omega+i\mathrm{a})^2)(\mathbf{p}-\mathbf{q})^2(\mathbf{p}-\mathbf{k})^2} \\ & + \frac{1}{4\omega^4}\frac{d^2+4d-4}{(d-3)^2}\int_{\mathbf{p},\mathbf{q}}\frac{1}{(\mathbf{q}^2-(\omega+i\mathrm{a})^2)(\mathbf{p}^2-(\omega+i\mathrm{a})^2)}. \end{aligned} \tag{A15}$$

When reducing tensor integral to scalar ones, the following results are needed, for $m, n \in \mathbb{N}$:

$$\begin{aligned} I_{\rm in}[1,1,1,1,0] &\simeq -[128\pi^2(\omega+i\mathrm{a})^2\epsilon^2]^{-1} + O(\epsilon^0), \\ I_{\rm in}[1,1,1,1,-n] &\simeq -\frac{[4(\omega+i\mathrm{a})^2]^n}{n}\frac{1}{128\pi^2(\omega+i\mathrm{a})^2\epsilon} + O(\epsilon^0) \quad \text{for} \quad n \geq 1, \\ I_{\rm in}[1,-m,1,1,-n] &\simeq -\frac{(-1)^m[(\omega+i\mathrm{a})^2]^{m+n}\Gamma(m+2n+1)}{64\pi^{3/2}\epsilon\Gamma(n+1)\Gamma(m+n+\frac{3}{2})} + O(\epsilon^0), \\ I_{\rm in}[1,1,-m,1,-n] &\simeq -\frac{[4(\omega+i\mathrm{a})^2]^{m+n}}{32\pi^2(m+n+1)\epsilon} + O(\epsilon^0), \\ I_{\rm in}[1,1,1,-m,-n] &\simeq I_{\rm in}[1,1,-m,1,-n] + O(\epsilon^0), \\ I_{\rm in}[1,1,0,0,-n] &\simeq O(\epsilon^0). \end{aligned} \tag{A16}$$

From there, one can compute the only unknown parameter involved in the following equation:

$$\int_{\mathbf{p},\mathbf{q}} \frac{q_{(i_1\ldots i_r)}}{\mathcal{D}_{(tail)^2}} \equiv \int_{\mathbf{p},\mathbf{q}} \frac{q_{(i_1\ldots i_r)}}{[\mathbf{q}^2-(\omega+i\mathrm{a})^2][\mathbf{p}^2-(\omega+i\mathrm{a})^2](\mathbf{p}-\mathbf{q})^2(\mathbf{p}-\mathbf{k})^2} \simeq \frac{\mathcal{A}_r}{\omega^2}k_{(i_1\ldots i_r)}, \tag{A17}$$

$k_{(i_1\ldots i_r)}$ being (still following the notation of Ref. [39]) the symmetric traceless (STF) combination of k^i's. In detail:

$$\begin{aligned} \frac{\mathcal{A}_r}{\omega^2}k_{(i_1\ldots i_r)} \times k_{i_1}\ldots k_{i_r} = \mathcal{A}_r C_r (\omega^2)^{r-1} &= \int_{\mathbf{p},\mathbf{q}} \frac{q_{(i_1\ldots i_r)}k_{i_1}\ldots k_{i_r}}{\mathcal{D}_{(\rm tail)^2}} \\ &= \sum_{j=0}^{[\frac{r}{2}]} b_{r,j}(\omega^2)^j \int_{\mathbf{p},\mathbf{q}} \frac{(\mathbf{q}^2)^j(\mathbf{q}\cdot\mathbf{k})^{r-2j}}{\mathcal{D}_{(\rm tail)^2}} \\ &= (\omega^2)^r \sum_{j=0}^{[\frac{r}{2}]}\sum_{a_1=0}^{r-2j} b_{r,j}(-2\omega^2)^{-a_1}\binom{r-2j}{a_1} I_{\rm in}[1,1,1,1,-a_1] \\ &\simeq -\frac{(\omega^2)^{r-1}C_r}{128\pi^2}\left[\frac{1}{\epsilon^2}-\frac{2H_r}{\epsilon}\right], \end{aligned} \tag{A18}$$

with H_r the harmonic number and

$$b_{r,j} \equiv \frac{r!}{4^j j!(r-2j)!(2-r-d/2)_j}, \qquad C_r \equiv \sum_{i=0}^{[\frac{r}{2}]} b_{r,i} = \frac{\Gamma(d+r-2)}{(d-3)!!(d+2r-4)!!},$$

$(a)_b$ being the Pochhammer symbol. Moreover, one needs to compute integrals like the one above, with the addition of up to four p_i's and up to two extra q_j's (not involved in the STF combination with the other q^{i_r}'s), and this can be achieved via a tedious but straightforward scalarization procedure.

For instance, for one extra p_i, one can write

$$\int_{\mathbf{p},\mathbf{q}} \frac{q_{(i_1\ldots i_r)}p_i}{\mathcal{D}_{(\mathrm{tail})^2}} = \frac{\mathcal{A}_r^{(p)}}{\omega^2} k_{(i_1\ldots i_r)k_i} + \mathcal{B}_r^{(p)}\delta_{i(i_1}k_{i_2}\cdots_{i_r)}, \tag{A19}$$

and two independent contractions are needed to solve the linear system. One is the same as above, while another can be obtained by contracting the index i with one of the STF indices. The integrals are just slightly more complicated with respect to the one needed in Eq. (A17). Adding extra factors to the integrand does not introduce insurmountable complications.

For one extra, non-STF, q factor, one can proceed in the same way:

$$\int_{\mathbf{p},\mathbf{q}} \frac{q_{(i_1\ldots i_r)}q_j}{\mathcal{D}_{(\mathrm{tail})^2}} = \frac{\mathcal{A}_r^{(q)}}{\omega^2} k_{(i_1\ldots i_r)k_j} + \mathcal{B}_r^{(q)}\delta_{j(i_1}k_{i_2}\cdots_{i_r)} \tag{A20}$$

and solve the associated linear system. Actually, by noticing that

$$q_{(i_1\ldots i_r)}q_j = q_{(i_1\ldots i_r j)} + \frac{r}{d+2r-2}\mathbf{q}^2\delta_{j(i_1}q_{i_2\ldots i_r)}, \tag{A21}$$

one can straightforwardly derive

$$\begin{aligned}
&\int_{\mathbf{p},\mathbf{q}} \frac{q_{(i_1\ldots i_r)}q_j}{\mathcal{D}_{(\mathrm{tail})^2}} \\
&= \int_{\mathbf{p},\mathbf{q}} \frac{q_{(i_1\ldots i_r j)}}{\mathcal{D}_{(tail)^2}} + \frac{r}{d+2r-2}\int_{\mathbf{p},\mathbf{q}} \mathbf{q}^2 \frac{\delta_{j(i_1}q_{i_2\ldots i_r)}}{\mathcal{D}_{(\mathrm{tail})^2}} \\
&= \frac{\mathcal{A}_{r+1}}{\omega^2}k_{(i_1\ldots i_r j)} + \frac{r}{d+2r-2}\mathcal{A}_{r-1}\delta_{j(i_1}k_{i_2}\cdots_{i_r)} \\
&= \frac{\mathcal{A}_{r+1}}{\omega^2}k_{(i_1\ldots i_r)j} + \frac{r}{d+2r-2}[\mathcal{A}_{r-1} - \mathcal{A}_{r+1}]\delta_{j(i_1}k_{i_2}\cdots_{i_r)}.
\end{aligned} \tag{A22}$$

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