

## Modified inertia as nonconservative Newtonian dynamics

Ahmad Shariati<sup>1,\*</sup> and Nosratollah Jafari<sup>2,†</sup>

<sup>1</sup>*Department of Physics, Faculty of Physics and Chemistry, Alzahra University, Tehran 19938 97973, Iran*

<sup>2</sup>*Department of Physics, Nazarbayev University, Kabanbay Batyr Ave 53, Nur-Sultan 010000, Kazakhstan*

 (Received 27 April 2021; revised 8 June 2021; accepted 16 September 2021; published 19 October 2021)

Modified Newtonian dynamics by Milgrom is a paradigm for explaining the rotation curves of spiral galaxies and various other large scale structures. This paradigm includes several different theories. Here we present Milgrom's modified inertia (MI) theory in terms of a simple and tractable nonconservative Newtonian dynamics, which is useful in obtaining observable predictions of MI. It is found that: 1) Modified inertia theory is equivalent to a Newtonian theory, with a nonconservative gravitational field, and dark matter density; 2) The tidal force in the equivalent Newtonian dynamics is nonconservative, and its effect on a binary system in free fall in the gravitational field of a spheroid is addressed. We also discuss attempts to restore conservations in MI.

DOI: [10.1103/PhysRevD.104.084070](https://doi.org/10.1103/PhysRevD.104.084070)

### I. INTRODUCTION

To solve the plateau of rotation curves of spiral galaxies there are two paradigms, dark matter (DM) and modified Newtonian dynamics (MOND), which was proposed by Milgrom [1,2] and has been vastly investigated by Milgrom himself [3–9] and various other authors [10,11]. For a review of MOND and dark matter we refer the reader to [12–16].

MOND is not a single theory, it refers to several theories. The basic idea behind them is that one can change Newtonian dynamics on the scale of galaxies, so that the plateau of the rotation curves could be explained with no need to add dark matter halos. The first version of MOND as proposed by Milgrom [1] is the *modified inertia* (MI) in the form  $m\mu(a)\mathbf{a} = \mathbf{h}$ , where  $\mathbf{h} = -\nabla\phi$  is the Newtonian gravitational field. Later, Bekenstein and Milgrom [10] studied the so-called *modified gravity* (MG) theories, which keep  $m\mathbf{a} = \mathbf{F}$  intact, but the gravitational field is not the Newtonian one. There appeared several modified gravity theories which tried to implement Milgrom's idea. For a comparison of these theories see Zhao and Famaey [17], where the authors use a generalized virial theorem to compare different modified gravity theories.

In this article, we would like to present a system which is mathematically equivalent to Milgrom's MI. The benefit of this equivalent system is to provide a framework for finding predictions of the MI—predictions that could be used for verification or refutation of the theory. This is just the first step towards simplifying MI, just like quasilinear MOND made MOND easier [17,18], with the possible advantage of tackling the nonconservativeness challenge of making MI

mathematically easier. The original idea of modifying Newtonian dynamics is to write the governing differential equations not as the usual Newtonian form  $\mathbf{F} = m\mathbf{a}$ , but as the equation  $\mathcal{F} = m\mathbf{a}\mu(a)$ , where  $\mu$  is a function characterizing the theory (to be discussed below). It should be noted that we are deliberately using  $\mathcal{F}$  instead of  $F$ ; while  $\mathcal{F}$  denotes the force field in the MOND framework,  $F$  denotes the force field in the Newtonian framework.

The function  $\mu(a)$ , called the interpolating function, is a monotonically increasing function of the absolute value of the acceleration  $a = |\mathbf{a}|$ , such that for large enough values of  $a$  (compared to some fundamental acceleration of the theory, denoted by  $a_0$ ),  $\mu(a) \simeq 1$ , and for very small values of  $a$ ,  $\mu(a) \simeq 0$ . Milgrom showed that this modification of the Newtonian dynamics, for  $a_0 \sim 10^{-10} \text{ m s}^{-2}$ , could account for the flatness of the rotation curves of spiral galaxies, with no need of introducing any extra *dark* matter.

To find which one of these paradigms is chosen by nature, various groups proposed or did experiments [18,19]. Gundlach *et al.* [20] have shown that, for accelerations as small as  $10^{-14} \text{ m s}^{-2}$ , the Newtonian equation  $\mathbf{F} = m\mathbf{a}$  is valid. Existence of galaxies without dark matter is another reason against MOND [21]. Though there are arguments against such reasoning [22], it seems reasonable that the rotation curves support standard matter [23]. McGaugh *et al.* [24] reported a correlation between the radial acceleration traced by rotation curves and the observed distribution of baryonic matter, which might be construed as a signal against the dark matter paradigm (since this means dark matter must somehow be correlated to baryonic matter). But we should note that MOND is not the only way to interpret this correlation. In another direction, recently Chae *et al.* [25] published evidence for the violation of the strong equivalence principle in favor of MOND.

\*a.shariati@alzahra.ac.ir

†nosratollah.jafari@nu.edu.kz

To interpret observations and to predict observable phenomena, we have to have a theoretical framework. In this article, we emphasize a theoretical argument, based on a transformation from a MOND differential equation of a test particle to Newtonian equations, which leads to various conclusions and predictions in the framework of MOND. We use a transformation to write the MI equation  $\mathbf{h} = \mu(a)\mathbf{a}$  in the Newtonian equivalent form  $\mathbf{a} = \mathbf{g}$ , where  $\mathbf{g}$  is obtained from  $\mathbf{h}$  unambiguously. We then use this equivalent version to study the consequences of modifying inertia, and to study the relation between MI and MG versions. We found that modifying inertia leads to the nonconservation of energy in a binary system which is in free fall in the halo of the galaxy. This result could not be derived by the method of Zhao and Famaey [17], the results of which are valid for the modified gravity theories.

## II. TRANSFORMING TO AN EQUIVALENT NEWTONIAN THEORY

The basic idea behind this article is that  $\mathbf{F} = m\mathbf{a}$  is a framework to write the dynamics [26]. Newtonian dynamics (ND) is based on the differential equation

$$\text{ND} \quad \mathbf{F} = m\mathbf{a}, \quad (1)$$

where  $\mathbf{a} = d^2\mathbf{r}/dt^2$  is the acceleration. The most important feature of this is that the differential equations governing a point particle are of second order, such that when put them in the form  $\mathbf{F} = m\mathbf{a}$ , the *Newtonian force*  $\mathbf{F}$  depends on the position and the velocity of the particle, and not on the acceleration  $\mathbf{a}$  itself. In classical electrodynamics, the reaction force of a radiating charged particle violates this assumption, but that is beyond our present considerations. In a gravitational field  $\mathbf{g}$ , the motion of a test particle is given by the equation  $\mathbf{a} = \mathbf{g}$ , where  $\mathbf{g}$  is determined by the field equations  $\nabla \times \mathbf{g} = 0$  and  $\nabla \cdot \mathbf{g} = -4\pi G\rho$ , with  $\rho$  being the mass density of the source.

The MI version of MOND states that the differential equations for a test particle of mass  $m$  are

$$\text{MOND} \quad \mathcal{F} = m\mathbf{a}\mu(a), \quad (2)$$

where the *interpolating* function  $\mu(a)$  is a dimensionless, smooth, positive, and monotonically increasing function of the absolute value of the acceleration  $a = |\mathbf{a}|$ , depending on a fundamental small acceleration  $a_0 \sim 10^{-10} \text{ m s}^{-2}$ , and having the following properties:

$$\lim_{a \rightarrow 0^+} \mu(a) = 0, \quad (3)$$

$$\lim_{a \rightarrow \infty} \mu(a) = 1, \quad (4)$$

$$\mu'(a) > 0 \quad \forall a > 0. \quad (5)$$

As Milgrom [1] stated explicitly: ‘‘The force field  $\mathcal{F}$  is assumed to depend on its sources and to couple to the body, in the conventional way.’’ Consider the MI equation for a gravitational field  $\mathbf{h}$ , where  $\mu(a)$  is the interpolating function which defines the MI, and where  $\mathbf{h}$  solves the usual Newtonian field equations. Thus, the MI version of MOND is given by the following system:

$$a\mu(a) = \mathbf{h}, \quad (6)$$

$$\nabla \cdot \mathbf{h} = -4\pi G\rho_m, \quad (7)$$

$$\nabla \times \mathbf{h} = 0. \quad (8)$$

Here,  $\rho_m$  is the *mass density* function in the MOND framework.

It is easy to see that  $\mathcal{F} = m\mathbf{a}\mu(a)$  can be transformed to  $\mathbf{F} = m\mathbf{a}$ . To do this, we first introduce the *pseudo-acceleration* field

$$\mathbf{h} = m^{-1}\mathcal{F}, \quad (9)$$

and write the MOND equations as

$$\mathbf{h} = a\mu(a). \quad (10)$$

Since  $\mathbf{h}$  and  $\mathbf{a}$  are parallel, we have  $h = a\mu(a)$ . Using the inverted interpolating function  $\nu(h)$  (see Appendix A), we write this as  $a = h\nu(h)$ . Multiplying it with  $\hat{\mathbf{a}} = \hat{\mathbf{h}}$  (the unit vector), we get

$$\mathbf{a} = h\nu(h). \quad (11)$$

We can also multiply by  $m$  to get

$$m\mathbf{a} = \mathcal{F}\nu\left(\frac{\mathcal{F}}{m}\right) = \mathbf{F}. \quad (12)$$

This is the usual Newton’s equation of motion, for *the acceleration field*

$$\mathbf{g} = h\nu(h). \quad (13)$$

Thus, the modified inertia equation  $m\mathbf{a}\mu(a) = \mathcal{F}$  could be written as  $m\mathbf{a} = m\mathbf{g}$ , where  $\mathbf{h} = m^{-1}\mathcal{F}$  is what we call the *pseudo-acceleration field*, and  $\mathbf{g} := h\nu(h)$  is *the acceleration field*.

Since we have good experience with the Newtonian equation  $\mathbf{a} = \mathbf{g}$ , and because it is completely equivalent to the modified inertia equation  $a\mu(a) = \mathbf{h}$ , we could now derive some useful information about MI theories.

Defining  $\mathbf{g}$  by (13), the MI version of the MOND equation of motion of a test particle is equivalent to the Newtonian one:

$$\mathbf{a} = \mathbf{g}. \quad (14)$$

Using (7), (8), and (13), it is easy to find field equations governing  $\mathbf{g}$ :

$$\nabla \cdot \mathbf{g} = -4\pi G \rho_m \nu(h) + \mathbf{h} \cdot \nabla \nu(h), \quad (15)$$

$$\nabla \times \mathbf{g} = -\mathbf{h} \times \nabla \nu(h). \quad (16)$$

So the dynamics of a test particle in the MI version of MOND is equivalent to the system of equations (14)–(16) where  $\mathbf{h}$  is the solution to (7) and (8).

### III. PHYSICAL IMPLICATIONS

We are now going to obtain information about the MI version of MOND using systems (7), (8), and (14)–(16).

#### A. Dark matter in disguise

We see that in the ND equivalent version, the mass density is being modified [multiplied by  $\nu(h)$ ], and we have an extra term in the right-hand side of  $\nabla \cdot \mathbf{g}$ , which could be interpreted as a *dark* mass density.

$$\rho_d = -\frac{1}{4\pi G} \mathbf{h} \cdot \nabla \nu(h) = -\frac{1}{8\pi G} \cdot \frac{\nu'(h)}{h} \mathbf{h} \cdot \nabla h^2. \quad (17)$$

For the simple form of the function  $\mu(a)$  [from (A9)], we get

$$\rho_d = \frac{1}{8\pi G} \cdot \frac{a_0}{h^2 \sqrt{h^2 + 4a_0 h}} \mathbf{h} \cdot \nabla h^2. \quad (18)$$

By assuming that the accelerations due to visible and dark matter are always parallel, Dunkel [27] showed that the MOND equations can be derived from classical Newtonian dynamics, provided one also takes into account the gravitational influence of a DM component. Sivaram [28] took a similar approach. The approach of the present article, however, is more general and shows that dark matter is an inevitable consequence of modifying inertia. This will be more clear in the following section.

#### B. Dynamics around a point particle

As an example, let us consider the acceleration around a point mass  $M$ . Here  $\mathbf{h} = -(GM/r^2)\hat{\mathbf{r}}$ , for which  $h = GM/r^2$ . Using the simple form of the MOND function  $\mu$ , we get

$$\nabla \cdot \mathbf{g} = -4\pi GM \delta(\mathbf{r}) - \frac{2a_0}{r} \left(1 + \frac{4a_0 r^2}{GM}\right)^{-1/2}, \quad (19)$$

$$\nabla \times \mathbf{g} = 0. \quad (20)$$

This clearly shows that accepting the modified inertia equation  $\mathbf{a}\mu(a) = \mathbf{h}$  for a point particle is equivalent to accepting an infinite dark matter, with density

$$\rho_d = \frac{a_0}{2\pi Gr} \left(1 + \frac{4a_0 r^2}{GM}\right)^{-1/2}. \quad (21)$$

Defining

$$b := \sqrt{\frac{GM}{4a_0}}, \quad (22)$$

this becomes

$$\rho_d = \frac{\rho_0}{s\sqrt{1+s^2}}, \quad (23)$$

where  $s = r/b$  and

$$\rho_0 = \frac{1}{\pi} \left(\frac{a_0}{GM}\right)^{3/2} M = \frac{1}{8\pi} \cdot \frac{M}{b^3}. \quad (24)$$

One should note the scalings:

$$b \propto M^{1/2}, \quad (25)$$

$$\rho_0 \propto M^{-1/2}, \quad (26)$$

$$\rho_0 b^3 \propto M. \quad (27)$$

For  $a_0 = 1.0 \times 10^{-10} \text{ m s}^{-2}$  and the solar mass  $M = M_\odot = 2.0 \times 10^{30} \text{ kg}$ , we get

$$b = 5.8 \times 10^{14} \text{ m} = 1.9 \times 10^{-2} \text{ pc}, \quad (28)$$

$$\rho_0 = 4.1 \times 10^{-16} \text{ kg m}^{-3} = 6.0 \times 10^3 M_\odot \text{ pc}^{-3}. \quad (29)$$

We see that for a point mass, the corresponding dark matter density  $\rho_d$  [given by (21) or (23)] behaves as  $r^{-2}$  for large  $r$ , so that  $\lim_{r \rightarrow \infty} r^2 \rho_d(r) = \rho_0$  and the mass content  $\int \rho_d d^3 r$  diverges linearly. Unlike conventional dark matter theories which could circumvent this infinity by stating that  $\lim_{r \rightarrow \infty} r^2 \rho_d(r) = 0$ , in MI the behavior of  $\rho_d$  for large  $r$  is dictated by  $\nu(h)$ , which is uniquely determined by the function  $\mu(a)$ . As far as the asymptotic behavior of  $\nu(h)$  for small values of  $h$  is  $\nu(h) \propto h^{-1/2}$ , we get  $\rho_d \propto r^{-2}$  for large  $r$ . Both the simple and standard forms of  $\mu(a)$  lead to this asymptotic behavior (A8). In fact, if  $\mu(a) \propto a$  for  $a \ll a_0$ , as was proposed explicitly by Milgrom [1], then it is easy to see that  $\nu(h) \propto h^{-1/2}$  for  $h \ll a_0$ . Therefore, there is no escape from this linear divergence of mass content in MI.

### C. Nonconservation of momentum

As was pointed out by Felten [29], momentum of a two body system is not conserved in MI. Using the equivalent system  $\mathbf{a} = \mathbf{g}$ , where  $\mathbf{g} = \mathbf{h}\nu(h)$ , we can easily see why this is so. Consider two bodies, with masses  $m_1$  and  $m_2$ , a distance  $r$  apart. In MI one assumes that at the position of  $m_2$  we have  $\mathbf{h}_2 = -Gm_1/r^2\hat{\mathbf{n}}$ , and at the position of  $m_1$  we have  $\mathbf{h}_1 = +Gm_2/r^2\hat{\mathbf{n}}$ , where  $\hat{\mathbf{n}}$  is the the unit vector joining 1 to 2. The forces therefore are

$$\mathbf{F}_{1\rightarrow 2} = -\frac{Gm_1m_2}{r^2}\nu_1\hat{\mathbf{n}}, \quad (30)$$

$$\mathbf{F}_{2\rightarrow 1} = +\frac{Gm_2m_1}{r^2}\nu_2\hat{\mathbf{n}}, \quad (31)$$

where  $\nu_1 = \nu(Gm_1/(r^2a_0))$ ,  $\nu_2 = \nu(Gm_2/(r^2a_0))$ . Since  $\nu_1 \neq \nu_2$ , Newton's third law is violated, and momentum is not conserved.

### D. Nonconservation of energy

If  $\rho_m(\mathbf{r})$  has spherical symmetry,  $\mathbf{h}$  is parallel to  $\hat{\mathbf{r}}$  and  $\mathbf{h} \times \nabla\nu(h)$  vanishes. But, in general, we do not have spherical symmetry. Consider for example the gravitational field of a spheroid. Relative to the center, the first two terms of the multipole expansion of the potential consists of a monopole of mass  $M$  and a quadrupole of moment  $\epsilon Md^2$ , where  $d$  is the length scale of the quadrupole moment, and  $\epsilon = \pm 1$  for prolate or oblate spheroids. In spherical-polar coordinates  $(r, \theta, \phi)$  we can write the Newtonian potential for large  $r$  (see Appendix B), from which we find

$$\nabla \times \mathbf{g} = -\frac{3}{4}\epsilon\sqrt{GMa_0} \cdot \frac{d^2}{r^4} \sin(2\theta)\hat{\mathbf{e}}_\phi + \mathcal{O}\left(\frac{1}{r^6}\right), \quad (32)$$

valid for  $r \gg 2b = \sqrt{GM/a_0}$  [see (B6)]. A nonvanishing curl of the  $\mathbf{g}$  field would imply some effects, because from  $\mathbf{a} = \mathbf{g}$ , one could easily get the work-kinetic energy theorem

$$\Delta\left(\frac{1}{2}mv^2\right) = m \int \mathbf{g} \cdot d\mathbf{r}. \quad (33)$$

If  $\nabla \times \mathbf{g} \neq 0$ , then the line integral depends on path, and in particular it does not vanish for a closed path. Let us define  $\delta\mathbf{g}$  by

$$\mathbf{g} = \mathbf{h} + \delta\mathbf{g}. \quad (34)$$

Since  $\nabla \times \mathbf{h} = 0$ , we have

$$\oint \nabla \times \mathbf{g} \cdot d\mathbf{r} = \oint \nabla \times \delta\mathbf{g} \cdot d\mathbf{r}. \quad (35)$$

Therefore, the work-kinetic energy theorem could be written as

$$\Delta\left(\frac{1}{2}mv^2 + V\right) = m \int \delta\mathbf{g} \cdot d\mathbf{r}, \quad (36)$$

where  $V$  is the usual Newtonian potential energy defined by  $m\mathbf{h} = -\nabla V$ . If  $\delta\mathbf{g}$  is small, we can consider the right-hand side as a perturbation and deduce some observable results.

### E. Effect on a binary system

Before getting to the subject, it should be noted that the investigation by Zhao *et al.* [18] on modified Kepler's law and two-body problem in MOND is in the context of a conservative theory given by a Lagrangian; however, the treatment we present here is in the context of MI.

In MI theory, consider two objects with masses  $m_1$  and  $m_2$  forming a binary and in free fall in the gravitational field  $\mathbf{h}$  of a galaxy. Transforming to the equivalent system [Eqs. (14)–(16)], and assuming that for the internal dynamics of the binary we can use ordinary Newtonian gravitation, we get the following differential equations:

$$m_1\ddot{\mathbf{r}}_1 = +Gm_1m_2\frac{\mathbf{r}}{r^3} + m_1\mathbf{g}(\mathbf{r}_1), \quad (37)$$

$$m_2\ddot{\mathbf{r}}_2 = -Gm_1m_2\frac{\mathbf{r}}{r^3} + m_2\mathbf{g}(\mathbf{r}_2), \quad (38)$$

where  $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ . It follows from these two equations that

$$\ddot{\mathbf{r}} = -G(m_1 + m_2)\frac{\mathbf{r}}{r^3} + \mathbf{a}_T, \quad (39)$$

where  $\mathbf{a}_T$  is the galactic tidal field

$$\mathbf{a}_T = \mathbf{r} \cdot \nabla \mathbf{g}(\mathbf{R}), \quad (40)$$

with  $\mathbf{R}$  being the center of mass of the binary.

In the Newtonian theory,  $\nabla \times \mathbf{g} = 0$ , and it follows that  $\nabla \times \mathbf{a}_T$  also vanishes, because

$$\nabla \times \mathbf{a}_T|_r = \nabla \times \mathbf{g}|_R. \quad (41)$$

In MI however, because of (16), the galactic tidal field is not conservative.

If  $\hat{\mathbf{n}}$  is the unit vector normal to the orbital plane, along the angular momentum of the binary, then the work done on the binary by the tidal field of the galaxy, in one revolution, would be

$$\Delta W = \mu \oint \mathbf{a}_T \cdot d\mathbf{r} = \mu \iint \nabla \times \mathbf{a}_T \cdot \hat{\mathbf{n}} da \quad (42)$$

$$= \mu \iint \nabla \times \mathbf{g} \cdot \hat{\mathbf{n}} da, \quad (43)$$

where  $\mu$  is the reduced mass of the binary. Consider a binary with elliptical orbit—semimajor axis  $s$  and

eccentricity  $\epsilon$ —and center of mass at  $(r_0, \theta_0, \phi_0)$ ; and let us approximate the gravitational field of the galaxy by a mass + quadrupole  $(M, \epsilon M d^2)$  at the origin. Let the angular momentum of the binary (which defines its orbital plane) make an angle  $\beta$  with  $\hat{e}_\phi(\phi_0)$ . Using (32), and noting that for  $s \ll r_0$  both  $\nabla \times \mathbf{g}$  and  $\hat{e}_\phi$  are almost constant over the orbit of the binary, we get

$$\Delta W = -\frac{3\epsilon}{4} \frac{m_1 m_2}{m_1 + m_2} \sqrt{GM a_0} \frac{d^2}{r_0^4} \left[ \pi s^2 \sqrt{1 - \epsilon^2} \right] \times \sin(2\theta_0) \cos \beta. \quad (44)$$

By Kepler's law, the frequency of the orbit is

$$f = \sqrt{\frac{G(m_1 + m_2)}{4\pi s^3}}. \quad (45)$$

Therefore, due to the tidal force being nonconservative in MI, the binary loses or gains energy with power

$$P_{\text{MI}} = -\frac{3\sqrt{\pi}\epsilon}{8} \sqrt{1 - \epsilon^2} (Gm_1 m_2) \sqrt{\frac{M}{m_1 + m_2}} \frac{d^2 s^{1/2}}{r_0^4} \times \sqrt{a_0} \sin(2\theta_0) \cos \beta. \quad (46)$$

This power has some peculiar properties:

- (a) It is proportional to  $\cos \beta$ , so that for  $\beta > 90^\circ$  the sign changes. In other words, the binary either loses energy or gains energy according to its sense of rotation around  $\hat{e}_\phi$ .
- (b) It is proportional to  $\sin(2\theta_0)$ , which means that there is a sign change at  $\theta_0 = 90^\circ$ .

$P_{\text{MI}}$  is proportional to  $s^{1/2}$ , which makes sense—it is a tidal effect. In comparison, the power of the gravitational wave radiation of the binary, as found by Peters and Mathews [30] (see also [31]), is proportional to  $s^{-5}$ . Thus:

$$P_{\text{GW}} = -\frac{32}{5} \frac{G^4 m_1^2 m_2^2 (m_1 + m_2)}{c^5 s^5} \frac{1 + \frac{73}{24}\epsilon^2 + \frac{37}{96}\epsilon^4}{(1 - \epsilon^2)^{7/2}}. \quad (47)$$

As an example, to see that this effect could be, in principle, observable, consider a binary with  $m_1 \simeq m_2$  and  $\epsilon \simeq 0$ . For this binary we have

$$P_{\text{GW}} \simeq -\left(\frac{m}{M_\odot}\right)^5 \left(\frac{s}{\text{au}}\right)^{-5} (4 \times 10^{13}) \text{ W}. \quad (48)$$

Now, suppose this binary is at a distance  $r_0 = 15$  kpc from the center of the Milky Way. Assuming  $M = 5 \times 10^{10} M_\odot$ ,  $d = 10$  kpc, and  $\epsilon = -1$  [see (B14)–(B16)], we have  $b \simeq 4$  kpc, and the condition  $r_0 \gg b$  is almost fulfilled, so we get

$$P_{\text{MI}} \simeq [\sin(2\theta_0) \cos \beta] \left(\frac{m}{M_\odot}\right)^{\frac{3}{2}} \left(\frac{s}{1 \text{ au}}\right)^{\frac{1}{2}} \left(\frac{a_0}{10^{-10} \text{ m s}^{-2}}\right)^{\frac{1}{2}} \times (2 \times 10^{14}) \text{ W}. \quad (49)$$

The nonconservation of energy in a binary system we just presented is a consequence of  $\nabla \times \mathbf{a}_T \neq 0$ , which is a consequence of  $\nabla \times \mathbf{g} \neq 0$ . For an  $N$ -body system like a globular cluster, this means violation of the virial theorem, which is valid in modified gravity theories (see [17]).

#### IV. MODIFIED INERTIA VS MODIFIED GRAVITY

The Poisson equation of the Newtonian gravity,  $\nabla^2 \phi = 4\pi G \rho$ , is obtained from the Lagrangian density

$$\mathcal{L}_N = -\rho \phi - \frac{1}{8\pi G} |\nabla \phi|^2. \quad (50)$$

Bekenstein and Milgrom [10] introduced the Lagrangian density

$$\mathcal{L}_{\text{BM}} = -\rho \psi - \frac{1}{8\pi G} \mathcal{F} \left( \frac{|\nabla \psi|^2}{a_0^2} \right), \quad (51)$$

where the functions  $\mathcal{F}$  and  $\mu$  are related by

$$\mu(h) = \mathcal{F}'(h^2/a_0^2), \quad \mathcal{F}'(x) = \frac{d\mathcal{F}}{dx}, \quad (52)$$

and  $\psi$  now satisfies the following equation:

$$\nabla \cdot [\mu(|\nabla \psi|/a_0) \nabla \psi] = 4\pi G \rho. \quad (53)$$

For  $|\nabla \psi| \gg a_0$ , we have  $\mu \simeq 1$ , and we get the usual Poisson equation for the Newtonian gravitational field; but for  $|\nabla \psi| \lesssim a_0$ , we have deviations from Newtonian gravity, consistent with rotation curves of spiral galaxies. This theory is called the modified gravity version of MOND.

Let us fix our terminology and notation. We have three models:

MI The modified inertia theory of Milgrom [1], summarized by Eqs. (6)–(8).

BM The modified gravity of Bekenstein and Milgrom [10], given by  $\mathcal{L}_{\text{BM}}$  [Eq. (51)].

NN The nonconservative Newtonian dynamics given by Eqs. (7), (8), (14)–(16).

In Sec. II, we showed that MI is equivalent to NN. But BM could not be equivalent to NN, because if we write the usual Lagrangian for the motion of a test particle in the BM model as

$$L = \frac{1}{2} m v^2 - m \psi, \quad (54)$$

we get the equation of motion

$$\mathbf{a} = -\nabla\psi. \quad (55)$$

But the equation of motion of the NN model is  $\mathbf{a} = \mathbf{g}$ , and we know that  $\mathbf{g} \neq -\nabla\psi$ , simply because  $\nabla \times \nabla\psi = 0$ , but  $\nabla \times \mathbf{g} \neq 0$ .

As far as  $\psi$  is independent of time, from (54) it follows that in BM the energy of a test particle is conserved. On the other hand, MI theory is equivalent to NN with  $\nabla \times \mathbf{g} \neq 0$  (16), which means the potential is not well defined and the energy is not conserved.

## V. SUMMARY AND CONCLUSION

The modified inertia of Milgrom states that if  $\rho_m$  is the mass distribution of a galaxy, the acceleration  $\mathbf{a}$  of a test particle (a star) is the solution of  $a\mu(a) = h$ , where  $\mu(a)$  is a function characterizing the theory (see Appendix A) and  $h$  is the gravitational field satisfying  $\nabla \cdot \mathbf{h} = -4\pi G\rho_m$ , and where  $\nabla \times \mathbf{h} = 0$ . Introducing the function  $\nu(h)$  (see Appendix A) we have  $a = h\nu(h)$ . Introducing  $\mathbf{g} = h\nu(h)$ , the dynamics of the particle are given by the Newtonian dynamics equation  $\mathbf{a} = \mathbf{g}$ , where  $\mathbf{g}$  satisfies

$$\nabla \cdot \mathbf{g} = -4\pi G\rho_m\nu(h) + \mathbf{h} \cdot \nabla\nu(h), \quad (56)$$

$$\nabla \times \mathbf{g} = -\mathbf{h} \times \nabla\nu(h). \quad (57)$$

This means that the modified inertia version of MOND is mathematically equivalent to an acceleration field with three features:

- (1) The mass density is modified,  $\rho_m \rightarrow \rho_m\nu(h)$ ;
- (2) There is a dark matter with density  $\rho_d = -\nu'(h)\mathbf{h} \cdot \nabla h / (4\pi G)$ ;
- (3) The acceleration field is nonconservative. This leads to nonconservativeness of the galactic tidal force which has some observable effects on the binaries and perhaps globular clusters of stars.

Besides, it was argued that the modified gravity theory of Bekenstein and Milgrom [10] is not exactly equivalent to the modified inertia theory of Milgrom [1]. It should be noted that the nonconservative Newtonian system given in this article is different from the quasilinear MOND of Milgrom [7], which is conservative.

Modifications introduced by Milgrom started a fruitful investigation by various researchers which still continues, and this is invaluable. What we are saying in this article is that some models could be interpreted in the framework (or paradigm) of Newtonian dynamics, which could lead to new insights.

### A. Challenges to form a conservative MI theory

From the first days of introducing MOND by Milgrom, there have been efforts to form a conservative MI theory.

One way to do this is to replace the standard action  $\frac{1}{2} \int v^2 dt$  by a more complicated action of the form  $A_m S[\mathbf{r}(t), a_0]$ , where  $A_m$  depends on the body related to the particle's mass, and  $S$  is a functional of the trajectory of the particle, but depending on the particle's entire history [3,4]. Milgrom demonstrated that, in the context of such theories, the simple MOND relation (2) is exact for circular orbits in an axisymmetric potential (although not for general orbits) [16]. Such theories are usually highly nonlocal [12,16]. Recently, Alzain [32] has tried to construct a relativistic theory implementing Milgrom's MI.

## ACKNOWLEDGMENTS

We would like to thank A.-H. Fatollahi for his valuable comments, which motivated the early version of this article. This work was partially supported by Alzahra University's research council, and partly by Nazarbayev University.

## APPENDIX A: THE INTERPOLATING FUNCTIONS

Let us review the transformation  $\mu(a) \rightarrow \nu(h)$  to invert the interpolating function [14,17,33,34].

The equation  $h = a\mu(a)$ , where  $\mu(a)$  satisfies (3)–(5), could be solved for  $a$ . The proof is a simple application of the inverse function theorem. Denoting the solution of  $h = a\mu(a)$  by  $a = f(h)$ , let us define  $\nu(h) = h^{-1}f(h)$ , so that we have

$$a = h\nu(h). \quad (A1)$$

Dividing  $a\mu(a) = h$  by  $a = h\nu(h)$ , one gets  $\mu(a)\nu(h) = 1$ , from which it follows that  $\nu(h)$  is a dimensionless, monotonically decreasing function, asymptotic to 1 for large  $h$ .

$$\lim_{h \rightarrow 0^+} \nu(h) = +\infty, \quad (A2)$$

$$\lim_{h \rightarrow \infty} \nu(h) = 1, \quad (A3)$$

$$\nu'(h) < 0 \quad \forall h > 0. \quad (A4)$$

The *simple* form of the interpolating function  $\mu(a)$  is (Fig. 1)

$$\mu(a) = \frac{a}{a + a_0}. \quad (A5)$$

Solving  $a\mu(a) = \frac{a^2}{a+a_0} = h$  for  $a$  one gets

$$\nu(h) := \frac{1}{2} \left( 1 + \sqrt{1 + \frac{4a_0}{h}} \right). \quad (A6)$$

For later use, we note that the asymptotic form of  $\nu(h)$  is

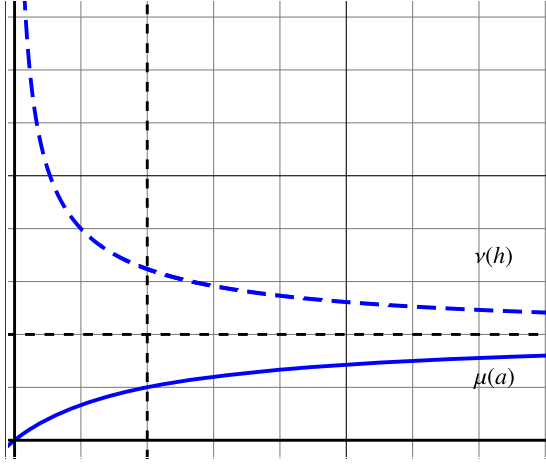


FIG. 1. Simple form of the interpolating functions  $\mu(a)$  (A5), solid; and  $\nu(h)$  (A6), dashed. The scale  $a_0$  is indicated by the vertical dashed line. The horizontal dashed line indicates 1 on the ordinate.

$$\nu(h) \simeq 1 + \frac{a_0}{h} \quad h \gg a_0, \quad (\text{A7})$$

$$\nu(h) \simeq \sqrt{\frac{a_0}{h}} \quad h \ll a_0, \quad (\text{A8})$$

$$\nu'(h) = -\frac{a_0}{h\sqrt{h^2 + 4a_0h}} \simeq -\frac{a_0^{1/2}}{2h^{3/2}} \quad h \ll a_0. \quad (\text{A9})$$

The *standard* form of the interpolating functions  $\mu(a)$  and  $\nu(h)$  is the pair

$$\begin{aligned} \mu(a) &= \frac{a}{\sqrt{a^2 + a_0^2}}, \\ \nu(h) &= \frac{1}{\sqrt{2}} \left( 1 + \sqrt{1 + \frac{4a_0^2}{h^2}} \right)^{1/2}. \end{aligned} \quad (\text{A10})$$

It should be noted that both simple and standard forms of  $\mu(a)$  lead to the same asymptotic behavior for  $\nu(h)$  for small  $h$ , and therefore the same asymptotic form for  $\nu'(h)$  for small  $h$ .

## APPENDIX B: A MASS + QUADRUPOLE SYSTEM

In spherical-polar coordinates  $(r, \theta, \varphi)$ , consider the potential

$$\psi(r, \theta) = -\frac{GM}{r} - \epsilon \frac{GMd^2}{r^3} \left( \frac{3}{2} \cos^2\theta - \frac{1}{2} \right), \quad (\text{B1})$$

where  $\epsilon = \pm 1$ . This potential describes a mass + quadrupole system. It is straightforward to find  $\mathbf{h} = -\nabla\psi$ , and then, using  $\mathbf{g} = \mathbf{h}\nu(h)$ , to find  $\mathbf{g}$ .

$$\mathbf{h} = h_r \hat{\mathbf{e}}_r + h_\theta \hat{\mathbf{e}}_\theta, \quad (\text{B2})$$

$$-\frac{h_r}{GM} = \frac{1}{r^2} + \frac{\epsilon d^2}{r^4} \cdot \frac{3}{2} (3\cos^2\theta - 1), \quad (\text{B3})$$

$$-\frac{h_\theta}{GM} = \frac{\epsilon d^2}{r^4} \frac{3}{2} \sin(2\theta), \quad (\text{B4})$$

$$\frac{h^2}{(GM)^2} = \frac{1}{r^4} + \epsilon \frac{d^2}{r^6} (3\cos^2\theta - 1) + \mathcal{O}\left(\frac{1}{r^8}\right). \quad (\text{B5})$$

Using either the simple or the standard form of the interpolating function, we get

$$\frac{\nu'(h)}{h} \simeq \frac{1}{2} \sqrt{\frac{a_0}{(GM)^5}} r^5, \quad r \gg \sqrt{\frac{GM}{a_0}} = 2b. \quad (\text{B6})$$

Now, using

$$\frac{h_r}{r} = -\frac{GM}{r^3} + \mathcal{O}\left(\frac{1}{r^5}\right), \quad (\text{B7})$$

$$\frac{\partial h^2}{\partial \theta} = \epsilon \frac{G^2 M^2 d^2}{r^6} 3 \sin(2\theta) + \mathcal{O}\left(\frac{1}{r^8}\right), \quad (\text{B8})$$

$$\frac{\partial h^2}{\partial r} = -\frac{G^2 M^2}{r^5} + \mathcal{O}\left(\frac{1}{r^7}\right), \quad (\text{B9})$$

we can find the leading term of  $\nabla \times \mathbf{g}$ .

$$\nabla \times \mathbf{g} = -\mathbf{h} \times \nabla \nu(h) \quad (\text{B10})$$

$$= -\frac{\nu'(h)}{2h} \mathbf{h} \times \nabla h^2 \quad (\text{B11})$$

$$= -\frac{\nu'(h)}{2h} \left( \frac{h_r}{r} \frac{\partial h^2}{\partial \theta} - h_\theta \frac{\partial h^2}{\partial r} \right) \hat{\mathbf{e}}_\varphi \quad (\text{B12})$$

$$= -\frac{3}{4} \epsilon \sqrt{GMa_0} \cdot \frac{d^2}{r^4} \sin(2\theta) \hat{\mathbf{e}}_\varphi + \mathcal{O}\left(\frac{1}{r^6}\right). \quad (\text{B13})$$

Using the values given by Sofue [35], we consider the following mass distribution for the Milky Way (excluding the halo):

- (i) A central black hole of mass  $M_{\text{bh}} = 3.6 \times 10^6 M_\odot$ .
- (ii) A spherical bulge of mass  $M_{\text{b}} = 9.2 \times 10^9 M_\odot$ .
- (iii) A disk of mass  $M_{\text{D}} = 4.0 \times 10^{10} M_\odot$  with exponential density of length scale  $a_{\text{D}} = 5.0$  kpc.

From these figures we get

$$M = 4.9 \times 10^{10} M_\odot, \quad (\text{B14})$$

$$\epsilon M d^2 = -5M_{\text{D}} a_{\text{D}}^2 = -5.0 \times 10^{12} M_\odot \text{ kpc}^2, \quad (\text{B15})$$

$$d = 10 \text{ kpc}. \quad (\text{B16})$$

- [1] M. Milgrom, *Astrophys. J.* **270**, 365 (1983).  
[2] M. Milgrom, *Astrophys. J.* **270**, 371 (1983).  
[3] M. Milgrom, *Ann. Phys. (N.Y.)* **229**, 384 (1994).  
[4] M. Milgrom, *Phys. Lett. A* **253**, 273 (1999).  
[5] M. Milgrom, *New Astron. Rev.* **46**, 741 (2002).  
[6] M. Milgrom, *Phys. Rev. D* **80**, 123536 (2009).  
[7] M. Milgrom, *Mon. Not. R. Astron. Soc.* **403**, 886 (2010).  
[8] M. Milgrom, *Phys. Rev. Lett.* **109**, 251103 (2012).  
[9] M. Milgrom, *Scholarpedia* **9**, 31410 (2014), revision #196523.  
[10] J. Bekenstein and M. Milgrom, *Astrophys. J.* **286**, 7 (1984).  
[11] J. D. Bekenstein, *Phys. Rev. D* **70**, 083509 (2004).  
[12] M. Milgrom, *Acta Phys. Pol. B* **32**, 3613 (2001).  
[13] Y. Sofue and V. Rubin, *Annu. Rev. Astron. Astrophys.* **39**, 137 (2001).  
[14] B. Famaey and S. McGaugh, *Living Rev. Relativity* **15**, 10 (2012).  
[15] M. Milgrom, *Stud. Hist. Philos. Sci. Part B* **71**, 170 (2020).  
[16] R. H. Sanders and S. S. McGaugh, *Annu. Rev. Astron. Astrophys.* **40**, 263 (2002).  
[17] H. Zhao and B. Famaey, *Phys. Rev. D* **81**, 087304 (2010).  
[18] H. Zhao, B. Li, and O. Bienaymé, *Phys. Rev. D* **82**, 103001 (2010).  
[19] A. Yu. Ignatiev, *Can. J. Phys.* **93**, 166 (2015).  
[20] J. H. Gundlach, S. Schlamminger, C. D. Spitzer, K.-Y. Choi, B. A. Woodahl, J. J. Coy, and E. Fischbach, *Phys. Rev. Lett.* **98**, 150801 (2007).  
[21] P. van Dokkum, S. Danieli, Y. Cohen, A. Merritt, A. J. Romanowsky, R. Abraham, J. Brodie, C. Conroy, D. Lokhorst, L. Mowla, E. O'Sullivan, and J. Zhang, *Nature (London)* **555**, 629 (2018).  
[22] P. Kroupa, H. Haghi, B. Javanmardi, A. Hasani Zonoozi, O. Müller, I. Banik, X. Wu, H. Zhao, and J. Dabringhausen, *Nature (London)* **561**, E4 (2018).  
[23] P. Kroupa, I. Banik, H. Haghi, A. Hasani Zonoozi, J. Dabringhausen, B. Javanmardi, O. Müller, X. Wu, and H. Zhao, *Nat. Astron.* **2**, 925 (2018).  
[24] S. S. McGaugh, F. Lelli, and J. M. Schombert, *Phys. Rev. Lett.* **117**, 201101 (2016).  
[25] K.-H. Chae, F. Lelli, H. Desmond, S. S. McGaugh, P. Li, and J. M. Schombert, *Astrophys. J.* **904**, 51 (2020).  
[26] F. Wilczek, *Phys. Today* **57**, No. 10, 11 (2004).  
[27] J. Dunkel, *Astrophys. J.* **604**, L37 (2004).  
[28] C. Sivaram, *Int. J. Mod. Phys. D* **26**, 1743010 (2017).  
[29] J. E. Felten, *Astrophys. J.* **286**, 3 (1984).  
[30] P. C. Peters and J. Mathews, *Phys. Rev.* **131**, 435 (1963).  
[31] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields*, Course of Theoretical Physics Vol. 2 (Pergamon Press, New York, 1975).  
[32] M. Alzain, *J. Astrophys. Astron.* **38**, 59 (2017).  
[33] A. Shariati and N. Jafari, [arXiv:0710.1411](https://arxiv.org/abs/0710.1411).  
[34] S. S. McGaugh, *Astrophys. J.* **683**, 137 (2008).  
[35] Y. Sofue, *Publ. Astron. Soc. Jpn.* **69**, R1 (2017).