Spherically symmetric black holes and affine-null metric formulation of Einstein's equations

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The definition of well-behaved coordinate charts for black hole spacetimes can be tricky, as they can lead for example to either unphysical coordinate singularities in the metric (e.g., $r = 2M$ in the Schwarzschild black hole) or to an implicit dependence of the chosen coordinates to physical relevant coordinates (e.g., the dependence of the null coordinates in the Kruskal metric). Here we discuss two approaches for coordinate choices in spherically symmetric spacetimes allowing us to explicitly discuss "solitary" and spherically symmetric black holes from a regular horizon to null infinity. The first approach relies on a construction of a regular null coordinate system (where regular is meant as being defined from the horizon to null infinity) given an explicit solution of the Einstein-matter equations. The second approach is based on an affine-null formulation of the Einstein equations and the respective characteristic initial value problem. In particular, we present a derivation of the Reissner-Nordström black holes expressed in terms of these regular coordinates.

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I. INTRODUCTION

The classical theory of general relativity (GR) predicts the existence of fascinating compact objects like black holes. They are, roughly speaking, regions of spacetime in which our understanding of the physical laws breaks down and from which no information can escape. That black holes are not just an academic mathematical solution of Einstein's field equations but a true astrophysical object is acknowledged in the 2020 Nobel prize. Not only the gravitational collapse of a compact object is an inevitable feature of nature but also a direct measurement of a black hole shadow of the supermassive black hole in the galaxy M87 has been obtained by the Event horizon telescope[1] The 2020 award was given to the two astrophysicists, Andrea Ghez and Rainer Genzel, and the mathematical physicist Roger Penrose. While the astrophysicists received the award for the (indirect) astronomical observation of the central black hole in the Milky Way, Penrose received it "for the discovery that black hole formation is a robust prediction of the general theory of relativity [1]." But Penrose's contributions to our understanding of black holes go further than this, as his research provided most of the mathematical tools we use nowadays to analyze black hole spacetimes. One of the defining properties of a black hole is the presence of an event horizon, a null hypersurface separating the interior of a black hole from an external observer.

Since the analysis of a spacetime involves the definition of a spacetime chart, adapted coordinates may be given in a way that the metric g_{ab} is either well defined or singular if it is evaluated at the horizon. On one hand, the classical example in a spherically symmetric spacetime for singular coordinates at the event horizon are the Schwarzschild coordinates or the Eddington-Finkelstein coordinates. In the first case, the metric components blow up at the event horizon located at the radius $r = 2M$, where M is the mass of the black hole; while in the second case, the null coordinate diverges at the horizon. On the other hand, one example of well defined coordinates in spherical symmetry at the black hole horizon is given by the Kruskal-Szekeres coordinates, which are globally well defined and only singular at the central singularity $r = 0$. If we are interested in studying matter fields in the vicinity of the horizon, we can see that it is of importance to have well-defined coordinates at the horizon as otherwise no proper statements on the physical behavior of those fields can be made. However, having well behaved coordinates at the horizon is one side of the story, only, because we do not only want to study fields at and near the horizon. We also want to know how these fields behave far

[1]Not to mention the 2017 Nobel prize given to The LIGO collaboration.

away from it, as this is the region where the external observer is making his/her measurements of the dynamical processes taking place in the horizon's neighborhood. In particular, an astronomical observer far away from the black hole measures electromagnetic (or gravitational) radiation coming from the near region of the black hole. This emitted radiation follows outgoing null geodesics and the astronomical observer receives the radiation at the asymptotic end of the outgoing null hypersurfaces generated by those geodesics.

Mathematically there are two ways to asymptotically analyze radiation fields; in the first approach, matter fields and the physical metric g_{ab} are expanded in the physical spacetime with respect to inverse powers of a suitable radial coordinate while the second approach employs the so-called Penrose compactification of spacetime [2]. This compactification consists in attaching a null boundary to the physical spacetime. Thereby an extended conformal spacetime manifold is built using a conformal metric $\hat{g}_{ab} = \Omega^2 g_{ab}$ in which Ω is a suitable conformal factor vanishing at the null boundary. The attached null boundary is called null infinity, $\mathcal{I}$, and a local Taylor series expansion of geometrical and physical quantities off $\mathcal{I}$ in the conformal spacetime allows one to mathematically analyse the radiation fields. There are in fact two such boundaries $\mathcal{I}^+$ and $\mathcal{I}^-$, also known as future null infinity and past null infinity. Indeed, an idealized astronomical observer would be placed at $\mathcal{I}^+$, in the far future of the black holes.

The first convincing understanding of nonlinear radiation fields in general relativity has been done by Bondi and collaborators [3,4]. They introduced a chart consisting of an (outgoing) null coordinate u (corresponding to the retarded time in Minkowski spacetime), an areal distance r and two spherical angles $x^A = (\theta, \phi)$. Furthermore they required the metric to approach a Minkowski metric in outgoing polar null coordinates for the (physical) spacetime metric g_{ab} that is expanded in inverse powers of r. In the asymptotic region $u = \text{const}$ are null hypersurfaces, whose generating rays are parametrized with r. This Bondi null coordinate u however is not well suited to study fields at the horizon of a black hole. As an example, we consider again charts in Schwarzschild spacetime. First, with the well known tortoise coordinate $r^*(r)$, the outgoing Eddington-Finkelstein coordinate $u = t - r^*$ takes the form $u = t - r$ for large values of r, where t is the inertial time of the asymptotic observer, but u is singular at the horizon $r = 2m$. Second, Kruskal-Szekeres coordinates are well defined at the horizon and they allow us to understand the conformal structure of the Schwarzschild spacetime; however, the coordinates have the caveat that the areal distance coordinate r is expressed as an implicit function in terms of the Kruskal's null coordinates and the standard flat space null coordinates. Because of this, the analysis of fields near the horizon and at large distances in his chart is difficult. Yet, there is another (less known) global representation of the Schwarzschild spacetime due to Israel [5,6] (and rediscovered by [7,8], see also Blau's online lecture notes for a complete discussion [9]) where the metric of a Schwarzschild black hole takes a simple and explicit form with rational functions

$$g_{ab}dx^a dx^b = -\frac{2y^2}{8m^2 - wy}dw^2 + 2dwdy - \left(2m - \frac{wy}{4m}\right)^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

Israel obtained this metric by analyzing the null geodesics in the standard Schwarzschild metric representation adopting the w coordinate to the null structure. The past and future horizons in the above metric are given by $y = 0$ and $w = 0$, respectively. Note that the radial coordinate $x^1 = y$ is an affine parameter of the null vectors generating the null hypersurfaces $w = \text{const}$, which is indicated by $g_{wy} = 1$. For an asymptotic analysis using Penrose's compactification scheme, introducing an inverse affine parameter $\Upsilon = (4m)/y$, a rescaling $w \to 4mw$ and a conformal factor $\Omega = \Upsilon/(4m)$, we then discover that the metric is given by the conformal metric

$$\hat{g}_{ab}dx^a dx^b = \Omega^2 g_{ab} d\hat{x}^a d\hat{x}^b = -2dwd\Upsilon - w^2(d\theta^2 + \sin^2\theta d\phi^2) + O(\ell) \quad (2)$$

which is well defined for $\Upsilon = 0$, i.e., at null infinity. We can see that the coordinate pair (w, y) consists of bona-fide coordinates so that it allows us to construct a coordinate chart at the horizon $y = 0$ as well as in the asymptotic region for $y \to \infty$. It is of interest to see whether such pair (w, y) exist in a general sense, so that it can be used to chart black hole spacetimes from a horizon to null infinity. The main purpose of this work is to give an affirmative answer for various spherically symmetric spacetimes. Thereby we present two possible scenarios for achieving this aim. In the first one we follow previous works of [10,11], where a regular null coordinate system is constructed based on geometrical restrictions (see Sec. II). In the second approach, we follow the affine-null metric formulation of Einstein equations [12–14], which is a formulation of Einstein equations with respect to an affine-null metric. This formulation shares similarities with the Bondi-Sachs metric approach of General Relativity in which the relevant field equations are cast into a hierarchical system [15]. We demonstrate with the example of the Reissner-Nordström solution that the two approaches lead to equivalent results.

The regular null coordinate framework of [10,11] is summarized in Sec. II. The following sections employ this framework for nonextremal (Sec. III A) and extremal spherically symmetric black holes (Sec. III B). We also show how it can be used to find a regular null coordinate version of the outgoing (Sec. IV A) and ingoing (Sec. IV B)

Vaidya solution. The spherically symmetric affine null metric formulation is discussed in Sec. V, where the Reissner-Nordström solution is derived using for the first time a characteristic initial value formulation to obtain the charged version of the metric (1).

To be in lines with [10,11], we follow the notation of Geroch, Held and Penrose (GHP) [16], and work exclusively with the negative signature convention −2 for the physical metric.

II. THE FRAMEWORK

Using the above mentioned framework, we introduce regular null coordinates based on the assumption that a suitable family of null surfaces are caustic free in a neighborhood of timelike infinity i^+ containing a portion of the black hole horizon $\mathcal{H}^+$ and future null infinity $\mathcal{I}^+$.

In this paper, we assume the spacetime to be spherically symmetric, asymptotically flat at future null infinity (M, g_{ab}) and containing a black hole; for more details about the formalism we refer to [10,11].

We choose a Bondi coordinate u in such a way that it coincides with the center of mass Bondi cuts in the regime $u \to \infty$ limit. In the past of an open set of future null infinity $(\mathcal{I}^+)$ defined by those points for which their Bondi retarded time u is in the range $u \in (u_0, \infty)$ we require there exists a smooth null function $w = w(u)$ such that $w = 0$ at the horizon $\mathcal{H}^+$, $\lim_{u\to\infty} w = 0$, $\dot{w} \equiv \frac{dw}{du} > 0$, and $w < 0$ for all u in the (exterior) region between $\mathcal{H}^+$ and $\mathcal{I}^+$. The kind of metrics satisfying these conditions are referred as solitary black holes (SBHs) in [10,11]. The associated conformal diagram is depicted in Fig. 1.

The null geodesic congruence defined by $\tilde{\ell} = du$ allows for the introduction of an affine parameter r used as a radial coordinate which is fixed by the requirement that it

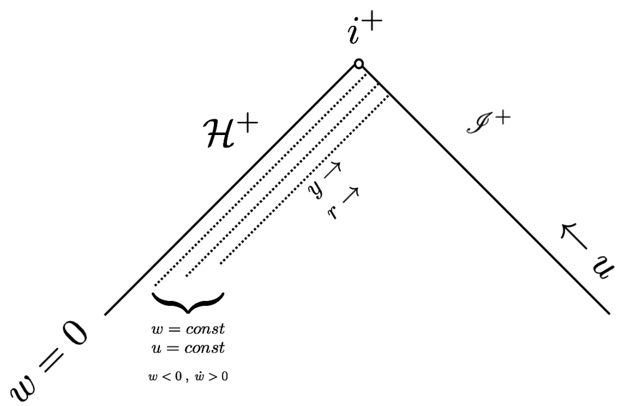

FIG. 1. In the past of an open set of future null infinity defined by those points for which their Bondi retarded time u is in the range $u \in (u_0, \infty)$ we require the existence of a regular null function w such that: $w = 0$ at the horizon $\mathcal{H}^+$, and $w < 0$ in the region of interest. In a vicinity of i^+, the null surfaces of constant retarded time u are smooth all the way up to the event horizon for SBHs.

coincides asymptotically with the luminosity distance [11]. The surfaces $(r, u) =$ constant are spheres which inherit natural spherical coordinates defined in the Bondi cuts at $\mathcal{I}^+$ which label null rays of the congruence. All this provides a coordinate system (u, r, x^A) with x^A coordinates of the two-dimensional sphere S^2. In a similar way, we define the one form $\ell_a \equiv (dw)_a$, then, the geodesic vector field ℓ^a is also tangent to the null congruence defined by $\tilde{\ell}^a$. It is therefore natural to introduce the affine function y through $\ell^a = (\frac{\partial}{\partial y})^a$. Therefore, the functions (w, y) can be used as coordinate functions in the region where the null congruence ℓ^a does not show caustics. The affine parameter y, for each null geodesic, can be chosen so that the 2-spheres $u =$ const, $r =$ const. coincide with the 2-spheres $w =$ const, $y =$ const, implying the following relationship between r and y:

$$r = \dot{w}y + r_0(w). \tag{3}$$

Hence, one has a new coordinate system (w, y, x^A) where x^A are again coordinates of S^2. We assume that r is a smooth function of (w, y) all the way up to the horizon.

Let us observe that from the null vector fields ℓ^a and $\tilde{\ell}^a$ one can construct null tetrads $(\ell^a, m^a, \bar{m}^a, n^a)$, and $(\tilde{\ell}^a, \tilde{m}^a, \bar{\tilde{m}}^a, \tilde{n}^a)$ adapted to the geometry of the coordinate system introduced above.[2] The freedom in this choice is reduced by choosing the vectors $m^a = \tilde{m}^a$, and tangent to the topological 2-spheres $(w, y) =$ constant $= (u, r)$.

From $w = w(u)$ it follows that $dw = \dot{w}du$ which implies the following relation between the two tetrads

$$\ell^a = \dot{w}\tilde{\ell}^a, \qquad n^a = \frac{1}{\dot{w}}\tilde{n}^a, \qquad m^a = \tilde{m}^a. \tag{4}$$

If we denote the five (complex) Weyl tensor spinor components [16] Ψ_N and $\tilde{\Psi}_N$ for $N \in \{0, 1, 2, 3, 4\}$ in each of the respective tetrads, then we get the following relations for the Weyl curvature scalars $\Psi_N = \dot{w}^{(2-N)}\tilde{\Psi}_N$. Similar relations are obtained for the Ricci curvature scalars, in particular $\Phi_{00} = \dot{w}^2\tilde{\Phi}_{00}$, $\Phi_{11} = \tilde{\Phi}_{11}$ and $\Phi_{22} = \dot{w}^{-2}\tilde{\Phi}_{22}$.

The above assumptions of regularity of the new coordinates at the horizon imply that the limit $r_H \equiv \lim_{w\to 0} r(w, y)$ exists and is constant [10,11].

Since by assumption $\dot{w}(w)$ admits a Taylor expansion around $w = 0$ we can write:

$$\dot{w} = a(w) = a_1 w + \mathcal{O}(w^2). \tag{5}$$

Assuming that $a_1 \neq 0$ the above equation can be integrated giving the important relation

[2]With the usual normalization $1 = \ell^a n_a = -m^a\bar{m}_a$ and $1 = \tilde{\ell}^a\tilde{n}_a = -\tilde{m}^a\bar{\tilde{m}}_a$ with all other respective scalar products being zero.

$$w(u) = -\exp(a_1(u-u_0)) + \mathcal{O}(\exp(2a_1 u)), \quad (6)$$

where $\exp(-a_1 u_0)$ is the rescaling freedom mentioned previously associated with the choice of origin for the Bondi retarded time u.

As shown in [10] a_1 has a clear geometrical meaning as follows from the properties of the vector field $\chi = \partial_u$:

(1) It is a smooth vector field that is a null geodesic generator at $\mathcal{I}^+$. As u is a Bondi coordinate it generates inertial time translations at future null infinity.
(2) It is a null geodesic generator of the horizon $\mathcal{H}^+$.
(3) At the horizon $\mathcal{H}^+$, χ satisfies the equation,

$$\chi^a \nabla_a \chi^b \equiv k_H \chi^b; \quad (7)$$

where k_H is a generalized surface gravity.
(4) The coefficient a_1 is the negative of the surface gravity k_H, i.e.,

$$a_1 = -k_H = \text{const.}$$

For a proof of these properties, we refer to [10].

Therefore, the family of spacetimes considered here admits a notion of surface gravity which coincides with the usual one in cases when the spacetime is stationary. Note that if we had taken $a_1 = 0$ above, one would have obtained $k_H = 0$. This situation corresponds to the special cases involving (in particular) the stationary extremal black holes. We will discuss this case in Sec. III B.

With this definition of surface gravity, the relation between the null coordinate w and the Bondi retarded time u in the case of $k_H \neq 0$ reads

$$w = -\exp(-k_H(u-u_0)) + \mathcal{O}(\exp(-2k_H u)). \quad (8)$$

III. THE FORMALISM APPLIED TO SPHERICALLY SYMMETRIC BLACK HOLES

In this section, we specify the regular coordinates (w, y) of the previously explained formalism for static, spherically symmetric spacetimes. We discuss the nonextremal and extremal solutions.

Consider a static and spherically symmetric metric in Bondi coordinates with a timelike Killing vector ∂_u,

$$ds^2 = f(r)du^2 + 2dudr - h(r)^2 d\Omega^2, \quad (9)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$, is the metric of an unit sphere written in standard spherical coordinates, (θ, ϕ), and $h(r)$ a smooth function which is nonvanishing at the horizon. The event horizon of this metric is located on the null surface $r =$ constant placed in the bigger root of $f(r) = 0$. We will call this root r_H and assume that $f(r)$ and $h(r)$ are regular functions at $r = r_H$. A null tetrad adapted to this coordinate system is given by:

$$\tilde{\ell}^a = \left(\frac{\partial}{\partial r}\right)^a, \quad (10)$$

$$\tilde{n}^a = \left(\frac{\partial}{\partial u}\right)^a - \frac{f(r)}{2}\left(\frac{\partial}{\partial r}\right)^a, \quad (11)$$

$$\tilde{m}^a = \frac{1}{\sqrt{2}h(r)}\left[\left(\frac{\partial}{\partial \theta}\right)^a + \frac{i}{\sin(\theta)}\left(\frac{\partial}{\partial \phi}\right)^a\right], \quad (12)$$

$$\tilde{\bar{m}}^a = \frac{1}{\sqrt{2}h(r)}\left[\left(\frac{\partial}{\partial \theta}\right)^a - \frac{i}{\sin(\theta)}\left(\frac{\partial}{\partial \phi}\right)^a\right]. \quad (13)$$

Using the GHP notation [16] the resulting expressions for the nonvanishing spin coefficients are

$$\tilde{\rho} = -\frac{1}{h}\frac{dh}{dr}, \qquad \tilde{\rho}' = \frac{f}{2h}\frac{dh}{dr},$$
$$\tilde{\epsilon}' = -\frac{1}{4}\frac{df}{dr}, \qquad \tilde{\beta} = \tilde{\beta}' = \frac{\sqrt{2}}{4h}\cot(\theta). \quad (14)$$

In particular, as it is well known, the expansion $\tilde{\rho}$ associated to the null congruence $\tilde{l}^a$ is not vanishing if evaluated at the horizon. This unphysical result is a consequence that this coordinate system is not regular there.

A. The nonextremal case ($k_H \neq 0$)

Now, we wish to make a coordinate transformation to regular coordinates on the horizon (w, y). To do that we assume that the black hole of interest is nonextremal, i.e., the case where

$$\dot{w} = a_1 w + O(w^2). \quad (15)$$

Moreover, we assume that the relation between $\dot{w}$ and w is exactly linear, giving

$$w = e^{a_1 u}, \quad (16)$$

$$r = a_1 w y + r_H. \quad (17)$$

The extremal case will be dealt with below. The previous equations imply

$$du = \frac{dw}{a_1 w}, \quad (18)$$

$$dr = a_1 y dw + a w dy. \quad (19)$$

By replacing these relations into Eq. (9), we get

$$ds^2 = \left(\frac{f(a_1wy + r_H)}{a_1^2w^2} + \frac{2y}{w}\right)dw^2 + 2dwdy - [h(a_1wy + r_H)]^2 d\Omega^2. \quad (20)$$

This is an exact expression for the class of metrics (9) in terms of regular coordinates (w, y). As follows from Eq. (17), in these coordinates the horizon $\mathcal{H}^+$ is placed at $w = 0$.

We would like to study these metrics in the limit $w \to 0$ at a neighborhood of the horizon. As of the regularity requirements, we can make an expansion of g_{ww} in terms of w. In a neighborhood of the horizon of size $a_1yw \ll r_H$ we have,

$$\begin{aligned} f(r_H + a_1yw) &= f(r_H) + f'(r_H)a_1yw \\ &\quad + \frac{1}{2}f''(r_H)a_1^2y^2w^2 + O(w^3) \\ &= f'(r_H)a_1yw + \frac{1}{2}f''(r_H)a_1^2y^2w^2 \\ &\quad + O(w^3), \end{aligned} \quad (21)$$

where we used $f(r_H) = 0$. Note that the regularity of the metric (20) at $w = 0$ requires

$$a_1 = -k_H = -\frac{f'(r_H)}{2}. \quad (22)$$

which is true for static and spherically symmetric metrics. This is an independent way of proving (7) which is valid in the spherically symmetric situation we are considering. Hence, the expansion in Eq. (21) becomes

$$f(r_H + a_1yw) = -2a_1^2yw + \frac{1}{2}f''(r_H)a_1^2y^2w^2 + O(w^3). \quad (23)$$

A similar expansion follows for $h(r)$, but starting with a nonzero constant term $h(r_H)$ (because the area of the horizon is not zero).

Therefore by replacing this expression into Eq. (20) we get

$$g_{ww} = \frac{1}{2}f''(r_H)y^2 + \frac{1}{6}f'''(r_H)a_1wy^3 + \cdots + \frac{1}{n!}f^{(n)}(r_H)(a_1w)^{n-2}y^n + O(w^{n-1}). \quad (24)$$

In particular the 4-dimensional spacetime metric at a vicinity of the horizon reads,

$$ds^2|_{r=r_H} = \frac{1}{2}f''(r_H)y^2dw^2 + 2dwdy - h^2(r_H)d\Omega^2 + O(w^3). \quad (25)$$

The spin coefficients associated to the new null coordinate system are

$$\rho = \dot{w}\tilde{\rho} = -\dot{w}\left[\frac{1}{h}\frac{dh}{dr}\right]\Bigg|_{r=\dot{w}y+r_H}, \quad (26)$$

$$\rho' = \frac{1}{\dot{w}}\tilde{\rho}' = \frac{1}{2\dot{w}}\left[\frac{f}{h}\frac{dh}{dr}\right]\Bigg|_{r=\dot{w}y+r_H}, \quad (27)$$

$$\begin{aligned} \epsilon' &= \left[\frac{\tilde{\epsilon}'}{\dot{w}} - \frac{1}{2}\tilde{n}^a\nabla_a(\dot{w}^{-1})\right]\Bigg|_{r=\dot{w}y+r_H} \\ &= \left[-\frac{1}{4\dot{w}}\frac{df}{dr} + \frac{k_H}{2\dot{w}}\right]\Bigg|_{r=\dot{w}y+r_H}, \end{aligned} \quad (28)$$

$$\beta = \tilde{\beta} = \left[\frac{\sqrt{2}}{4h}\cot(\theta)\right]\Bigg|_{r=\dot{w}y+r_H}. \quad (29)$$

Note in particular, that now $\rho = 0$ at the horizon (as it should be), and even when ρ' and ϵ' have a factor $\dot{w}^{-1}$ in their expressions, they are regular at the horizon as it can be seen by studying their limit using Eqs. (21) and (22). For completeness we give expressions for the nonvanishing scalar curvatures,

$$\Phi_{00} \coloneqq \frac{\dot{\omega}^2}{h}\frac{d^2h}{dr^2}, \quad (30)$$

$$\Phi_{11} = -\frac{1}{8h^2}\left[\frac{d^2f}{dr^2}h^2 + 2 - 2f\left(\frac{dh}{dr}\right)^2\right], \quad (31)$$

$$\Phi_{22} = \frac{f^2}{4\dot{\omega}^2h}\frac{d^2h}{dr^2}, \quad (32)$$

$$\Lambda = -\frac{1}{24h^2}\left[\frac{d^2f}{dr^2}h^2 + 4h\frac{dh}{dr}\frac{df}{dr} + 4fh\frac{d^2h}{dr^2} - 2 + 2f\left(\frac{dh}{dr}\right)^2\right], \quad (33)$$

$$\Psi_2 = \frac{1}{12h^2}\left[\frac{d^2f}{dr^2}h^2 - 2fh\frac{d^2h}{dr^2} - 2h\frac{dh}{dr}\frac{df}{dr} - 2 + 2f\left(\frac{dh}{dr}\right)^2\right]. \quad (34)$$

Note that taking into account the Eq. (23) all these curvature scalars are regular at $w = 0$, including Φ_{22}.

As an example let us consider a Reissner-Nordström black hole,

$$f(r) = 1 - \frac{2m}{r} + \frac{Q^2}{r^2}, \quad (35)$$

$$h(r) = r, \quad (36)$$

$$r_H = m + (m^2 - Q^2)^{\frac{1}{2}}, \quad (37)$$

$$k_H = \frac{r_H^2 - Q^2}{2r_H^3} \quad (38)$$

we get from (20)

$$ds^2 = \frac{(1 - 2k_H(r + r_H))y^2}{r^2} dw^2 + 2dwdy - r^2 d\Omega^2, \quad (39)$$

with $r = -k_H wy + r_H$. In these coordinates, $y = 0$ corresponds to the nonexpanding null hypersurface $\mathcal{H}^-$ being the past null horizon. The future horizon $\mathcal{H}^+$ is at $w = 0$.

In the case $Q = 0$, we obtain the Schwarzschild solution

$$ds^2 = -\frac{2k_H y^2}{-k_H wy + 2m} dw^2 + 2dwdy - (r_H - k_H wy)^2 d\Omega^2, \quad (40)$$

with corresponding $k_H = (2r_H)^{-1}$. This is exactly the solution found by Israel [see Eq. (1)] [5,6] (and rediscovered by Pajerski and Newman [7] as well as Klöbsch and Strobl [8], who also obtained the Reissner-Norström metric in these coordinates using a connection between this metric and highly symmetric solutions of particular two-dimensional generalized dilaton gravity models.)

As noted by Blau [9], the Schwarzschild metric as expressed in Eq. (40) admits the isometry $\tilde{w} = \lambda w$, $\tilde{y} = \lambda^{-1} y$ which corresponds to the timelike Killing vector $\chi = -k_H(w\partial_w - y\partial_y)$. This property is also shared by the more general metric given by Eq. (20).

As a final remark, let us note that our construction of regular coordinates $\{w, y\}$ for static and spherically symmetric asymptotically flat spacetimes is not restricted to metrics which are solutions of the Einstein's equations, as long as the assumptions given by the regularity requirements are satisfied.

B. The extremal case $k_H = 0$

In many situations, for certain choices of parameters that describe a black hole, an extremal solution can be obtained where the surface gravity is zero. Let us consider for example the extremal case with $f = (1 - \frac{r_H}{r})^2$. An extremal Reissner-Nordström solution with $Q = m$ falls into this family. In such a case, as the surface gravity is zero, (and therefore $a_1 = 0$) a naive ansatz for Eq. (5) would be to consider that $\dot{w} = a_2 w^2$, however it can be checked that the resulting expression for the metric in $\{w, y\}$ coordinates is not regular at $w = 0$. In the next subsections we will present two alternative approaches to solve this problem.

1. *Approach I: The direct construction of an analytic null function w*

The extreme Reissner-Nordström metric can be expressed [17] by:

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega^2; \quad (41)$$

where

$$f(r) = \left(1 - \frac{m}{r}\right)^2. \quad (42)$$

It is customary to define the tortoise coordinate

$$r^* = \int_{r_1}^{r} \frac{dr}{f}; \quad (43)$$

which for the extremal Reissner-Nordström metric gives

$$r^* = r + 2m \ln\left(\frac{r - m}{m}\right) - \frac{m^2}{r - m} + c_1. \quad (44)$$

Then, it is usual to define the (outgoing/ingoing) null coordinates,

$$u = t - r^*, \quad (45)$$

and

$$v = t + r^*. \quad (46)$$

In general one can consider other families of null coordinates where the metric can be expressed as:

$$ds^2 = -4f(r)A(u)B(v)dudv - r^2 d\Omega^2, \quad (47)$$

with

$$dr = -f(r)(A(u)du + B(v)dv); \quad (48)$$

which for the previous case of (45) and (46) one has to take

$$A(u) = -B(v) = \frac{1}{2}; \quad (49)$$

and note that in this case one has

$$dr^* = \frac{1}{f} dr = -\frac{1}{2}(du - dv). \quad (50)$$

We now look for a null coordinate w which is regular near the horizon, with

$$ds^2 = -f(r)A(w)B(v)dwdv - r^2 d\Omega^2. \quad (51)$$

Recall that the null radial geodesic equation is [17] [p. 216, Eq. (70)]:

$$\frac{dr}{d\lambda} = \pm E; \quad (52)$$

so that along these null geodesics the radial coordinates is proportional to the affine parameters. In particular for the incoming null radial geodesics one has

$$\frac{dr}{d\lambda} = -E(v); \tag{53}$$

where we have the freedom to choose for different v's different constants $E > 0$.

Let us consider then the incoming null geodesics, that is with $dv = 0$. Then, in the integral form one must have

$$r^* = -\int \alpha'(w)dw; \tag{54}$$

where $\alpha'(w) \equiv \frac{d\alpha}{dw}$; so that

$$r^* = r + 2m \ln\left(\frac{r-m}{m}\right) - \frac{m^2}{r-m} + c_1 = -\alpha(w). \tag{55}$$

Since r behaves as the affine parameter, we can think of λ as given by $E(v)\lambda = -(r-m)$, and as we want w to be regular at the horizon, $r = m$, we take, along a null geodesic in $v = v_0$, $E(v_0) = E_0 > 0$ and $w = \lambda$, so that we set

$$\frac{\alpha}{m} = -2\ln\left(-\frac{E_0}{m}w\right) - \frac{m}{E_0 w} + \frac{E_0}{m}w; \tag{56}$$

since in this way we capture the two terms with divergent behaviors, and where we have divided by m the original expression to deal with quantities without units.

Then, recalling the relation between r^* and u at constant v, we have

$$\frac{\alpha}{m} = -2\ln\left(-\frac{E_0}{m}w\right) - \frac{m}{E_0 w} + \frac{E_0}{m}w = \frac{u-u_0}{2m}; \tag{57}$$

where without loss of generality we can take $u_0 = 0$, so that after differentiation with respect to u we obtain

$$\dot{w} = \frac{1}{2}\frac{E_0 w^2}{(E_0 w - m)^2}. \tag{58}$$

with a Taylor expansion around $w = 0$,

$$\dot{w} = \frac{E_0}{2m^2}\left(w^2 + 2\frac{E_0}{m}w^3\right) + \mathcal{O}(w^4). \tag{59}$$

Note that $w = 0$ corresponds to $u \to \infty$, $\dot{w} > 0$ in the exterior region of the black hole and it has an analytic expansion in powers of w. Equation (57) [or equivalently (58)] define our desired w coordinate.

Relation with coordinate y.—With respect to the coordinate y, for an incoming null geodesic, contained in the hypersurface $v = v_0$, one will have a functional dependence of the form $y_v(w) = y_v(\lambda)$.

The general relation between r and y is of the form:

$$r = \dot{w}(y - y_0(w)) + \tilde{r}_0(w) = \dot{w}y + r_0(w); \tag{60}$$

where we know that in general $r_H \equiv r_0(w=0)$ is the radius of the horizon.

Then, along the incoming null geodesic, contained in the hypersurface $v = v_0$, one will have

$$r = \dot{w}(y_v(w) - y_0(w)) + \tilde{r}_0(w) = m - E_0 w; \tag{61}$$

due to the previous relation between affine parameter and radial coordinate.

Let us note that at this stage we have the freedom to choose $y_0(w)$ and the function $\tilde{r}_1(w)$, in $\tilde{r}_0(w) = r_H + \tilde{r}_1(w)$, with $\lim_{w\to 0}\tilde{r}_1(w) = 0$, and $r_H = m$.

Choice (a) Let us consider the choice of $y_0(w)$ so that:

$$\dot{w}y_0(w) = \tilde{r}_1(w); \tag{62}$$

then we would have

$$r = \dot{w}y + r_H; \tag{63}$$

so that when we take $y = 0$ one would have $r = r_H$, that is the past horizon $\mathcal{H}^-$. Note that in this case $y = 0$ implies an incoming null geodesic contained in $\mathcal{H}^-$.

Let us note that although this seems to be a natural choice, it might involve a singular definition for the coordinate y; which is related to the fact that $\mathcal{H}^-$ can not be taken as the initial incoming null hypersurface, used in the previous mechanism to define the coordinate w, since it is outside the manifold covered by v.

Choice (b) From the previous discussion one is tempted to consider

$$\tilde{r}_1(w) = -E_0 w. \tag{64}$$

Then, for incoming null geodesic, contained in the hypersurface $v = v_0$, one will have from (61) that

$$\dot{w}(y_v(w) - y_0(w)) = 0; \tag{65}$$

so that in particular, by taking $y_0(w) = 0$ one would have that at the original incoming null geodesic

$$y_v(w) = 0; \tag{66}$$

that is we choose in this way that the value 0 of coordinate y is at the original incoming null geodesics. Then we would have

$$r = \dot{w}y + r_H - E_0 w. \tag{67}$$

Note that for Choice (a) one has $r_0 = r_H = m$; while Choice (b) one has $r_0 = r_H - E_0 w = m - E_0 w$.

Relation with the $W \coloneqq g_{ww}$ component.—Using $dw = \dot{w}du$, one has

$$f du^2 + 2dudr = \frac{f}{\dot{w}^2}dw^2 + \frac{2}{\dot{w}}dwdr - r^2 d\Omega^2. \tag{68}$$

And using the relation of $r(w, y)$ one has

$$ds^2 = \left(\frac{f}{\dot{w}^2} + \frac{2}{\dot{w}}\left(\frac{d\dot{w}}{dw}y + \frac{dr_0}{dw}\right)\right)dw^2 + 2dwdy - r^2 d\Omega^2; \tag{69}$$

which shows the g_{ww} dependence on the choice of gauges y_0 and r_0 or equivalently $\tilde{r}_0$. Note that

$$W = \frac{(\dot{w}y + r_0(w) - m)^2}{\dot{w}^2(\dot{w}y + r_0(w))^2} + \frac{2}{\dot{w}}\left(\frac{E_0 mw}{(m - E_0 w)^3}y + \frac{dr_0}{dw}\right); \tag{70}$$

so that for Choice (a) one has

$$W = \frac{y^2}{(\dot{w}y + m)^2} + \frac{2}{\dot{w}}\frac{E_0 mw}{(m - E_0 w)^3}y. \tag{71}$$

However, it is not well behaved at $\mathcal{H}^+$, $W = 4y/w + y^2/m^2 + 4E_0 y/m + \mathcal{O}(w)$. Therefore, it does not serve our purposes.

Notwithstanding, for Choice (b) one has

$$\begin{aligned} W &= \frac{(\dot{w}y - E_0 w)^2}{\dot{w}^2(\dot{w}y + m - E_0 w)^2} + \frac{2}{\dot{w}}\left(\frac{E_0 mw}{(m - E_0 w)^3}y - E_0\right) \\ &= -\frac{4(2w^4E_0^4 - 5w^3E_0^3m + 3w^2E_0^2m^2 + wE_0m^3 - w^3yE_0^2 - m^4)my^2}{(-E_0w + m)(-2w^3E_0^3 + 6w^2E_0^2m - 6wE_0m^2 + 2m^3 + w^2E_0y)^2}; \end{aligned} \tag{72}$$

which is well behaved at a neighborhood of $\mathcal{H}^+$, $W \approx y^2/m^2 + 6y^2E_0w/m^3 + \mathcal{O}(w^2)$. Note also that $W|_{y=0} = W_{,y}|_{y=0} = 0$, in agreement with the condition that the hypersurface $y = 0$ be null with w as an affine parameter of its null vector generator $n = \partial_w$ (cf. Sec. V).

Hence, the coordinate system (w, y) related to the Bondi coordinates (u, r) by

$$u = 2m\left(-2\ln\left(-\frac{E_0}{m}w\right) - \frac{m}{E_0 w} + \frac{E_0}{m}w\right), \tag{73}$$

$$r = \frac{1}{2}\frac{E_0 w^2}{(E_0 w - m)^2}y + m - E_0 w; \tag{74}$$

is the kind of coordinates that accomplishes the requirements of Sec. II.

2. Approach II: Requiring regularity of W at the horizon

We can also consider a more general transformation and see the necessary conditions for the metric to be regular at $w = 0$. First, let as note that at the considered case $f(r_H) = f'(r_H) = 0$. Let us assume that

$$r = \dot{w}y + r_0(w) = a(w)y + r_0(w), \tag{75}$$

and make and expansion of $a(w)$ and $r_0(w)$ of the form

$$a(w) = a_2 w^2 + a_3 w^3 + \mathcal{O}(w^4), \tag{76}$$

$$r_0(w) = r_H + r_1 w + \mathcal{O}(w^2), \tag{77}$$

with $a_2 \neq 0$. Therefore, by replacing $du = dw/\dot{a}(w)$ and (75) in (9) and expanding in w we obtain

$$ds^2 = W dw^2 + 2dydw - r^2 d\Omega^2, \tag{78}$$

with W given by

$$W = \frac{f}{a(w)^2} + \frac{2}{a(w)}\left(\frac{da(w)}{dw}y + \frac{dr_0}{dw}\right). \tag{79}$$

Taking into account the relation Eqs. (76) and (77), we obtain the following expansion of W in powers of w:

$$W = \frac{(a_2 r_1)^2 f''(r_H) + 4r_1 a_2^3}{w^2 a_2^4} - \frac{-24a_2^5 y + 12a_3 r_1 a_2^3 + (6a_3 a_2^2 r_1^2 - 6a_2^4 r_1 y)f''(r_H) - a_2^3 r_1^3 f'''(r_H)}{6wa_2^5} + \mathcal{O}(w^0), \tag{80}$$

with $f''(r_H) = \frac{2}{r_H^2}$, $f'''(r_H) = -\frac{12}{r_H^3}$.

For the metric to be regular at $w = 0$, the $\mathcal{O}(w^{-2})$ and $\mathcal{O}(w^{-1})$ contributions must vanish. The $\mathcal{O}(w^{-2})$ term is zero if $r_1 = 0$ or $r_1 = -\frac{4a_2}{f''(r_H)} = -2r_H^2 a_2$. However, the solution $r_1 = 0$ must be discarded because in that case the $\mathcal{O}(w^{-1})$ term of (80) cannot be made zero. Taking knowledge of this, we solve for a_3 from the requirement of vanishing of the $\mathcal{O}(w^{-1})$ term, resulting in

$$a_3 = -\frac{4}{3}\frac{f'''(r_H)}{(f''(r_H))^2}a_2^2 = 4r_H a_2^2.$$

Hence, we have obtained that a family of regular coordinates at the neighborhood of $w = 0$ is determined by

$$a(w) = a_2(w^2 + 4r_H a_2 w^3) + \mathcal{O}(w^4), \tag{81}$$

$$r_0(w) = r_H - 2a_2 r_H^2 w + \mathcal{O}(w^2), \tag{82}$$

with a_2 a free parameter. Note that the transformation given by (58) [with expansion (59)] and (74) are compatible with the expansions (81) and (82) by setting $a_2 = \frac{E_0}{2r_H^2}$. Of course, there exist other possibilities that satisfy these relations. However, as we will see in Sec. V, these coordinates naturally appear in the affine-null metric formulation of the Einstein-Maxwell equations. An alternative proposal for the transformation $\{u, r\} \to \{w, y\}$ that can be checked to satisfy the relations (81)–(82) and which is also global in the sense that admit maximal extension of the metric can be found in [8].

To end this section, it is worthwhile to mention that at the horizon $\mathcal{H}^+$, the Killing vector $\chi^a = (\partial_u)^a$ expressed in the $\{w, y\}$ basis coordinates reads $\chi^a|_{\mathcal{H}^+} = -r_1(\partial_y)^a = 2a_2 r_H^2 \ell^a$. In particular, χ^a never vanishes at $\mathcal{H}^+$, in agreement with the well-known result that extremal black holes do not have bifurcated Killing horizons [18].

IV. VAIDYA SPACETIMES

Here we present the outgoing and ingoing Vaidya solutions in the regular coordinates (w, y).

A. The retarded Vaidya spacetimes

Expressions for the outgoing Vaidya metric in regular (w, y)-type coordinates have already been discussed in the literature by Israel [5] and Fayos *et al.* [19]. However, our coordinate expression is not exactly the same and for completeness we present the form of this solution in the framework of Sec. II.

The Vaidya metric in Bondi coordinates is known as [20]

$$ds^2 = \left(1 - \frac{2m(u)}{r}\right)du^2 + 2dudr - r^2 d\Omega^2 \tag{83}$$

In order to obtain the well behaved coordinates satisfying the requirements of Sec. I we define

$$w = -\exp(-k_H u), \tag{84}$$

with $k_H = 1/(4m_0)$ where $m_0 = \lim_{u\to\infty} m(u)$ and $r = \dot{w}y + 2m_0$ which gives

$$r(w, y) = 2m_0 - \frac{yw}{4m_0}. \tag{85}$$

In the new coordinates, the metric becomes

$$ds^2 = \left(\frac{4m_0}{w}\right)^2\left(2 - \frac{2m(w)}{r(w,y)} - \frac{r(w,y)}{2m_0}\right)dw^2 + 2dwdy - r^2(w,y)d\Omega^2, \tag{86}$$

and the Ricci tensor that follows from Einstein's equations is given by,

$$R_{ab} = \frac{8m_0}{wr(w,y)^2}\frac{dm}{dw}\ell_a\ell_b = \rho_{\text{out}}\ell_a\ell_b, \tag{87}$$

with ρ_{out} representing the energy density of outgoing (scalar) radiation. Physically, its divergence would signal the presence of a firewall of outgoing radiation at the horizon detected as a physical divergence of ρ_{out} for any observers crossing the horizon. In principle there is not a constraint on the dependence of $m(w)$ with w, however, in our setting, as explained in Sec. II, we require regularity of the metric at the horizon. Such regularity therefore imposes that $m(w) = m_0 + w^2 m_2 + O(w^3)$. In terms of the Bondi mass $m(u)$ at $\mathcal{I}^+$ representing a fall-off of the form $\exp(-2k_H u)$ to the Schwarzschild geometry.

The previous result has an intuitive meaning: in order for outgoing Vaidya radiation to escape to $\mathcal{I}^+$ for late $u \to \infty$ it has to be sent with increasingly high local energy from the vicinity of the black hole horizon where the geometry imposes an ever increasing redshift.

This effect has been pointed out before by Israel [5] in terms of null coordinates that are very similar to the ones used here (See also Fayos *et al.* where global extensions of this metric are discussed [19]). This effect is also reminiscent of the Christensen-Fulling regularity conditions of the stress tensor in semiclassical studies of QFT on the Schwarzschild background [21].

B. Ingoing Vaidya: Collapsing null shell

As a second application for Vaidya spacetimes we consider a collapsing null shell. We assume that the spacetime inside the shell is Minkowskian charted with the standard double null coordinates, the retarded time $u_M = t_M - r_M$, the advanced time $v_M = t_M + r_M$ and spherical angles $x_M^A = (\theta, \phi)$ so that the metric is given by,

$$ds^2 = dv_M du_M - r_M^2 q_{AB} dx_M^A dx_M^B. \qquad (88)$$

with

$$2r_M = (v_M - u_M). \qquad (89)$$

The shell collapses (Fig. 2) at the advanced time $v_M = v_i$ and its metric outside is described by a Schwarzschild metric in double null coordinates (u, v, x^A)

$$ds^2 = \left(1 - \frac{2m}{r}\right) dv du - r^2 q_{AB} dx^A dx^B \qquad (90)$$

where r is the areal radius function related to u and v by

$$2r^* = v - u, \qquad (91)$$

with r^* the tortoise coordinate

$$r^* = r + 2m \ln\left(\frac{r}{4m} - 1\right). \qquad (92)$$

We require matching at the shell, i.e.: $x_M^A = x^A$, $v = v_M = v_i$ and the continuity of the areal radius function

$$r(u, v_i) = r_M(u_M, v_i) = \frac{v_i - u_M}{2}. \qquad (93)$$

From the transformation (91), the metric (90) can be rewritten in terms of the Bondi coordinates (outgoing Eddington-Finkelstein coordinates) as

$$ds^2 = \left(1 - \frac{2m}{r}\right) du^2 + 2dr du - r^2 q_{AB} dx^A dx^B. \qquad (94)$$

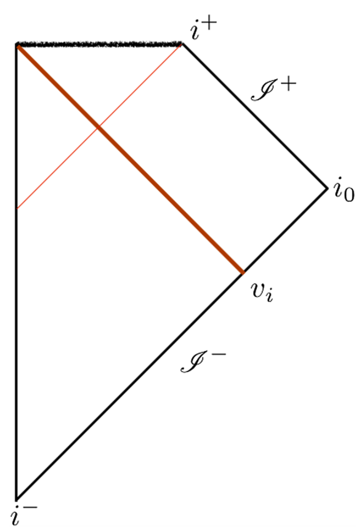

FIG. 2. Penrose's diagram for the collapsing null shell. For values $v < v_i$ the spacetime is Minkowski, $v = v_i$ is the null shell and $v > v_i$ is the conformal diagram of a Schwarzschild black hole.

By replacing (91) and (93) into (92) (evaluated at $v = v_i$) we obtain

$$2r_* = (v_i - u) = (v_i - u_M) + 4m \ln\left(\frac{v_i - u_M}{4m} - 1\right). \qquad (95)$$

Hence,

$$u = u_M - 4m \ln\left(\frac{v_i - 4m - u_M}{4m}\right). \qquad (96)$$

Now we define a new global coordinate

$$w \equiv u_M - v_i + 4m, \qquad (97)$$

where $w < 0$ outside the black hole horizon. In terms of w, (96) reads

$$u = w + v_i - 4m - 4m \ln\left(-\frac{w}{4m}\right). \qquad (98)$$

From this relation we find

$$\frac{dw}{du} = \dot{w} = \frac{1}{(1 - \frac{4m}{w})} = -\frac{k_H w}{1 - k_H w} = -k_H w(1 + k_H w + [k_H w]^2 + \cdots) \qquad (99)$$

with $k_H = 1/4m$, the surface gravity of the resulting Schwarzschild black hole. Note that (99) is positive for $w < 0$. The expansion in (99) shows the regularity at $w = 0$. Applying the coordinate transformation (98) to (94) while using (99) yields

$$ds^2 = \left(1 - \frac{2m}{r}\right)\left(1 - \frac{4m}{w}\right)^2 dw^2 + 2dr\left(1 - \frac{4m}{w}\right) dw - r^2 d\Omega^2. \qquad (100)$$

From Eq. (3) and taking $r_0(w) = r_H = 2m$, we obtain

$$r = \frac{y}{(1 - \frac{4m}{w})} + 2m \qquad (101)$$

whose total differential is

$$dr = \frac{dy}{(1 - \frac{4m}{w})} - \frac{4my dw}{w^2(1 - \frac{4m}{w})^2}. \qquad (102)$$

Note that from (96), w and u_M only differ by a constant, and therefore the associated affine parameters can also be chosen to agree, i.e., $r_M = y$. Moreover, using this identification between r_M and y and taking into account Eqs. (93) and (97), we see that the null shell $v_M = v_i$ is described in the $\{w, y, x^A\}$ coordinates by $y = 2m - \frac{w}{2}$. This gives us the final form of the metric,

$$ds^2 = \begin{cases} \frac{-y(w^2-12mw+32m^2-8my)}{(4m-w)(yw+2mw-8m^2)}dw^2 + 2dydw - \left(\frac{yw}{w-4m}+2m\right)^2 d\Omega^2 & y > \frac{4m-w}{2} \\ dw^2 + 2dwdy - y^2 d\Omega^2 & y \leq \frac{4m-w}{2} \end{cases} \tag{103}$$

In the bottom we have the Minkowski metric where we can recognize $y = r_M$ is the affine parameter. The metric turns into the top one at the position of the shell. Notice that r_M (the Minkowski radius) does not coincide with the affine parameter corresponding to the Bondi null hypersurfaces $u =$ constant that here we call r.

V. EINSTEIN-MAXWELL FIELDS: A NULL AFFINE CHARACTERISTIC FORMULATION

In the previous sections, we have seen how to construct null regular coordinates starting from known spacetime solutions of the field equations written in Bondi type coordinates. In fact, the considered approach is valid regardless of the validity of Einstein's equations, as long as the requirements demanded in the introduction are met. A valid question is whether these coordinates can be obtained naturally by a direct solution of the field equations, without the need to previously go through another (Bondi) coordinate system. The answer to this question is affirmative and leads toward the affine-null metric formulation of Einstein equations [12]. In particular, we present as an example, and for the first time in the literature, how to obtain the Reissner-Nordström solution directly at these regular coordinates starting with a characteristic formulation of Einstein's equations, i.e.; giving certain data on a certain 2-sphere and on two null surfaces that intersect it orthogonally.

A. The characteristic initial value formulation

In a spherically symmetric spacetime charted with coordinates $x^a = (w, y, x^A)$ consider a family of null hypersurfaces $\mathcal{N}_w = \{w = \text{const}\}$. The surface forming rays of $\mathcal{N}_w$ are parametrized with an affine parameter $x^1 = y$. Suppose there is a designated null hypersurface $\mathcal{B}$ whose generators are orthogonal to those of $\mathcal{N}_w$. At the common intersections Σ_w of $\mathcal{N}_w$ and $\mathcal{B}$, we set $y = 0$ (Fig. 3). As any 4-dimensional spherically symmetric manifold can be represented as a product of two dimensional spacetimes and round 2-spheres whose total surface areas are given by $4\pi r^2$ using a radius function $r(x^a)$, we choose the area function r such that the common intersections of $\mathcal{N}_w$ and $\mathcal{B}$ have the area $4\pi r^2(w, y = 0)$ for every value of w. The coordinates x^A are the standard spherical angles of the spheres.

A four dimensional metric with signature -2 adapted to the above is given by

$$g_{ab}dx^a dx^b = Wdw^2 + 2dwdy - r^2 q_{AB}dx^A dx^B, \tag{104}$$

and its inverse metric is given by

$$g^{ab}\partial_a\partial_b = 2\partial_w\partial_y - W\partial_y^2 - \frac{q^{AB}\partial_A\partial_B}{r^2}, \tag{105}$$

where $q_{AB} = \text{diag}(1, \sin^2\theta)$, $W|_{y=0} = 0$ because the coordinate surface $y = 0$ is the null hypersurface $\mathcal{B}$. We remark that the affine parameter y has the gauge freedom $y \to A(w) + B(w)y$. On $\mathcal{B}$ we have the additional freedom to choose w as an affine parameter along its generators giving the further condition $W_{,y} = 0$[3] [14,22]. Ingoing (n) and outgoing (ℓ) null vectors in terms of the metric fields are

$$\ell^a = (\partial_y)^a, \qquad n^a = (\partial_w)^a + \frac{W}{2}(\partial_y)^a, \tag{106}$$

whose expansion rates are[4]

$$\Theta_\ell = \nabla_a \ell^a = \partial_y \ln r^2, \tag{107}$$

$$\Theta_n = \nabla_a n^a = \partial_w \ln r^2 + \frac{(Wr^2)_y}{2r^2}. \tag{108}$$

We consider the existence of a covector field $A_a = (A_w, A_y, 0, 0)$ forming the Faraday tensor $F_{ab} = 2A_{[b,a]}$ of an electromagnetic field in vacuum. The Faraday tensor has the gauge freedom in $A_a \to A_a + \chi_{,a}$ with the real scalar field χ allowing us to choose $A_y = 0$ everywhere, by selecting a χ such that [15,23]

$$\chi(w, y) = -\int_0^y A_y(w, \tilde{y})d\tilde{y}. \tag{109}$$

Hence, we have in the adapted null gauge

$$A_a = \alpha(w, y)dw. \tag{110}$$

Here, we formulate a metric-based spherically symmetric characteristic initial value problem with respect to Σ_w, $\mathcal{B}$

[3]On $\mathcal{B}$, the null vector $n^a|_{\mathcal{B}} = (\partial_w)^a$ satisfies the geodesic equation $n^b\nabla_b n^a = -\frac{1}{2}W_{,y}n^a$.

[4]In order to preserve notation of previous works [14] (where $y = \lambda$), in this section we will not make use of the GHP notation. However, note that in general Θ_ℓ and Θ_n are related to the GHP scalars ρ, ρ', ϵ and ϵ' by: $\Theta_\ell = -(\rho + \bar{\rho}) + \epsilon + \bar{\epsilon}$, $\Theta_n = -(\rho' + \bar{\rho}') + \epsilon' + \bar{\epsilon}'$ respectively. In particular, in the case that ℓ, n are affine parametrized, $\epsilon + \bar{\epsilon} = \epsilon' + \bar{\epsilon}' = 0$.

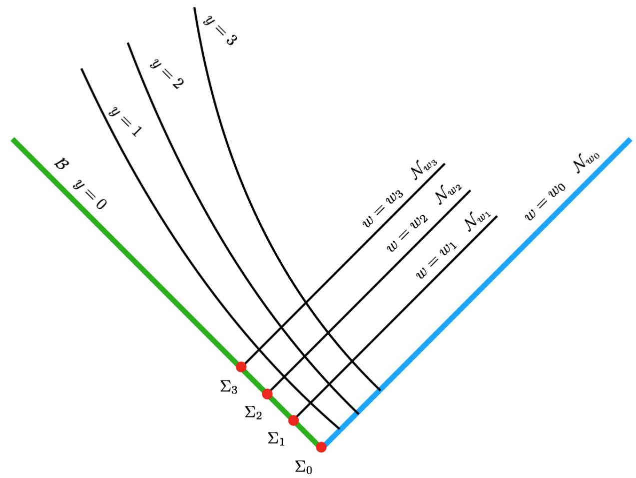

FIG. 3. A graphical representation of the geometry for the spherically symmetric characteristic initial value formulation. The boundary surface $\mathcal{B}$ where $y = 0$ is given in green while the initial data surface $\mathcal{N}_0$ where $w = w_0$ is light blue. Every point in this diagram represents a two-dimensional sphere, in particular the red dots correspond to the two-dimensional common cross sections of the family of null hypersurfaces $\mathcal{N}_w$ with the boundary $\mathcal{B}$. In fact any other intersections of surfaces $y =$ const and $w =$ const are two-dimensional spaces. Note that in general, the surfaces $y =$ const are not null hypersurfaces, like in a double null formulation where the surfaces $y =$ const would be displayed parallel to one another.

and $\mathcal{N}_w$ for the Einstein-Maxwell equations for the metric (104). This is an extension of [12,14] to the Einstein-Maxwell case in spherical symmetry, while a formulation for an Einstein-scalar field system can be found in [13]. We provide initial boundary values for the metric and Maxwell field on a null hypersurface $w_0 =$ const, on $\mathcal{B}$ given by $y = 0$ and on the intersection Σ_{w_0} characterized by $w = w_0$ with w_0 and $y = 0$. The field equations are

$$R_{ab} = 8\pi\left(T_{ab} - \frac{1}{2}g_{ab}T^c_c\right), \qquad \nabla_a T^{ab} = 0,$$

where R_{ab} is the Ricci tensor and T_{ab} is the energy momentum tensor (for negative metric signature)

$$T_{ab} = -\frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F^{cd}F_{cd}\right), \qquad T^c_c = 0,$$

determined by the Maxwell field F_{ab}.

The divergence free condition of the energy momentum tensor gives the vacuum Maxwell equations $\nabla_a F^{ab} = 0$ which can be grouped into a hypersurface equation

$$0 = \frac{1}{r^2}(r^2\alpha_{,y})_{,y}, \tag{111}$$

assumed to hold everywhere on the family $\mathcal{N}_w$ and a supplementary equation on $\mathcal{B}$

$$0 = \frac{1}{r^2}(r^2\alpha_{,y})_{,w}\Big|_{\mathcal{B}}. \tag{112a}$$

The supplementary equation holds everywhere provided the hypersurface equation are fulfilled everywhere [23].

Similarly, the twice contracted Bianchi identities allow us to group the Einstein equations into one supplementary equation on $\mathcal{B}$, one hypersurface equation with no w–derivatives and one evolution equation, respectively, for the metric fields [12,14]

$$0 = \left(-\frac{2r_{,ww}}{r}\right)\Big|_{\mathcal{B}}, \tag{112b}$$

$$0 = -\frac{2r_{,yy}}{r}, \tag{112c}$$

$$0 = [y + 2rr_{,w} - Wrr_{,y}]_{,y} - r^2\alpha^2_{,y} \tag{112d}$$

Provided the hypersurface equation (112c) and evolution equation (112d) hold on the family $\mathcal{N}_w$, the supplementary equation (112b) holds on $\mathcal{N}_w$, provided it is fulfilled on $\mathcal{B}$ [23].

The particular grouping of the Maxwell and Einstein equations allows us to set up a characteristic initial boundary value problem for a family of null hypersurfaces $w > w_0 =$ const with an initial null hypersurface $\mathcal{N}_{w=w_0}$, the null boundary surface $\mathcal{B}$ and the common intersection of Σ_{w_0} of $\mathcal{N}_{w_0}$ and $\mathcal{B}$, (Fig. 3).

Since the spacetime is spherically symmetric, the shear tensors of two null hypersurfaces $\mathcal{B}$ and $\mathcal{N}_{w_0}$ must vanish and the intrinsic metrics on those null hypersurfaces must have the form $-r(w, y)^2 q_{AB}$. Therefore, only the expansion rate of $\mathcal{N}_w$ and $\mathcal{B}$ have nontrivial physical meaning. Since the two expansion rates evaluated on $\mathcal{B}$ and $\mathcal{N}_w$ determine the first derivatives of the conformal factor r, and as r completely determines the intrinsic properties of the two null hypersurfaces, it is natural to take r and the fields[5]

$$N \coloneqq r_{,w}, \qquad \Theta \coloneqq r_{,y}, \tag{113}$$

as fundamental metric variables for a characteristic initial value formulation of the Einstein–Maxwell equations. Regarding the Maxwell field, only the derivatives $\alpha_{,a}$ of the vector potential A_a come into question as fundamental matter variables. Indeed, calculation of F_{ab} reveals that only $\alpha_{,y}$ is needed to set up F_{ab} in the chosen null gauge.

[5]The variable N is called ρ in [12–14], we have chosen a different name here to not confuse it with the spin coefficients.

In fact, because Σ_{w_0} is the common intersection of $\mathcal{B}$ and $\mathcal{N}_{w_0}$, the fundamental matter and metric variables should be prescribed as independent constants on Σ_{w_0},

$$A \coloneqq \alpha_{,y}|_{\Sigma_{w_0}}, \qquad r_0 \coloneqq r|_{\Sigma_{w_0}},$$
$$N_0 = N|_{\Sigma_{w_0}}, \qquad \Theta_0 = \Theta|_{\Sigma_{w_0}}, \tag{114}$$

Due to the spherical symmetry, it is possible to relate (before evaluation of the field equations on the boundary) A with the charge Q on Σ_{w_0} in the following way; The charge distribution on the cross section Σ_{w_0} is calculated according to

$$Q \coloneqq Q|_{\Sigma_{w_0}} = \int_{\Sigma_{w_0}} j^a\Sigma_a = \frac{1}{4\pi}\int_{\Sigma_{w_0}} F^{ab}{}_{;b}d\Sigma_a$$
$$= \frac{1}{8\pi}\int_{\Sigma_{w_0}} F^{ab}d\Sigma_{ab}, \tag{115}$$

where Q is the total charge. Using the surface element of the 2-surface Σ_{w_0}, $d\Sigma_{ab} = 2\ell_{[a}n_{b]}\sqrt{\det(g_{AB})}d\theta d\phi$, we find

$$Q = \frac{1}{4\pi}\int_{\Sigma_{w_0}} F^{ab}r^2\sin\theta d\theta d\phi = r_0^2 A, \tag{116}$$

Thus, we set

$$\alpha_{,y}|_{\Sigma_{w0}} = A = \frac{Q}{r_0^2}, \tag{117}$$

hereafter.

Following [12–14], introduction of

$$Y = W - \frac{2r_{,w}}{r_{,y}}, \tag{118}$$

casts (112d) into a hypersurface equation

$$0 = 1 - (r\Theta Y)_{,y} - r^2\alpha_{,y}^2. \tag{119}$$

The hypersurface equation (112c) shows that its integration requires the knowledge of $r|_{\mathcal{B}}$ and $\Theta|_{\mathcal{B}}$, which are not known on $\mathcal{B}$ (the supplementary equations (112b) only allows one to propagate N along $\mathcal{B}$). But if (112d) is evaluated on $\mathcal{B}$, we have

$$0 = [1 + 2(r_{,w}r_{,y} + rr_{,wy}) - r^2\alpha_{,y}^2]|_{\mathcal{B}}, \tag{120}$$

allowing us to propagate Θ along the boundary, i.e.,

$$(2r\Theta)_{,w}|_{\mathcal{B}} = (-1 + r^2\alpha_{,y}^2)|_{\mathcal{B}}. \tag{121}$$

Moreover, (120) allows us to find algebraically the mixed derivative

$$\mu = r_{,wy} \tag{122}$$

everywhere on $\mathcal{B}$ provided we know r, Θ, N and $\alpha_{,y}$ on $\mathcal{B}$,

$$\mu|_{\mathcal{B}} = \left.\frac{-1 - 2N\Theta + r^2\alpha_{,y}^2}{2r}\right|_{\mathcal{B}}, \tag{123}$$

In fact, evaluation of (123) on the cross section $y = 0$ and $w = w_0$ gives us

$$\mu_0 \coloneqq \mu|_{\Sigma_{w_0}} = \frac{1}{2r_0}\left(\frac{Q^2}{r_0^2} - 1 - 2N_0\Theta_0\right). \tag{124}$$

where we used (114) and (117). Taking the w derivative of Eq. (112c), we obtain an equation to propagate μ off the boundary $\mathcal{B}$

$$\mu_{,y} = 0. \tag{125}$$

We further note that the definition of $\mu = r_{,wy} = N_{,y}$ serves as an additional hypersurface equation to propagate N off the boundary $\mathcal{B}$,

$$N_{,y} = \mu. \tag{126}$$

Now we are in position to spell out the basic equations for the spherically symmetric Einstein-Maxwell system in a metric null-affine formulation. Given the data (114) and (117) on a cross section Σ_{w_0}, the boundary data are determined by the hierarchical set of equations

$$r_{,ww}|_{\mathcal{B}} = N_{,w}|_{\mathcal{B}} = 0, \tag{127a}$$

$$(r^2\alpha_{,y})_{,w}|_{\mathcal{B}} = 0, \tag{127b}$$

$$(2r\Theta)_{,w}|_{\mathcal{B}} = (-1 + r^2\alpha_{,y}^2)|_{\mathcal{B}}, \tag{127c}$$

together with the constraint (123) and the initial value for Y,

$$Y|_{\mathcal{B}} = -\left.\frac{2N}{\Theta}\right|_{\mathcal{B}} \tag{128}$$

determined from (118) evaluated on $\mathcal{B}$. The hypersurface equations are summarized by the following hierarchy

$$r_{,yy} = \Theta_{,y} = 0, \tag{129a}$$

$$(r^2\alpha_{,y})_{,y} = 0, \tag{129b}$$

$$(r\Theta Y)_{,y} = 1 - r^2\alpha_{,y}^2, \tag{129c}$$

$$\mu_{,y} = 0, \tag{129d}$$

$$N_{,y} = \mu. \tag{129e}$$

The missing metric field W is found from the definition (118).

B. Solution of the hierarchy of equations

We now solve this system with initial values (114) and (117) and begin with the boundary equations (127). Its general solution is

$$r(w,0) = r_0 + N_0(w - w_0), \tag{130a}$$

$$N(w,0) = N_0, \tag{130b}$$

$$\alpha_{,y}(w,0) = \frac{Q}{r^2(w,0)}, \tag{130c}$$

$$\Theta(w,0) = \frac{1}{r(w,0)}\left[r_0\Theta_0 + \frac{1}{2}\left(\frac{Q^2}{r_0 r(w,0)} - 1\right)(w - w_0)\right], \tag{130d}$$

together with the relations from the algebraic constraints

$$\mu(w,0) = \frac{1}{2r(w,0)}\left[-1 - 2N_0\Theta(w,0) + \frac{Q^2}{r^2(w,0)}\right] \tag{130e}$$

$$Y(w,0) = -\frac{2N_0}{\Theta(w,0)} \tag{130f}$$

With these boundary values at hand, the solution of the hypersurface equations (129) is

$$\Theta(w,y) = \Theta(w,0) \coloneqq \Theta(w), \tag{131a}$$

$$r(w,y) = r_0 + N_0(w - w_0) + \Theta(w)y \tag{131b}$$

$$\alpha_{,y}(w,y) = \frac{Q}{r^2(w,y)} \tag{131c}$$

$$Y(w,y) = \frac{-2N_0 r(w,0)}{r(w,y)\Theta(w)} + \frac{y}{r(w,y)\Theta(w)}\left(1 - \frac{Q^2}{r(w,0)r(w,y)}\right)] \tag{131d}$$

$$\mu(w,y) = \mu(w,0) \coloneqq \mu(w), \tag{131e}$$

$$N(w,y) = \mu(w)y + N_0. \tag{131f}$$

Having calculated Θ, Y and N we are in position to find the missing metric field

$$W(w,y) = \frac{y[1 + 2\mu(w)r(w,y) + 2N_0\Theta(w)]}{r(w,y)\Theta(w)} - \frac{yQ^2}{r^2(w,y)r(w,0)\Theta(w)}, \tag{132}$$

which not only vanishes on $\mathcal{B}$, thus assuring that $\mathcal{B}$ is a null hypersurface, but also satisfies the other gauge condition $W_{,y}|_{\mathcal{B}} = 0$.

In a spherically symmetric spacetime the mass of a system is given by the Misner-Sharp mass, m, defined covariantly via

$$1 - \frac{2m(r)}{r} = -g^{ab}r_{,a}r_{,b}, \tag{133}$$

where the minus sign in the right-hand side comes from the signature of the spacetime. The Misner-Sharp mass is a special case for the quasilocal Hawking mass on a given 2-surface Σ [24][6]

$$m_H(\Sigma) = \sqrt{\frac{A}{16\pi}}\left(1 + \frac{1}{16\pi}\int_\Sigma \Theta_\ell\Theta_n dS\right), \tag{134}$$

where A is its area, dS its surface area element and Θ_ℓ and Θ_n are the expansion rates of two orthogonal null vectors that are orthogonal to Σ. In spherical symmetry the Hawking mass and Misner-Sharp mass coincide, which can be seen by evaluation of the two on Σ_0 resulting in

$$\begin{aligned} m|_{\Sigma_0} = m_H(\Sigma_0) &= \frac{r}{2}(1 + g^{ab}r_{,a}r_{,b})|_{\Sigma_0} \\ &= \frac{r_0}{2}(1 + 2N_0\Theta_0) \end{aligned} \tag{135}$$

while using that

$$\Theta_\ell|_{\Sigma_0} = 2\frac{\Theta_0}{r_0}, \qquad \Theta_n|_{\Sigma_0} = 2\frac{N_0}{r_0}. \tag{136}$$

From Eqs. (132), (133) and (135) we can also recover the well known relation between the quasilocal Misner-Sharp mass $m|_{\Sigma_0}$ and the Bondi mass m_B. More precisely, on an arbitrary null hypersurface $w = \text{const}$, the Misner-Sharp mass (133) gives in the "proper" asymptotic limit the constant Bondi mass $m_B(w) = m_B = \text{const}$,

$$\begin{aligned} m_B(w) &\coloneqq \lim_{\substack{y\to\infty \\ w=\text{const}}} \frac{r}{2}(1 + g^{ab}r_{,a}r_{,b}) \\ &= \frac{r_0}{2}(1 + 2N_0\Theta_0) + \frac{Q^2}{2r_0} \\ &= m|_{\Sigma_0} + \frac{Q^2}{2r_0} \equiv m_B = \text{const}. \end{aligned} \tag{137}$$

From now on, we will refer to the Bondi mass m_B simply as m.

Until now the two surfaces $\mathcal{B}$ and $\mathcal{N}_{w_0}$ have been arbitrary. With different choices for the corresponding

[6]The original reference gives a general form, (134) is its specialization to maximal symmetry as e.g., found in [25].

initial data we can obtain the nonextremal and extremal Reissner-Nordström spacetimes. Let us consider these separate cases.

1. The nonextremal Reissner-Nordström metric

Let us choose now the data on Σ_0 such that $N_0 = 0$, and $\Theta_0 \neq 0$. As follows of (130a), this choice implies that $r(w,0) = r_0 = \text{const} \neq 0$. Then, evaluation of (131e) shows the constancy of the field μ everywhere

$$\mu_0 = \mu(w) = -\frac{r_0^2 - Q^2}{2r_0^3}. \tag{138}$$

Consequently, the null hypersurface $\mathcal{B}$ is free of expansion, i.e., $\Theta_n|_{\mathcal{B}} = 0$.

With respect to this expansion-free null boundary, we see that the metric functions reduce to

$$r = r_0 + y[\Theta_0 + \mu_0(w - w_0)], \tag{139}$$

$$W = \frac{2y(\mu_0^2(w - w_0) + \mu_0\Theta_0) + 4r_0\mu_0 + 1}{\{r_0 + [\Theta_0 + \mu_0(w - w_0)]y\}^2} y^2. \tag{140}$$

Also note that from (131a) and (130d) follows the existence of a null hypersurface $w = \hat{w}$ in $\mathcal{N}_w$ such that $\Theta(\hat{w}, y) = 0$, meaning

$$\Theta(\hat{w}, y) = \frac{1}{r_0}\left[r_0\Theta_0 + \frac{1}{2}\left(-1 + \frac{Q^2}{r_0^2}\right)(\hat{w} - w_0)\right] \stackrel{!}{=} 0, \tag{141}$$

which implies

$$\hat{w} = \frac{-\Theta_0 + \mu_0 w_0}{\mu_0}. \tag{142}$$

The coordinate transformation $w \to \tilde{w} = w - \hat{w}$ shifts the affine parameter w of the geodesics on $\mathcal{B}$ such that the metric components have a particular simple form

$$r = r_0 + \mu_0 w y, \tag{143}$$

$$W = \frac{1 + 2\mu_0(r + r_0)}{r^2} y^2, \tag{144}$$

independent of the values of Θ_0 and w_0 where we have dropped the tilde to avoid a further complication of the notation.

Using Eq. (137) we can relate the constant r_0 with the Bondi mass m of the metric:

$$\frac{Q^2 + r_0^2}{2r_0} = m. \tag{145}$$

with solutions

$$r_0(\epsilon) = m + \epsilon\sqrt{m^2 - Q^2}, \tag{146}$$

where $\epsilon = \pm 1$. We see that $r_0(\epsilon = 1)$ is nothing else than the value(s) r_H^+ of the event horizon of the Reissner-Nordström solution. Note also, that the value of μ_0 given by (138) agrees with the value of minus the surface gravity, $-k_H$, of this spacetime. Indeed, this solution is exactly the same as (40) presented in the Sec. III A which was obtained using the coordinate transformation from Bondi to the regular ones $\{w, y\}$.

2. The extremal Reissner-Nordström

For the extremal case $Q = m$ we can proceed in two different ways.

Data I: Choosing again $N_0 = 0$ and $\Theta_0 \neq 0$ on Σ_0, we can use the results of the previous Subsec. V B 1 obtaining for the extremal case:

$$\mu(w) = \mu_0 = 0, \tag{147}$$

$$r_0 = m, \tag{148}$$

$$r = m + \Theta_0 y, \tag{149}$$

$$W = \frac{y^2}{r^2}, \tag{150}$$

$$\Theta(w, y) = \Theta_0. \tag{151}$$

We recognize that the resulting coordinates (w, y) are basically the Bondi coordinates (u, r) after a rescaling of u and a reparametrization of the affine parameter r: $w = \Theta_0 u$, $r = m + \Theta_0 y$, with metric

$$ds^2 = \left(1 - \frac{m}{r}\right)^2 du^2 + 2dudr - r^2 d\Omega^2. \tag{152}$$

However, we know that this coordinate system is not regular at the horizon $\mathcal{H}^+$, in particular there is not a $w = \hat{w}$ value where $\Theta(\tilde{w}, y) = 0$. Hence, in order to find a regular coordinate system at $\mathcal{H}^+$ we must set the initial data differently.

Let us also remark that in the limit $\Theta_0 \to 0$ on Σ_0 we obtain from Eqs. (148)–(151),

$$r = m = Q, \tag{153}$$

$$W = \frac{y^2}{m^2} = \frac{y^2}{Q^2}, \tag{154}$$

$$\alpha = \frac{y}{Q}, \tag{155}$$

giving as a limit the metric

$$ds^2 = \frac{y^2}{Q^2}dw^2 + 2dwdy - Q^2 d\Omega^2, \qquad (156)$$

with electromagnetic potential vector $A_a = \frac{y}{Q}dw$.

This metric is the well-known Bertotti-Robinson solution which represents a shear and expansion free conformally flat solution of the Einstein-Maxwell equations with uniform electromagnetic field [26–29].

Data II: Let us choose both N_0 and Θ_0 different from zero on Σ_0. From Eq. (137) we obtain for the extremal case

$$\Theta_0 = -\frac{(m-r_0)^2}{2N_0 r_0^2}. \qquad (157)$$

After replacing this expression into the Eqs. (130a), (130d) and (131a) we obtain,

$$\Theta(w,y) = \Theta(w,0) = -\frac{[N_0(w-w_0) - m + r_0]^2}{2N_0[r_0 + N_0(w-w_0)]^2}. \qquad (158)$$

Therefore, there exists a null hypersurface $w = \hat{w}$ in $\mathcal{N}_w$ such that $\Theta(\hat{w}, y) = 0$, which implies

$$\hat{w} = \frac{m - r_0 + N_0 w_0}{N_0}. \qquad (159)$$

As in V B 1, the coordinate transformation $w \to \tilde{w} = w - \hat{w}$ shifts the affine parameter w of the geodesics on $\mathcal{B}$ such that from the set of equations (130a)–(130f), (131a)–(131f) and (132) we obtain for the metric components r and W the following expressions (dropping again the tilde in w for simplicity in the notation):

$$r = m + N_0 w - \frac{N_0 w^2 y}{2(N_0 w + m)^2}, \qquad (160)$$

$$W = -\frac{4(2w^4N_0^4 + 5w^3N_0^3m + 3w^2N_0^2m^2 - wN_0m^3 - w^3yN_0^2 - m^4)my^2}{(N_0w+m)(2w^3N_0^3 + 6w^2N_0^2m + 6wN_0m^2 + 2m^3 - w^2N_0y)^2}, \qquad (161)$$

which agrees with the solution presented in Sec. III B after the identification $N_0 = -E_0$.[7] Note again, that near the horizon $\mathcal{H}^+$, r and W behave as:

$$r = m + \mathcal{O}(w), \qquad (162)$$

$$W = \frac{y^2}{m^2} + \mathcal{O}(w), \qquad (163)$$

in correspondence with the well-known result that an extremal Reissner-Nordström black hole near the horizon looks like the Bertotti-Robinson metric Eq. (156) [30].

VI. SUMMARY AND OUTLOOK

In this work, we have discussed two different approaches to study coordinate charts of black hole spacetimes that are regular at the black hole horizon and at large distances toward null infinity. In particular, we discuss these charts in spherical symmetry. The first approach consists in defining a suitable coordinate pair (w, y), which obeys geometric conditions as specified in Sec. II. Of particular importance in the construction of this pair is the Bondi coordinate pair (u, r) where u takes the value of the retarded time in the limit toward null infinity and r is the affine parameter on surfaces $u = \text{const}$. The pair (w, y) is then constructed by remapping the affine parameter r of the surfaces $u = \text{const}$ to another parameter y. Unlike (u, r), (w, y) can be defined from the horizon to null infinity when caustics are not present. Within this formalism, we find (i) regular representations of static and spherically symmetric black holes, including solutions for extremal/nonextremal Reissner Nordström black holes, (ii) the outgoing Vaidya metric depending on a mass function $m(w)$ that has an extremum at $w = 0$ and (iii) an ingoing Vaidya solution of a collapsing shell. All those solutions are explicit with respect to the (w, y) pair, allowing thus to write exemplary black hole spacetimes using null coordinates in an explicit, rather than implicit, way from the horizon to null infinity.

In the second approach, we have started out with a general affine-null coordinate system. In this affine-null chart, we set up a characteristic initial value problem[8] for the field equations in spherical symmetry for an Einstein-Maxwell system. The derived equations form a hierarchical system of equations that can be solved with data given on a common intersection of two null hypersurfaces. There are in principle four free parameters that can be specified in the spherically symmetric case discussed here. Nevertheless, after fixing the values of the expansion rates of the in and outgoing null vectors at the common intersection, the final solution provides us with the Reissner-Nordström black hole whose line element is given by (20) for the

[7] This identification follows from the definition of N in Eqs. (113) and (53) (with $w \equiv \lambda$); both valued at Σ_0.

[8] A past value problem is set up in a similar way.

nonextremal case, and determined by the metric components Eqs. (160) and (161) in the extremal situation.

For the nonextremal case, we can pursue with a compactification procedure for (40) like for (1). If we make the rescaling $w \to k_H w$ and $y \to (k_H \Upsilon)^{-1}$ with k_H the surface gravity of Reissner-Nordström black hole together with a suitable conformal factor $\Omega = k_H \Upsilon$, the resulting conformal metric has the expansion at $\mathcal{I}^+$, i.e., at $\Upsilon = 0$, like (2).

To our knowledge, our presentation is the first to construct a Reissner-Nordström black hole in Israel like coordinates directly by solving the Einstein equations using a characteristic initial value problem, while other approaches relied on coordinate transformations [6] or using two-dimensional generalized dilaton gravity models [8] (also see for discussion [9]).

Both of the two approaches have their regime in which they are most useful. The framework starting out with a Bondi coordinate u is useful (as we have demonstrated) if particular solutions of the Einstein equations are known. Then, the null coordinate w and its affine parameter y can be directly constructed locally. As we have restricted the framework to spherical symmetry, further work is needed to extend it to more general spacetimes, for example axisymmetric spacetimes. Regarding the coordinates, we expect a condition not only linking r with y, but also also additional relationships between the angular coordinates parametrizing the cuts $u = \text{const}$ at different values of r with those of cuts of w for given values of y. In particular, in the study of the Kerr metric in [31], the existence of a double null family of hypersurfaces was used to define the corresponding null coordinates. However, to define the complete coordinate system, one has to solve two issues. First, the pair of null functions have a functional dependence on a scalar defined as the solution of a particular partial differential equation that depends on the radial and nonsymmetric angular coordinate. Second, the original Boyer-Lindquist symmetric angular coordinate must be changed [31] to a new one that is well behaved near and at the horizon. Therefore, in extending the present work to axis-symmetric spacetimes, one has to be prepared to address these issues. Indeed, in their discussion of the gravitational wave memory effect of boosted Kerr-Schild black holes [32,33], the authors have shown in [32] using a Penrose compactification scheme on uncharged Kerr-Schild metrics, that an asymptotic Bondi frame can only be properly constructed if the angular coordinates issue is taken into account. It remains to be of further study, how the condition of Sec. II need to be specified for more general systems.

The second approach is most useful numerically, i.e., when we wish to find a solution of the Einstein equation given certain initial values. Recently, Crespo and collaborators [13] have used an affine-null formulation in spherical symmetry with a scalar field to study the Choptuik critical solution. Therein, the authors have integrated the Einstein-scalar field equations numerically, also using a hierarchical set of equations.[9]

We would like to make a few comments about possible generalizations of the second approach to more general spacetimes. In [14] it was shown that, for vacuum Einstein's equations, a null affine formulation can be set as a hierarchical system. However its extension to spacetimes containing matter fields could be nontrivial. Consider, as one example, the Einstein scalar field equations for a general affine null metric g_{ab} and a massless scalar field Φ. In this case, we have the nonzero contravariant components of the metric g^{ya} and g^{AB} with $|g^{wy}| = 1$, and $\sqrt{\det g_{AB}} = r^2$ [14]. The initial data required to solve the field equations for g_{ab} and Φ is given by Φ and the transverse traceless part of $h^{AB}_{,y}$ on a given null hypersurface $\mathcal{N}_{w_0}$ (for the following discussion we do not need the boundary data). For the vacuum case, it was shown in [12,14] that the main equations form a hierarchical system. If the main equations also contain matter terms, this is also formally true, as it can be inferred from a similar analysis using Bondi-Sachs coordinates [34–37]. Nevertheless, the main equations, become coupled because of the additional requirement of a divergence-free energy momentum tensor. In particular, for the example of a massless scalar field, we must satisfy the scalar wave equation $\nabla^a \nabla_a \Phi = 0$. This equation has nontrivial coupling terms between the metric components and derivatives of Φ, which should be uncoupled in order to restore the hierarchy. Notwithstanding, it is worthwhile to remark that for perfect fluids with a Newtonian limit the metric variables of a Bondi-Sachs metric can be obtained via a hierarchical integration of initial data consisting of the fluid variables and the Newtonian potential [34–36]. It remains to be seen if this is also possible in the null affine formulation and more generally, if it can be achieved in other situations or under other symmetry assumptions.

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[9]Their equations, however, are different from ours, because the matter terms enter into the r–hypersurface equation (112c) rather than the W–hypersurface equation (112d) as in our case.

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