Ultracompact rotating gravastars and the problem of matching with Kerr spacetime

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A number of authors provided arguments that a rotating gravastar is a good candidate for a source of the Kerr metric. These arguments were based on the second order perturbation analysis. In the following paper, we construct a perturbative solution of the rotating gravastar up to the third perturbation order and show that once we demand finiteness of the Kretschmann scalar expansion, it cannot be continuously matched with the Kerr spacetime.

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I. INTRODUCTION

Gravastars, proposed by Mazur and Mottola [1] as an alternative to black holes, have been studied extensively in the recent years [2–9]. One of the issues concerning gravastars is to find a rotating gravastar solution. So far only perturbative versions of such a solution exist [10–12]. These studies indicate that in the ultracompact limit [13] the rotating gravastar can be a source of the Kerr metric (i.e., I, Love, Q numbers tend to those of Kerr in this limit). Similar perturbation-type sources (thin shells) of the Kerr metric were studied earlier by, e.g., [14–16]. On the other hand, constructing perturbation sources of the Kerr metric have been criticized by Krasiński [17].

In this work, we take perturbation approach to check if the matching of the gravastar with the Kerr spacetime survives at higher orders. It means that we want to construct a rotating analogue of [13] with the Kerr spacetime outside. We use slightly different framework to [10–12] and instead of solving Einstein equations both for interior and exterior, we *a priori* assume that an exterior solution is the Kerr metric. Then we seek for an interior solution and try to match it with the Kerr metric.

Most of the work on rotating gravastars was based on Hartle's structure equations [18] (see also [19–21]). Hartle's framework allows to study slowly rotating perfect fluid objects up to the second order in the angular momentum. To go beyond the second order, we find it easier to follow Rostworowski [22], who provided a nonlinear extension of Regge-Wheeler and Zerilli formalisms. Formalism given by [22] is dedicated to (Λ-) vacuum spacetimes and can be easily adapted to our needs. The difference between Hartle's framework and our approach is only on the level of ansatz on metric perturbation form and they are physically equivalent within the range of applicability of Hartle's framework. We find that [22] provides a very powerful tool for dealing with nonlinear perturbations. Although in the present article we describe perturbation analysis only up to the third order, we solved Einstein equations up to the sixth order to calculate the Kretschmann scalar and we think it's possible to go further if needed.

The paper is organized as follows: in Secs. II–IV we provide preliminaries, in Sec. V we discuss the matching, in Sec. VI we expand the Kerr metric, in Secs. VII and VIII we solve interior Einstein equations and try to match interior and exterior metrics and in Sec. IX we summarize and discuss our calculations.

II. BACKGROUND SOLUTION

As a background, we take the ultracompact gravastar model [13]. In static coordinates $(t,\ r,\ u,\ \varphi)$, where $u = \cos\theta$, its metric is given by:

$$\bar{g} = f(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2\left(\frac{du^2}{1-u^2} + (1-u^2)d\varphi^2\right), \tag{1}$$

where

$$f(r) = \begin{cases} \frac{1}{4}\left(1 - \frac{r^2}{4M^2}\right) & r \le R, \\ 1 - \frac{2M}{r} & r > R, \end{cases} \tag{2}$$

$$h(r) = \begin{cases} 1 - \frac{r^2}{4M^2} & r \le R, \\ 1 - \frac{2M}{r} & r > R. \end{cases} \tag{3}$$

An induced metric is continuous across the (null) matching surface $r = 2M$. There is a nonzero stress-energy tensor induced on this shell, see [13] for the details. The exterior metric is a solution to vacuum Einstein equations and the interior metric is a solution to Einstein equations with a

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cosmological constant $\Lambda = \frac{3}{4M^2}$. Both interior and exterior metrics are singular at $r = 2M$. To keep them regular, also in higher perturbation orders, we use Eddington–Finkelstein (EF) coordinates (v, r, u, φ). Interior metric in EF coordinates reads:

$$\bar{g} = \frac{1}{4}\left(1 - \frac{r^2}{4M^2}\right)dv^2 + drdv + r^2\left(\frac{du^2}{1-u^2} + (1-u^2)d\varphi^2\right). \tag{4}$$

and exterior metric in EF coordinates reads:

$$\bar{g} = \left(1 - \frac{2M}{r}\right)dv^2 + 2drdv + r^2\left(\frac{du^2}{1-u^2} + (1-u^2)d\varphi^2\right). \tag{5}$$

III. POLAR EXPANSION

In a spherically symmetric background, in 3 + 1 dimensions, vector and tensor components split into two sectors: polar and axial (for the details see e.g., [23–27]). Symmetric tensors have 7 polar and 3 axial components. Below we list the expansion of the components of symmetric tensors in axial symmetry (P_ℓ denotes the ℓth Legendre polynomial). In the polar sector we have:

$$S_{ab}(r,u) = \sum_{0\le\ell} S_{\ell ab}(r)P_\ell(u), \qquad a,b = v,r, \tag{6}$$

$$S_{au}(r,u) = -\sum_{1\le\ell} S_{\ell au}(r)\partial_u P_\ell(u), \qquad a = v,r, \tag{7}$$

$$\frac{1}{2}\left((1-u^2)S_{uu}(r,u) + \frac{S_{\varphi\varphi}(r,u)}{(1-u^2)}\right) = \sum_{0\le\ell} S_{\ell+}(r)P_\ell(u), \tag{8}$$

$$\frac{1}{2}\left((1-u^2)S_{uu}(r,u) - \frac{S_{\varphi\varphi}(r,u)}{(1-u^2)}\right) = \sum_{2\le\ell} S_{\ell-}(r)(-\ell(\ell+1)P_\ell(u) + 2u\partial_u P_\ell(u)). \tag{9}$$

In the axial sector we have:

$$S_{a\varphi}(r,u) = \sum_{1\le\ell} S_{\ell a\varphi}(r)(-1+u^2)\partial_u P_\ell(u), \qquad a = v,r, \tag{10}$$

$$S_{u\varphi}(r,u) = \sum_{2\le\ell} S_{\ell u\varphi}(r)(\ell(\ell+1)P_\ell(u) - 2u\partial_u P_\ell(u)). \tag{11}$$

IV. METRIC PERTURBATIONS

We assume that there exists an exact, stationary and axially symmetric solution to Einstein equations, which we expand into series in a parameter a (which will be an angular momentum per unit mass of a an exterior metric) around the static metric (2):

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \sum_{i=1}^{\infty} \frac{a^i}{i!}\,{}^{(i)}h_{\mu\nu} \tag{12}$$

After perturbation expansion we polar–expand metric perturbations according to (6)–(11). Thus, apart from the perturbation index i, all perturbations gain an index ℓ corresponding to the ℓth Legendre polynomial.

For axial perturbations we take:

$${}^{(i)}h_\ell = \begin{pmatrix} 0 & 0 & 0 & {}^{(i)}h_{\ell v\varphi}(r)(-1+u^2)\partial_u P_\ell(u) \\ 0 & 0 & 0 & {}^{(i)}h_{\ell r\varphi}(r)(-1+u^2)\partial_u P_\ell(u) \\ 0 & 0 & 0 & 0 \\ {}^{(i)}h_{\ell v\varphi}(r)(-1+u^2)\partial_u P_\ell(u) & {}^{(i)}h_{\ell r\varphi}(r)(-1+u^2)\partial_u P_\ell(u) & 0 & 0 \end{pmatrix}. \tag{13}$$

Using the gauge freedom, we set ${}^{(i)}h_{\ell u\varphi}(r) = 0$, what corresponds to the Regge-Wheeler (RW) gauge.

For the polar perturbations we take:

$${}^{(i)}h_\ell = \begin{pmatrix} {}^{(i)}h_{\ell vv}(r)P_\ell(u) & {}^{(i)}h_{\ell vr}(r)P_\ell(u) & 0 & 0 \\ {}^{(i)}h_{\ell vr}(r)P_\ell(u) & {}^{(i)}h_{\ell rr}(r)P_\ell(u) & 0 & 0 \\ 0 & 0 & {}^{(i)}h_{\ell+}(r)\frac{P_\ell(u)}{1-u^2} & 0 \\ 0 & 0 & 0 & {}^{(i)}h_{\ell+}(r)(1-u^2)P_\ell(u) \end{pmatrix}. \tag{14}$$

Using the gauge freedom, we set ${}^{(i)}h_{\ell ru} = {}^{(i)}h_{\ell vu} = {}^{(i)}h_{\ell -} = 0$, what also corresponds to the RW gauge. Note that in [18] there are no ${}^{(i)}h_{\ell vr}$ and ${}^{(i)}h_{\ell r\varphi}$ coefficients in the metric ansatz. This fact arises from the fact that Hartle uses static coordinates. For EF coordinates in the background both ${}^{(i)}h_{\ell vr}$ and ${}^{(i)}h_{\ell r\varphi}$ turn out to be nonzero in most cases.

In the interior, we solve perturbation Einstein equations with a cosmological constant $\Lambda = \frac{3}{4M^2}$. For a given order i and a given multipole ℓ, they have the following form:

$$\delta^{(i)}G_{\ell\mu\nu} + \frac{3}{4M^2}{}^{(i)}h_{\ell\mu\nu} = {}^{(i)}S_{\ell\mu\nu}, \tag{15}$$

where $\delta^{(i)}G_{\ell\mu\nu}$ denotes the components of the Einstein tensor expansion built of metric perturbations of order i. ${}^{(i)}S_{\mu\nu}$ denotes a source for the ith order Einstein equations consisting of metric perturbations of orders lower that i. We provide an explicit form of Eqs. (15) in the Appendix A.

V. MATCHING INTERIOR WITH EXTERIOR

We match the exterior metric with the interior metric on a three-dimensional hypersurface located at $r^\pm = r_b^\pm$, where "+" and "−" stand for exterior and interior, respectively. From the first Israel junction condition ([28,29]) we demand the continuity of the induced metric at the matching hypersurface:

$$[[\mathfrak{g}_{ab}]] = 0, \tag{16}$$

where $[[E]] = E^+(r_b^+) - E^-(r_b^-)$. Following [11], we introduce intrinsic coordinates on the three-dimensional hypersurface: $y^a = (V, U, \Phi)$. Then we express interior and exterior coordinates $x^{\pm\mu}$ on a hypersurface in terms of y^a:

$$x^{-\mu}|_{r_b^-} = (A^- V, r_b^-(U), F^-(U), \Phi), \tag{17}$$

$$x^{+\mu}|_{r_b^+} = (A^+ V, r_b^+(U), F^+(U), \Phi), \tag{18}$$

where $r_b^\pm(U) = 2M + \frac{a^2}{M^2}\eta^\pm(U) + \mathcal{O}(a^4)$, $F^\pm(U) = U + \frac{a^2}{M^2}\lambda^\pm(U) + \mathcal{O}(a^4)$. We expand $\eta^\pm$ into $\eta^\pm(U) = \eta_0^\pm + \eta_2^\pm P_2(U)$.

The metric induced on this hypersurface is given by:

$$\mathfrak{g}_{ab}^\pm = \begin{pmatrix} (A^\pm)^2 g_{vv}^\pm & A^\pm g_{vr}^\pm r_b^{\pm\prime}(U) + A^\pm g_{vu}^\pm F^{\pm\prime}(U) & A^\pm g_{v\varphi}^\pm \\ A^\pm g_{vr}^\pm r_b^{\pm\prime}(U) + A^\pm g_{vu}^\pm F^{\pm\prime}(U) & (F^{\pm\prime}(U))^2 g_{uu}^\pm + (r_b^{\pm\prime}(U))^2 g_{rr}^\pm + 2F^{\pm\prime}(U) r_b^{\pm\prime}(U) g_{ru}^\pm & F^{\pm\prime}(U) g_{u\varphi}^\pm + r_b^{\pm\prime}(U) g_{r\varphi}^\pm \\ A^\pm g_{v\varphi}^\pm & F^{\pm\prime}(U) g_{u\varphi}^\pm + r_b^{\pm\prime}(U) g_{r\varphi}^\pm & g_{\varphi\varphi}^\pm \end{pmatrix}. \tag{19}$$

Using the freedom of choice of coordinates V, U, Φ, we set $F^+(U) = U$ and $A^+ = 1$ (see, e.g., [10]). For simplicity, we denote $A^- = A$.

The location of the matching hypersurface is not known *a priori* and $\eta^\pm(U)$ and $\lambda^-(U)$ are unknown functions that need to be found. Our procedure of matching interior and exterior metrics for a given perturbation order is the following:

(1) We solve perturbation Einstein equations for the interior. These solutions contain two constants per ℓ in every perturbation order, but most of these constants need to be set to zero to keep the Kretschmann scalar expansion regular at $r = 0$ and $r = 2M$. However, this is not straightforward to apply, because in our case the singularities in the expansion of the Kretschmann scalar occur in higher perturbation orders than the singularities of the metric itself (in the opposition to the exterior case, e.g., Raposo *et al.* [30]). Therefore, to settle constants in the third order, we solved Einstein equations up to the sixth perturbation order to study behavior of the Kretschmann scalar. Since these expressions are too long to be listed in this paper, we make them available in the *Mathematica* notebook [31].

(2) We act with the general gauge transformation on the interior metric, and then we solve matching conditions (16) for constants arising from Einstein equations, for $\eta^\pm(U)$, $\lambda(U)$, and for gauge components. Finding a proper gauge is a part of the matching problem and using the result of Bruni *et al.* [32], we are able to control the impact of the gauge from the lower perturbation order on the metric functions in the higher perturbation order.

(3) If the matching is successful, we go to the higher perturbation order.

The second junction condition tells about the energy content of the matching hypersurface—already in the background solution there is a thin shell located at $r = 2M$ (since this is a null hypersurface, second junction condition needs to be modified, see [29,13] for the details). However, in the next sections we show that even the first junction condition is not possible to fulfill, therefore we do not find it necessary to discuss second junction condition at all.

VI. KERR METRIC EXPANSION

As an exterior metric, we take the Kerr solution. In the advanced EF coordinates it reads:

$$ds^2 = -\left(1-\frac{2Mr}{a^2u^2+r^2}\right)dv^2 + 2dvdr + \frac{a^2u^2+r^2}{1-u^2}du^2 + (1-u^2)\left(\frac{2a^2Mr(1-u^2)}{a^2u^2+r^2}+a^2+r^2\right)d\varphi^2 + \frac{4aMr(1-u^2)}{a^2u^2+r^2}dvd\varphi + 2a(1-u^2)drd\varphi. \quad (20)$$

Since we solve the interior equations in RW gauge, we prefer to use the Kerr metric in RW gauge as well. To do this, we expand (20) into series in a up to the 3rd order, and then act with the gauge transformations (B1)–(B3) to move to the RW gauge. Finally, we obtain:

$$\begin{aligned} ds^2 = &-\left(\left(1-\frac{2M}{r}\right) - \frac{a^2M(u^2(6M^2-Mr-3r^2)-2M^2+Mr+r^2)}{r^5}\right)dv^2 \\ &+\left(\frac{2a^2M(1-3u^2)}{r^3}\right)dr^2 + \left(\frac{r^2}{1-u^2}+\frac{a^2M(3u^2-1)(2M+r)}{r^2(u^2-1)}\right)du^2 \\ &+\left(r^2(1-u^2)+\frac{a^2M(u^2-1)(3u^2-1)(2M+r)}{r^2}\right)d\varphi^2 \\ &+2\left(1+\frac{a^2M(3u^2-1)(M+r)}{r^4}\right)dvdr + 2\left(\frac{a^3M(1-u^2)(5u^2-1)(9M+5r)}{5r^4}\right)drd\varphi \\ &+2\left(\frac{2aM(1-u^2)}{r} - \frac{a^3M(u^2-1)(M^2(6u^2-2)+M(r-5ru^2)+r^2(1-5u^2))}{r^5}\right)dvd\varphi + \mathcal{O}(a^4), \end{aligned} \quad (21)$$

For simplicity, we omit "+" and "−" coordinate superscripts and use them only when it is necessary to differentiate the interior from the exterior. We expand (21) into series in a. Below we list nonzero components of this expansion after the polar decomposition.

$$\begin{aligned} &{}^{(1)}h^+_{1v\varphi} = -\tfrac{2M}{r}, && {}^{(2)}h^+_{2+} = -\tfrac{4M(2M+r)}{r^2}, \\ &{}^{(2)}h^+_{0vv} = \tfrac{4M^2}{3r^4}, && {}^{(3)}h^+_{1v\varphi} = \tfrac{24M^3}{5r^5}, \\ &{}^{(2)}h^+_{2vv} = \tfrac{4M(6M^2-Mr-3r^2)}{3r^5}, && {}^{(3)}h^+_{3v\varphi} = \tfrac{4M(-6M^2+5Mr+5r^2)}{5r^5}, \\ &{}^{(2)}h^+_{2vr} = \tfrac{4M(M+r)}{r^4}, && {}^{(3)}h^+_{3r\varphi} = -\tfrac{4M(9M+5r)}{5r^4}. \\ &{}^{(2)}h^+_{2rr} = -\tfrac{8M}{r^3}, && \end{aligned} \quad (22)$$

VII. INTERIOR SOLUTION

A. The first order

1. Axial $\ell = 1$

For $\ell = 1$ there is no $h_{u\varphi}$ component and we can use the remaining gauge freedom to set ${}^{(1)}h^-_{1r\varphi} = 0$. Linearized Einstein equation are homogeneous (A1)–(A3) and yield:

$${}^{(1)}h^-_{1v\varphi} = \Omega_{11}r^2 + \frac{\Pi_{11}}{r} \quad (23)$$

where Ω_{11} and Π_{11} are arbitrary constants. We set $\Pi_{11} = 0$ to make the Kretschmann scalar expansion regular at r = 0, therefore we are left with ${}^{(1)}h^-_{1v\varphi} = \Omega_{11}r^2$. It turns out that this solution is a pure gauge, but we will discuss it later.

B. The second order

1. Polar $\ell = 0$

For $\ell = 0$ there are no h_-, h_{vu}, h_{ru} components in the polar decomposition and we have an additional gauge freedom, which we use to set ${}^{(2)}h^-_{0vr}$, ${}^{(2)}h^-_{0+}$ to zero.

The only nonzero variables left are ${}^{(2)}h^-_{0vv}$ and ${}^{(2)}h^-_{0rr}$. Solution to Einstein equations (A4)–(A10) with $\ell = 0$ and with sources (A13)–(A15) reads:

$$ {}^{(2)}h^-_{0vv} = \frac{4r^2\Omega_{11}^2}{3} - \frac{c_{20}(r^2-4M^2)}{64M^4} + \frac{d_{20}}{r}, \tag{24} $$

$$ {}^{(2)}h^-_{0rr} = \frac{c_{20}}{r^2-4M^2}. \tag{25} $$

where c_{20} and d_{20} are arbitrary constants. This solution is singular at $r = 0$ and $r = 2M$. To avoid singularity in the Kretschmann scalar expansion at $r = 0$, we set $d_{20} = 0$. Singularity at $r = 2M$ can be removed using a transformation generated by a gauge vector ${}^{(2)}\xi_0$ (${}^{(2)}\xi_{0v} = \frac{c_{20}((r^2-4M^2)\tanh^{-1}(\frac{r}{2M})+2Mr)}{64M^3}$, ${}^{(2)}\xi_{0r} = \frac{c_{20}\tanh^{-1}(\frac{r}{2M})}{8M}$, ${}^{(2)}\xi_{0u} = 0$, ${}^{(2)}\xi_{0\varphi} = 0$), what yields:

$$ {}^{(2)}h^-_{0vv} = \frac{4r^2\Omega_{11}^2}{3} + \frac{c_{20}}{16M^2}, \tag{26} $$

$$ {}^{(2)}h^-_{0vr} = 0, \tag{27} $$

$$ {}^{(2)}h^-_{0rr} = 0, \tag{28} $$

$$ {}^{(2)}h^-_{0+} = \frac{c_{20}r^2}{4M^2}. \tag{29} $$

2. Polar $\ell = 2$

Solution to Einstein equations (A4)–(A10) with $\ell = 2$ and with sources (A16)–(A19) reads:

$$ {}^{(2)}h^-_{2vv} = \frac{(r^2-4M^2)^2}{128M^4}\,{}^{(2)}h^-_{2rr} - \frac{4}{3}r^2\Omega_{11}^2, \tag{30} $$

$$ {}^{(2)}h^-_{2vr} = -\frac{1}{4}\left(1-\frac{r^2}{4M^2}\right){}^{(2)}h^-_{2rr}, \tag{31} $$

$$ {}^{(2)}h^-_{2rr} = \frac{c_{22}}{16M^4r^3} + \frac{d_{22}(3(r^2-4M^2)^2\coth^{-1}(\frac{2M}{r}) + 2Mr(5r^2-12M^2))}{32M^3r^3(r^2-4M^2)^2}, \tag{32} $$

$$ {}^{(2)}h^-_{2+} = \frac{c_{22}(4M^2+r^2)}{128M^6r} + \frac{d_{22}(3M(4M^2+r^2)\coth^{-1}(\frac{2M}{r}) - 2r(3M^2+r^2))}{256M^6r}, \tag{33} $$

where c_{22} and d_{22} are arbitrary constants. To avoid singularity in the Kretschmann scalar expansion at $r = 0$ and $r = 2M$ we need to set $c_{22} = 0$, $d_{22} = 0$, what yields:

$$ {}^{(2)}h^-_{2vv} = -\frac{4}{3}r^2\Omega_{11}^2, \tag{34} $$

$$ {}^{(2)}h^-_{2vr} = 0, \tag{35} $$

$$ {}^{(2)}h^-_{2rr} = 0, \tag{36} $$

$$ {}^{(2)}h^-_{2+} = 0. \tag{37} $$

C. The third order

1. Axial $\ell = 1$

The solution to Einstein equations (A1)–(A3) with $\ell = 1$ reads:

$$ {}^{(3)}h^-_{1v\varphi} = \Omega_{31}r^2 + \frac{\Pi_{31}}{r}. \tag{38} $$

To avoid singularity in the Kretschmann scalar expansion at r = 0, we set $\Pi_{31} = 0$.

2. Axial $\ell = 3$

Solution to Einstein equations (A1)–(A3) with $\ell = 3$ reads:

$$ {}^{(3)}h^-_{3v\varphi} = \frac{(r^2-4M^2)}{r^3}\Pi_{33} + \frac{(-120M^4r + 20M^2r^3 + 60(4M^5 - M^3r^2)\coth^{-1}(\frac{2M}{r}) + r^5)}{3r^3}\Omega_{33}, \tag{39} $$

$$ {}^{(3)}h^-_{3r\varphi} = \frac{8M^2}{r^3}\Pi_{33} + \frac{8M^2(\frac{r(-120M^4+20M^2r^2+r^4)}{r^2-4M^2} - 60M^3\coth^{-1}(\frac{2M}{r}))}{3r^3}\Omega_{33}, \tag{40} $$

where Ω_{33} and Π_{33} are arbitrary constants. Singularities at $r=0$ and $r=2M$ lead to the singularity in the Kretschmann scalar expansion, therefore $\Omega_{33}=0$, $\Pi_{33}=0$.

VIII. MATCHING

A. First order

Before matching, we act with the general gauge transformation on the interior metric. Although we consider stationary metrics, we take gauge vectors that depend on v coordinate. It might happen that acting with gauge vectors depending on v explicitly, we obtain metric independent of v (we discuss such a case in Sec. IX). From the matching conditions (16) we have:

$$\frac{{}^{(1)}h^{+}_{1v\varphi}(2M)}{A} - {}^{(1)}h^{-}_{1v\varphi}(2M) = -\partial_v {}^{(1)}\xi_{1\varphi}(v,2M), \quad (41)$$

To keep transformed metric v-independent, we use (B4) and (B5) and obtain a condition:

$$ {}^{(1)}\xi_{1\varphi} = q_{11} v r^2 + {}^{(1)}\gamma_{1\varphi}(r), \quad (42)$$

where q_{11} is an arbitrary constant and γ_1 is an arbitrary function of r. From (41) we obtain:

$$\Omega_{11} = -\frac{1}{4AM^2} + q_{11}. \quad (43)$$

B. Second order

We act with the most general second order gauge transformation (B1)–(B2) on the interior metric. To keep transformed metric v-indepedent, we use (B7)–(B13) and obtain conditions:

$$ {}^{(2)}\xi_{0v} = -4M^2 f q_{20} v + {}^{(2)}\gamma_{0v}(r), \quad (44)$$

$$ {}^{(2)}\xi_{0r} = 8M^2 q_{20} v + {}^{(2)}\gamma_{0r}(r), \quad (45)$$

$$ {}^{(2)}\xi_{2v} = {}^{(2)}\gamma_{2v}(r), \quad (46)$$

$$ {}^{(2)}\xi_{2r} = {}^{(2)}\gamma_{2r}(r), \quad (47)$$

$$ {}^{(2)}\xi_{2u} = {}^{(2)}\gamma_{2u}(r), \quad (48)$$

where q_{20} is an arbitrary constant and ${}^{(i)}\gamma_{\ell\mu}$ are functions of r.

Matching conditions (16) yield:

$$ {}^{(2)}h^{+}_{0vv}(2M) - A^2\,{}^{(2)}h^{-}_{0vv}(2M) = \frac{A^2\eta_0^- + 2\eta_0^+}{2M^3} + \frac{16}{3}A^2M^2 q_{11}(q_{11}-2\Omega_{11}) + \frac{A^2}{2M}{}^{(2)}\gamma_{0v}(2M), \quad (49)$$

$$ {}^{(2)}h^{+}_{2vv}(2M) - A^2\,{}^{(2)}h^{-}_{2vv}(2M) = \frac{A^2\eta_2^- + 2\eta_2^+}{2M^3} - \frac{16}{3}A^2M^2 q_{11}(q_{11}-2\Omega_{11}) + \frac{A^2}{2M}{}^{(2)}\gamma_{2v}(2M), \quad (50)$$

$$2\eta_2^+ - \eta_2^- A = AM^2\,{}^{(2)}\gamma_{2v}(2M), \quad (51)$$

$$[[{}^{(2)}h_{0+}(2M)]] = -\frac{8(\eta_0^+ - \eta_0^-)}{M} + 8\lambda'(U) + 8M\,{}^{(2)}\gamma_{0v}(2M), \quad (52)$$

$$[[{}^{(2)}h_{2+}(2M)]] = -\frac{8(\eta_2^+ - \eta_2^-)}{M} + 8M\,{}^{(2)}\gamma_{2v}(2M) - 6\,{}^{(2)}\gamma_{2u}(2M), \quad (53)$$

$$[[{}^{(2)}h_{2-}(2M)]] = {}^{(2)}\gamma_{2u}(2M) + \frac{16U\lambda(U) + 8(1-U^2)\lambda'(U)}{3(U^2-1)^2}. \quad (54)$$

After plugging solutions to perturbation equations into (49)–(54), we obtain:

$$\eta_0^- - = -M^2\,{}^{(2)}\gamma_{0v}(2M) - \frac{4MU}{3}\lambda_1 - \frac{M}{8}c_{20} - \frac{M}{6} - \frac{1}{4}M(3U^2-1)\,{}^{(2)}\gamma_{2u}(2M), \quad (55)$$

$$\eta_2^- = -\frac{M}{3} - M^2\,{}^{(2)}\gamma_{2v}(2M) + \frac{1}{2}M\,{}^{(2)}\gamma_{2u}(2M), \quad (56)$$

$$\eta_0^+ = -\frac{M}{6} + \frac{2\lambda_1 MU}{3} + \frac{1}{8}M(3U^2-1)\,{}^{(2)}\gamma_{2u}(2M), \quad (57)$$

$$\eta_2^+ = \frac{M}{6} - \frac{1}{4}M\,{}^{(2)}\gamma_{2u}(2M), \quad (58)$$

$$A = -1, \quad (59)$$

$$\lambda(U) = \lambda_1(U^2-1) + \frac{3}{8}U(U^2-1)\,{}^{(2)}\gamma_{2u}(2M). \quad (60)$$

where λ_1 is an arbitrary constant. To keep η_0^- independent of U, we have to set $\lambda_1=0$ and ${}^{(2)}\gamma_{2u}(2M)=0$, what leads to:

$$\eta_0^- - = -M^{2(2)}\gamma_{0v}(2M) - \frac{M}{8}c_{20} - \frac{M}{6}, \tag{61}$$

$$\eta_2^- = -\frac{M}{3} - M^{2(2)}\gamma_{2v}(2M), \tag{62}$$

$$\eta_0^+ = -\frac{M}{6}, \tag{63}$$

$$\eta_2^+ = \frac{M}{6}, \tag{64}$$

$$A = -1, \tag{65}$$

$$\lambda(U) = 0, \tag{66}$$

$${}^{(2)}\gamma_{2u}(2M) = 0. \tag{67}$$

C. Third order

Again, we act with the most general third order gauge transformation (B1)–(B3) on the interior metric. To keep transformed metric v-indepedent, we use (B4)–(B6) and obtain conditions:

$${}^{(3)}\xi_{1\varphi} = q_{31}r^2 v + {}^{(3)}\gamma_{1\varphi}(r), \tag{68}$$

$${}^{(3)}\xi_{3\varphi} = {}^{(3)}\gamma_{3\varphi}(r), \tag{69}$$

where q_{31} is an arbitrary constant and ${}^{(i)}\gamma_{\ell\mu}$ are functions of r. Using (43) and (61)–(67), third order matching conditions (16) yield:

$${}^{(3)}h^+_{1v\varphi}(2M) - A\,{}^{(3)}h^-_{1v\varphi}(2M) = \frac{3(5c_{20}+8)}{20M^2} + 3c_{20}q_{11} - 192M^4 q_{11}q_{20} + M^2 4(q_{31} - 12q_{20}), \tag{70}$$

$${}^{(3)}h^+_{3v\varphi}(2M) - A\,{}^{(3)}h^-_{3v\varphi}(2M) = \frac{3}{10M^2}, \tag{71}$$

$$5M^{2(3)}\xi_{3,\varphi}(2M) = 6^{(2)}\gamma_{2r}(2M)(4M^2q_{11}+1) + 2(3M^{(2)}\gamma_{2v}(2M)+1)(M^{(1)}\gamma'_{1\varphi}(2M) - {}^{(1)}\gamma_{1\varphi}(2M)). \tag{72}$$

Condition (72) can be fulfilled just by setting all the gauge components to zero. Setting $\xi_{2u}=0$ and plugging (38)–(40) into (70), we obtain:

$$\Omega_{31} = \frac{9}{80M^4} + q_{31} + \frac{3(4M^2q_{11}+1)(c_{20}-64M^4q_{20})}{16M^4}. \tag{73}$$

However, (71) does not have any free parameters and it cannot be fulfilled (we obtain contradiction $-\frac{3}{10M^2}=0$). That makes impossible to match interior with exterior in the third order.

IX. DISCUSSION AND SUMMARY

Although we found the matching impossible, it is interesting to know what is the interior solution we obtained. The regular interior solution up to the third order reads:

$$ds^2 = \begin{pmatrix} -\frac{1}{4}\left(1-\frac{r^2}{4M^2}\right) + a^2\left(\frac{c_{20}}{32M^2} + r^2(1-u^2)\Omega_{11}^2\right) & \frac{1}{2} & 0 & \frac{1}{6}ar^2(u^2-1)(6\Omega_{11}+a^2\Omega_{31}) \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{r^2}{1-u^2} + \frac{a^2c_{20}r^2}{8M^2(1-u^2)} & 0 \\ \frac{1}{6}ar^2(u^2-1)(6\Omega_{11}+a^2\Omega_{31}) & 0 & 0 & r^2(1-u^2) + \frac{a^2c_{20}r^2(1-u^2)}{8M^2} \end{pmatrix}. \tag{74}$$

It turns out that this is an exact solution to Einstein equations—a gauge–transformed de Sitter space. To see this, let us take the gauge vector with components:

$${}^{(1)}\xi_1 = (0,0,0,r^2\Omega_{11}v), \tag{75}$$

$${}^{(2)}\xi_0 = \left(-\frac{c_{20}r}{16M^2} + \frac{c_{20}(r^2-4M^2)}{128M^4}v, \frac{c_{20}v}{16M^2}, 0, 0\right), \tag{76}$$

$${}^{(2)}\xi_2 = (0,0,0,0), \tag{77}$$

$$ {}^{(3)}\xi_1 = \left(0, 0, 0, \left(r^2\Omega_{13} - \frac{3c_{20}r^2\Omega_{11}}{8M^2}\right)v\right), \tag{78} $$

$$ {}^{(3)}\xi_3 = (0, 0, 0, 0). \tag{79} $$

Acting with those vectors on (74) (using formulas (B1)–(B3), we obtain

$$ ds^2 = \begin{pmatrix} -\frac{1}{4}\left(1 - \frac{r^2}{4M^2}\right) & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{r^2}{1-u^2} & 0 \\ 0 & 0 & 0 & -r^2(u^2-1) \end{pmatrix}, \tag{80} $$

what is exactly the background de Sitter metric, so all perturbations we obtained are a pure gauge. One can ask, if allowing for a change in the background density does not affect this result, but the answer is no. We repeated the calculation allowing for the perturbations of density and pressure (within the equation of state $p = -\rho$), but they do not change the conclusions.

We would also like to comment on a recent article [33] that concerns the same problem as our work. Authors of [33] use Hartle formalism to match the rotating gravastar with the Kerr black hole up to the second perturbation order. They succeed to do that (as we do in the second order), but there are two main differences between our approaches. The first difference is the choice of the matching surface. We do not fix the matching surface and we treat it as a variable to be found. Authors of [33] fix the matching surface to be the horizon of the Kerr black hole. It seems to be contradictory to our results, because we do not have a freedom to perform matching on the horizon, but there comes the second difference between our papers. We dismiss solutions which produce singularities both at $r = 0$ and at $r = 2M$, whereas authors of [33] allow for the solutions which have a singularity at $r = 0$ in the second perturbation order. Because of that, they have additional freedom in the interior solution and they are able to match it with Kerr on the horizon. The justification they make for allowing such a singular solution is the possibility that the singularity is not real, but it appears as an artefact of the perturbative expansion. This argument touches the sensitive point of the perturbative expansion. It may happen that a function which is not singular at some point, in this case at $r = 0$, has singular expansion coefficients when expanded in the perturbation parameter (see Summary and Discussion in [33]). Authors of [33] do not determine whether such a scenario is the origin of the singularity they allow for. On the other hand, we cannot exclude that the singular terms in the Kretschmann scalar that we put to zero are such artificial singularities. If this is the case and if we did not set $c_{22} = 0$, we would be able to match solutions in the third order. Unfortunately, this ambiguity seems to be an inherent limitation of the perturbation theory.

To sum up, we made an attempt to match the ultracompact rotating gravastar with the Kerr metric using the nonlinear perturbation theory. The solution we choose is a general solution to the perturbation equations around a static gravastar that does not produce the singularities in the Kretschmann scalar expansion. Although the matching can be performed up to the second order, in the third order it is is no longer possible. What is more, the interior of the ultracompact rotating gravastar is just the de Sitter metric. Since some of the proposed sources of the Kerr metric are based on the second perturbation order calculations, we find it necessary to check if these results survive at the higher perturbation orders.

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APPENDIX A: EINSTEIN EQUATIONS

Einstein equations (15) of order i divide into two parts: the homogeneous part $\delta G_{\ell\mu\nu}$ consisting of metric perturbations of order i and sources ${}^{(i)}S_{\ell\mu\nu}$ consisting of metric perturbations of orders j ($j < i$). These equations need to be solved order by order: after solving Einstein equations up to order i one can construct explicit form of $i + 1$ order source.

1. Homogeneous part

In the axial sector in the RW gauge, there are two nonzero variables: ${}^{(i)}h_{\ell v\varphi}$ and ${}^{(i)}h_{\ell r\varphi}$ (for simplicity, we omit i and ℓ indices in formulas (A1)–(A10)). Homogeneous part of Einstein equations reads (where we introduce $E_{\mu\nu} = \delta G_{\mu\nu} + \frac{3}{4M^2}h_{\mu\nu}$):

$$ 2i!r^2E_{v\varphi} = (2f + \ell(\ell+1) - 2)h_{v\varphi} - r^2fh''_{v\varphi}, \tag{A1} $$

$$ 2i!r^2E_{r\varphi} = 2r^2h''_{v\varphi} - 4h_{v\varphi} + (\ell(\ell+1) - 2)h_{r\varphi}, \tag{A2} $$

$$ 2i!E_{u\varphi} = fh'_{r\varphi} + 2h'_{v\varphi} + f'h_{r\varphi}. \tag{A3} $$

In the polar sector in the RW gauge, there are four nonzero variables: ${}^{(i)}h_{\ell vv}$, ${}^{(i)}h_{\ell vr}$, ${}^{(i)}h_{\ell rr}$, ${}^{(i)}h_{\ell +}$. Homogeneous part of Einstein equations reads:

$$8i!r^4E_{vv} = 2f^3r^3h'_{rr} + 8f^2r^3h'_{vr} - 2f^2r^2h''_{+} + 4fr^2(2rf' + 2f + \ell(\ell+1))h_{vr} + f(2rf' + \ell(\ell+1) - 2)h_{+} + fr(2f - rf')h'_{+} + f^2r^2(4rf' + 2f + \ell(\ell+1))h_{rr} + 4r^2(2f + \ell(\ell+1))h_{vv} + 8fr^3h'_{vv}, \tag{A4}$$

$$4i!r^4E_{vr} = -2f^2r^3h'_{rr} + (-2rf' - \ell(\ell+1) + 2)h_{+} - fr^2(4rf' + 2f + \ell(\ell+1))h_{rr} - 2r^2(4rf' + 4f + \ell(\ell+1))h_{vr} + r(rf' - 2f)h'_{+} - 8fr^3h'_{vr} + 2fr^2h''_{+} - 8r^3h'_{vv} - 8r^2h_{vv}, \tag{A5}$$

$$2i!r^4E_{rr} = r^2(2rf' + \ell(\ell+1))h_{rr} + 2fr^3h'_{rr} + 8r^3h'_{vr} - 2r^2h''_{+} + 4rh'_{+} - 4h_{+}, \tag{A6}$$

$$2i!E_{vu} = h_{vr}f' + fh'_{vr} + 2h'_{vv}, \tag{A7}$$

$$4i!r^3E_{ru} = r^2(rf' + 2f)h_{rr} - 4r^3h'_{vr} + 8r^2h_{vr} - 2rh'_{+} + 4h_{+}, \tag{A8}$$

$$4i!r^2E_{+} = -4r^2(4rf' + 4f + \ell(\ell+1) - 4)h_{vr} - fr^3(rf' + 2f)h'_{rr} - 4r^3(rf' + 2f)h'_{vr} + 2r(rf' - 2f)h'_{+} + 4(f - rf')h_{+} - r^2(4f^2 + f(6rf' + \ell(\ell+1) - 4) + r^2f'^2)h_{rr} + 2fr^2h''_{+} - 8r^4h''_{vv} - 16r^3h'_{vv}, \tag{A9}$$

$$4i!E_{-} = fh_{rr} + 4h_{vr}. \tag{A10}$$

2. Sources

Below we list the nonzero components of sources for Einstein equations. Sources for the ith order perturbation equations can be found in the following way (see, e.g., appendix A of [34]). Let us assume that we already know the solution to perturbation Einstein equations up to the ith order (it consists of metric perturbations ${}^{(j)}h_{\mu\nu}$ with $j \le i$):

$$\tilde{g}_{\mu\nu} = \bar{g}_{\mu\nu} + \sum_{j=1}^{i}\sum_{\ell} {}^{(j)}h_{\ell\mu\nu}\frac{a^j}{j!}. \tag{A11}$$

Using this solution we can calculate the Einstein tensor $G_{\mu\nu}(\tilde{g})$, which satisfies the ith order perturbation equations and contributes to the $i+1$th (and higher) order perturbation equations. Finally, the source of the order $i+1$ is given by:

$${}^{(i+1)}S_{\mu\nu} = [i+1](-G_{\mu\nu}(\tilde{g})), \tag{A12}$$

where $[k](\cdots)$ denotes the kth order expansion of a given quantity. Although in most cases expressions for the sources ${}^{(i+1)}S_{\mu\nu}$ are complicated, their construction is a purely algebraic task and can be easily performed using computer algebra. Below we list nonzero components of ith order sources in terms of explicit solutions ${}^{(j)}h_{\mu\nu}$ found for lower orders.

The source for the second order:

$${}^{(2)}S_{0vv} = 4\left(1 - \frac{r^2}{4M^2}\right)\Omega_{11}^2, \tag{A13}$$

$${}^{(2)}S_{0vr} = -8\Omega_{11}^2, \tag{A14}$$

$${}^{(2)}S_{0+} = -16\Omega_{11}^2, \tag{A15}$$

$${}^{(2)}S_{2vv} = \left(\frac{r^2}{M^2} - 8\right)\Omega_{11}^2, \tag{A16}$$

$${}^{(2)}S_{2vr} = 8\Omega_{11}^2, \tag{A17}$$

$${}^{(2)}S_{2vu} = \frac{8}{3}r\Omega_{11}^2, \tag{A18}$$

$${}^{(2)}S_{2+} = 16r^2\Omega_{11}^2. \tag{A19}$$

The sources for the third order are zero.

APPENDIX B: GAUGE TRANSFORMATIONS

Consider a gauge transformation induced by a gauge vector $\xi = \sum_{i=0}^{\infty}\frac{a^i}{i!}{}^{(i)}\xi$. According to [32], metric perturbations transform in the following way:

$${}^{(1)}h_{\mu\nu} \to {}^{(1)}h_{\mu\nu} + \pounds_{{}^{(1)}\xi}\bar{g}_{\mu\nu}, \tag{B1}$$

$$ {}^{(2)}h_{\mu\nu} \to {}^{(2)}h_{\mu\nu} + (\pounds_{{}^{(2)}\xi} + \pounds^2_{{}^{(1)}\xi})\bar{g}_{\mu\nu} + 2\pounds_{{}^{(1)}\xi}{}^{(1)}h_{\mu\nu}, \tag{B2} $$

$$ {}^{(3)}h_{\mu\nu} \to {}^{(3)}h_{\mu\nu} + (\pounds^3_{{}^{(1)}\xi} + 3\pounds_{{}^{(1)}\xi}\pounds_{{}^{(2)}\xi} + \pounds_{{}^{(3)}\xi})\bar{g}_{\mu\nu} + 3(\pounds^2_{{}^{(1)}\xi} + \pounds_{{}^{(2)}\xi}){}^{(1)}h_{\mu\nu} + 3\pounds_{{}^{(1)}\xi}{}^{(2)}h_{\mu\nu}, \tag{B3} $$

where $\pounds_{{}^{(i)}\xi}$ denotes a Lie derivative with respect to ${}^{(i)}\xi$.

An explicit form of (B1)–(B3) for a gauge vector of order i acting on a metric components of order i reads (for clarity, we omit i indices, dots and primes correspond to derivatives with respect to v and r, respectively):

$$ h_{\ell v\varphi} \to h_{\ell v\varphi} - \dot{\xi}_\varphi, \tag{B4} $$

$$ h_{\ell r\varphi} \to h_{\ell r\varphi} + \frac{2\xi_\varphi}{r} - \xi'_\varphi, \tag{B5} $$

$$ h_{\ell u\varphi} \to h_{\ell u\varphi} + \xi_\varphi, \tag{B6} $$

$$ h_{\ell vv} \to h_{\ell vv} - \frac{1}{4}(f\xi_r + 2\xi_v)f' + 2\dot{\xi}_v, \tag{B7} $$

$$ h_{\ell vr} \to h_{\ell vr} + \frac{1}{2}f'\xi_r + \xi'_v + \dot{\xi}_r, \tag{B8} $$

$$ h_{\ell rr} \to h_{\ell rr} + 2\xi'_r, \tag{B9} $$

$$ h_{\ell +} \to h_{\ell +} + 2rf\xi_r - \ell(\ell+1)\xi_u + 4r\xi_v, \tag{B10} $$

$$ h_{\ell -} \to h_{\ell -} - \xi_u, \tag{B11} $$

$$ h_{\ell vu} \to h_{\ell vu} - \xi_v - \dot{\xi}_u, \tag{B12} $$

$$ h_{\ell ru} \to h_{\ell ru} - \xi_r + \frac{2}{r}\xi_u - \xi'_u. \tag{B13} $$

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