Deflection and gravitational lensing of null and timelike signals in general asymptotically (anti–)de Sitter spacetimes

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The deflection and gravitational lensing of light and massive particles in arbitrary static, spherically symmetric and asymptotically (anti–)de Sitter spacetimes are considered in this work. We first proved that for spacetimes whose metric satisfies certain conditions, the deflection of null rays with fixed closest distance will not depend on the cosmological constant Λ, while that of timelike signals and the apparent angle in gravitational lensing still depend on Λ. A two-step perturbative method is then developed to compute the change of the angular coordinate and total travel time in the weak field limit. The results are quasipower series of two small quantities, with the finite distance effect of the source/detector naturally taken into account. These results are verified by applying to some known asymptotically de Sitter spacetimes. Using an exact gravitational lensing equation, we solved the apparent angles of the images and time delays between them and studied the effect of Λ on them. It is found that generally, a small positive Λ will decrease the apparent angle of images from both sides of the lens and increase the time delay between them. The time delay between signals from the same side of the lens but with different energy however, will be decreased by Λ.

I. INTRODUCTION

Deflection of light by a massive object was one of the most striking predictions of General Relativity (GR). Its confirmation [1] not only greatly advanced the acceptance of GR among the scientific community, but also laid the foundation for the gravitational lensing (GL) phenomena. GL nowadays has become an important tool in astronomy and cosmology, due to the fact that GL observables can be connected to the properties of the source, the lens, the messengers and the spacetime the messengers went through. It was used to investigate properties of supernova [2], coevolution of galaxies and supermassive black holes (SMBHs) [3], cosmological parameters [4,5] and dark matter and energy [6,7] and to find exoplanets [8,9].

Regarding the messengers in the GL, light rays were the only practical choice. However, with the discovery of the SN 1987A supernova neutrinos [10,11] and more recent blazar TXS 0506 + 056 neutrinos [12,13], and the important observation of gravitational waves (GWs) after 2015 [14,15], it was clear that neutrinos and GWs can both be the messengers in GL. These messengers are becoming particularly important partially due to the discoveries of

GL of supernova [16,17] and multimessenger GRB-GW events [18], and partially because the angular and time resolutions of their detectors are advancing rapidly. Meanwhile, this calls for the more careful investigation of the deflection and GL of massive timelike signals, in hope of better understanding some key problems in astronomy and particle physics, such as the supernova mechanism and neutrino properties [19–21].

In recent years, the deflection and GL of both null and timelike signals in asymptotic flat spacetimes have been considered theoretically by many authors using different methods, such as the perturbative method [19,20,22–25] and Gauss-Bonnet theorem method [26–30], and in different limits, including both the weak field limit (WFL) and strong field limit [23,31–34]. With the establishment of the existence of dark energy and dark matter, people are becoming increasingly interested in the deflection and GL in asymptotically (anti–)de Sitter (dS) spacetimes. Many authors studied the deflection of light rays in the Schwarzschild-dS (SdS) spacetime [35–37]. Li *et al.* studied the strong GL of light rays in a brane world black hole (BH) [38]. Panpanich *et al.* investigated the deflection angle of light rays in a massive gravity [39]. Seçuk *et al.* computed the deflection at a particular detector location in the Ressner-Nordstrom(RN)-dS-Monopole spacetime [40]. Heydari-Fard studied the deflection angle in a novel 4D Gauss-Bonnet-de Sitter spacetime using the Rindler-Ishak

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method. However, the above investigations of deflection and GL are all done for light rays. There are still few works that considered the deflection of timelike rays in (a)dS spacetimes. Previous research either mainly concentrated on the influence of Λ on some general features of the massive particles' trajectory, such as its perihelion shift [41–44] or innermost stable circular orbits [45], or at most studied deflection of timelike rays in a particular spacetime [46]. In other words, a general method that can deal with arbitrary static and spherically symmetric (SSS) and asymptotically (a)dS spacetimes, like what Ref. [31] did to the deflection in the strong field limit, is still lacking. In this work, we attempt to establish a perturbative method that can fulfill this task. This is an extension of our previous work in the asymptotically flat spacetimes [24,47]. Moreover, we will also prove a result regarding the (in) dependence of the deflection angle on the cosmological constant Λ.

This work is organized as follows. In Sec. II, we lay out the basic metric, geodesics and definition of the deflection, and show the condition that the deflection angle is (in) dependent on the cosmological constant. In Sec. III, the perturbative method developed in Refs. [24,47] is extended to the case of arbitrary SSS and asymptotically (a)dS spacetimes. Both the deflection angle and total travel time for null and timelike rays are computed, with the finite distance effect of the source and detector taken into account. These results are applied to four particular dS spacetimes in Sec. IV to verify their correctness. In Sec. V, the apparent angles of the GL images are solved from a new GL equation applicable to deflection angles with finite distance effect. The time delays between images from opposite sides of the lens and between images with different energies are also computed. The effects of Λ to these observables are carefully studied. Section. VI concludes the work with a short discussion. Throughout the work we use the natural unit $G = c = 1$ and spacetime signature $(-, +, +, +)$.

II. DEFLECTION IN ASYMPTOTICALLY dS SPACETIMES

We start from the most general SSS and asymptotically dS spacetime with metric

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where (t, r, θ, ϕ) are the coordinates and A, B, C are metric functions depending on r only. Without losing any generality, for SSS spacetimes we can choose the metric functions A and B to satisfy $A = 1/B$, which is allowed (at least locally) by doing necessary changes of coordinates in r. For SSS and asymptotically dS spacetimes therefore we can set the metric functions to the following form:

$$A(r) = \frac{1}{B(r)} = f(r) - \frac{\Lambda r^2}{3} g(r), \quad (2)$$

where Λ is the cosmological constant. In the following, we will focus on the asymptotically dS case, i.e., assuming $\Lambda > 0$. However, as one can verify, all methods and results in this work are also valid to the asymptotically anti-dS cases. In Eq. (2), we will assume that $f(r)$ and $g(r)$ can be asymptotically expanded as inverse power series of r, implying that the spacetime is asymptotically flat if Λ was zero.

From the geodesic equations associated with metric (1), the following equations can be obtained:

$$\frac{d\phi}{dr} = \sqrt{\frac{B}{C}} \frac{L}{\sqrt{(E^2/A - \kappa)C - L^2}}, \quad (3)$$

$$\frac{dt}{dr} = \frac{\sqrt{BC}}{A} \frac{E}{\sqrt{(E^2/A - \kappa)C - L^2}}, \quad (4)$$

where $\kappa = 0, 1$ respectively for null and timelike signals and

$$E = A(r)\frac{dt}{d\tau}, \qquad L = C(r)\frac{d\phi}{d\tau} \quad (5)$$

are two first integral constants, with τ being the affine parameter or proper time. In asymptotically flat spacetimes, they are interpreted respectively as the energy and angular momentum per unit mass measured by asymptotic observers. Here we will just treat them as two constants characterizing the test particle. L can be related to E and the minimal radius r_0 using the radial equation (4), i.e., $dr/dt|_{r=r_0} = 0$, to find

$$L = \sqrt{C(r_0)(E^2/A(r_0) - \kappa)}. \quad (6)$$

In other words, among the three variables L, E, r_0, only two of them are free. If desired, r_0 can be solved from Eq. (6) as a function of L, E, κ, and Λ. In asymptotically flat spacetimes, this r_0 can be related to the impact parameter b of the trajectory. Although in the asymptotically dS spacetimes, we can still define an *effective* impact parameter b_{eff} for purpose of comparison using Eq. (6) for L

$$b_{\text{eff}} = \frac{L}{\sqrt{E^2 - \kappa}} = \frac{\sqrt{C(r_0)(E^2/A(r_0) - \kappa)}}{\sqrt{E^2 - \kappa}}, \quad (7)$$

the interpretation of $|b|$ as the distance from the asymptotic line to the lens center is in general spoiled. The reason is that for asymptotically dS spacetimes with $\Lambda > 0$, no timelike signal will be able to go beyond the cosmological horizon located at $r_H \approx \sqrt{\Lambda/3}$. Another important reason one should restrain from using this b_{eff} is that it makes the
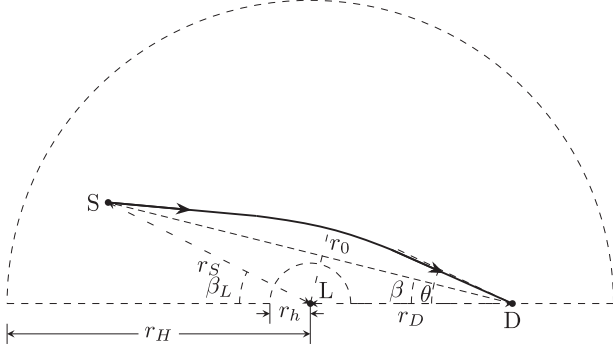
FIG. 1. The deflection and GL of signals in asymptotically dS spacetimes. S, L, D represent the source, lens and detector respectively. The large dashed circle and the smaller dashed circle are respectively the cosmological horizon with radius r_H and black hole event horizon with radius r_h. β_L and β label the angular location of the source with respect to the lens and detector respectively. The observed apparent angle is labeled as θ.

analysis of the influence of Λ obscured because as revealed by Eq. (7), $b_{\rm eff}$ itself is also Λ dependent.

Using Eqs. (3) and (4), the change of the angular coordinate $\Delta\phi$ and the travel time Δt from a source at radius r_s to a detector at radius r_d (see Fig. 1) can be computed as

$$\Delta\phi = \left[\int_{r_0}^{r_s} + \int_{r_0}^{r_d}\right]\sqrt{\frac{B}{C}}\frac{L}{\sqrt{(E^2/A - \kappa)C - L^2}}\mathrm{d}r, \quad (8)$$

$$\Delta t = \left[\int_{r_0}^{r_s} + \int_{r_0}^{r_d}\right]\frac{\sqrt{BC}}{A}\frac{E}{\sqrt{(E^2/A - \kappa)C - L^2}}\mathrm{d}r. \quad (9)$$

Throughout this work, we will assume that the source, lens and detector are all static so that r_s and r_d will not change.

In principle, for general metric functions A, B and C, these two integrations cannot be carried out to closed forms and therefore approximations have to be made. In this work, we will show that there exists a systematical way to perturbatively handle these integrals so that the deflection angle and travel time for both null and timelike signals can be computed to any desired accuracy for any SSS and asymptotically (a)dS spacetimes. Moreover, the method can take into account the finite distance effect of r_s and r_d naturally. Before this however, we first prove in Sec. II A that in some special cases of the spacetime and signal, the deflection $\Delta\phi$ for fixed r_0 is indeed independent of Λ.

A. (In)dependence of $\Delta\phi$ on Λ for timelike(null) signals

In this subsection, we will first show that for asymptotically dS spacetimes with $A(r)$ and $B(r)$ given by Eq. (2), if the metric functions satisfy

$$C(r) = h \cdot r^2 g(r), \quad (10)$$

where h is a Λ and r independent constant, then for a null signal with fixed minimal radius r_0 and L, E, its angular coordinate change $\Delta\phi$ as defined in Eq. (8) will be independent of Λ. We will further show that the apparent angle θ (see Fig. 1) against the lens-detector axis in GL as measured by a detector with fixed r_d however, does change as Λ varies. Moreover, we will also show that for timelike signals, not only the apparent angle θ but also the $\Delta\phi$ itself will change as Λ changes.

We start from substituting Eq. (6) into Eq. (8) and setting $\kappa = 0$ for null signals, yielding

$$\Delta\phi = \left[\int_{r_0}^{r_s} + \int_{r_0}^{r_d}\right]\sqrt{\frac{A(r)B(r)C(r_0)}{C(r)[A(r_0)C(r) - A(r)C(r_0)]}}\mathrm{d}r$$

$$= \left[\int_{r_0}^{r_s} + \int_{r_0}^{r_d}\right]\frac{r_0}{r}\sqrt{\frac{g(r_0)}{g(r)}}$$

$$\times \sqrt{\frac{1}{h[f(r_0)r^2g(r) - f(r)r_0^2g(r_0)]}}\mathrm{d}r, \quad (11)$$

where in the second step metric functions (2) and (10) are used. It is from Eq. (11) we observe that $\Delta\phi$ does not explicitly contain or depend on Λ for fixed r_s, r_d and r_0. After integration, $\Delta\phi$ becomes a function of the following form:

$$\Delta\phi = \Delta\phi(r_s, r_d, r_0, \text{other parameters}), \quad (12)$$

where "other parameters" include all the particle parameters such as its energy E and spacetime parameters such as its mass M and charge Q, but not the cosmological constant Λ.

We emphasize that this is a general result applicable to all metrics satisfying Eqs. (2) and (10). Such metrics certainly include the familiar ones like SdS or RNdS spacetimes. Moreover, this independence holds regardless of the values of r_s, r_d or r_0 as long as they are physically allowed. In other words, this is also applicable to source/detector near the cosmological horizon (i.e., $r_s, r_d \to r_H$), or to signals that experience lensing in the strong field limit (i.e., $r_0 \to r_{\rm photon\ sphere}$). We also point out that the independence of $\Delta\phi$ on Λ for fixed r_0 was previously recognized for null rays in the SdS spacetime [48].

On the other hand, for timelike signals with $\kappa = 1$, substituting Eq. (6) and the metric functions (2) and (10) into Eq. (8), we obtain

$$\Delta\phi = \left[\int_{r_0}^{r_s} + \int_{r_0}^{r_d}\right] dr \frac{r_0}{r} \frac{\sqrt{g(r_0)}}{\sqrt{g(r)}}$$
$$\times \sqrt{\frac{E^2 - f(r_0) + \frac{\Lambda r_0^2}{3} g(r_0)}{E^2 h[f(r_0)g(r)r^2 - f(r)g(r_0)r_0^2] - h[r^2 g(r) - r_0^2 g(r_0)][f(r) - \frac{\Lambda r^2}{3} g(r)][f(r_0) - \frac{\Lambda r_0^2}{3} g(r_0)]}}. \tag{13}$$

Here no further simplification like in the null case is possible. It is then clear that this $\Delta\phi$ will unavoidably depend on the value of Λ. Similarly, for the total travel time, substituting Eqs. (6), (2) and (10) and $\kappa = 0$ into Eq. (9), we have

$$\Delta t = \left[\int_{r_0}^{r_s} + \int_{r_0}^{r_d}\right] \frac{\sqrt{hr^2 g(r)[f(r_0) - \frac{\Lambda r_0^2}{3} g(r_0)]}}{f(r) - \frac{\Lambda r^2}{3} g(r)}$$
$$\times \frac{E}{\sqrt{E^2 h[f(r_0)g(r)r^2 - f(r)r(r_0)r_0^2] - h(r^2 g(r) - r_0^2 g(r_0))[f(r) - \frac{\Lambda r^2}{3} g(r)][f(r_0) - \frac{\Lambda r_0^2}{3} g(r_0)]}} dr. \tag{14}$$

It is also clear that the total travel time and consequently the time delay between the images will all depend on Λ, even for light rays. For timelike rays, we can verify similarly to $\Delta\phi$ in Eq. (13) that the dependence of Δt on Λ is even more apparent.

B. Dependence of the apparent angle on Λ

The apparent angle of the light ray observed by a static observer against the observer-lens axis however is not $\Delta\phi$ but θ, as illustrated in Fig. 1. As pointed out by Bhadra *et al.* [49], θ is indeed dependent on Λ for light rays in the SdS spacetime. In this part, we will briefly demonstrate that this is generally true for other asymptotically dS spacetimes too. The angle θ in the SSS spacetimes described by metric (1) can be computed using the projection of the four velocity of the signal

$$\frac{dx_{\text{sig}}^\mu}{d\tau} = \left[\frac{E}{A}, \sqrt{E^2 - \frac{L^2 A}{C} - \frac{\kappa}{B}}, 0, \frac{L}{C}\right] \tag{15}$$

onto that of the static observer [24,50]

$$\frac{dx_{\text{obs}}^\mu}{d\tau} = \left[\frac{1}{\sqrt{A(r_d)}}, 0, 0, 0\right]. \tag{16}$$

It is straightforward to show that the resultant apparent angle θ for metric (1) is

$$\theta = \arcsin\left(\frac{L}{\sqrt{E^2 - \kappa A(r_d)}} \sqrt{\frac{A(r_d)}{C(r_d)}}\right)$$
$$= \arcsin\left(\sqrt{\frac{E^2 - \kappa A(r_0)}{E^2 - \kappa A(r_d)}} \sqrt{\frac{A(r_d)C(r_0)}{A(r_0)C(r_d)}}\right), \tag{17}$$

where in the second step Eq. (6) is substituted for L.

The apparent angle (17) at a freely varying b_{eff} and a particular observer location r_b determined by the alignment of the source, lens and observer (see Eq. (15) of Ref. [46] for its value in SdS spacetime), was previously computed for light rays and timelike signals in SdS spacetime in Refs. [35–37] and [46] respectively. Although in these works, θ was called the "deflection angle" and this choice of r_b is not well justified [49], nevertheless we plugged $r_d = r_b$ into Eq. (17) and verified that Eq. (17) will correctly yield the corresponding values in these cases, i.e., Eq. (15) of Ref. [35], Eq. (17) of Ref. [36] (with $a = 0$), Eq. (31) of Ref. [37] and Eq. (18) of Ref. [46]. In Ref. [51], the apparent angle, also called the deflection angle there, in dilaton–de Sitter spacetime was computed for null rays. We also plugged in the corresponding r_d given in Eq. (2.22) of that paper into Eq. (17) and verified that this yields its Eq. (4.3) to the order M^2/r_0^2, MD/r_0^2, $r_0^3\Lambda/M$ and $r_0^2\Lambda$ [see Eq. (48) for the meaning of these parameters].

For light signal with $\kappa = 0$, Eq. (17) simplifies to

$$\theta = \arcsin\left(\sqrt{\frac{A(r_d)C(r_0)}{A(r_0)C(r_d)}}\right). \tag{18}$$

In a GL configuration, if Λ is changed while holding source and detector static, i.e., r_s, r_d and β_L in Fig. 1 unchanged, then it is clear from Fig. 1 that $\Delta\phi = \pi \pm \beta_L$ is also kept as a constant. From Eq. (12), this in turn means that for null signal, as Λ varies, r_0 should not change either. Then immediately using Eq. (18) we see that θ will vary since all variables in the metric function $A(r_0)$ or $A(r_d)$ are unchanged except Λ.

For timelike signal, we have shown previously that unlike null signal, its $\Delta\phi$ will change with the variation of Λ. For its apparent angle θ, it is also seen from Eq. (17)

by setting $\kappa = 1$ that it will depend on the parameter Λ through $A(r_0)$ and $A(r_d)$, just as in the case of null signal.

Finally we also note there exist other definitions of the deflection angle in studies in asymptotically dS spacetimes. For example Ref. [52] defined a deflection angle α_{SdS} as the difference between the inner angles of a geodesic triangle in the SdS spacetime and in a comparison dS spacetime. Moreover, as pointed out before, the deflection angle in Refs. [35–37,46] is actually the apparent angle in this work. Furthermore, if the spacetime is expanding, as in ΛCDM standard cosmology, a deflection angle can also be defined [53]. Because of the difference in these definitions, however, the conclusion about the effect of Λ on these deflection angles might be different from the SSS case considered here.

III. PERTURBATIVE DEFLECTION IN THE WFL

In this section, we will resume computing $\Delta\phi$ and Δt according to Eqs. (8) and (9) in the WFL for asymptotically dS spacetimes. In this limit, we can safely assume that

$$M \ll r_0 \ll \{r_s, r_d\} < r_H \approx \sqrt{3/\Lambda}, \qquad (19)$$

where M is the mass of the spacetime.

A. The change of angular coordinate $\Delta\phi$

Starting from definition (8), our goal here is to find a perturbative way to systematically expand the integrand, which should then allow us to carry out the integration and find an approximation of $\Delta\phi$. The main difficulty in perturbative expansion in the asymptotically dS spacetime in the WFL, comparing to that in asymptotically flat spacetime, is that we cannot straightforwardly carry out the large r_0/M (or b_{eff}/M) expansion because the source or observer cannot be at a radius larger than the cosmological horizon r_H. Therefore in this work, we use a trick to hold r_0/r_H small while doing the large r_0 expansion. We first define a small variable ε

$$\varepsilon \equiv r_0\sqrt{\Lambda} \sim r_0/r_H \quad \text{so that } r_0 = \frac{\varepsilon}{\sqrt{\Lambda}}. \qquad (20)$$

Then we can make a change of variable in Eq. (8) from r to the new integral variable u,

$$r \to u = \frac{r}{r_0} \quad \text{or equivalently } r = ur_0 = \frac{u\varepsilon}{\sqrt{\Lambda}}. \qquad (21)$$

Using this, the integral limits of Eq. (8) and the Jacobian become

$$r_0 \to 1, \qquad r_{s,d} \to \frac{r_{s,d}\sqrt{\Lambda}}{\varepsilon} \equiv u_{s,d}, \qquad (22)$$

$$dr \to \frac{\varepsilon}{\sqrt{\Lambda}} du. \qquad (23)$$

Substituting Eq. (6) into Eq. (8) and using the change of variables (21) for all r, $\Delta\phi$ becomes

$$\Delta\phi = \left[\int_1^{u_s} + \int_1^{u_d}\right] \sqrt{\frac{B\left(\frac{u\varepsilon}{\sqrt{\Lambda}}\right)}{C\left(\frac{u\varepsilon}{\sqrt{\Lambda}}\right)}} \frac{\sqrt{\left[\frac{E^2}{A\left(\frac{\varepsilon}{\sqrt{\Lambda}}\right)} - \kappa\right]C\left(\frac{\varepsilon}{\sqrt{\Lambda}}\right)}}{\sqrt{\left[\frac{E^2}{A\left(\frac{u\varepsilon}{\sqrt{\Lambda}}\right)} - \kappa\right]C\left(\frac{u\varepsilon}{\sqrt{\Lambda}}\right) - \left[\frac{E^2}{A\left(\frac{\varepsilon}{\sqrt{\Lambda}}\right)} - \kappa\right]C\left(\frac{\varepsilon}{\sqrt{\Lambda}}\right)}} \frac{\varepsilon}{\sqrt{\Lambda}} du. \qquad (24)$$

Since it is assumed that the spacetime is asymptotically flat if Λ were zero, we can set the $f(r)$, $g(r)$ and $C(r)$ in Eq. (2) to the following expansion form at large r:

$$f(r) = \sum_{n=0} \frac{f_n}{r^n}, \quad g(r) = \sum_{n=0} \frac{g_n}{r^n}, \quad C(r) = r^2 \sum_{n=0} \frac{c_n}{r^n}. \qquad (25)$$

Without losing any generality, we can always set $f_0 = g_0 = c_0 = 1$ by a scaling of coordinates t, r and constant Λ. Moreover, we can always identify $f_1 = -2M$ where M is the Arnowitt-Deser-Misner mass of the spacetime.

To perturbatively compute $\Delta\phi$ in Eq. (24), we propose a two-step perturbative method, to expand the integrand for small Λ first and then for small ε. According to Eq. (20), expanding around small Λ is essentially equivalent to a large r_0 expansion, while the subsequent small ε expansion makes

sure that r_0/r_H is still kept small. We emphasize that in order for the method to work and to have a proper interpretation of the result, it is important to carry out the expansion in the above order. Substituting Eqs. (2) and (25) into (24) and carrying out the said expansion, $\Delta\phi$ is found to be

$$\Delta\phi = \left[\int_1^{u_s} + \int_1^{u_d}\right] \sum_{n=0}^{\infty} \frac{\Lambda^{\frac{n}{2}}}{\varepsilon^n(u+1)^n}$$
$$\times \left\{\sum_{m=0}^{\infty} \varepsilon^{2m}\left[\sum_{k=1-n-2\delta_{m0}}^{n+2m-1} p_{n,m,k} \frac{u^k}{\sqrt{u^2-1}}\right]\right\} du, \qquad (26)$$

where $p_{n,m,k}$ are the expansion coefficients determined by the metric function coefficients and δ_{m0} is the Kronecker delta function. The first several $p_{n,m,k}$ are given by

$$p_{0,0,-1} = 1, \tag{27a}$$

$$p_{0,1,1} = \frac{-\kappa}{6(E^2 - \kappa)}, \tag{27b}$$

$$p_{1,0,-2} = p_{1,0,-1} = -\frac{f_1 + c_1}{2}, \tag{27c}$$

$$p_{1,0,0} = -\frac{E^2(f_1 - c_1) + \kappa c_1}{2(E^2 - \kappa)}, \tag{27d}$$

$$p_{1,1,0} = \frac{E^4(2g_1 - 2c_1)}{12(E^2 - \kappa)^2}$$
$$+ \frac{\kappa[E^2(3c_1 + 5f_1 - 4g_1) - c_1 - 3f_1 + 2g_1]}{12(E^2 - \kappa)^2}, \tag{27e}$$

$$p_{1,1,1} = \frac{\kappa[E^2(c_1 + f_1 - 2g_1) - c_1 - 3f_1 + 2g_1]}{12(E^2 - \kappa)^2}, \tag{27f}$$

$$p_{1,1,2} = \frac{\kappa[E^2(f_1 - c_1) + c_1]}{12(E^2 - \kappa)^2}. \tag{27g}$$

The higher-order ones can be obtained easily too but are more lengthy and therefore not presented here except in some specific spacetimes in Sec. IV. Note that since the dimension of f_n, g_n and c_n are M^n where M is the mass of the spacetime, from Eq. (27) we see that the dimension of $p_{n,m,k}$ is also M^n. One would expect that if f_n, g_n and c_n are comparable to or smaller than M^n, $p_{n,m,k}$ will also be smaller than or at most comparable to M^n.

Now the computation of $\Delta\phi$ in Eq. (26) relies on the integrability of the following integrals:

$$I_{n,k}(u_{s,d}) \equiv \int_1^{u_{s,d}} \frac{u^k}{(u+1)^n \sqrt{u^2 - 1}} du \tag{28}$$

$$= \int_0^{\alpha_{s,d}} \frac{\cos^{n-k-1}\alpha}{(\cos\alpha + 1)^n} d\alpha, \tag{29}$$

where in the second step a change of variables $u = \sec\alpha$ is used, and $\alpha_{s,d} \equiv \sec^{-1}(u_{s,d})$ are defined. We can prove that the integral (29) can always be carried out for general n and k. Their results are elementary functions given in Eq. (A6). Later, to study the finite distance effect of $r_{s,d}$ and to solve the lensing equations, we can also expand $I_{n,k}$'s in the large $u_{s,d}$ limit, whose leading order results are presented in Eqs. (A8) and (A9). Substituting $I_{n,k}$ back into Eq. (26) and replacing ε by r_0 using (20), we have finally

$$\Delta\phi = \sum_{j=s,d} \sum_{n=0,m=0}^{\infty} \Delta\phi_{n,m}, \tag{30}$$

where the contribution from order $1/r_0^n$ and $(r_0^2\Lambda)^m$ is

$$\Delta\phi_{n,m} = \frac{(r_0^2\Lambda)^m}{r_0^n} \sum_{k=1-n-2\delta_{m0}}^{n+2m-1} p_{n,m,k} I_{n,k}\left(\frac{r_j}{r_0}\right). \tag{31}$$

A few comments are in order for result (30). Firstly, this result is a quasipower series of two variables, namely $1/r_0$ and $r_0^2\Lambda$. The simultaneous appearance of r_0^2 and Λ in $(r_0^2\Lambda)^m$ makes sure that the large r_0 limit will not cause divergence in $\Delta\phi$ since $r_0 < r_H \sim \sqrt{3/\Lambda}$. Secondly, note from the expression (27) that for null signal and if $c_n = h \cdot g_n$ for all $n \geq 0$ (h was fixed to 1 when setting $c_0 = g_0 = 1$), all $p_{n,m,k}$ with $m \geq 1$ indeed will reduce to zero. Since the $m \geq 1$ terms in $\Delta\phi$ are exactly the terms containing Λ, the above is just a manifestation of the independence of $\Delta\phi$ on Λ for null signals in spacetime satisfying Eq. (10), as proven in Sec. II A. It is also clear that even for timelike rays with $\kappa = 1$, the $m \geq 1$ terms are generally suppressed in the ultrarelativistic limit ($E \to \infty$) by the $1/E^{2m}$ factor and therefore the effect of Λ will be less obvious.

Further noticing the order estimation (A8), for the k summation in Eq. (30) we see that for each fixed m and n, the term with highest index $k = n + 2m - 1$ will dominate. Therefore keeping only these terms, $\Delta\phi$ in Eq. (30) should roughly be

$$\Delta\phi \approx \sum_{j=s,d} \left[\sum_{n=0}^{\infty} \frac{L_{n,n-1}}{r_0^n} p_{n,0,n-1} \right.$$
$$+ \sum_{n=0,m=1}^{\infty} \frac{1}{2m-1} \frac{1}{r_0^n} (r_0^2\Lambda)^m p_{n,m,n+2m-1} \left(\frac{r_j}{r_0}\right)^{2m-1} \right]$$
$$= \sum_{j=s,d} \left[\sum_{n=0}^{\infty} \frac{L_{n,n-1}}{r_0^n} p_{n,0,n-1} \right.$$
$$+ \frac{r_0}{r_{s,d}} \sum_{n=0,m=1}^{\infty} \frac{1}{2m-1} \frac{1}{r_0^n} (r_{s,d}^2\Lambda)^m p_{n,m,n+2m-1} \right], \tag{32}$$

where $L_{n,n-1}(n \geq 0)$ are r_0 and Λ independent constants given in Eq. (A8). Since the coefficients $p_{n,m,k}$ are of dimension M^n, if the coefficients f_n, g_n and c_n are not much larger than M^n, then as an order estimation we should have

$$\mathcal{O}(\Delta\phi) = \sum_{n=0}^{\infty} \left(\frac{M}{r_0}\right)^n \cdot \mathcal{O}(1)$$
$$+ \frac{r_0}{r_{s,d}} \sum_{n=0,m=1}^{\infty} \frac{M^n}{r_0^n} (r_{s,d}^2\Lambda)^m \cdot \mathcal{O}(1). \tag{33}$$

Clearly, $\Delta\phi$ is a quasiseries of product of two series of two small parameters, M/r_0 and $r_{s,d}^2\Lambda$ respectively. The dominance of the terms depends on the detailed numerical values of these parameters. In Sec. V, we will consider two limiting cases: the first being M/r_0 as large as possible and

$r_{s,d}^2\Lambda$ is small, and the second case being $r_{s,d}^2\Lambda$ as large as possible and M/r_0 small.

B. The total travel time

For the total travel time (9), we can carry out a procedure similar to steps from Eqs. (19)–(30). After the change of variables (21) and expansion of Λ and then ε, Δt becomes

$$\Delta t = \left[\int_1^{u_s} + \int_1^{u_d}\right] \sum_{n=0}^{\infty} \frac{\Lambda^{\frac{n-1}{2}}}{\varepsilon^{n-1}(u+1)^n}$$
$$\times \left\{\sum_{m=0}^{\infty} \varepsilon^{2m}\left[\sum_{k=1-n}^{\max[n-2,0]+2m+1} q_{n,m,k}\frac{u^k}{\sqrt{u^2-1}}\right]\right\}du, \quad (34)$$

where $q_{n,m,k}$ are related to the coefficients of the metric functions. The first few of them are

$$q_{0,0,1} = \frac{E}{\sqrt{E^2-\kappa}}, \quad (35a)$$

$$q_{0,1,1} = -\frac{E^3}{6(E^2-\kappa)^{\frac{3}{2}}}, \quad (35b)$$

$$q_{0,1,2} = 0, \quad (35c)$$

$$q_{0,1,3} = \frac{E(2E^2-3\kappa)}{6(E^2-\kappa)^{\frac{3}{2}}}, \quad (35d)$$

$$q_{1,0,0} = \frac{E(c_1-3f_1)}{2(E^2-\kappa)^{\frac{1}{2}}}, \quad (35e)$$

$$q_{1,0,1} = -\frac{Ef_1(2E^2-3\kappa)}{2(E^2-\kappa)^{\frac{3}{2}}}, \quad (35f)$$

and higher-order ones can be similarly obtained without any difficulty. The qualitative difference of Eq. (34) from Eq. (26) is that the beginning order of $\sqrt{\Lambda}/\varepsilon$ is from -1. This is expected because Δt should carry a dimension of $[\sqrt{\Lambda}/\varepsilon]^{-1} = [r_0]$.

Equation (34) can also be integrated, yielding the result

$$\Delta t = \sum_{j=s,d} \sum_{n=0,m=0}^{\infty} \Delta t_{n,m}, \quad (36)$$

where

$$\Delta t_{n,m} = \frac{(r_0^2\Lambda)^m}{r_0^{n-1}} \sum_{k=1-n}^{\max[n-2,0]+2m+1} q_{n,m,k} I_{n,k}\left(\frac{r_j}{r_0}\right), \quad (37)$$

with $I_{n,k}$ given by Eq. (A6). In particular, using the first few $I_{n,k}$ in Eq. (A7), to the order $\mathcal{O}(r_0^3\Lambda)$, the total time is

$$\Delta t = \sum_{i=s,d}\left\{q_{0,0,1}r_0\sqrt{\left(\frac{r_i}{r_0}\right)^2-1} + r_0^3\Lambda\left[q_{0,1,1}\sqrt{\left(\frac{r_i}{r_0}\right)^2-1} + \frac{q_{0,1,3}}{3}\sqrt{\left(\frac{r_i}{r_0}\right)^2-1}\left(\left(\frac{r_i}{r_0}\right)^2+2\right)\right] + q_{1,0,0}\sqrt{\frac{r_i/r_0-1}{r_i/r_0+1}}\right.$$
$$\left. + q_{1,0,1}\left[\ln\left(\sqrt{\left(\frac{r_i}{r_0}\right)^2-1}+\frac{r_i}{r_0}\right) - \sqrt{\frac{r_i/r_0-1}{r_i/r_0+1}}\right]\right\} + \mathcal{O}\left(\frac{M}{r_0}, r_0^5\Lambda^2\right),$$
$$= \sum_{i=s,d}\frac{E}{\sqrt{E^2-\kappa}}\left\{r_0\left(\frac{r_i}{r_0}-\frac{r_0}{2r_i}\right) + \frac{r_0^3\Lambda}{6(E^2-\kappa)}\left[\frac{(2E^2-3\kappa)r_i^3}{3r_0^3} - \frac{3\kappa r_i}{2r_0} - \frac{(2E^2-9\kappa)r_0}{8r_i}\right]\right.$$
$$\left. + \frac{(E^2-\kappa)c_1-E^2f_1}{2(E^2-\kappa)}\left(1-\frac{r_0}{r_i}\right) - \frac{f_1(2E^2-3\kappa)}{2(E^2-\kappa)}\ln\left(\frac{2r_i}{r_0}\right)\right\} + \mathcal{O}\left(\frac{M}{r_0}, r_0^5\Lambda^2, \frac{r_0^2}{r_i^2}\right), \quad (38)$$

where in the second step the $q_{n,m,k}$ given in Eq. (35) and the expansion of $I_{n,k}$ in Eq. (A9) are used. The first, second and last two terms in the curl brackets are respectively due to the $\Delta t_{0,0}, \Delta t_{0,1}$ and $\Delta t_{1,0}$ terms in Eq. (36). Later in Sec. V B, we will show that the total time truncated to the above order will determine the leading three orders of the time delay between the GL images. One particularly interesting limit of Δt is its ultrarelativistic limit. Expanding Eq. (38) in the infinite E limit, we obtain

$$\Delta t \approx \sum_{i=s,d} \left\{ r_i - \frac{r_0^2}{2r_i} + \frac{\Lambda}{3}\left(\frac{r_i^3}{3} - \frac{r_0^4}{8r_i}\right)\right.$$

$$+ \frac{c_1 - a_1}{2}\left(1 - \frac{r_0}{r_i} + \frac{r_0^2}{2r_i^2}\right) - a_1 \ln\frac{8r_i}{r_0}$$

$$+ \frac{1}{E^2}\left[\frac{r_i}{2} - \frac{r_0^2}{4r_i} + \frac{\Lambda}{4}\left(-r_0^2 r_i + \frac{r_0^4}{2r_i}\right)\right.$$

$$\left.\left. + \frac{c_1 - 3a_1}{4}\left(1 - \frac{r_0}{r_i} + \frac{r_0^2}{r_i^2}\right)\right]\right\}$$

$$+ \mathcal{O}\left(\frac{M}{r_0}, r_0^5\Lambda^2, \frac{r_0^2}{r_i^2}, \frac{1}{E^4}\right). \tag{39}$$

The first two lines represent the leading orders of the travel time of null rays and the third and fourth lines are the relativistic corrections. They will be useful when considering the time delay between different kinds of signals in Sec. V B.

IV. APPLICATION TO ASYMPTOTICALLY dS SPACETIMES

To check the validity of the perturbative method presented in Sec. III, especially the change of the angular coordinate (30), in this section we will compute $\Delta\phi$ in a few known asymptotically dS spacetimes.

A. Schwarzschild-dS spacetime

We first consider the SdS spacetime case, which has a metric of Eq. (1) with (2) and

$$f(r) = 1 - \frac{2M}{r}, \qquad g(r) = 1, \qquad C(r) = r^2. \tag{40}$$

From this, reading off the coefficients f_i, g_i and c_i in Eq. (25) and substituting into Eq. (27) and further into (30), we immediately obtain $\Delta\phi$ in SdS spacetime for both null and timelike signals, with finite distance effect taken into account. To the order $\mathcal{O}(M/r_0)^2$ and $\mathcal{O}(r_0^2\Lambda)^1$, we have

$$\Delta\phi_S = \sum_{j=s,d}[\Delta\phi_{S,0,0} + \Delta\phi_{S,0,1} + \Delta\phi_{S,1,0} + \Delta\phi_{S,1,1}$$

$$+ \Delta\phi_{S,2,0}] + \mathcal{O}\left[\left(\frac{M}{r_0}\right)^3, (r_0^2\Lambda)^2\right], \tag{41}$$

where the contributions from each order are

$$\Delta\phi_{S,0,0} = I_{0,-1}, \tag{42a}$$

$$\Delta\phi_{S,0,1} = -(r_0^2\Lambda)\frac{\kappa}{6(E^2-\kappa)}I_{0,1}, \tag{42b}$$

$$\Delta\phi_{S,1,0} = \frac{M}{r_0}\left(I_{1,-2} + I_{1,-1} + \frac{E^2}{E^2-\kappa}I_{1,0}\right), \tag{42c}$$

$$\Delta\phi_{S,1,1} = \frac{M}{r_0}(r_0^2\Lambda)$$

$$\times\frac{\kappa[(3-5E^2)I_{1,0} + (3-E^2)I_{1,1} - E^2 I_{1,2}]}{6(E^2-\kappa)^2}, \tag{42d}$$

$$\Delta\phi_{S,2,0} = \frac{M^2}{r_0^2}\left[\frac{3}{2}I_{2,-3} + 3I_{2,-2} + \frac{3(3E^2-\kappa)}{2(E^2-\kappa)}I_{2,-1}\right.$$

$$\left.+ \frac{E^2(3E^2-5\kappa)}{(E^2-\kappa)^2}I_{2,0} + \frac{E^2(3E^2-4\kappa)}{2(E^2-\kappa)^2}I_{2,1}\right]. \tag{42e}$$

Note the higher orders are obtained but not presented here due to their length. To see the finite distance effect of the source and detectors more clearly, using the expansions (A9) for $I_{n,k}$, the $\Delta\phi_{S,n,m}$ become to the leading orders of r_0/r_i $(i = s, d)$

$$\Delta\phi'_{S,0,0} = \frac{\pi}{2} - \frac{r_0}{r_j}, \tag{43a}$$

$$\Delta\phi'_{S,0,1} = -(r_0^2\Lambda)\frac{\kappa}{6(E^2-\kappa)}\left(\frac{r_j}{r_0} - \frac{r_0}{2r_j}\right), \tag{43b}$$

$$\Delta\phi'_{S,1,0} = \frac{M}{r_0}\left[1 + \frac{E^2}{E^2-\kappa}\left(1 - \frac{r_0}{r_j}\right)\right], \tag{43c}$$

$$\Delta\phi'_{S,1,1} = \frac{M}{r_0}(r_0^2\Lambda)$$

$$\times\frac{\kappa\{2r_0 r_j[3\kappa\ln(\frac{2r_j}{r_0}) - 5E^2] - 2E^2 r_j^2\}}{12(E^2-\kappa)^2 r_0 r_j}, \tag{43d}$$

$$\Delta\phi'_{S,2,0} = \frac{M^2}{r_0^2}\frac{[E^4(15\pi-16) + 2E^2\kappa(4-9\pi) + 3\pi\kappa]}{8(E^2-\kappa)^2}. \tag{43e}$$

Here we kept the result to different orders of (r_0/r_j) in different $\Delta\phi_{S,n,m}$ because when they blend into $\Delta\phi_S$, a combined second order of the small quantities M/r_0, $r_0^2\Lambda$ and r_0/r_j can be achieved. Setting $\Lambda = 0$ in Eqs. (42) and (43) reduces them to the Schwarzschild results obtained previously for both timelike and null rays [23]. On the other hand, setting $\kappa = 0$, the $(r_0^2\Lambda)^n$ $(n \geq 1)$ terms automatically disappear from $\Delta\phi_S$ and the result becomes that of null rays [23]. If one is interested in the deflection of null rays expressed in terms of b_{eff}, then replacing r_0 by b_{eff} using (7) and setting $\kappa = 0$ in Eq. (43), Eq. (5) of Ref. [54] can be recovered. On the other hand, if one is interested in the null deflection from a source to an observer both located asymptotically at the cosmological horizon, then setting $\kappa = 0$ and $r_s = r_d = r_H$ in Eq. (43), Eq. (55) of Ref. [55] is obtained.

To verify the correctness of these results, in Fig. 2(a) we plot the contributions from different $\Delta\phi'_{S,n,m}$ in Eq. (43). Note that in this figure as well as in Fig. 2(b), some orders
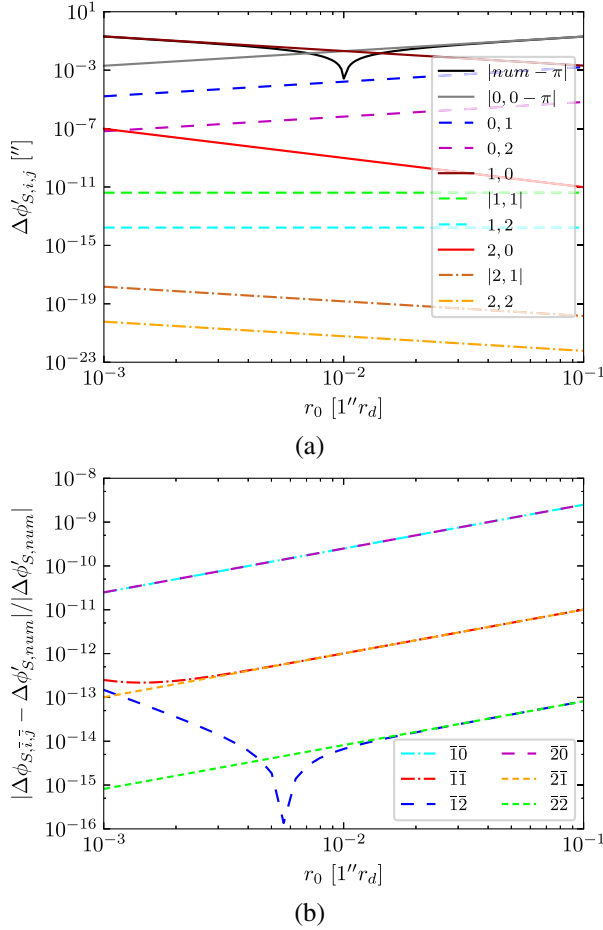
(a)

(b)

FIG. 2. $\Delta\phi'_S$ using Eq. (43). (a) The curve with legend "n,m" corresponds to $\Delta\phi'_{S,n,m}$ and "|n,m|" to $|\Delta\phi'_{S,n,m}|$. (b) The curve with $\bar{n}\,\bar{m}$ corresponds to $|\Delta\phi'_{S,n,m} - \Delta\phi'_{S,num}|/|\Delta\phi'_{S,num}|$.

higher than shown in Eq. (43) are also plotted for illustration purpose. In cosmology, it is known through Planck 2018 observation that the cosmological constant is quite small, $\Lambda_c \leq 1.06 \times 10^{-52}$ [m^{-2}] [56]. To see the effect of Λ clearly among the contributions from all terms, we have to use very large $r_0, r_{s,d}$ and relatively small M. Therefore we chose a toy dwarf galaxy whose distance is $r_d = 13.4$ billion ly [57] with a mass $M = 10^6 \, M_\odot$ as the lens. For energy E, as pointed out in Sec. III, the infinite E limit will force $\Delta\phi$ to approach its value of null rays and become independent of Λ in the SdS case. Therefore in the plots, we will only set E to a relatively small value, $E = 6$, which corresponds to the $v/c = 0.99$ if the spacetime was asymptotically flat. From Fig. 2(a), it is seen that for terms not involving Λ, i.e., $\Delta\phi'_{S,n,0}$ ($n = 1, 2$) plotted using the brown and red solid curves, the contribution for each n decreases monotonically as r_0 increases, and the two terms are separated by about M/r_0. These are expected

features similar to the perturbative deflection angle of Schwarzschild spacetime [23]. For the terms containing Λ, one observes that for $\Delta\phi'_{S,0,i}$ ($i = 1, 2$) plotted using blue and magenta dashed curves, their sizes increase with the increase of r_0. For $\Delta\phi'_{S,1,i}$ ($i = 1, 2$) plotted using green and cyan dashed curves however, their sizes are almost independent of r_0 and then for $\Delta\phi'_{S,2,i}$ ($i = 1, 2$) using light and darker dot-dashed curves, their sizes decrease monotonically as r_0 increases. These features can be understood from the general order estimation Eq. (32), where we knew that $\Delta\phi_{S,n,m}$ ($m \geq 1$) to the leading order should be proportional to r_0^{1-n}. Among the terms dependent on Λ, clearly, $\Delta\phi_{S,0,1}$ is the largest. Comparing to the largest Λ independent term $\Delta\phi_{S,1,0}$, it is seen that for the given choice of parameters $(M, r_{s,d}, \Lambda)$, $\Delta\phi_{S,0,1}$ only becomes comparable when r_0 reaches about $8['']\times r_d$.

To study how good is the perturbative result (41) in approximating the true deflection angle, we first defined a truncated $\Delta\phi_{S,\bar{n},\bar{m}}$ from (41):

$$\Delta\phi_{S,\bar{n},\bar{m}} = \sum_{j=s,d}\sum_{n=0}^{\bar{n}}\sum_{m=0}^{\bar{m}} \Delta\phi_{S,n,m}, \qquad (44)$$

where $\bar{n}$ and $\bar{m}$ are the truncation orders of n and m respectively. And then in Fig. 2(b), its relative difference $|\Delta\phi_{S,\bar{n},\bar{m}} - \Delta\phi_{S,num}|/|\Delta\phi_{S,num}|$ from the numerical deflection angle $\Delta\phi_{S,num}$ obtained using numerical integration from the original definition (8) was plotted. It is seen that as both $\bar{n}$ and $\bar{m}$ increase, the percentage difference between the truncated $\Delta\phi_{\bar{n}\,\bar{m}}$ and the $\Delta\phi_{S,num}$ keeps decreasing. Note that because the $\Delta\phi_{S,n,m}(m \geq 1)$ terms are negative, for some $\bar{n}, \bar{m}$ and at some particular r_0, these truncated $\Delta\phi_{S,\bar{n},\bar{m}}$ might cross the numerical value $\Delta\phi_{S,num}$ and therefore the difference at that point is zero.

B. RNdS spacetime

RNdS spacetime is described by the metric (1) with (2) and

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \qquad g(r) = 1, \qquad C(r) = r^2. \qquad (45)$$

As in the case of SdS spacetime, reading off the coefficients f_i, g_i and c_i in Eq. (25) from this and substituting them into Eqs. (27) and (30), the change of the angular coordinate in the RNdS spacetime $\Delta\phi_R$ to the order $(M/r_0)^2$ and $(r_0^2\Lambda)^1$ becomes

$$\Delta\phi_{\rm R} = \sum_{j=s,d}\left[\Delta\phi_{\rm S,0,0} + \Delta\phi_{\rm S,0,1} + \Delta\phi_{\rm S,1,0} + \Delta\phi_{\rm S,1,1}\right.$$

$$+ \frac{M^2}{r_0^2}\left\{-\frac{1}{2}(\hat{Q}^2 - 3)I_{2,-3} - (\hat{Q}^2 - 3)I_{2,-2} - \frac{1}{2}[E^2(2\hat{Q}^2 - 9) + \kappa(3 - \hat{Q}^2)]I_{2,-1}\right.$$

$$\left.-\frac{E^2}{(E^2 - \kappa)^2}[E^2(\hat{Q}^2 - 3) + \kappa(5 - \hat{Q}^2)]I_{2,0} - \frac{E^2}{2(E^2 - \kappa)^2}[E^2(\hat{Q}^2 - 3) + \kappa(4 - \hat{Q}^2)]I_{2,1}\right\}\right]$$

$$+ \mathcal{O}\left[\left(\frac{M}{r_0}\right)^3, (r_0^2\Lambda)^2\right], \tag{46}$$

where $\hat{Q} \equiv Q/M$ and $\Delta\phi_{\rm S,n,m}$ are given in Eq. (42). It is seen that the effect of the electric charge on the deflection angle emerges from order $1/r_0^2$ for the $\Lambda = 0$ case. The large source/detector distance expansion of $\Delta\phi_{\rm R}$ is

$$\Delta\phi_{\rm R}' = \sum_{j=s,d}\left\{\Delta\phi_{\rm S,0,0}' + \Delta\phi_{\rm S,0,1}' + \Delta\phi_{\rm S,1,0}' + \Delta\phi_{\rm S,1,1}'\right.$$

$$\left.+ \frac{M^2}{r_0^2}\frac{1}{8(E^2 - \kappa)^2}[E^4(15\pi - 3\pi\hat{Q}^2 - 16) + 2E^2\kappa(2\pi\hat{Q}^2 - 9\pi + 4) + (3 - \hat{Q}^2)\pi\kappa]\right\}$$

$$+ \mathcal{O}\left[\left(\frac{M}{r_0}\right)^3, (r_0^2\Lambda)^2, \left(\frac{r_0}{r_{s,d}}\right)^2\right], \tag{47}$$

where $\Delta\phi_{\rm S,n,m}'$ are given in Eqs. (43). We can easily verify that when setting $\hat{Q} = 0$, the second-order terms proportional to $(M/r_0)^2$ in Eqs. (46) and (47) reduce to $\Delta\phi_{\rm S,2,0}$ in Eq. (42e) and $\Delta\phi_{\rm S,2,0}'$ in Eq. (43e). On the other hand, taking $\Lambda = 0$, Eqs. (46) and (47) reduce to the corresponding results in RN spacetime for both null and timelike rays [22,24].

From Eq. (47) it is clear that the spacetime charge Q does not appear in the first-order terms of $(r_0^2\Lambda)$. Therefore to this order, the effect of Λ on $\Delta\phi_{\rm R}$ is the same as its effect in SdS spacetime. As we saw in Sec. IV A for the SdS case, the first order of $(r_0^2\Lambda)$ is already very small and therefore we will not pursue to study the second-order terms proportional to $(r_0^2\Lambda)^2$ in the RNdS spacetime.

C. Dilaton–dS spacetime

The metric of the dilaton–dS spacetime is given by Eq. (1) with (2) and [58]

$$f(r) = 1 - \frac{2M}{r}, \qquad g(r) = 1 - \frac{2D}{r},$$

$$C(r) = r^2\left(1 - \frac{2D}{r}\right), \tag{48}$$

where D is the dilaton charge. When $D = 0$, this reduces to the SdS spacetime metrics. Reading off the coefficients f_i, g_i and c_i in Eq. (25) from this and substituting into Eqs. (27) and (30), we have $\Delta\phi_{\rm D}$ in this spacetime to the leading orders

$$\Delta\phi_{\rm D} = \sum_{j=s,d}[\Delta\phi_{\rm D,0,0} + \Delta\phi_{\rm D,0,1} + \Delta\phi_{\rm D,1,0} + \Delta\phi_{\rm D,1,1}$$

$$+ \Delta\phi_{\rm D,2,0}] + \mathcal{O}\left[\left(\frac{M}{r_0}\right)^3, (r_0^2\Lambda)^2\right], \tag{49}$$

where

$$\Delta\phi_{\rm D,0,0} = \Delta\phi_{\rm S,0,0}, \tag{50a}$$

$$\Delta\phi_{\rm D,0,1} = \Delta\phi_{\rm S,0,1}, \tag{50b}$$

$$\Delta\phi_{\rm D,1,0} = \frac{M}{r_0}\left[(1 + \hat{D})(I_{1,-2} + I_{1,-1})\right.$$

$$\left.+ \left(\frac{E^2}{E^2 - \kappa} - \hat{D}\right)I_{1,0}\right], \tag{50c}$$

$$\Delta\phi_{\rm D,1,1} = \frac{M}{r_0}(r_0^2\Lambda)\frac{\kappa}{6(E^2 - \kappa)}\times\left[\left(\hat{D} - \frac{5E^2 - 3}{E^2 - \kappa}\right)I_{1,0}\right.$$

$$\left.+ \left(\hat{D} - \frac{E^2 - 3}{E^2 - \kappa}\right)I_{1,1} + \left(\hat{D} - \frac{E^2}{E^2 - \kappa}\right)I_{1,2}\right], \tag{50d}$$

$$\Delta\phi_{\rm D,2,0} = \frac{M^2}{r_0^2}\left\{\frac{1}{2}(3\hat{D}^2 + 2\hat{D} + 3)I_{2,-3} + (3\hat{D}^2 + 2\hat{D} + 3)I_{2,-2}\right.$$

$$+ \left[\frac{\hat{D}^2}{2} - \frac{\hat{D}E^2}{E^2 - \kappa} + \frac{3(3E^2 - \kappa)}{2(E^2 - \kappa)}\right]I_{2,-1}$$

$$+ \left[-3\hat{D}^2 + \frac{\hat{D}\kappa}{E^2 - \kappa} + \frac{E^2(3E^2 - 5\kappa)}{(E^2 - \kappa)^2}\right]I_{2,0}$$

$$\left.+ \left[-\frac{\hat{D}^2}{2} - \frac{\hat{D}E^2}{E^2 - \kappa} + \frac{E^2(3E^2 - 4\kappa)}{2(E^2 - \kappa)^2}\right]I_{2,1}\right\}, \tag{50e}$$

where $\Delta\phi_{S,0,0}$ and $\Delta\phi_{S,0,1}$ are given in Eqs. (42a) and (42b) and $\hat{D} \equiv D/M$. The finite distance expansions of these contributions are

$$\Delta\phi'_{D,0,0} = \Delta\phi'_{S,0,0}, \tag{51a}$$

$$\Delta\phi'_{D,0,1} = \Delta\phi'_{S,0,1}, \tag{51b}$$

$$\Delta\phi'_{D,1,0} = \frac{M}{r_0}\left[\left(\hat{D} - \frac{E^2}{E^2 - \kappa}\right)\frac{r_0}{r_j} + \frac{2E^2 - \kappa}{E^2 - \kappa}\right], \tag{51c}$$

$$\Delta\phi'_{D,1,1} = \frac{M}{r_0}(r_0^2\Lambda)\frac{\kappa}{6(E^2 - \kappa)}\left\{\left[\hat{D} - \frac{E^2}{(E^2 - \kappa)}\right]\frac{r_j}{r_0}\right.$$
$$\left. + \hat{D} - \frac{5E^2 - 3\ln(\frac{2r_j}{r_0})}{E^2 - \kappa}\right\}, \tag{51d}$$

$$\Delta\phi'_{D,2,0} = \frac{M^2}{r_0^2}\left[-\frac{1}{8}\pi\hat{D}^2 - \frac{\hat{D}((3\pi - 8)E^2 + (4 - \pi)\kappa)}{4(E^2 - \kappa)}\right.$$
$$\left. + \frac{(15\pi - 16)E^4 + (8 - 18\pi)E^2\kappa + 3\pi\kappa}{8(E^2 - \kappa)^2}\right]. \tag{51e}$$

where $\Delta\phi'_{S,0,0}$ and $\Delta\phi'_{S,0,1}$ are given in Eqs. (43a) and (43b). When the dilaton charge is zero, Eqs. (50) and (51) reduce to the corresponding SdS results (42) and (43) respectively.

Unlike the charge parameter Q in the RNdS spacetime, the dilaton charge D in $\Delta\phi_D$ appears from the first order of M/r_0, i.e., in the $\Delta\phi_{D,1,m}$ terms in Eq. (50) and $\Delta\phi'_{D,1,m}$ terms in Eq. (51). For the effect of Λ in which we are more interested however, because the metric functions (48) of the dilaton–dS spacetime also satisfy the condition (10), it is seen from Eqs. (50b) and (50d) that when $\kappa = 0$, the $(r_0^2\Lambda)$ terms will also vanish completely, as predicted in Sec. II A. Therefore, the effect of Λ on $\Delta\phi_D$ of null rays with fixed r_0 will be absent too.

D. Brane world BH spacetime

The line element of the brane world BH is given by Eq. (1) with (2) and [59]

$$f(r) = 1 - \frac{2M}{r}, \quad g(r) = 1 + \frac{2\hat{\beta}}{r} + \frac{\hat{\beta}^2}{r^2}, \quad C(r) = r^2, \tag{52}$$

with $\Lambda = \alpha^2, \hat{\beta} \equiv \beta/\alpha$ and α, β are some parameters that have not been well constrained. When $\hat{\beta} = 0$, clearly this reduces to the SdS spacetime. Note that this metric does not satisfy condition (10) and therefore we would expect that even for light rays, the $\Delta\phi(r_0)$ will depend on Λ. This is indeed the case, as we will see in Eq. (56).

Reading off the coefficients and substituting them into Eqs. (27) and (30), the change of the angular coordinate in this spacetime becomes

$$\Delta\phi_B = \sum_{j=s,d}[\Delta\phi_{B,0,0} + \Delta\phi_{B,0,1} + \Delta\phi_{B,1,0} + \Delta\phi_{B,1,1}$$
$$+ \Delta\phi_{B,2,0}] + \mathcal{O}\left[\left(\frac{M}{r_0}\right)^3, (r_0^2\Lambda)^2\right], \tag{53}$$

where the contributions from each order are

$$\Delta\phi_{B,0,0} = \Delta\phi_{S,0,0}, \tag{54a}$$

$$\Delta\phi_{B,0,1} = \Delta\phi_{S,0,1}, \tag{54b}$$

$$\Delta\phi_{B,1,0} = \Delta\phi_{S,1,0}, \tag{54c}$$

$$\Delta\phi_{B,1,1} = \frac{M}{r_0}(r_0^2\Lambda) \times \left\{\left[-\frac{\kappa(5E^2 - 3)}{6(E^2 - \kappa)^2} + \frac{\hat{\beta}}{3M}\right]I_{1,0}\right.$$
$$- \left[\frac{\kappa(E^2 - 3)}{6(E^2 - \kappa)^2} + \frac{\kappa\hat{\beta}}{3M(E^2 - \kappa)}\right]I_{1,1}$$
$$\left. - \frac{\kappa E^2}{6(E^2 - \kappa)^2}I_{1,2}\right\}, \tag{54d}$$

$$\Delta\phi_{B,2,0} = \Delta\phi_{S,2,0}, \tag{54e}$$

where $\Delta\phi_{S,n,m}$ are given in Eq. (42). Note for the terms in Eq. (53) that do not involve $(r_0^2\Lambda)$, i.e., terms except $\Delta\phi_{B,n,1}$, their forms are exactly as in the SdS case because in the case that Λ is not present, substituting Eq. (52) into Eq. (2) leads exactly to the SdS spacetime.

Again, to see the finite distance effect, using the expansions (A9) for the first few $I_{n,k}$'s, the $\Delta\phi_{B,n,m}$ become to the leading orders of $r_0/r_i (i = s, d)$

$$\Delta\phi'_{B,0,0} = \Delta\phi'_{S,0,0}, \tag{55a}$$

$$\Delta\phi'_{B,0,1} = \Delta\phi'_{S,0,1}, \tag{55b}$$

$$\Delta\phi'_{B,1,0} = \Delta\phi'_{S,1,0}, \tag{55c}$$

$$\Delta\phi'_{B,1,1} = \frac{M}{r_0}(r_0^2\Lambda) \times \left\{-\frac{\kappa E^2}{6(E^2 - \kappa)^2}\frac{r_j}{r_0} + \frac{E^2\frac{\hat{\beta}}{M}}{3(E^2 - \kappa)}\right.$$
$$\left. + \frac{\kappa[((2 - 2E^2)\frac{\hat{\beta}}{M} + 3)\ln(\frac{2r_j}{r_0}) - 5E^2]}{6(E^2 - \kappa)^2}\right\}, \tag{55d}$$

$$\Delta\phi'_{B,2,0} = \Delta\phi'_{S,2,0}, \tag{55e}$$

where again $\Delta\phi'_{S,n,m}$ are given in Eq. (43). After changing from r_0 to b_{eff}, this result agrees with Eq. (25) of Ref. [60] to the order $\Lambda\hat{\beta}b_{eff}$. A key difference of Eq. (55) from the previous SdS, RNdS and dilaton–dS spacetime results is that due to the breaking of condition (10) by metric (52), even for the null signal the effect of the cosmological

constant Λ will still be present. That is, for light, setting κ to zero in Eq. (55d) we have

$$\Delta\phi'_{B,1,1}(\kappa=0) = \frac{M}{r_0}(r_0^2\Lambda)\frac{\hat{\beta}}{3M} = \frac{\hat{\beta}r_0\Lambda}{3}. \qquad (56)$$

In addition, this result itself also shows that our perturbative method for computing $\Delta\phi$ can work not only for metrics satisfying (10) but also those breaking it.

In Figs. 3(a) and 3(b), we plot the contribution from each order of Eq. (55) for $(\kappa=1, E=6)$ and $\kappa=0$ respectively. We chose the parameter $\hat{\beta}=0.01''r_d$. Other parameters are the same as in Fig. 2. Comparing plots 3(a) and 3(b), we observe that unlike the case of $\Delta\phi_S$, some of the $\Delta\phi_{B,n,m}$ ($m\geq 1$) terms proportional to Λ^m also survived after setting $\kappa=0$, e.g., the $\Delta\phi_{B,1,1}$, $\Delta\phi_{B,2,1}$ and $\Delta\phi_{B,2,2}$ terms.

V. THE GRAVITATIONAL LENSING IN ASYMPTOTICALLY dS SPACETIMES

A. The GL apparent angles

In order to study the GL effect in asymptotically dS spacetimes, we should first build a proper lensing equation
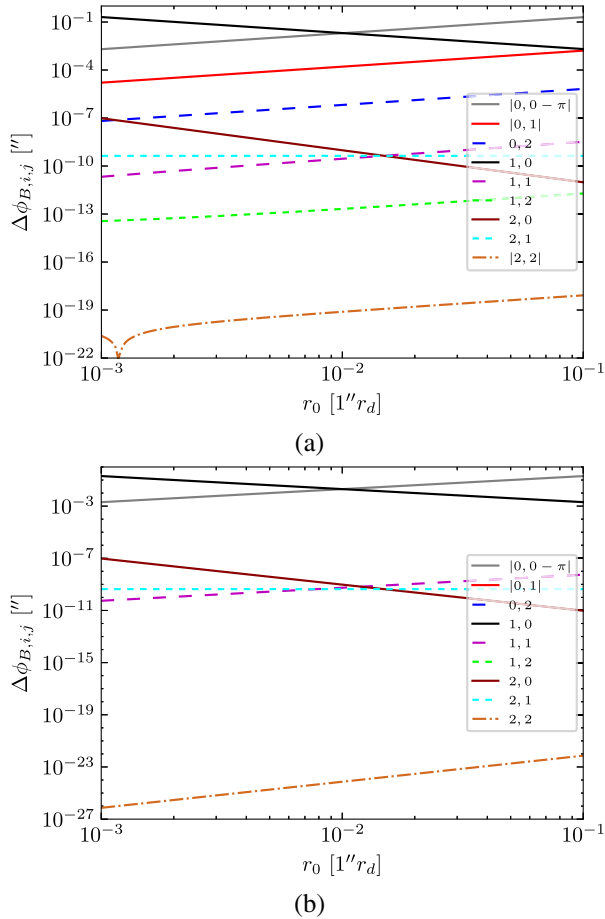

(a)

(b)

FIG. 3. $\Delta\phi_B$ using Eq. (55). (a) $\kappa=1$, $E=6$; (b) $\kappa=0$. Other parameters are the same as in Fig. 2.

(LE), from which the apparent angles of the lensed images can be solved for any given angular position of the lens, i.e., β_L or β in Fig. 1. Traditionally, this LE was built using deflection angle for source and observer located at spacial infinity, together with some approximate geometric relation. However in the case that spatial infinity cannot be reached as in asymptotically dS spacetimes, this is not possible. In contrast, if $\Delta\phi$ with finite distance effect is known, we can actually build an exact LE without having to set the source/detector to some approximate locations. This exact LE is just the relation between the change of angular coordinate $\Delta\phi$ given in Eq. (30) and the source angular location β_L (see Fig. 1):

$$\pi \pm \beta_L = \Delta\phi(r_0), \qquad (57)$$

where the $+$ and $-$ signs correspond respectively to the trajectories rotating anticlockwise and clockwise. From this equation, the closest distance r_0 can be solved, after which using Eqs. (6) and (17) the apparent angle θ can be obtained.

Since $\Delta\phi$ in (30) after the large $r_{s,d}/r_0$ expansion of $I_{n,k}$ is a rational function of r_0, the analytical solution to Eq. (57) will be difficult if we keep too high or too many orders of r_0 in $\Delta\phi$. Therefore we will truncate $\Delta\phi$ in Eq. (57) to the highest solvable order. We will analyze the GL in two limiting cases. Case (1) is when M/r_0 is as large as possible while $r_{s,d}^2\Lambda$ is as small as possible, and case (2) is when M/r_0 is as small as possible while $r_{s,d}^2\Lambda$ is as large as possible.

In case (1), such as in the GL by the SgrA* SMBH, we would have a typical $\mathcal{O}(M/r_0) = 10^{-6} \sim 10^{-5}$, and if $\Lambda = \Lambda_c$ is used, $r_{s,d}^2\Lambda_c = 6.7 \times 10^{-12}$. Then the $n=0$, 1 and $m=0$ terms in $\Delta\phi$ are much larger than the $n\geq 2$, $m=0$ and $n\geq 0$, $m\geq 1$ terms and therefore keeping only the former terms, the LE (57) becomes

$$\pi \pm \beta_L = \sum_{j=s,d}\left[p_{0,0,-1}I_{0,-1}\left(\frac{r_j}{r_0}\right) + \frac{1}{r_0}\sum_{k=-2}^{0}p_{1,0,k}I_{1,k}\left(\frac{r_j}{r_0}\right)\right]. \qquad (58)$$

To solve r_0, we will directly use the large $r_{s,d}/r_0$ expansions of $I_{0,-1}$ and $I_{1,k}(k=-2,-1,0)$ given in Eq. (A9). Then the above equation becomes a quadratic equation of r_0:

$$\pi \pm \beta_L = \sum_{j=s,d}\left\{\frac{\pi}{2} - \frac{r_0}{r_j} + \frac{1}{r_0}\left[p_{1,0,-2}\left(2-\frac{\pi}{2}\right)\right.\right.$$
$$\left.\left. + p_{1,0,-1}\left(\frac{\pi}{2}-1\right) + p_{1,0,0}\right]\right\}$$
$$= \sum_{j=s,d}\left[\frac{\pi}{2} - \frac{r_0}{r_j} - \frac{1}{r_0}\frac{(2E^2-\kappa)f_1}{2(E^2-\kappa)}\right], \qquad (59)$$

where in the second step $p_{0,0,-1}$ and $p_{1,0,k}$ in Eq. (27) were substituted. Solving this, we find two allowed r_0: r_{0+} and r_{0-} for signals from the counterclockwise and clockwise directions respectively:

$$r_{0\pm} = \frac{1}{2(r_d + r_s)}\left[\mp r_d r_s \beta_L + \sqrt{r_d^2 r_s^2 \beta_L^2 - \frac{4r_d r_s(r_d + r_s)(2E^2 - \kappa)f_1}{E^2 - \kappa}}\right]. \quad (60)$$

Case (2) can be realized by the GL of more distant objects so that $r_{s,d}$ and consequently $r_{s,d}^2\Lambda$ can be larger. In this case, the $n = 0, m = 0, 1$ terms of $\Delta\phi$ could be more important than the $n = 1, m = 0$ term which is certainly

much larger than the $n \geq 2$ terms. Therefore the LE (57) becomes

$$\pi \pm \beta_L = \sum_{j=s,d}\left[p_{0,0,-1}I_{0,-1}\left(\frac{r_j}{r_0}\right) + \frac{1}{r_0}\sum_{k=-2}^{0} p_{1,0,k}I_{1,k}\left(\frac{r_j}{r_0}\right)\right.$$
$$\left. + (r_0^2\Lambda)\sum_{k=-1}^{1} p_{0,1,k}I_{0,k}\left(\frac{r_j}{r_0}\right)\right]. \quad (61)$$

Here we kept the term proportional to $1/r_0$ because this does not spoil the solvability of the equation. Using the large $r_{s,d}/r_0$ expansion of $I_{0,k}(k = -2, -1, 0)$ and $I_{1,k}(k = -1, 0, 1)$ in Eq. (A9), this also becomes a quadratic equation of r_0,

$$\pi \pm \beta_L = \sum_{j=s,d}\left\{\frac{\pi}{2} - \frac{r_0}{r_j} + \frac{1}{r_0}\left[p_{1,0,-2}\left(2 - \frac{\pi}{2}\right) + p_{1,0,-1}\left(\frac{\pi}{2} - 1\right) + p_{1,0,0}\right] + r_0 r_j \Lambda p_{0,1,1}\right\}$$
$$= \sum_{j=s,d}\left[\frac{\pi}{2} - \frac{r_0}{r_j} - \frac{1}{r_0}\frac{(2E^2 - \kappa)f_1}{2(E^2 - \kappa)} - \frac{\kappa r_0 r_j \Lambda}{6(E^2 - \kappa)}\right], \quad (62)$$

where in the second step the $p_{n,m,k}$ were substituted by Eq. (27). From this, again r_0 can be readily solved as

$$r_{0\pm} = \frac{1}{2(r_d + r_s)[1 + \kappa r_s r_d \Lambda/[6(E^2 - \kappa)]]}\left[\mp r_d r_s \beta_L + \sqrt{r_d^2 r_s^2 \beta_L^2 - \frac{4r_d r_s(r_d + r_s)(2E^2 - \kappa)f_1}{E^2 - \kappa}\left(1 + \frac{\kappa r_s r_d \Lambda}{6(E^2 - \kappa)}\right)}\right]. \quad (63)$$

The β_L in this and the former result (60) can be replaced by the more commonly used β in Fig. 1, by using the following geometrical relation between them:

$$r_s \sin\beta_L = (r_d + r_s \cos\beta_L)\tan\beta. \quad (64)$$

This equation allows us to rewrite any formula involving β_L into a formula using β. Solving this to the first order and substituting into Eq. (63), $r_{0\pm}$ become

$$r_{0\pm} = \frac{r_d}{2[1 + \kappa r_s r_d \Lambda/[6(E^2 - \kappa)]]}\left[\mp\beta + \sqrt{\beta^2 - \frac{4r_s(2E^2 - \kappa)f_1}{r_d(r_d + r_s)(E^2 - \kappa)}\left(1 + \frac{\kappa r_s r_d \Lambda}{6(E^2 - \kappa)}\right)}\right]. \quad (65)$$

Clearly, Eq. (65) is a more general result than Eq. (60), which is the asymptotically flat spacetime result: when $\Lambda \to 0$, Eq. (65) reduces to Eq. (60). In order for the effect of Λ on $r_{0\pm}$ to be apparent, from Eq. (65) we see that we should have for $\kappa = 1$ the following condition:

$$r_s r_d \Lambda \gtrsim 6(E^2 - 1). \quad (66)$$

Noting the value of Λ_c, this means only for source and lens that are of cosmic distance can the effect of Λ be important to $r_{0\pm}$. When this is the case, expanding (63) in small Λ, one arrives at

$$r_{0\pm}(\Lambda \to 0) = r_{0\pm}(\Lambda = 0) + \Lambda \times \left[\frac{r_d r_s \beta_L}{2(r_d + r_s)}\right.$$
$$\left. - \frac{r_d^2 r_s^2 \beta_L^2 - \frac{2r_d r_s(r_d + r_s)(2E^2 - \kappa)f_1}{E^2 - \kappa}}{2(r_d + r_s)\sqrt{r_d^2 r_s^2 \beta_L^2 - \frac{4r_d r_s(r_d + r_s)(2E^2 - \kappa)f_1}{E^2 - \kappa}}}\right], \quad (67)$$

where $r_{0\pm}(\Lambda = 0)$ is just the $r_{0\pm}$ in Eq. (60). It is not too difficult by using typical parameter values to numerically verify that the square bracket part is actually negative, and therefore a positive Λ will decrease both r_{0+} and r_{0-}.

Having obtained $r_{0\pm}$, substituting it into Eq. (17), we can get the apparent angles of the two images,

$$\theta_\pm = \arcsin\left(\sqrt{\frac{E^2 - \kappa A(r_{0\pm})}{E^2 - \kappa A(r_d)}}\sqrt{\frac{A(r_d)C(r_{0\pm})}{A(r_{0\pm})C(r_d)}}\right). \quad (68)$$

Since $r_d \gg r_0 \gg |f_1| = 2M$, and if $\mathcal{O}(f_n, g_n, c_n) \le M^n$, then these apparent angles can be further expanded in terms of the ratios among $\{r_d, r_0, M, \Lambda\}$. Substituting Eqs. (2), (25) into (68), carrying out the expansion in terms of $\Lambda, r_{0\pm}/r_d, M/r_{0\pm}$ and keeping only to the first order for each of these, the apparent angles become

$$\theta_\pm = \frac{r_{0\pm}}{r_d} - \frac{\Lambda r_{0\pm} r_d E^2}{6(E^2 - \kappa)} + \mathcal{O}\left[\left(\frac{r_0}{r_d}\right)^2, \Lambda r_d M, \Lambda^2\right], \quad (69)$$

where $r_{0\pm}$ is in Eq. (63). Further carrying out the small Λ expansion in $r_{0\pm}$ using (67), the effect of Λ on $\theta_\pm$ becomes more apparent,

$$\theta_\pm = \frac{r_{0\pm}(\Lambda = 0)}{r_d} + \left[\frac{C_\Lambda}{r_d} - \frac{r_{0\pm}(\Lambda = 0)E^2}{6(E^2 - \kappa)}\right]\Lambda$$
$$+ \mathcal{O}\left[\left(\frac{r_0}{r_d}\right)^2, \Lambda r_d M, \Lambda^2\right], \quad (70)$$

where C_Λ is the coefficient of Λ in the second term of Eq. (67). Since both terms in the square bracket are negative, it is clear then $\theta_\pm$ will also decrease as Λ increases. We point out that if $\Lambda = 0$ as in asymptotically flat spacetimes, the $r_{0\pm}(\Lambda = 0)/r_d$ term can be reexpressed using Eq. (60) in terms of the angle β, and $\theta_\pm(\Lambda = 0)$ become

$$\theta_\pm(\Lambda = 0) = \frac{r_{0\pm}(\Lambda = 0)}{r_d} = \frac{1}{2}\left(\sqrt{\beta^2 + 4\theta_E^2} \mp \beta\right), \quad (71)$$

where the Einstein angular radius $\theta_E(\Lambda = 0)$ is

$$\theta_E(\Lambda = 0) = \sqrt{-\frac{4r_s(2E^2 - \kappa)f_1}{r_d(r_d + r_s)(E^2 - \kappa)}}. \quad (72)$$

Equation (71) agrees perfectly with the apparent angles of conventional point mass lensing [61].

In Fig. 4, we plot $r_{0\pm}$ in (63) and the two associated apparent angles $\theta_\pm$ in (68) as functions of β_L and Λ. The coefficient f_1 is set to $-2M$ and other parameters are the same as in Fig. 2. We see from Fig. 4(a) that with this parameter setting, the value of $r_{0\pm}(\beta_L \to 0)/r_d$, which is roughly the Einstein radius (in asymptotic flat spacetime), is 0.010 [″] and the minimal radius r_{0+} (or r_{0-}) decreases (or increases) as β_L increases, as expected from Fig. 1. From Fig. 4(c) we see however, the exact value of θ_E (defined as the value of $\theta_\pm$ at $\beta_L = 0$) is at 0.0066 [″],
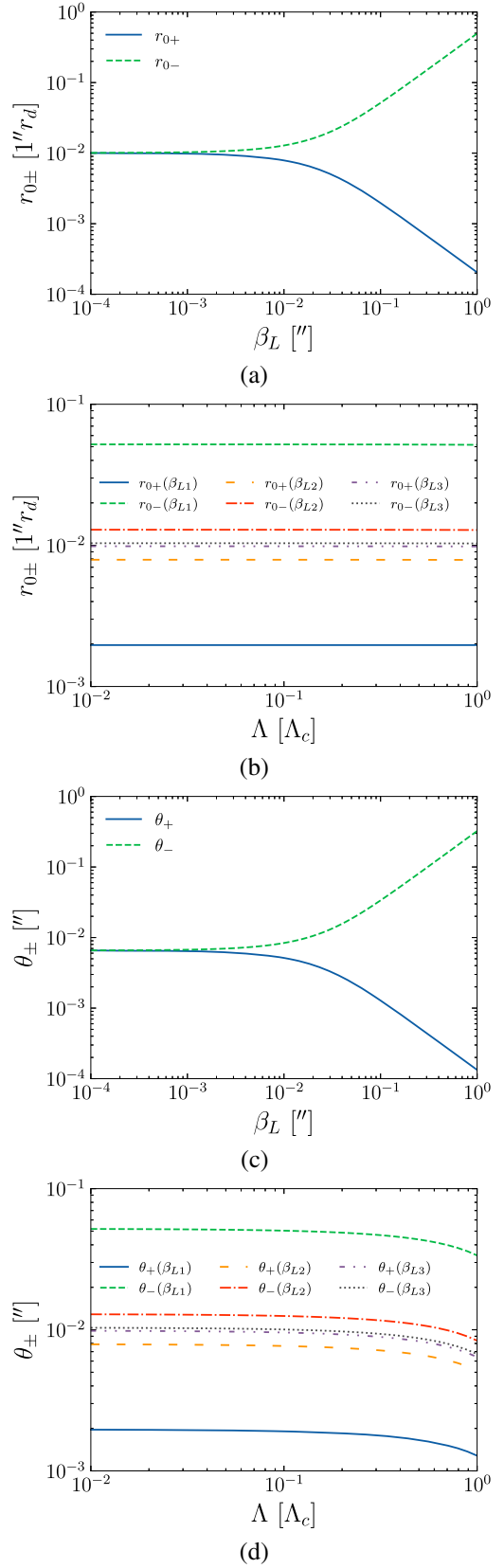

(a)

(b)

(c)

(d)

FIG. 4. The minimal radii $r_{0\pm}$ and apparent angles $\theta_\pm$ for two trajectories from two sides of the lens, as functions of β_L [(a) and (c)] and Λ [(b) and (d)]. The Λ in (a) and (c) are set to Λ_c. The chosen β_L in are respectively $\beta_{L,1} = 10^{-3}$, $\beta_{L,2} = 10^{-2}$ and $\beta_{L,3} = 10^{-1}$ [″].

slightly different from the value estimated from Fig. 4(a). The reason is that the estimation $\theta_E \approx r_{0\pm}(\beta_L \to 0)/r_d$ only works well when $\Lambda = 0$, as indicated by Eqs. (70) and (71), and the extra term proportional to Λ in Eq. (70) decreases this value.

On the other hand, from Fig. 4(b) we observe that for all three chosen β_L ($\beta_{L,1} = 10^{-3}, \beta_{L,2} = 10^{-2}$ and $\beta_{L,3} = 10^{-1}$ [″]), $r_{0\pm}$ are (almost) independent of Λ. As pointed out earlier, this is because for the chosen r_s, r_d, Λ and E, the condition (66) is still far from being reached, which consequently suppressed the effect of Λ. As for the dependence of $\theta_\pm$ on Λ, we see from Fig. 4(d) that unlike $r_{0\pm}$, the apparent angles from both sides of the lens for the same three representative β_L decrease very apparently as Λ increases, as predicted by Eq. (70).

B. Time delay between GL images

With $r_{0\pm}$ known in Eq. (65), we can now utilize the total travel time (38) to compute the time delay between GL images. If the images are formed by the rays of the same energy E but from different sides of the lens, then such time delay is

$$\Delta^2 t_\pm \equiv \Delta t(r_{0+}(E), E) - \Delta t(r_{0-}(E), E), \quad (73)$$

where both the minimal radii $r_{0\pm}$ and total travel time explicitly depend on the signal energy. On the other hand, if we consider the time delay between signals from the same side of the lens but having different energies, such as the time delay between light (or GW) and neutrinos, or neutrinos of different mass eigenstates, then such time delay is

$$\Delta^2 t_{12} \equiv \Delta t(r_{0\xi}(E_1), E_1) - \Delta t(r_{0\xi}(E_2), E_2), \quad (74)$$

where $\xi = \pm$ represents the signals from the "+" or "−" side of the lens.

For the time delay $\Delta^2 t_\pm$, substituting Eq. (65) into (38), and then further into (73) and simplifying the result, we find

$$\Delta^2 t_\pm = \sum_{i=s,d} \frac{E}{\sqrt{E^2 - \kappa}} \frac{2l^2 \sqrt{\eta + 1}}{r_i} + \frac{El^2 \sqrt{\eta + 1}}{6(E^2 - \kappa)^{\frac{3}{2}}} \frac{\Lambda}{r_i}$$
$$\times [(2E^2 - 9\kappa)l^2(\eta + 2) + 6\kappa r_i^2]$$
$$+ \frac{E}{2(E^2 - \kappa)^{\frac{3}{2}}} \left\{ [(E^2 - \kappa)c_1 - E^2 f_1] \frac{2l}{r_i} \right.$$
$$\left. - f_1(2E^2 - 3\kappa) \ln \frac{\sqrt{\eta + 1} + 1}{\sqrt{\eta + 1} - 1} \right\}$$
$$+ \mathcal{O}\left(\frac{r_i}{b^2}, \frac{r_i r_0^2 \Lambda}{b}, r_i (r_0^2 \Lambda)^2 \right), \quad (75)$$

where the auxiliary parameters l and η are

$$l = \frac{r_d \beta}{2[1 + \kappa r_s r_d \Lambda / [6(E^2 - \kappa)]]}, \quad (76)$$

$$\eta = -\frac{4f_1 r_s (2E^2 - \kappa)}{r_d(r_s + r_d)\beta^2(E^2 - \kappa)} \left[1 + \frac{\kappa r_s r_d \Lambda}{6(E^2 - \kappa)} \right]. \quad (77)$$

The first, second and third terms in Eq. (75) are due to the $\Delta t_{0,0}, \Delta t_{0,1}$ and $\Delta t_{1,0}$ orders in the total travel times (38) respectively. Therefore we can expect that the contribution from first term to the time delay should be larger than other terms. Note that in the case of $\Lambda = 0$, Eq. (75) agrees with the time delay in asymptotically flat SSS spacetimes [47]. The dependence of (75) on Λ however is not only through the second term since there also exist Λ dependence in l and η. To isolate the effect of Λ, we can further expand (75) in the small Λ limit, and find to the leading order of Λ

$$\Delta^2 t_\pm \approx \sum_{i=s,d} \frac{E}{\sqrt{E^2 - \kappa}} \left(\frac{2l_0^2 \sqrt{\eta_0 + 1}}{r_i} - f_1 \ln \frac{\sqrt{\eta_0 + 1} + 1}{\sqrt{\eta_0 + 1} - 1} \right)$$
$$+ \frac{\Lambda E}{(E^2 - \kappa)^{\frac{3}{2}}} \left[\frac{-\kappa l_0^2 r_s r_d(3\eta_0 + 4)}{6\sqrt{\eta_0 + 1} r_i} + \frac{\kappa f_1 r_s r_d}{6\sqrt{\eta_0 + 1}} \right.$$
$$\left. + \sqrt{\eta_0 + 1} \left(\kappa l_0^2 r_i + \frac{l_0^4(\eta_0 + 2)(E^2 - \kappa)}{3r_i} \right) \right]$$
$$+ \mathcal{O}(\Lambda^2), \quad (78)$$

where

$$l_0 = l(\Lambda \to 0) = \frac{r_d \beta}{2}, \quad (79)$$

$$\eta_0 = \eta(\Lambda \to 0) = -\frac{8f_1 r_s}{r_d(r_s + r_d)\beta^2}. \quad (80)$$

The first, second and third terms of the $\mathcal{O}(\Lambda^1)$ order come respectively from the $\Delta t_{0,0}$ order, the $\Delta t_{1,0}$ order and $\Delta t_{0,1}$ order in the total travel time.

Here it is important to note that in the time delay $\Delta^2 t_\pm$ in Eq. (75), three nontrivial spacetime parameters, c_1, f_1 and Λ are involved. However the first term of the $\Delta t_{0,1}$ order given in the curl bracket, which contains the only occasion of c_1, can be shown to be always much smaller than the logarithmic term. And therefore in the expansion in Eq. (78), the c_1 disappears in the leading order of $\Delta^2 t_\pm$. Therefore one can conclude that for arbitrary asymptotically dS spacetimes, the time delay $\Delta^2 t_\pm$ depends only on f_1 and Λ, to the leading order.

In Fig. 5, we plot the time delay (75) and its three terms $\Delta^2 t_{\pm 0,0}, \Delta^2 t_{\pm 0,1}$ and $\Delta^2 t_{\pm 1,0}$ corresponding to contributions from order $\Delta t_{0,0}, \Delta t_{0,1}$ and $\Delta t_{1,0}$ in the total travel time, as functions of β and Λ. Note that for comparison purpose, in Fig. 5(b), we have allowed Λ to go beyond Λ_c to about $100\Lambda_c$. We observe from Fig. 5(a) that as β decreases to zero, all contributions decrease monotonically to zero, as
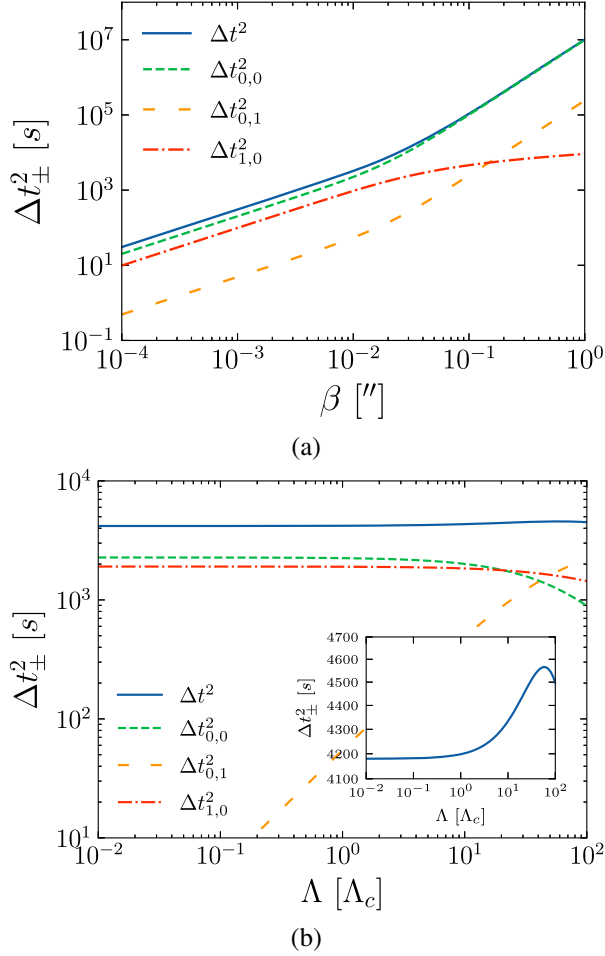
(a)

(b)

FIG. 5. Time delay $\Delta^2 t_\pm$ in Eq. (75) and its various terms as a functions of β for $\Lambda = \Lambda_c$ (a) and as functions of Λ for $\beta = 0.01['']$ (b). The inset shows the total $\Delta^2 t_\pm$ using linear scale for y axis. Other parameters are the same as in Fig. 4.

expected for the time delay between signals from the two sides of the lens. When β is smaller than $0.007[''] \sim \theta_E$, the $\Delta^2 t_{\pm 0,0}$ and $\Delta^2 t_{\pm 1,0}$ contribute similarly. When β increases beyond θ_E however, the $\Delta^2 t_{\pm 0,1}$ increases much faster than $\Delta^2 t_{\pm 1,0}$ and surpasses it at about $\beta \approx 0.23['']$. On the other hand, from Fig. 5(b) we observe that as Λ increases to $\sim 10\Lambda_c$, the $\Delta^2 t_{\pm 0,1}$ term increases almost linearly as dictated by the second term of Eq. (75) and its contribution surpasses the $\Delta^2 t_{\pm 1,0}$ and $\Delta^2 t_{\pm 0,0}$ terms when $\Lambda = 50.6\Lambda_c$ and $\Lambda = 41.1\Lambda_c$ respectively. Moreover, we also note that for Λ smaller than $10\Lambda_c$, superficially both the $\Delta^2 t_{\pm 0,0}$ and $\Delta^2 t_{\pm 1,0}$ terms are almost invariant as Λ changes. However, from Eq. (78) it is known that indeed these two terms also contain Λ dependence comparable to the $\Delta^2 t_{\pm 0,1}$ term in at least some parameter space spanned by β, Λ and E etc. Indeed, as Λ increases larger than $10\Lambda_c$, their dependence on Λ becomes very apparent. Furthermore, in the three terms proportional to Λ in Eq. (78), we can show that for any reasonable set of parameters $\{r_d, M, \beta\}$, the third term

will always be positive and larger than the first two terms and therefore the total $\Delta^2 t_\pm$ should increase as the small Λ increase. This is also seen in the inset of Fig. 5(b), in which $\Delta^2 t_\pm$ increased until $\Lambda \approx 59\Lambda_c$, beyond which the small Λ expansion (78) broke down.

For the time delay between signals from the same side, substituting $r_{0\pm}$ in Eq. (65) with the same sign but different energies E_1 and E_2 into (38), and then further into (74), the value of $\Delta^2 t_{12}$ can be obtained. Since the signals we consider are usually relativistic or null, we can actually substitute (65) directly into (39) and then expanding it in the large energy limit and directly find, to the leading order of $1/E^2$,

$$\Delta^2 t_{12} = \sum_{i=s,d} \left(\frac{1}{E_1^2} - \frac{1}{E_2^2} \right) \left[\frac{r_i}{2} - \frac{r_i r_{0\xi\infty}^2 \Lambda}{4} \right] + \mathcal{O}\left(\frac{r_i}{E^4} \right), \quad (81)$$

where

$$r_{0\xi\infty} \equiv r_{0\xi}(E \to \infty) = l_0(\sqrt{\eta_0 + 1} - \xi), \quad \xi = \pm 1. \quad (82)$$

It is seen that generally, the existence of a positive cosmological constant will decrease the time delay $\Delta^2 t_{12}$. The second and first terms in the square bracket of Eq. (81) have a ratio of $r_{0\xi\infty}^2 \Lambda/2$. Unfortunately, for the currently known Λ_c, $r_{0\xi\infty}^2 \Lambda_c \ll 1$ and therefore the second term involving Λ will be much smaller than the first term. Let us take the time delay between different mass eigenstates $|\nu_i\rangle$ ($i = 1, 2, 3$) of supernova neutrinos as an example. Assuming that the average energy of the mass eigenstates is $\langle E \rangle = 10$ [MeV], then the energies per unit mass of the first mass eigenstates $|\nu_j\rangle$ and the second mass eigenstate $|\nu_k\rangle$ are respectively $E_1 = \langle E \rangle / m_j$ and $E_2 = \langle E \rangle / m_k$. Substituting them into Eq. (81), then the time delay between them simplifies to

$$\Delta^2 t_{12} = \sum_{i=s,d} \frac{\Delta m_{jk}^2}{\langle E \rangle^2} \left[\frac{r_i}{2} - \frac{r_i r_{0\xi\infty}^2 \Lambda}{4} \right] + \mathcal{O}\left(\frac{r_i}{E^4} \right). \quad (83)$$

Since the mass square differences of neutrino mass eigenstates are [62]

$$\Delta m_{12}^2 = -7.53 \times 10^{-5} \text{ [eV}^2], \quad (84)$$

and for normal ordering

$$\Delta m_{23}^2 = -2.453 \times 10^{-3} \text{ [eV}^2] \quad (85)$$

and inverse ordering

$$\Delta m_{23}^2 = 2.546 \times 10^{-3} \text{ [eV}^2], \quad (86)$$

then for the time delay between the mass eigenstates $|\nu_1\rangle$ (or roughly $|\nu_2\rangle$) and $|\nu_3\rangle$, one can compute the first and

second terms in the square bracket of Eq. (83) using the same $r_{s,d}$ and M as in Fig. 4. They yield at most 10.7 [s] and 2.14×10^{-8} [s] for $\beta \approx 10['']$ and $\xi = -1$. Only if $r_{0\xi\infty}^2 \Lambda$ were larger, which requires a larger $r_{s,d}$, M and β, or the resolution of the time delay is much better than $\mathcal{O}(10^{-8})$ [s], will the effect of Λ on $\Delta^2 t_{12}$ be explicitly distinguishable.

VI. CONCLUSION

In this paper, we studied the deflection and GL of null and timelike rays in general SSS and asymptotically (a)dS spacetimes. We first showed that if the metric function satisfies a simple condition, Eq. (10), then the change of the angular coordinate for fixed r_0 will actually not depend on the parameter Λ for null rays, although that of timelike rays (13), the apparent angles of the GL images (17), and the total travel time (14) will still depend on Λ explicitly. A two-step perturbative method is then developed to calculate the change of the angular coordinate and the result is expressed into a quasipower series form of both M/r_0 and Λ, i.e., Eq. (30), with the coefficients of the series determined by the asymptotic expansion coefficients of the metric functions. The finite distance effect of the source/detector is also naturally taken into account. The methodology is applied to SdS, RNdS, dilaton–dS and brane-world–dS spacetimes to find the deflection in them. The total travel time is computed using the same method and expressed in a similar quasiseries, Eq. (36).

Using the obtained deflection, an exact GL equation is used to solve the allowed minimal radius (65) from each side of the lens and the corresponding apparent angles. It is found that generally, due to the smallness of currently known Λ, only for source of very large distance can the

effect of Λ be apparent on these quantities. In this case, as Λ increases, both apparent angles will decrease as Λ increases (see Fig. 4). The time delays $\Delta^2 t_{\pm}$ between the images from two sides and $\Delta^2 t_{12}$ between signals from the same side are also obtained in Eqs. (75) and (81). We found that qualitatively a small positive Λ will cause the former to increase while the latter to decrease.

We emphasize the generality of the perturbative method developed in this work: besides the four spacetimes considered in Sec. IV, it can be applied to other interesting asymptotically dS spacetimes for the computation of both deflection and time delay for both null and timelike rays, with finite distance effect naturally taken into account. Moreover, although in the examples in this work, we explicitly consider the case of asymptotically de Sitter spacetimes with $\Lambda > 0$, indeed both the method and relevant results are valid for asymptotically anti–de Sitter spacetimes. Finally, a generalization of the method to more realistic spacetime such as the equatorial plane of Kerr–dS or Kerr-Newmann–dS spacetimes is also possible.

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APPENDIX: INTEGRATION FORMULAS

To carry out the integrals in Eq. (29), we first need to do partial fraction decomposition to the integrand. When $k + 1 < 0$, we have

$$
\begin{aligned}
\frac{\cos^{n-k-1}\alpha}{(\cos\alpha + 1)^n} &= \sum_{i=0}^{n} \frac{C_{n-k-1}^i (-1)^{n-k-i-1}}{(1+\cos\alpha)^{n-i}} + \sum_{i=1}^{-k-1}\sum_{j=0}^{i} C_{n-k-1}^{i+n} C_i^j (-1)^{k+i+1} \cos^j\alpha, \\
&= \sum_{i=0}^{n} \frac{C_{n-k-1}^i (-1)^{n-k-i-1}}{(2\cos^2\frac{\alpha}{2})^{n-i}} + \sum_{i=1}^{-k-1}\sum_{j=0}^{i} C_{n-k-1}^{i+n} C_i^j (-1)^{k+i+1} \cos^j\alpha.
\end{aligned} \tag{A1}
$$

When $0 \le k + 1 \le n$, we have

$$
\frac{\cos^{n-k-1}\alpha}{(\cos\alpha + 1)^n} = \sum_{i=0}^{n-k-1} \frac{C_{n-k-1}^i (-1)^{n-k-i-1}}{(2\cos^2\frac{\alpha}{2})^{n-i}}. \tag{A2}
$$

When $n < k + 1$, we have

$$
\frac{\cos^{n-k-1}\alpha}{(\cos\alpha + 1)^n} = (-1)^{k-n+1} \sum_{i=1}^{n} \frac{C_{k-i}^{n-i}}{(2\cos^2\frac{\alpha}{2})^i} + \sum_{j=1}^{k-n+1} \frac{(-1)^{k-n-j+1} C_{k-j}^{k-n-j+1}}{\cos^j\alpha}. \tag{A3}
$$

Equation (29) is integrable if the terms on the right-hand sides of these equations are integrable. Their integration can be done using Eqs. **2.513** 3, 4 and **2.519** 1, 2 in Ref. [63] and the results for $l > 0$ are

$$\int \frac{d\alpha}{\cos^{2l}\alpha} = \frac{\sin\alpha}{2l-1}\left[\sec^{2l-1}\alpha + \sum_{k_l=1}^{l-1}\frac{2^{k_l}(l-1)(l-2)\ldots(l-k_l)}{(2l-3)(2l-5)\ldots(2l-2k_l-1)}\sec^{2l-2k_l-1}\alpha\right],\tag{A4a}$$

$$\int \frac{d\alpha}{\cos^{2l+1}\alpha} = \frac{\sin\alpha}{2l}\left[\sec^{2l}\alpha + \sum_{k_l=1}^{l-1}\frac{(2l-1)(2l-3)\ldots(2l-2k_l+1)}{2^{k_l}(l-1)(l-2)\ldots(l-k_l)}\sec^{2l-2k_l}\alpha\right] + \frac{(2l-1)!!}{2^l l!}\ln\,\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right),\tag{A4b}$$

$$\int \cos^{2l}\alpha d\alpha = \frac{1}{2^{2l}}C_{2l}^l\alpha + \frac{1}{2^{2l-1}}\sum_{k_l=0}^{l-1}C_{2l}^{k_l}\frac{\sin\left[(2l-2k_l)\alpha\right]}{2l-2k_l},\tag{A4c}$$

$$\int \cos^{2l+1}\alpha d\alpha = \frac{1}{2^{2l}}\sum_{k_l=0}^{l}C_{2l+1}^{k_l}\frac{\sin\left[(2l-2k_l+1)\alpha\right]}{2l-2k_l+1}.\tag{A4d}$$

Corresponding to Eqs. (A4a) and (A4b), and (A4c) and (A4d), we then define two functions $F_n(\alpha)$ and $G_n(\alpha)$ as

$$F_0(\alpha) = \alpha \tag{A5a}$$

$$F_1(\alpha) = \ln\,\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) \tag{A5b}$$

$$F_n(\alpha) = \frac{\sin\alpha}{n-1}\left[\sec^{n-1}\alpha + \sum_{i=1}^{\left[\frac{n}{2}\right]-1}\frac{(n-2)(n-4)\ldots(n-2i)}{(n-3)(n-5)\ldots(n-2i-1)}\sec^{n-2i-1}\alpha\right]$$
$$+ \begin{cases} 0, & n \text{ is even and } n > 0, \\ \frac{(n-2)!!}{(n-1)!!}\ln\,\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right), & n \text{ is odd and } n > 1, \end{cases} \tag{A5c}$$

$$G_0(\alpha) = \alpha \tag{A5d}$$

$$G_1(\alpha) = \sin\alpha \tag{A5e}$$

$$G_n(\alpha) = \frac{1}{2^{n-1}}\sum_{i=0}^{\left[\frac{n+1}{2}\right]-1}C_n^i\frac{\sin\left[(n-2i)\alpha\right]}{n-2i} + \begin{cases} \frac{1}{2^n}C_n^{\frac{n}{2}}\alpha, & n \text{ is even and } n > 0, \\ 0, & n \text{ is odd and } n > 1. \end{cases} \tag{A5f}$$

So the result of the integral Eq. (29) is

$$I_{n,k} = \int_0^{\alpha_j}\frac{\cos^{n-k-1}\alpha}{(\cos\alpha+1)^n}d\alpha \quad (\alpha_j = \sec^{-1}u_j, j = s, d)$$

$$= \begin{cases} \sum_{i=0}^{n}\frac{C_{n-k-1}^i(-1)^{n-k-i-1}}{2^{n-i-1}}F_{2n-2i}(\frac{\alpha_j}{2}) + \sum_{i=1}^{-k-1}\sum_{j=0}^{i}C_{n-k-1}^{n+i}C_i^j(-1)^{k+i+1}G_j(\alpha_j), & k+1 < 0, \\ \sum_{i=0}^{n-k-1}\frac{C_{n-k-1}^i(-1)^{n-k-i-1}}{2^{n-i-1}}F_{2n-2i}(\frac{\alpha_j}{2}), & 0 \le k+1 < n, \\ (-1)^{k-n+1}\sum_{i=1}^{n}\frac{C_{k-i}^{n-i}}{2^{i-1}}F_{2i}(\frac{\alpha_j}{2}) + \sum_{i=1}^{k-n+1}(-1)^{k-n-i+1}C_{k-i}^{k-n-i+1}F_i(\alpha_j), & n < k+1. \end{cases} \tag{A6}$$

When n, k are small, the first few $I_{n,k}$ that will be used in the main text can be given explicitly:

$$I_{0,1}(u_j) = \sqrt{u_j^2 - 1}, \tag{A7a}$$

$$I_{0,2}(u_j) = \frac{u_j\sqrt{u_j^2 - 1}}{2} + \frac{1}{2}\ln\left(\sqrt{u_j^2 - 1} + u_j\right), \tag{A7b}$$

$$I_{0,3}(u_j) = \frac{1}{3}\sqrt{u_j^2 - 1}(u_j^2 + 2), \tag{A7c}$$

$$I_{1,0}(u_j) = \frac{u_j - 1}{\sqrt{u_j^2 - 1}}, \tag{A7d}$$

$$I_{1,1}(u_j) = \ln\left(\sqrt{u_j^2 - 1} + u_j\right) - \frac{\sqrt{u_j^2 - 1}}{u_j + 1}. \tag{A7e}$$

The large u_j expansion of $I_{n,k}(u_j)$ can also be worked out to the first nontrivial order as

$$I_{n,k} \approx \begin{cases} \frac{u_{s,d}^{k-n}}{k-n} = \frac{1}{k-n}\left(\frac{r_{s,d}}{r_0}\right)^{k-n}, & k > n \\ \ln(2u_j) = \ln\left[\frac{2r_{s,d}}{r_0}\right], & k = n, \\ \frac{(n-k-1)!}{2^k(2n-2k-1)!!} \\ \times {}_2F_1(\frac{1}{2},-k;\frac{1}{2}+n-k;-1) \equiv L_{n,k}, & k < n. \end{cases} \tag{A8}$$

where ${}_2F_1$ is the Gaussian hypergeometric function and for notation simplicity, we have denoted the result for the case $k < n$ as $L_{n,k}$. For the first few n and k that will be used, their expansions can be worked to even higher orders,

$$I_{0,-1} = \frac{\pi}{2} - \frac{1}{u_j} + \mathcal{O}\left(\frac{1}{u_j^2}\right), \tag{A9a}$$

$$I_{0,0} = \ln(2u_j) + \mathcal{O}\left(\frac{1}{u_j^2}\right), \tag{A9b}$$

$$I_{0,1} = u_j - \frac{1}{2u_j} + \mathcal{O}\left(\frac{1}{u_j^2}\right), \tag{A9c}$$

$$I_{0,2} = \frac{u_j^2}{2} + \frac{\ln 2u_j}{2} - \frac{1}{4} + \mathcal{O}\left(\frac{1}{u_j^2}\right), \tag{A9d}$$

$$I_{0,3} = \frac{1}{3}\sqrt{u_j^2 - 1}(u_j^2 + 2), \tag{A9e}$$

$$I_{1,-2} = 2 - \frac{\pi}{2} + \mathcal{O}\left(\frac{1}{u_j^2}\right), \tag{A9f}$$

$$I_{1,-1} = \frac{\pi}{2} - 1 + \mathcal{O}\left(\frac{1}{u_j^2}\right), \tag{A9g}$$

$$I_{1,0} = 1 - \frac{1}{u_j} + \mathcal{O}\left(\frac{1}{u_j^2}\right), \tag{A9h}$$

$$I_{1,1} = \ln(2u_j) - 1 + \frac{1}{u_j} + \mathcal{O}\left(\frac{1}{u_j^2}\right), \tag{A9i}$$

$$I_{1,2} = u_j - \ln(2u_j) + 1 - \frac{3}{2u_j} + \mathcal{O}\left(\frac{1}{u_j^2}\right), \tag{A9j}$$

$$I_{2,-3} = \frac{7\pi}{4} - \frac{16}{3} + \mathcal{O}\left(\frac{1}{u_j^2}\right), \tag{A9k}$$

$$I_{2,-2} = \frac{10}{3} - \pi + \mathcal{O}\left(\frac{1}{u_j^2}\right), \tag{A9l}$$

$$I_{2,-1} = \frac{\pi}{2} - \frac{4}{3} + \mathcal{O}\left(\frac{1}{u_j^2}\right), \tag{A9m}$$

$$I_{2,0} = \frac{1}{3} + \mathcal{O}\left(\frac{1}{u_j^2}\right), \tag{A9n}$$

$$I_{2,1} = \frac{2}{3} - \frac{1}{u_j} + \mathcal{O}\left(\frac{1}{u_j^2}\right). \tag{A9o}$$

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