

New topological Gauss-Bonnet black holes in five dimensions

Yuxuan Peng^{✉*}

Department of Physics, School of Science, East China University of Technology, Nanchang, Jiangxi 330013, China and CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China

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We investigate vacuum static black hole solutions of Einstein-Gauss-Bonnet gravity with a negative cosmological constant in five dimensions. These are solutions with horizons of nontrivial topologies. The first one possesses a horizon with the topology $S^1 \times H^2$, and a varying Gauss-Bonnet coupling constant α . By looking into its thermodynamic properties, we find that its specific heat capacity with fixed volume is negative; therefore, it is thermodynamically unstable. The second one is equipped with a so-called “Sol manifold” as its horizon, and interestingly, the product of the Gauss-Bonnet coupling constant α and the cosmological constant Λ is fixed. For the second solution, the total energy is a constant and the entropy vanishes. These results expand our knowledge of both topological black holes in higher dimensions and the property of higher curvature corrections of gravitational theories.

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I. INTRODUCTION

Black holes are both simple and complicated objects in nature. The smooth event horizon of a stationary black hole can only have spherical topology; therefore, they are simple. However, this simplicity is ensured only when the spacetime dimension is four and the dominant energy condition (DEC) is satisfied—this is the topology theorem by Hawking in 1972 [1]. In the year 1994 Chruściel and Wald [2] proposed a topology theorem without the condition of smoothness. Black hole horizons can be complicated while those two conditions are violated, i.e., one can consider higher dimensions or break the DEC.

In five dimensions, there are not only the famous “Myers-Perry” black holes [3] with spherical topology S^3 but also “black rings” found by Emparan and Reall [4] with topology $S^1 \times S^2$. Actually, these two topologies are the only possibilities for five-dimensional asymptotically flat stationary black hole horizons according to the generalization [5] to higher dimensions of Hawking’s topology theorem. In dimensions $D > 5$ there are much more complicated black hole horizons, including S^{D-2} and $S^1 \times S^{D-3}$ topologies [6,7].

The DEC can be broken by introducing the negative cosmological constant Λ . The asymptotically anti-de Sitter (AdS, the maximally symmetric space with negative curvature) black holes are solutions with negative Λ and they can have horizons of three types: spherical with positive curvature, torus with flat geometry, and hyperbolic space with negative curvature [8–10]. These three

types appear in arbitrary dimensions. If one considers generic black holes that are not asymptotically AdS, with negative Λ , there are even black hole horizons with arbitrary genus $g > 1$ [11]. These are spacetimes locally equal to pure AdS spacetimes. Moreover, some static, plane symmetric solutions and cylindrically symmetric solutions of Einstein-Maxwell equations with a negative cosmological constant are investigated by Cai and Zhang [12].

It is interesting to explore the possibility of certain nontrivial horizon types in the presence of a negative Λ . If certain types were proven to be forbidden, this might help establishing new topology theorems; if certain types turned out to exist, these would be novel discoveries worth noticing. There is an important example, Ref. [13], which obtained nontrivial black holes in five dimensions. These black holes have three-dimensional horizons of different types. These types all belong to the eight three-dimensional “model geometries” classified by Thurston [14]. They are Euclidean E^3 , spherical S^3 , hyperbolic H^3 , $S^1 \times S^2$, $S^1 \times H^2$, and the so-called “Sol,” “Nil,” and “ SL_2R ” geometries. They all admit homogeneous metrics. For more details of these eight manifolds one can refer to [13–15]. These manifolds should be compact if one uses them to construct black hole horizons, and the topologies and the compactification of these eight manifolds are also described in [13]. The main results are Einsteinian solutions of black holes with Sol and Nil horizons.¹

¹According to [13], there had already been black hole horizons for the first five geometries except $S^1 \times S^2$, while “ SL_2R ” horizons are still unknown.

*yxpeng@alumni.itp.ac.cn

What we do is explore the effect of higher curvature corrections on horizon topology. To be exact, we consider five-dimensional Einstein-Gauss-Bonnet (EGB) gravity theory, a special case of the Lovelock gravity theory [16], the general second-order covariant gravity theory in dimensions higher than four, with a negative Λ . Moreover, the Gauss-Bonnet term can be regarded as corrections from the heterotic string theory [17,18]. There had already been spherical, Euclidean, and hyperbolic black hole horizons in this theory, shown by Cai [19]. What we found is a black hole with a $S^1 \times H^2$ horizon and one with a Sol-manifold horizon. These are static vacuum black holes. We then discuss the thermodynamic properties of these solutions, and interestingly, the Sol-manifold solution has zero entropy and constant mass. This may be caused by the fact that in the Sol-manifold solution the Gauss-Bonnet coupling constant has a fixed value. We include some discussion at the end of this paper. We hope our results may help further the understanding of black holes in higher curvature gravity theories and help explore possible topology theorems in generic cases.

II. THE TOPOLOGICAL BLACK HOLE SOLUTIONS

The action of EGB gravity theory with a negative cosmological constant reads

$$S = \frac{1}{16\pi G} \int d^{d+1}x \sqrt{-g} [R - 2\Lambda + \alpha(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})], \quad (2.1)$$

where G is the Newton constant and the cosmological constant Λ , the Gauss-Bonnet coupling denoted by α . The equations of motion are

$$\begin{aligned} & R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} \\ & - \alpha \left(4R_{\mu\rho}R^{\rho}_{\nu} - 2RR_{\mu\nu} + 4R^{\rho\sigma}R_{\mu\rho\nu\sigma} - 2R_{\mu}{}^{\rho\sigma\tau}R_{\nu\rho\sigma\tau} \right) \\ & + \frac{1}{2}g_{\mu\nu} \left(R^2 - 4R_{\rho\sigma}R^{\rho\sigma} + R_{\rho\sigma\tau\pi}R^{\rho\sigma\tau\pi} \right) \\ & = 0. \end{aligned} \quad (2.2)$$

A. The first solution with a constant curvature subspace

As we will see, the horizon topology of the first black hole solution is $S^1 \times H^2$. The metric of the first solution is assumed to be a warped product of a three-dimensional static black hole BH_3 and a two-dimensional hyperbolic space Σ_2 with constant negative curvature,

$$ds^2 = -V(r)dt^2 + \frac{1}{V(r)}dr^2 + r^2d\xi^2 + ad\Sigma_2^2, \quad (2.3)$$

with $V(r)$ an unknown function of coordinate r , a a positive constant, and $d\Sigma_2^2$ the line element of the 2D hyperbolic space,

$$d\Sigma_2^2 = d\theta^2 + \sinh^2\theta d\phi^2. \quad (2.4)$$

The new black hole solution in five dimensions has

$$V(r) = -\frac{\Lambda r^2}{3} - M, \quad a = \frac{3}{-2\Lambda} - 2\alpha, \quad (2.5)$$

where α is the Gauss-Bonnet parameter and Λ is the negative cosmological constant. The constant M is related to the total energy of the black hole and is related to the horizon radius r_h by the relation

$$M = -\frac{\Lambda r_h^2}{3}. \quad (2.6)$$

The solution above with $\alpha \rightarrow 0$ gives the Eq. (II.18) in Ref. [13].

B. The second solution with nontrivial horizon topology

The horizon of the second black hole solution is the so-called Sol manifold [13–15]. The Sol manifold is described by

$$ds^2 = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2. \quad (2.7)$$

The ansatz metric of the whole spacetime is

$$\begin{aligned} ds^2 = & -V(r)dt^2 + \frac{1}{V(r)}dr^2 + f(r)e^{2z}dx^2 \\ & + g(r)e^{-2z}dy^2 + h(r)dz^2, \end{aligned} \quad (2.8)$$

which gives the metric on a constant $\{t, r\}$ surface up to some constant. Here $V(r)$, $f(r)$, $g(r)$, and $h(r)$ are functions of r to be determined.

After putting the ansatz into the equations of motion, we obtain a solution as follows:

$$V(r) = -\frac{\Lambda r^2}{3} - M, \quad g(r) = r^2, \quad h(r) = \frac{r^2}{M}. \quad (2.9)$$

The metric is then

$$\begin{aligned} ds^2 = & -\left(-\frac{\Lambda r^2}{3} - M\right)dt^2 + \frac{1}{-\frac{\Lambda r^2}{3} - M}dr^2 \\ & + f(r)e^{2z}dx^2 + r^2e^{-2z}dy^2 + \frac{r^2}{M}dz^2. \end{aligned} \quad (2.10)$$

For the solution to possess a regular limit when $M \rightarrow 0$, one can rescale the coordinate $z \rightarrow \sqrt{M}z$ and get the metric

$$ds^2 = -\left(-\frac{\Lambda r^2}{3} - M\right) dt^2 + \frac{1}{-\frac{\Lambda r^2}{3} - M} dr^2 + f(r)e^{2\sqrt{M}z} dx^2 + r^2 e^{-2\sqrt{M}z} dy^2 + r^2 dz^2. \quad (2.11)$$

Here the coupling constant α is fixed to satisfy

$$\alpha = -\frac{3}{4\Lambda}, \quad (2.12)$$

M is an integration constant, and $f(r)$ an unfixed function² even when the equations of motion are satisfied. The horizon radius r_h also satisfies Eq. (2.6).

This solution is quite different from the Sol-manifold solution of Einstein gravity in [13]. Interestingly, when $f(r)$ and $g(r)$ are both set to be r^2 , the horizon manifold can be arbitrary as shown in Ref. [20].

III. THERMODYNAMICS OF THE NONTRIVIAL SOLUTIONS

In this section we study the thermodynamics of the black hole solutions given above.

A. The first solution

For the first solution (2.5), we can identify the cosmological constant with the thermodynamic pressure $P = -\Lambda/(8\pi G)$. This identification had been applied to explore the extended phase space of asymptotically AdS black holes [21].

The temperature can be obtained by the semiclassical method of removing the conical singularity of the near-horizon geometry

$$T = \frac{V'(r)}{4\pi} \Big|_{r \rightarrow r_h} = -\frac{\Lambda r_h}{6\pi} = \frac{4GPr_h}{3}, \quad (3.1)$$

and the entropy can be obtained by applying the Wald entropy formula [22,23]

$$\begin{aligned} S &= -2\pi \int_{\text{horizon}} \sqrt{\hat{g}} d^{d-1}x \frac{\partial L}{\partial R_{abcd}} \epsilon_{ab} \epsilon_{cd} \\ &= -\frac{3r_h \Omega (1 + 4\alpha\Lambda)}{8G\Lambda} \\ &= \frac{3r_h \Omega (1 - 32\pi\alpha GP)}{64\pi G^2 P}, \end{aligned} \quad (3.2)$$

where all the hatted quantities are intrinsic quantities on the horizon cross section on which the integral is defined and ϵ_{ab} is the natural volume element on the tangent space orthogonal to the cross section. The constant Ω is the volume of the hyperbolic subspace.

²One can also set $f(r) = r^2$ and leave $g(r)$ undetermined. These are the same solutions with different signs of z .

If we consider the first law of thermodynamics $dE = TdS$ without including Λ or P as a variable, we will obtain the expression for the total energy

$$E = \frac{r_h^2 \Omega (1 + 4\alpha\Lambda)}{32\pi G}. \quad (3.3)$$

However, when Λ or P is a thermodynamic variable, this quantity should be interpreted as the total enthalpy

$$\begin{aligned} H &= \frac{r_h^2 \Omega (1 + 4\alpha\Lambda)}{32\pi G} \\ &= \frac{3r_h \Omega (1 - 32\pi\alpha GP)}{64\pi G^2 P}. \end{aligned} \quad (3.4)$$

After rewriting $H(r_h, P)$ as the function of S and P , we arrive at

$$H(S, P) = \frac{128\pi G^3 P^2 S^2}{9\Omega - 288\pi\alpha GP\Omega}, \quad (3.5)$$

and the first law of thermodynamics becomes

$$dH = TdS + VdP, \quad (3.6)$$

with V being the thermodynamic volume

$$\begin{aligned} V &= \frac{256\pi G^3 P S^2 (1 - 16\pi\alpha GP)}{9\Omega (1 - 32\pi\alpha GP)^2} \\ &= \frac{1}{16} r_h^2 \Omega \left(\frac{1}{\pi GP} - 16\alpha \right) \\ &= \frac{9T^2 \Omega (1 - 16\pi\alpha GP)}{256\pi G^3 P^3}. \end{aligned} \quad (3.7)$$

The last equation above gives the equation of state of the black hole. One can see that the derivative of V with respect to P is always negative,

$$\left(\frac{\partial V}{\partial P} \right)_T = \frac{9T^2 \Omega (32\pi\alpha GP - 3)}{256\pi G^3 P^4} < 0, \quad (3.8)$$

for all T , since $1 - 32\pi\alpha GP > 0$ must be satisfied in Eq. (3.2).

The specific heat capacity C_p with fixed pressure is

$$\begin{aligned} C_p &= T \left(\frac{\partial S}{\partial T} \right)_P \\ &= \frac{9T \Omega (1 - 32\pi\alpha GP)}{256\pi G^3 P^2} > 0, \end{aligned} \quad (3.9)$$

and the specific heat capacity C_v with volume fixed is

$$C_v = C_p - T \left(\frac{\partial P}{\partial T} \right)_V \left(\frac{\partial V}{\partial T} \right)_P$$

$$= \frac{9T\Omega}{256\pi G^3 P^2 (32\pi\alpha GP - 3)} < 0, \quad (3.10)$$

since $1 - 32\pi\alpha GP > 0$. So this system is thermodynamically unstable in the sense of specific heat capacity with fixed volume, C_v .

B. The second solution

After applying the Wald entropy (3.2) formula to the second solution (2.11), we found that the entropy vanishes, since the integrand

$$\frac{\partial L}{\partial R_{abcd}} \epsilon_{ab} \epsilon_{cd} \propto \frac{(3M + \Lambda r^2) f'(r)}{2\Lambda r f(r)} \quad (3.11)$$

vanishes on the horizon, i.e., at the point $r = r_h$. According to the first law of thermodynamics $dE = TdS$, the total energy³ (i.e., the mass) of this solution is just a constant,

$$S = 0, \quad E = \text{const.} \quad (3.12)$$

This implies that the integration constant M in this solution does not stand for the total energy or mass.

IV. CONCLUSION AND DISCUSSION

In this paper we present two novel five-dimensional black hole solutions in Einstein-Gauss-Bonnet theory with a negative cosmological constant. They are simple vacuum static black holes with nontrivial horizons. Their horizons correspond to two of the eight types of three-dimensional ‘‘Thurston model geometries’’ in the literature [14,15].

The horizon topology of the first black hole solution is $S^1 \times H^2$, and it allows an extended phase interpretation of thermodynamics by including the cosmological constant as the thermodynamic pressure. We derive the heat capacities of this solution and find that it is thermodynamically unstable. This solution has the form of the so-called ‘‘warped product,’’ the perturbations of which can be studied in the formalism provided by Cai and Cao [24]. In this formalism, the hyperbolicity and causality of this solution can be investigated in the same way as Cao and Wu [25]. In that paper they analyzed various spacetimes, including dynamical spacetimes.

The second black hole has the ‘‘Sol manifold’’ as its horizon. The Gauss-Bonnet coupling constant α of this solution is fixed as $\alpha = -3/(4\Lambda)$. Curiously, there is an unfixed function $f(r)$ in the metric in Eq. (2.8). It seems

³For the second solution we do not address the extended phase space, so we only worry about the total energy instead of the enthalpy.

that there may be redundant degrees of freedom in this solution. Moreover, by applying the Wald entropy formula we find that its entropy vanishes, so its total energy is a constant due to the first law of thermodynamics. This phenomenon is thought provoking. In Ref. [26] some black hole solutions with zero mass and entropy have been presented in the context of Lovelock gravity theory. The authors fix the coupling constant to such a critical value that the kinetic fluctuations around the background spacetime vanish, so there are no excitations of the background. Since the entropy is related to the quantum degrees of freedom of the black hole, if the fluctuations are forbidden, the entropy is expected to vanish. This is why the entropy vanishes in their case. This is closely related to the fact that the effective gravitational constant $G_{\text{eff}} \rightarrow \infty$ [27]. Our case is similar: the coupling constant α is fixed, and the entropy is zero. However, our coefficient choice is different from that paper. This is an intriguing fact, implying that there might be more than one critical value. Our coefficient choice (2.12) is the same as the paper on dimensionally continued gravity [28,29] where they found the critical value of the coupling constant actually forbids kinetic fluctuations around the AdS backgrounds. The connection of the entropy and the kinetic fluctuations is quite interesting and deserves further investigation.

Among the eight model geometries [14,15] in three dimensions, the black holes with S^3 , H^3 , and R^3 horizons already exist in both Einstein and Gauss-Bonnet gravity. However, with the exception of the $S^1 \times H^2$ and the Sol-geometry horizons, we have not found static black holes with $S^1 \times S^2$, Nil-geometry, or SL_2R horizons in Gauss-Bonnet gravity yet, although such black holes (except SL_2R) have been found in Einstein gravity. To try to find or to rule out these solutions are important future directions. To generalize our results to charged cases is also interesting.

Another interesting direction is about the connection between quantum information and gravity. The concept of quantum complexity denotes the computation cost of reaching a certain quantum state, its holographic dual [30–33] as well as its precise definition (still unclear) on the quantum theory side had received much attention in recent years [34–57]. The late-time growth rate of complexity C is proportional to the product of the temperature and entropy $\dot{C} = TS$. It will be interesting to look into the solutions with vanishing entropy and see the behavior of the complexity.

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