Long-wavelength nonlinear perturbations of a complex scalar field

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We study the evolution of nonlinear superhorizon perturbations in a universe dominated by a complex scalar field. The analysis is performed adopting the gradient expansion approach, in the constant mean curvature slicing. We derive general solutions valid to second order in the ratio H^{-1}/L for scalar field inhomogeneities of size L subject to an arbitrary canonical potential. We work out explicit solutions for the quadratic and the quartic potentials and discuss their relevance in setting initial conditions required for the simulations of primordial black hole formation.

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I. INTRODUCTION

For decades, cosmologists have been interested in scalar fields (SFs) and their role in the evolution of our Universe [1,2]. The dynamics of such fields is usually described by the Einstein-Klein-Gordon (EKG) system of equations, which can be regarded as the relativistic generalization of Schrödinger-Poisson (SP) or Gross-Pitaesvkii-Poisson (GPP) systems (the second case arising when a self-interaction between particles is considered). The system was initially studied in the context of boson stars [3–15], inferred from the axion field—a pseudo–Nambu-Goldstone boson of the Peccei-Quinn phase transition—which was originally proposed to solve the strong *CP* problem in QCD. Such SFs have shown a variety of dynamical properties in a cosmological context, proving useful as important components of the Universe, such as dark matter [scalar field dark matter (SFDM); see Refs. [16–24] and also Refs. [25–31] for comprehensive reviews of this model], a variety of models acting as dark energy (DE) [32–34], such as quintessence [35–38], or phantom DE [39–42]. Tachyonic instabilities arise ubiquitously in SF models [43–45] and, most successfully, the majority of inflationary models [46–54] (see also Ref. [55] for a comprehensive review), among other possible realizations as components of the matter sector.

In the context of dark matter, it is usually assumed that the SFDM could be a real or complex SF minimally coupled to gravity. In the simplest scenario, the SFDM is a real scalar field (RSF) subject to a quadratic or mass potential. It has been demonstrated that the model is able to reproduce all the predictions of the standard cosmological model—the so-called Λ-*cold dark matter* (ΛCDM) model—with the advantage of solving some of the problems at small scales that ΛCDM implies, namely, the overproduction of satellite dwarf galaxies within the local group [56–58] and the *cusp-core* problem [59,60]. Additional phenomenology appears when modifications of the simplest model are considered. For example, the complex scalar field (CSF) presents a cosmological evolution at early times that departs from that of a real field [61]. Moreover, its perturbations present a wider instability band than its real field counterpart [62]. Despite all the success of SFDM, formal solutions for large scales (super-Hubble modes) are scarce and largely required for initial conditions in simulations of structure formation. This is a strong motivation for the present work.

A phase analogous to a universe dominated by SFDM arises in the primordial Universe, at much higher energy scales, in the reheating period—the transition period from inflation to the standard hot big bang cosmology [63–67]. A few reheating models propose a stage where one or more scalar fields oscillate around the bottom of their potential. Just as in SFDM, the universe dominated by a fast-oscillation field evolves effectively as a dust-dominated space. In such a scenario, primordial compact structures may form, analogous to those present in the SFDM model, but at much earlier times. With this in mind, Ref. [68] (see also Refs. [69,70]) argues that this "primordial structure formation" process could be completely analogous to the structure formation in SFDM, since from a computational point of view the only difference between both processes is

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given by the region in the parameter space of the model (and one may argue that the matter power spectrum differs in each scenario). In this way, several of the results of SFDM have been adopted for reheating models, showing that the reheating process could have taken place in a universe filled with inhomogeneities as a result of the fragmentation of the inflaton and formation of inflaton clusters.

A strong motivation for the present work is the possibility of structure formation during the reheating period, as well as the analogous process in the SFDM model. The inhomogeneities seeding the structure right after primordial inflation start their evolution on scales above the cosmological horizon (super-Hubble modes). Thus, the complete dynamics cannot be captured through a Newtonian system. In order to provide adequate solutions from the more general EKG system, it proves useful to adopt a gradient expansion of the evolution equations (although higher-order perturbative KG equations are available [71]). The gradient expansion formalism features an advantage over the standard picture (the well-known linear theory of cosmological perturbations): the fact that the former is not restricted to configurations with small amplitudes (linear). Instead, the only requirement of the gradient expansion is for inhomogeneities to be larger than the cosmological horizon. This description is, thus, useful to model any kind of superhorizon perturbations, including those which enter the cosmological horizon with a large amplitude, typical of configurations which form primordial black holes (PBHs).

In this paper, we study the initial evolution stages of CSF inhomogeneities as solutions of the EKG system, which serve as initial conditions for structure formation at, say, the reheating period. While historically the inflaton has been assumed as a RSF, several models consider instead a complex field (see, for example, Refs. [72–82]). Complex fields appear naturally in extensions of the standard model of particle physics, and they could have played an important role in the evolution of the Universe at high energies. Furthermore, due to the well-known differences between real and complex SFDM, it is natural to explore the particularities that a CSF inflaton bring to the primordial structure formation. Our study is relevant to the origin of complex scalar field structures in a cosmological background, when the evolution of inhomogeneities can be accounted for by the gradient expansion formalism [83], in which spatial derivatives are assumed to be small compared to time derivatives. Specifically, an expansion parameter ϵ for coordinate derivatives is introduced in the EKG system of equations, which is then solved order by order in powers of this parameter. Our purpose is, thus, to formulate nonlinear solutions for a CSF which can later be used as initial conditions for general relativistic evolution codes which work with the EKG system, instead of the SP or GPP. The gradient expansion for a RSF has been previously studied in the literature in Ref. [84] (see also Refs. [85,86] for the generalization to a general kinetic term for the RSF and Refs. [87,88] for the multifield case). As for the CSF, long-wavelength, but subhorizon and linear perturbations of a CSF in the context of $P(X)$ theories has been studied in [89]. We intend here to generalize such analysis to the case of a complex field.

The paper is organized as follows. In Sec. II, we present the energy-momentum tensor and the evolution equations of a CSF subject to an arbitrary potential $V(|\varphi|^2)$. In that same section, we rewrite the energy-momentum tensor of the CSF as a perfect fluid, with a word of warning. Thereafter, we adopt for our calculation the perfect fluid representation of the CSF. In Sec. III, we present the constraints and evolution equations we work with. Specifically, we present the EKG system in the so-called $3+1$ formalism to then decompose our system of equations in a cosmological conformal decomposition. In Sec. IV, we introduce the gradient expansion formalism to derive the $O(\epsilon^2)$ equations valid for a CSF. We then impose the *constant mean curvature* (CMC) slicing to simplify our equations as the standard practice dictates [84,90–92]. In that same section, we present the solutions of our system of equations which describe the evolution of superhorizon inhomogeneities of a CSF. In Sec. V, we apply our findings to some simple realizations of the CSF potential, namely, a quadratic and a quartic potential, while Sec. VI is where we draw our conclusions.

II. COMPLEX SCALAR FIELD IN A PERFECT FLUID FORM

Through this work, we consider a minimally coupled CSF φ subject to a generic potential $V(|\varphi|^2)$. Its energy-momentum tensor is given by

$$T_{\mu\nu} = \frac{1}{2}\nabla_\mu\varphi^*\nabla_\nu\varphi + \frac{1}{2}\nabla_\nu\varphi^*\nabla_\mu\varphi - g_{\mu\nu}\left[\frac{1}{2}\nabla^\sigma\varphi^*\nabla_\sigma\varphi + V(|\varphi|^2)\right]. \quad (1)$$

The equation of motion for φ is given by the KG equation:

$$\nabla_\mu\nabla^\mu\varphi = 2V'(|\varphi|^2)\varphi \Rightarrow \quad \partial_\mu(\sqrt{-g}\partial^\mu\varphi) = 2\sqrt{-g}V'(|\varphi|^2)\varphi, \quad (2)$$

where we have defined $V'(|\varphi|^2)$ as

$$V'(|\varphi|^2) \equiv \frac{dV(|\varphi|^2)}{d|\varphi|^2}, \quad (3)$$

with $g \equiv \mathrm{Det}(g_{\mu\nu})$, and $g_{\mu\nu}$ is the 4-metric tensor. Additionally, the CSF satisfies the 4-current conservation equation

$$\nabla_\mu\mathcal{J}^\mu = 0, \quad \text{where } \mathcal{J}^\mu \equiv -i[\varphi^*\nabla^\mu\varphi - \varphi\nabla^\mu\varphi^*]. \quad (4)$$

We can rewrite the energy-momentum tensor (1) in a perfect fluid form:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}, \tag{5}$$

if we follow the correspondences

$$p = -\left(\frac{1}{2}\nabla^\sigma\varphi^*\nabla_\sigma\varphi + V(|\varphi|^2)\right),$$
$$\rho = -\frac{1}{2}\nabla^\sigma\varphi^*\nabla_\sigma\varphi + V(|\varphi|^2), \tag{6a}$$

and with u_μ satisfying the condition

$$u_\mu u_\nu = \frac{\frac{1}{2}\nabla_\mu\varphi^*\nabla_\nu\varphi + \frac{1}{2}\nabla_\mu\varphi\nabla_\nu\varphi^*}{-\nabla^\sigma\varphi^*\nabla_\sigma\varphi}. \tag{6b}$$

From this last correspondence, one can verify the normalization $u^\mu u_\mu = -1$.

The hydrodynamic representation is a common practice in the study of cosmological scalar fields. It was first introduced by Madelung in Ref. [93]. In his work, Madelung managed to rewrite the Schrödinger equation—which results from applying the nonrelativistic limit to the KG equation—in a Euler-like system of equations for an irrotational fluid with an additional quantum potential arising from the finite value of $\hbar$. Since then, this hydrodynamic representation has been used in many contexts to study the CSF such as a cosmological component in the background [94–96], its linear perturbations [58,97,98], or in stationary configurations describing galaxies [30,99–101], among others. In the present work, we find it convenient to use this representation to study nonlinear superhorizon perturbations for the CSF. We have, in fact, verified that the correspondence is valid up to third order in the gradient expansion featured here. The reader must be warned, however, that the scalar field is not exactly a fluid, and the analogy must be verified for each case. In particular, the equation of state is an ill-defined concept for a scalar field, in that one cannot, in general, write the pressure as a linear function of the matter density, much less propose a given relation. Instead, one has to solve the KG equation [Eq. (2)], then construct the associated pressure and density, and only then derive a relation between them (see [96] as an example of this procedure). With this in mind, it is clear that, despite the fact that in the first instance we strongly rely on the hydrodynamical representation of the scalar field, at some point in our analysis, when seeking for solutions that govern the cosmological evolution of nonlinear cosmological perturbations for the CSF, it will be necessary to restore the equations to their form in terms of the field variables, in order to close the system (in the absence of an equation of state). The details of this procedure are presented below.

III. BASIC EQUATIONS

A. 3+1 formalism

Before presenting the system of equations that describe a CSF in a cosmological context, we start by rewriting the EKG equations in a convenient form. That is, we shall rewrite the EKG equations in a $3+1$ formalism, where the space-time line element is written in the following form:

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt). \tag{7}$$

Here α, β^i, and γ_{ij} are the lapse function, shift vector, and spatial metric, respectively. Latin indices range from 1 to 3 denoting space coordinates and are dropped and raised by γ_{ij} and γ^{ij}, unless otherwise specified, whereas Greek indices range from 0 to 3 denoting space-time coordinates. The space-time metric and its inverse are given by

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_j & \gamma_{ij} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^j}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix}. \tag{8}$$

Then $g = -\alpha^2\gamma$, where $\gamma \equiv \mathrm{Det}(\gamma_{ij})$. The covariant and contravariant components of the normal unit vector to the $t = \text{const}$ hypersurface Σ are given by

$$n_\mu = (-\alpha, 0, 0, 0), \qquad n^\mu = \left(\frac{1}{\alpha}, -\frac{\beta^i}{\alpha}\right), \tag{9}$$

respectively. On the other hand, the projection tensor to Σ is defined as $h^\mu_\nu \equiv \gamma^\mu_\nu + n^\mu n_\nu$.

In general, we can decompose the stress-energy tensor for the matter field $T_{\mu\nu}$ in the following form:

$$T_{\mu\nu} = E n_\mu n_\nu + J_\mu n_\nu + J_\nu n_\mu + S_{\alpha\beta} h^\alpha_\mu h^\beta_\nu, \tag{10}$$

where

$$E \equiv T_{\mu\nu} n^\mu n^\nu, \qquad J_\alpha \equiv -T_{\mu\nu} h^\mu_\alpha n^\nu, \qquad S_{\alpha\beta} \equiv T_{\mu\nu} h^\mu_\alpha h^\nu_\beta. \tag{11}$$

With these new definitions, the Einstein equations $G_{\mu\nu} = 8\pi T_{\mu\nu}$ can be written in the following set of equations:

(i) the Hamiltonian constraint $G^{\mu\nu} n_\mu n_\nu = 8\pi T^{\mu\nu} n_\mu n_\nu$ and momentum constraint $G^{\mu\nu} n_\mu h_{\nu i} = 8\pi T^{\mu\nu} n_\mu h_{\nu i}$.—

$$\mathcal{R} + K^2 - K_{ij}K^{ij} = 16\pi E \tag{12}$$

and

$$D_j K^j_i - D_i K = 8\pi J_i, \tag{13}$$

respectively, where D_i and $\mathcal{R}$ denote the covariant derivative and Ricci scalar, respectively, with respect to γ_{ij}, K_{ij} is the extrinsic curvature of Σ, defined as

$$K_{ij} \equiv -h_i^{\mu} h_j^{\nu} n_{\mu;\nu} = -\frac{1}{2\alpha}(\partial_t \gamma_{ij} - D_j\beta_i - D_i\beta_j), \tag{14}$$

and $K \equiv \gamma^{ij} K_{ij}$. In the above expression, and for the rest of this work, a semicolon denotes covariant derivative with respect to $g_{\mu\nu}$. Observe also that the above expression can be rewritten as

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_j\beta_i + D_i\beta_j, \tag{15}$$

which represents an evolution equation for the spatial part of the metric.

(ii) the evolution equations $G^{\mu\nu} h_{\mu i} h_{\nu j} = 8\pi T^{\mu\nu} h_{\mu i} h_{\nu j}$ for the metric variables.—

$$\begin{aligned}\partial_t K_{ij} &= \alpha(\mathcal{R}_{ij} + K K_{ij}) - 2\alpha K_{il} K^l_j \\ &\quad - 8\pi\alpha\left[S_{ij} + \frac{1}{2}\gamma_{ij}(E - S^l_l)\right] - D_j D_i \alpha \\ &\quad + (D_j\beta^m) K_{mi} + (D_i \beta^m) K_{mj} + \beta^m D_m K_{ij}.\end{aligned} \tag{16}$$

As usual, the lapse α and shift β^i remain undetermined due to the covariant character of the theory, and one can choose them as they better suit the specifics of the scenario under study.[1]

Similarly, we can decompose the CSF equation of motion in a $3+1$ form. To do this, we could proceed in two different ways, which would be to work directly with the field representation of the energy-momentum tensor and, therefore, rewrite the KG equation (2) in this $3+1$ representation, while a second possibility would be to rely on the perfect fluid representation of the CSF. In this work, we proceed by adopting the second option. We must emphasize that the hydrodynamic representation of the CSF cannot always be applied to all kind of metrics. However, as mentioned earlier, we verified that this equivalence exist in our gradient expansion approximation up to the order of $O(\epsilon^3)$ (see later).

Given that our energy-momentum tensor is now written in the perfect fluid form, we can define a 3-velocity v^i as $v^i = u^i/u_0$. We can express then u^μ and u_μ as

$$u^0 = [\alpha^2 - (v_k + \beta_k)(v^k + \beta^k)]^{-1/2}, \quad u^i = u^0 v^i, \tag{17a}$$

$$u_0 = -u^0[\alpha^2 - \beta_k(v^k + \beta^k)], \quad u_i = u^0(v_i + \beta_i). \tag{17b}$$

[1]In the cosmological case, it is natural to choose a constant lapse and a zero shift vector (see [102]), whereas in a black hole scenario the $1 + \log$ lapse and the Gamma driver shift are better suited for the problem (see [103] for a discussion on this subject).

From the conservation equation $\nabla_\mu T^\mu_\nu = 0$, we obtain a Euler-like system of differential equations (the same that are valid for a perfect fluid):

$$u^\mu \partial_\mu \rho + \frac{\rho + p}{\alpha\sqrt{\gamma}} \partial_\mu(\alpha\sqrt{\gamma} u^\mu) = 0, \tag{18a}$$

$$\begin{aligned}&\frac{1}{\sqrt{\gamma}}\partial_t[\sqrt{\gamma}(\rho + p)\alpha u^0 u_i] + D_j[(\rho+p)\alpha u^0 v^j u_i] \\ &\quad = -\alpha\partial_i p - (\rho + p)\alpha u^0[\alpha u^0 \partial_i \alpha - u_j D_i \beta^j].\end{aligned} \tag{18b}$$

The first of these two equations represents the conservation law of mass, whereas the second one is the conservation law of momentum. The different quantities defined in Eq. (11) are then expressed as

$$E = (\rho + p)(\alpha u^0)^2 - p, \tag{19a}$$

$$J_i = (\rho + p)\alpha u^0 u_i, \tag{19b}$$

$$S_{ij} = (\rho + p) u_i u_j + p\gamma_{ij}, \tag{19c}$$

where we have used that $u_\mu u_\nu n^\mu n^\nu = u^\mu u^\nu n_\mu n_\nu = (\alpha u^0)^2$ in the above expression and ρ and p are given in terms of the field variables as shown in Eq. (6a).

B. Cosmological conformal decomposition

In this section, we show how to rewrite the above equations in a cosmological context. To this end, we review the cosmological conformal decomposition as described in Refs. [90,91]. The idea of this decomposition is to assume an asymptotically spatially flat Friedmann universe; in such a case, the spatial metric γ_{ij} is decomposed as $\gamma_{ij} = \psi^4 a^2(t)\tilde{\gamma}_{ij}$, where $\tilde{\gamma} \equiv \mathrm{Det}(\tilde{\gamma}_{ij})$ is time independent and equal to $\eta \equiv \mathrm{Det}(\eta_{ij})$, with η_{ij} a time-independent metric of the flat tree space. The function $a(t)$ is a scale factor of a reference universe, which we can assume to be at our location, whereas ψ^4 encrypts deviations from the Friedmann universe. Also, the extrinsic curvature is decomposed as

$$K_{ij} = A_{ij} + \frac{\gamma_{ij}}{3} K, \tag{20}$$

and then A_{ij} is traceless by definition. Additionally, a new tensor $\tilde{A}_{ij}$ is also defined as

$$A^{ij} = \psi^{-4} a^{-2} \tilde{A}^{ij}, \qquad A_{ij} = \psi^4 a^2 \tilde{A}_{ij}. \tag{21}$$

With this new decomposition, we can rewrite $\mathcal{R}_{ij}$ as follows:

$$\mathcal{R}_{ij} = \tilde{R}_{ij} + \mathcal{R}^{\psi}_{ij}, \tag{22}$$

where

$$\mathcal{R}^{\psi}_{ij} \equiv -\frac{2}{\psi}\tilde{D}_i\tilde{D}_j\psi - \frac{2}{\psi}\tilde{\gamma}_{ij}\tilde{\triangle}\psi + \frac{6}{\psi^2}\tilde{D}_i\psi\tilde{D}_j\psi - \frac{2}{\psi^2}\tilde{\gamma}_{ij}\tilde{D}_k\psi\tilde{D}^k\psi,$$

$$\tilde{R}_{ij} \equiv \frac{1}{2}[-\bar{\triangle}\tilde{\gamma}_{ij} + \bar{D}_j\bar{D}^k\tilde{\gamma}_{ki} + \bar{D}_i\bar{D}^k\tilde{\gamma}_{kj} + 2\bar{D}_k(f^{kl}C_{lij}) - 2C^l_{kj}C^k_{il}],$$

$$f^{kl} \equiv \tilde{\gamma}^{kl} - \eta^{kl}, \qquad C^k_{ij} \equiv \frac{1}{2}\tilde{\gamma}^{kl}(\bar{D}_i\tilde{\gamma}_{jl} + \bar{D}_j\tilde{\gamma}_{il} - \bar{D}_l\tilde{\gamma}_{ij}).$$

In the above expressions, $\tilde{D}_i$ ($\bar{D}_i$) is the covariant derivative with respect to $\tilde{\gamma}_{ij}$ (η_{ij}), and $\tilde{\triangle} \equiv \tilde{\gamma}^{ij}\tilde{D}_i\tilde{D}_j$ ($\bar{\triangle} \equiv \eta^{ij}\bar{D}_i\bar{D}_j$). Also, indices for quantities with a tilde (hat) are lowered and raised by $\tilde{\gamma}_{ij}$ and $\tilde{\gamma}^{ij}$ (η_{ij} and η^{ij}), respectively. Then, we have

$$\mathcal{R}^{\psi} \equiv \gamma^{ij}\mathcal{R}^{\psi}_{ij} = -\frac{8}{\psi^5 a^2}\tilde{\triangle}\psi, \tag{23}$$

$$\mathcal{R}^{\psi}_{ij} - \frac{1}{3}\gamma_{ij}\mathcal{R}^{\psi} = -\frac{2}{\psi}\left[\tilde{D}_i\tilde{D}_j\psi - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{\triangle}\psi\right] + \frac{6}{\psi}\left[\tilde{D}_i\psi\tilde{D}_j\psi - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{D}^k\psi\tilde{D}_k\psi\right]. \tag{24}$$

With all these new definitions, Eqs. (12), (13), and (16) are rewritten as follows.

(i) The Hamiltonian and momentum constraints:

$$\mathcal{R}^k_k - \tilde{A}_{ij}\tilde{A}^{ij} + \frac{2}{3}K^2 = 16\pi E, \tag{25}$$

$$D_j\tilde{A}^j_i - \frac{2}{3}D_iK = 8\pi J_i. \tag{26}$$

With the conformal decomposition, the above equations can be transformed into the form

$$\tilde{\triangle}\psi \equiv \frac{\tilde{R}^k_k}{8}\psi - 2\pi\psi^5 a^2 E - \frac{\psi^5 a^2}{8}\left(\tilde{A}_{ij}\tilde{A}^{ij} - \frac{2}{3}K^2\right), \tag{27}$$

$$\tilde{D}^j(\psi^6\tilde{A}_{ij}) - \frac{2}{3}\psi^6\tilde{D}_iK = 8\pi J_i\psi^6. \tag{28}$$

(ii) The evolution equations for the metric variables:

$$(\partial_t - \mathcal{L}_\beta)\tilde{A}_{ij} = \frac{1}{a^2\psi^4}\left[\alpha\left(\mathcal{R}_{ij} - \frac{\gamma_{ij}}{3}\mathcal{R}\right) - \left(D_iD_j\alpha - \frac{\gamma_{ij}}{3}D_kD^k\alpha\right)\right] + \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j) - \frac{2}{3}(\bar{D}_k\beta^k)\tilde{A}_{ij} - \frac{8\pi\alpha}{a^2\psi^4}\left(S_{ij} - \frac{\gamma_{ij}}{3}S^k_k\right), \tag{29}$$

$$(\partial_t - \mathcal{L}_\beta)\psi = -\frac{\partial_t a}{2a}\psi + \frac{\psi}{6}(-\alpha K + \bar{D}_k\beta^k), \tag{30}$$

$$(\partial_t - \mathcal{L}_\beta)K = \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) - D_kD^k\alpha + 4\pi\alpha(E + S^k_k), \tag{31}$$

with $\mathcal{L}_\beta$ representing the Lie derivative along β^i.

The evolution equation for the spatial metric (15) yields

$$(\partial_t - \mathcal{L}_\beta)\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} - \frac{2}{3}\tilde{\gamma}_{ij}\bar{D}_k\beta^k. \tag{32}$$

The Euler-like system of Eqs. (18a) and (18b) are then rewritten as

$$(\partial_t + v^i\partial_i)\rho + \frac{\rho + p}{(a\psi^2)^3\alpha u^0}\{\partial_t[(a\psi^2)^3\alpha u^0] + \partial_i[(a\psi^2)^3\alpha u^0 v^i]\} = 0, \tag{33a}$$

$$\frac{1}{(a\psi^2)^3}\partial_t[(a\psi^2)^3(\rho+p)\alpha u^0 u_i] + D_j[(\rho+p)\alpha u^0 v^j u_i] + \partial_i p + (\rho+p)((\alpha u^0)^2\partial_i\alpha - \alpha u^0 u_j D_i\beta^j) = 0. \quad (33b)$$

The system from (27) to (33) is then the differential equations that govern the nonlinear evolution for a CSF in a cosmological context and for the gradient expansion at the order studied in this work. Observe then that, thanks to the perfect fluid representation of the CSF, all these equations are the same that are valid for a perfect fluid. The particular way to move from the solutions that we will obtain for the perfect fluid variables and the field variables will be presented later in this work.

C. Understanding the cosmological conformal decomposition

Given that this formalism is not the standard for describing cosmological inhomogeneities, in this section we provide a more familiar description of the above system. In particular, we make contact with the standard approach by showing how the well-known equations for a cosmological CSF are reproduced at the background level (we leave the treatment of inhomogeneities for the next section, in the context of a gradient expansion).

We start by considering a flat Friedmann-Lemaître metric

$$ds^2 = -dt^2 + a^2(t)\eta_{ij}dx^i dx^j. \quad (34)$$

This metric fits the $3+1$ representation (7) with $\alpha = 1$, $\beta^i = 0$, and $\gamma_{ij} = a^2(t)\eta_{ij}$. Furthermore, the cosmological conformal decomposition takes in this case $\psi = 1$ and $\tilde{\gamma}_{ij} = \eta_{ij}$. Observe that this also implies that $\mathcal{R}_{ij} = 0$ given that $\mathcal{R}^{\psi}_{ij}$ has only terms of the form $D_i\psi$, and $\tilde{R}_{ij}$ can be taken as dependent solely on factors of the form $\hat{D}_i\eta_{jk}$, which by definition are zero. Additionally, the different quantities in Eq. (19) reduce to $E = \rho$, $J_i = 0$, and $S_{ij} = pa^2(t)\eta_{ij}$, where we have considered that $u_i = 0$ (the comoving velocity u_i vanishes at the background level) and $\alpha u^0 = 1$, which can be easily seen from Eq. (6b) once we impose that $\varphi = \varphi(t)$.

We now focus on the system of Eqs. (27)–(33), which describe the evolution of the CSF in a cosmological context. First, observe that Eq. (32) implies that $A_{ij} = 0$. Similarly, Eq. (30) implies that $K = -3\partial_t a/a \equiv -3H$, where H is the well-known Hubble parameter. On the other hand, Eq. (27), which follows from the Hamiltonian constraint, reduces to the Friedmann equation

$$H^2 = \frac{8\pi}{3}\rho, \quad (35)$$

whereas Eq. (31) reduces to the acceleration equation

$$\frac{\partial_t^2 a}{a} = -\frac{8\pi}{3}[\rho + 3p], \quad (36)$$

where in the above expression $\partial_t^2 \equiv \partial^2/\partial t^2$. Similarly, from the mass conservation equation (33), it follows that

$$\partial_t\rho + 3H(\rho + p) = 0. \quad (37)$$

The reader can verify that all the other equations provide no extra information. Thus, Eqs. (35)–(37) constitute the system to describe a universe dominated by a CSF at the background level. In fact, it is well known that only two of these equations are independent of each other. Of course, this system is well known to describe a universe dominated by a perfect fluid, and the reason we arrived at this system is because we are working with the perfect fluid representation of the CSF. In order to close the system of equations, we might be tempted to define an equation of state for the CSF, as is typically done when working with perfect fluids. However, as we mentioned earlier, this fluid representation of the scalar field is only an auxiliary representation, so the definition of an equation of state should be taken with caution. We remark that this problem does not arise in the field representation, where the current conservation equation complements the system (we shall elaborate more about the field variables in the following section). For the meantime, we have shown that our formalism is capable of reproducing the usual equations that describe a CSF at the background level.

IV. LONG-WAVELENGTH SOLUTIONS FOR THE COMPLEX SCALAR FIELD

A. Gradient expansion: Basic assumptions

The idea behind the gradient expansion formalism [83] is to consider only those configurations with a scale larger than the cosmological horizon size. To this end, we attach to the spatial derivatives a fictitious parameter ϵ, which is typically associated to the scale L of the inhomogeneities as $\epsilon \equiv H^{-1}/L$. This yields a hierarchy of orders of ϵ associated to each variable in the system [as shown in Eq. (38) below]. Then the system of differential equations is solved order by order in powers of ϵ. The formalism, thus, restricts the validity of solutions to scales L much larger than the Hubble horizon, i.e., $L \gg H^{-1}$, so that $\epsilon \ll 1$ is guaranteed at all times. In this case, the lowest-order terms in ϵ are sufficient to describe superhorizon inhomogeneities which, on the other hand, are not restricted in amplitude.

Based on the above description, the gradient expansion requires additional considerations. First, it is assumed that ψ acquires the value of one in some asymptotic region of the universe (close to spatial infinity). This makes $a(t)$ the asymptotic scale factor of the universe and justifies our notation. A second requirement is that when $\epsilon \to 0$ (a value particularly reached at spatial infinity) the universe becomes locally homogeneous and isotropic, i.e., a Friedmann universe, which in our case is assumed to be flat. This imposes the asymptotic values $\alpha = O(\epsilon^0)$, $\beta^i = O(\epsilon)$, and $\partial_t\tilde{\gamma}_{ij} = O(\epsilon)$ in the limit $\epsilon \to 0$ [hereafter, $f = O(\epsilon^n)$ means that f is at least of the order of ϵ^n and admits higher-order contributions]. We guarantee such limits through the more restrictive conditions $\beta^i = O(\epsilon^3)$ and $\partial_t\tilde{\gamma}_{ij} = O(\epsilon^2)$. The condition $\alpha = O(\epsilon^0)$ implies that α must be a scale-independent quantity at the lowest order in the gradient expansion (consistent with the cosmological solution presented earlier). We can absorb at this order the time dependence by rescaling the time coordinate without loss of generality and set $\alpha = 1$. On the other hand, imposing $\beta^i = O(\epsilon^3)$ is a matter of coordinate choice, which can be fulfilled, in general. Finally, demanding $\partial_t\tilde{\gamma}_{ij} = O(\epsilon^2)$ is a choice inherited from inflationary solutions. Indeed, as pointed out in Ref. [84], taking $\partial_t\tilde{\gamma}_{ij} = O(\epsilon)$ introduces a decaying mode at this order, which affects all quantities at $O(\epsilon^2)$. That same reference shows that the condition $\partial_t\tilde{\gamma}_{ij} = O(\epsilon^2)$ is satisfied for fluctuations arising from vacuum perturbations. Therefore, given that one of the main motivations of this work is to study perturbations valid for a reheating period, we expect that such a requirement should be sufficiently general for our case.

Since our energy-momentum tensor of the CSF has been expressed in the form of a perfect fluid, the hierarchy in powers of ϵ of the quantities defined in previous sections is defined from that established for a perfect fluid. That is, by assuming $\beta^i = O(\epsilon^3)$ and $\partial_t\tilde{\gamma}_{ij} = O(\epsilon^2)$, we have [91]

$$\bar{\Psi} = O(\epsilon^0), \qquad \tilde{A}_{ij}, f_{ij}, \kappa, \Phi, \Psi, \partial_t\Psi, \delta, \delta p = O(\epsilon^2), \qquad v_i + \beta_i = O(\epsilon^3), \qquad \alpha u^0 = 1 + O(\epsilon^6), \tag{38}$$

where in the above expression $\partial_t\bar{\Psi} = 0$ and we have defined

$$f_{ij} \equiv \tilde{\gamma}_{ij} - \eta_{ij}, \qquad \kappa \equiv \frac{K\text{-}\bar{K}}{\bar{K}}, \qquad \Phi \equiv \alpha - 1, \qquad \delta p \equiv p - \bar{p}, \qquad \delta \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}}, \qquad \psi = \bar{\Psi}(1 + \Psi). \tag{39}$$

In all these expressions, quantities with a bar are used to refer to $O(\epsilon^0)$, i.e., background, quantities. The above estimations can be translated to the field variables, in which case we obtain

$$\delta\varphi \equiv \varphi - \bar{\varphi} = O(\epsilon^2), \qquad \partial_t\delta\varphi = O(\epsilon^2). \tag{40}$$

B. Leading-order equations

1. Order $O(\epsilon^0)$

The first step to construct the leading-order solutions valid for long-wavelength perturbations for a CSF is to reproduce the $O(\epsilon^0)$ equations that govern its evolution. The reader may find some of the following expressions redundant with those of Sec. III C; however, we present the results here to be consistent with the gradient expansion formalism and in order to show the additional equations required to describe the field variables of the CSF.

First at all, from Eq. (30) we have

$$\bar{K} = -3H + O(\epsilon^2). \tag{41}$$

If now we use the Hamiltonian constraint (27) and Eq. (19), we obtain

$$H^2 = \frac{8\pi}{3}\bar{\rho} + O(\epsilon^2), \tag{42}$$

$\bar{\rho}$ being written in terms of the field representation as

$$\bar{\rho} = \frac{|\partial_t\bar{\varphi}|^2}{2} + V(|\bar{\varphi}|^2) + O(\epsilon^2), \tag{43}$$

and $V(|\bar{\varphi}|^2)$ is the part of the potential $V(|\varphi|^2)$ that is $O(\epsilon^0)$. From Eqs. (31), (19a), and (19c) follows

$$\frac{\partial_t^2 a}{a} = -\frac{8\pi}{3}[\bar{\rho} + 3\bar{p}] + O(\epsilon^2), \tag{44}$$

where in the above expression $\bar{p}$ is written in terms of the field variables as

$$\bar{p} = \frac{|\partial_t\bar{\varphi}|^2}{2} - V(|\bar{\varphi}|^2) + O(\epsilon^2). \tag{45}$$

Finally, the fluid equation (33) is reduced to

$$\partial_t\bar{\rho} + 3H(\bar{\rho} + \bar{p}) = O(\epsilon^2). \tag{46}$$

If we express the above expression in terms of the field variables, it reduces to

$$\partial_t^2\bar{\varphi} + 3H\partial_t\bar{\varphi} + 2V'(|\bar{\varphi}|^2)\bar{\varphi} = O(\epsilon^2), \tag{47}$$

which is the well-known evolution equation for a CSF in a FL background. In the above expression, $V'(|\bar{\varphi}|^2)$ is the part of $V'(|\varphi|^2)$ that is $O(\epsilon^0)$ [see Eq. (3)]. To complement the equations at the background level, we need to specify some information on the phase of the scalar field. This information can be obtained from the 4-current conservation equation (4), which at $O(\epsilon^0)$ reads (see the Appendix):

$$\partial_t[a^3(\bar{\varphi}^*\partial_t\bar{\varphi} - \bar{\varphi}\partial_t\bar{\varphi}^*)] = O(\epsilon^2). \tag{48}$$

Notice that this last equation is not presented for a RSF (or it is trivially satisfied) and represents a key difference between our scenario and that of a RSF.

2. Order $O(\epsilon^2)$

Let us continue by presenting the equations that are valid up to $O(\epsilon^2)$. We adopt the CMC slicing, in which $K = \bar{K}$ and, thus, $\kappa = 0$.

In the CMC slicing, the system (27)–(33) is expressed as follows: The Hamiltonian constraint (27) is reduced to

$$\bar{\Delta}\,\bar{\Psi} = -2\pi\bar{\Psi}^5 a^2\bar{\rho}\delta + O(\epsilon^5). \tag{49}$$

Additionally, the momentum constraint (28) implies that $J_i = O(\epsilon^3)$. While the background Hamiltonian constraint reduces to the Friedmann equation, at the second order it represents a constraint equation for $\bar{\Psi}$, which encodes deviations from a Friedmann universe.

The evolution equations for the metric variables are

$$6\partial_t\Psi - 3H\Phi = O(\epsilon^4), \tag{50}$$

$$\Phi(\bar{\rho} + \bar{p}) + \delta p = -\frac{\bar{\rho}\delta}{3} + O(\epsilon^4), \tag{51}$$

$$\partial_t f_{ij} = -2\tilde{A}_{ij} + O(\epsilon^4), \tag{52}$$

$$\begin{aligned}\partial_t\tilde{A}_{ij} + 3H\tilde{A}_{ij} = \frac{1}{a^2\bar{\Psi}^4}\Bigg[&-\frac{2}{\bar{\Psi}}\left(\bar{D}_i\bar{D}_j\bar{\Psi} - \frac{1}{3}\eta_{ij}\bar{\Delta}\,\bar{\Psi}\right)\\ &+\frac{6}{\bar{\Psi}^2}\left(\bar{D}_i\bar{\Psi}\bar{D}_j\bar{\Psi} - \frac{1}{3}\eta_{ij}\bar{D}^k\bar{\Psi}\bar{D}_k\bar{\Psi}\right)\Bigg]\\ &+ O(\epsilon^4).\end{aligned} \tag{53}$$

Note that Eq. (51) shows no time derivatives. It is the result of taking the evolution of the extrinsic curvature trace, Eq. (31) in the CMC gauge, where derivatives of the extrinsic curvature trace are zero.

Finally, the mass and momentum conservation equations, in terms of the fluid variables, are

$$\bar{\rho}\partial_t\delta + (\bar{\rho} + \bar{p})(6\partial_t\Psi + D_i v^i) + 3H(\delta p - \bar{p}\delta) = O(\epsilon^4) \tag{54a}$$

and

$$\frac{1}{a^3}\partial_t[a^3(\bar{\rho} + \bar{p})u_i] + \partial_i[\delta p + (\bar{\rho} + \bar{p})\Phi] = O(\epsilon^5), \tag{54b}$$

respectively.

Observe that by substituting Eqs. (46) and (50) in the above we obtain

$$\partial_t(a^2\bar{\rho}\delta) = O(\epsilon^4),$$
$$\frac{1}{a^3}\partial_t[a^3(\bar{\rho} + \bar{p})u_i] = \frac{1}{3}\bar{\rho}\partial_i\delta + O(\epsilon^5).$$

From this equation, we derive, in the next section, the expression for the peculiar velocity. Note that, up to this point, the above set of differential equations is valid for any energy-momentum tensor which can be written in a perfect fluid form. However, as we mentioned earlier, the hydrodynamic representation of the scalar field is incomplete, so we must complement these equations with those that apply to the field variables.

From Eq. (6a), we have

$$\begin{aligned}\delta p = &\frac{1}{2}(\partial_t\bar{\varphi}^*\partial_t\delta\varphi + \partial_t\bar{\varphi}\partial_t\delta\varphi^*) - \Phi\partial_t\bar{\varphi}^*\partial_t\bar{\varphi}\\ &- V'(|\bar{\varphi}|^2)(\bar{\varphi}\delta\varphi^* + \bar{\varphi}^*\delta\varphi) + O(\epsilon^4).\end{aligned} \tag{55}$$

Using the constraint (51) and substituting Eq. (47) for the background field, we finally obtain

$$\frac{1}{a^3}\partial_t[a^3(\partial_t\bar{\varphi}^*\delta\varphi + \partial_t\bar{\varphi}\delta\varphi^*)] = -\frac{2}{3}\bar{\rho}\delta + O(\epsilon^4), \tag{56}$$

which is a constraint for the field fluctuation. Equivalently, from Eq. (6a), we also have $p = \rho - 2V(|\varphi|^2)$, and then

$$\delta p = \delta\rho - 2V'(|\bar{\varphi}|^2)(\bar{\varphi}\delta\varphi^* + \bar{\varphi}^*\delta\varphi) + O(\epsilon^4). \tag{57}$$

Using the above expression and (51), we can express Φ in terms of the field variables as

$$2V'(|\bar{\varphi}|^2)[\bar{\varphi}\delta\varphi^* + \bar{\varphi}^*\delta\varphi] = \frac{4}{3}\bar{\rho}\delta + \Phi(\bar{\rho} + \bar{p}) + O(\epsilon^4). \tag{58}$$

Finally, we can use the 4-current conservation equation (4) to obtain a particular constraint for the field variables (see the Appendix):

$$\begin{aligned}&\partial_t\{a^3[\partial_t(\bar{\varphi}^*\delta\varphi - \bar{\varphi}\delta\varphi^*) - 2(\partial_t\bar{\varphi}^*\delta\varphi - \partial_t\bar{\varphi}\delta\varphi^*)]\}\\ &\quad - a^3\partial_t\left(\frac{\Phi}{a^3}\right)[a^3(\bar{\varphi}^*\partial_t\bar{\varphi} - \bar{\varphi}\partial_t\bar{\varphi}^*)] = O(\epsilon^4),\end{aligned} \tag{59}$$

which, again, it is not present in the case of a real SF.

C. Solutions

It is well known that for a homogeneous and isotropic CSF there are two well-defined regimes in which the system can be simplified. As pointed out in Refs. [61,94,96], for the case in which the oscillations of the CSF are slower than the Hubble expansion ($\omega \ll H$), the CSF is equivalent to a stifflike fluid ($\bar{\rho} = \bar{p}$), a behavior expected in the early evolution of the CSF. On the other hand, when the expansion rate of the universe is much larger than the period of oscillations of the CSF ($\omega \gg H$), the CSF is dependent on its particular potential. This last behavior is of particular interest, since it is precisely in this regime that CSFs work for different scenarios (e.g., a reheating phase in the early Universe or a dark matter candidate at later stages). Since the latter regime is of wider interest, we will focus on finding solutions in this fast-oscillating regime and also comment on the solutions found in the *slow-oscillating* regime.

1. The fast-oscillating regime ($\omega \gg H$)

(a) Order $O(\epsilon^0)$.—Equation (48) can be solved immediately. For this, it is convenient to rewrite the background CSF $\bar{\varphi}$ in the polar form:

$$\bar{\varphi} = |\bar{\varphi}| e^{i\theta}, \tag{60}$$

where $|\bar{\varphi}|$ and θ are time-dependent functions. Substituting the above expression in Eq. (48), we obtain

$$\partial_t[2ia^3|\bar{\varphi}|^2\omega] = O(\epsilon^2). \tag{61}$$

Here $\omega \equiv d\theta/dt$ is the angular oscillation frequency of the CSF. Then

$$\omega = \frac{Q}{a^3|\bar{\varphi}|^2} + O(\epsilon^2), \tag{62}$$

where Q is the charge of the CSF and it is related to the conservation of total number of particles [22,61,94,104,105]. Recall that for a real field the previous equation is trivially satisfied, which means that the RSF particle is also its antiparticle.

At the background level, we find it more convenient to solve Eq. (47) instead of the fluid equation (46) due to the lack of an equation of state, as already argued. If we replace Eq. (60) in Eq. (47), we obtain

$$\partial_t^2|\bar{\varphi}| - |\bar{\varphi}|\omega^2 + 3H\partial_t|\bar{\varphi}| + 2V'(|\bar{\varphi}|^2)|\bar{\varphi}| = O(\epsilon^2). \tag{63}$$

Following the same procedure as in Refs. [61,94,96], the fast-oscillating regime applies by demanding the conditions

$$\omega \gg H \quad \text{and} \quad \frac{\partial_t|\bar{\varphi}|}{|\bar{\varphi}|} \ll \omega. \tag{64}$$

In this case, Eq. (63) reduces to

$$|\bar{\varphi}|\omega^2 - 2V'(|\bar{\varphi}|^2)|\bar{\varphi}| = O(\epsilon^2). \tag{65}$$

Here $\omega = \pm\sqrt{2V'(|\bar{\varphi}|^2)} + O(\epsilon^2)$ account for the two solutions of the oscillation frequency of the CSF. Observe that, from this last expression and Eq. (62), we have

$$\pm\sqrt{2V'(|\bar{\varphi}|^2)}|\bar{\varphi}|^2 = \frac{Q}{a^3} + O(\epsilon^2); \tag{66}$$

that is, the solution of $|\bar{\varphi}|$ is given in terms of the scale factor once $V(|\bar{\varphi}|^2)$ [and, thus, $V'(|\bar{\varphi}|^2)$] is specified. With this at hand, we can immediately find the value of $V(|\bar{\varphi}|^2)$ and $V'(|\bar{\varphi}|^2)$ in terms of the scale factor.

From the above equation and (60), we have

$$\bar{\varphi} = \frac{C}{[2V'(|\bar{\varphi}|^2)]^{1/4}a^{3/2}} \exp\left[i\int\sqrt{2V'(|\bar{\varphi}|^2)}dt\right] + O(\epsilon^2), \tag{67}$$

where C is a complex constant that fulfills the condition $Q = |C|^2$, and then $C = \sqrt{Q}e^{i\theta_0}$, with θ_0 a global phase. Also, from Eq. (60), it follows immediately that

$$\partial_t\bar{\varphi} = \left(i\sqrt{2V'(|\bar{\varphi}|^2)} + \frac{\partial_t|\bar{\varphi}|}{|\bar{\varphi}|}\right)\bar{\varphi}. \tag{68}$$

We use the above expressions to calculate the perfect fluid variables. In that case, we obtain from Eqs. (43) and (45)

$$\bar{\rho} = V'(|\bar{\varphi}|^2)|\bar{\varphi}|^2 + V(|\bar{\varphi}|^2) + O(\epsilon^2) \quad \text{and}$$
$$\bar{p} = V'(|\bar{\varphi}|^2)|\bar{\varphi}|^2 - V(|\bar{\varphi}|^2) + O(\epsilon^2). \tag{69}$$

This implies that the Hubble parameter H evolves from Eq. (42) as

$$H^2 = \frac{8\pi}{3}[V'(|\bar{\varphi}|^2)|\bar{\varphi}|^2 + V(|\bar{\varphi}|^2)] + O(\epsilon^2). \tag{70}$$

We see that, in the regime of fast oscillations, the dynamics produced by the CSF (the evolution of its energy density or the expansion of the universe) is intimately related to its potential. As we mentioned earlier, this is a well-known result.

Order $O(\epsilon^2)$.—Given that $\bar{\Psi} = O(\epsilon^0)$ and $\partial_t\bar{\Psi} = 0$, we have that

$$\bar{\Psi} = L^{(0)}(x^k), \tag{71}$$

where $L^{(0)}(x^k)$ is an arbitrary function of the spatial coordinate x^k and we have used the superscript (n) to denote quantities that scale as ϵ^n. Recall that configurations are subject to a size larger than the horizon. Equation (53) can be easily solved to obtain

$$\tilde{A}_{ij} = p_{ij}^{(2)} \frac{1}{a^3} \int_0^a \frac{d\tilde{a}}{H(\tilde{a})} + O(\epsilon^4), \tag{72}$$

where

$$p_{ij}^{(2)}(x^k) \equiv \frac{1}{\bar{\Psi}^4}\left[-\frac{2}{\bar{\Psi}}\left(\bar{D}_i\bar{D}_j\bar{\Psi} - \frac{1}{3}\eta_{ij}\bar{\triangle}\bar{\Psi}\right) + \frac{6}{\bar{\Psi}^2}\left(\bar{D}_i\bar{\Psi}\bar{D}_j\bar{\Psi} - \frac{1}{3}\eta_{ij}\bar{D}^k\bar{\Psi}\bar{D}_k\bar{\Psi}\right)\right],$$

and with H given by Eq. (42). The integration constant in Eq. (72) was omitted, since it yields a decaying mode, irrelevant for our purposes. Equation (52) yields immediately

$$f_{ij} = -2\int_0^a \frac{d\tilde{a}A_{ij}(\tilde{a})}{\tilde{a}H(\tilde{a})} + O(\epsilon^4), \tag{73}$$

where the constant of integration is dropped in order that $f_{ij} \to 0$ when $t \to 0$.

Next, the conservation equation (55) is easily integrated. The density is integrated as

$$\delta = \frac{\bar{\rho}_0 a_0^2}{\bar{\rho}a^2} R^{(2)}(x^k) + O(\epsilon^4), \tag{74}$$

with $R^{(2)}(x^k)$ an arbitrary function of the spatial coordinates x^k, restricted only to represent a fluctuation of size larger than the cosmological horizon, and with $\bar{\rho}_0 a_0^2$ a constant. Using the above expression, the right-hand side equation in (55) yields the peculiar velocity:

$$u_i = \frac{\bar{\rho}_0 a_0^2}{3a^3(\bar{\rho}+\bar{p})}\partial_i R^{(2)}(x^k)\int_0^a \frac{d\tilde{a}}{H(\tilde{a})} + O(\epsilon^3). \tag{75}$$

The functions $L^{(0)}(x^k)$ and $R^{(2)}(x^k)$ are related by Eq. (49) as

$$R^{(2)}(x^k) = -\frac{\bar{\Delta}L^{(0)}(x^k)}{2\pi\bar{\rho}_0 a_0^2 (L^{(0)}(x^k))^5} + O(\epsilon^5). \tag{76}$$

Let us now solve Eq. (56) with the aid of Eq. (74). In this case, we obtain

$$\partial_t\bar{\varphi}^*\delta\varphi + \partial_t\bar{\varphi}\delta\varphi^* = -\frac{2\bar{\rho}_0 a_0^2}{3a^3}R^{(2)}(x^k)\int_0^a \frac{d\tilde{a}}{H(\tilde{a})} + O(\epsilon^4). \tag{77}$$

Observe that in the fast-oscillating regime we can approximate $\partial_t\bar{\varphi} \simeq i\sqrt{2V'(|\bar{\varphi}|^2)}\bar{\varphi}$, from Eq. (68). Then

$$\bar{\varphi}^*\delta\varphi - \bar{\varphi}\delta\varphi^* \simeq -\frac{i}{\sqrt{2V'(|\bar{\varphi}|^2)}}\frac{2\bar{\rho}_0 a_0^2}{3a^3}R^{(2)}(x^k)\int_0^a \frac{d\tilde{a}}{H(\tilde{a})} + O(\epsilon^4). \tag{78}$$

Equivalently, from Eq. (58), we have

$$\partial_t\bar{\varphi}^*\delta\varphi - \partial_t\bar{\varphi}\delta\varphi^* \simeq -\frac{i}{\sqrt{2V'(|\bar{\varphi}|^2)}}\left[\frac{4}{3}\frac{\bar{\rho}_0 a_0^2}{a^2}R^{(2)}(x^k) + \Phi(\bar{\rho}+\bar{p})\right] + O(\epsilon^4). \tag{79}$$

Using these two expressions and Eq. (61) in Eq. (59), we obtain immediately

$$\partial_t\left\{a^3\left[\partial_t\left(\frac{1}{\sqrt{2V'(|\bar{\varphi}|^2)}}\frac{2\bar{\rho}_0 a_0^2}{3a^3}R^{(2)}(x^k)\int_0^a \frac{d\tilde{a}}{H(\tilde{a})}\right) - \frac{2}{\sqrt{2V'(|\bar{\varphi}|^2)}}\left(\frac{4}{3}\frac{\bar{\rho}_0 a_0^2}{a^2}R^{(2)}(x^k) + \Phi(\bar{\rho}+\bar{p})\right)\right]\right\} + 2Qa^3\partial_t\left(\frac{\Phi}{a^3}\right) = O(\epsilon^4). \tag{80}$$

By noticing from Eqs. (69) and (66) that $\bar{\rho}+\bar{p} \simeq 2V'(|\bar{\varphi}|^2)|\bar{\varphi}|^2 = \sqrt{2V'(|\bar{\varphi}|^2)}Q/a^3$, and using $\partial_t = aH\partial_a$, the above expression is rewritten as

$$aH\partial_a\left\{a^3\left[aH\partial_a\left(\frac{1}{\sqrt{2V'(|\bar{\varphi}|^2)}}\frac{2\bar{\rho}_0 a_0^2}{3a^3}R^{(2)}(x^k)\int_0^a \frac{d\tilde{a}}{H(\tilde{a})}\right) - \frac{2}{\sqrt{2V'(|\bar{\varphi}|^2)}}\frac{4}{3}\frac{\bar{\rho}_0 a_0^2}{a^2}R^{(2)}(x^k)\right]\right\} + 2Q\partial_t\Phi - 2Q\partial_t\Phi - 6QH\Phi = O(\epsilon^4). \tag{81}$$

Then

$$\Phi \simeq \frac{a}{6Q}\partial_a\left\{a^3\left[aH\partial_a\left(\frac{1}{\sqrt{2V'(|\bar{\varphi}|^2)}}\frac{2\bar{\rho}_0 a_0^2}{3a^3}R^{(2)}(x^k)\int_0^a \frac{d\tilde{a}}{H(\tilde{a})}\right) - \frac{2}{\sqrt{2V'(|\bar{\varphi}|^2)}}\left(\frac{4}{3}\frac{\bar{\rho}_0 a_0^2}{a^2}R^{(2)}(x^k)\right)\right]\right\} + O(\epsilon^4). \tag{82}$$

We can use this last result to solve Eq. (50):

$$\Psi = \frac{1}{2}\int_0^a \Phi \frac{d\tilde{a}}{\tilde{a}} = \frac{a^3}{12Q}\left[aH\partial_a\left(\frac{1}{\sqrt{2V'(|\bar{\varphi}|^2)}}\frac{2\bar{\rho}_0 a_0^2}{3a^3}R^{(2)}(x^k)\int_0^a \frac{d\tilde{a}}{H(\tilde{a})}\right) - \frac{2}{\sqrt{2V'(|\bar{\varphi}|^2)}}\left(\frac{4}{3}\frac{\bar{\rho}_0 a_0^2}{a^2}R^{(2)}(x^k)\right)\right] + O(\epsilon^4). \tag{83}$$

Finally, by adding Eqs. (77) and (79), we can obtain the solution for $\delta\varphi$:

$$\delta\varphi = -\frac{1}{\partial_t\bar{\varphi}^*}\left[\frac{2\bar{\rho}_0 a_0^2}{3a^3}R^{(2)}(x^k)\left(\int_0^a \frac{d\tilde{a}}{H(\tilde{a})} + \frac{i2a}{\sqrt{2V'(|\bar{\varphi}|^2)}}\right) + \frac{i\Phi(\bar{\rho}+\bar{p})}{\sqrt{2V'(|\bar{\varphi}|^2)}}\right] + O(\epsilon^4), \tag{84}$$

where Φ is given by Eq. (82).

2. The slow-oscillating regime ($\omega \ll H$)

Order $O(\epsilon^0)$.—In this regime, the solution obtained in Eq. (62) holds valid given that Eq. (61) is general and independent of any approximation. On the other hand, following again Refs. [61,94,96], in the slow-oscillating regime we consider $\omega \ll H$ and $\partial_t|\bar{\varphi}|/|\bar{\varphi}| \gg \omega$, in which case Eq. (63) is rewritten as

$$\partial_t^2|\bar{\varphi}| + 3H\partial_t|\bar{\varphi}| = O(\epsilon^2). \tag{85}$$

The above expression can be immediately integrated to obtain

$$\partial_t|\bar{\varphi}| = \partial_t|\bar{\varphi}|_0\left(\frac{a_0}{a}\right)^3 + O(\epsilon^2), \tag{86}$$

where $\partial_t|\bar{\varphi}|_0$ is a constant.

The energy and pressure in this case are given by

$$\bar{\rho} \simeq \bar{p} \simeq \frac{(\partial_t|\bar{\varphi}|)^2}{2} + O(\epsilon^2) = \frac{(\partial_t|\bar{\varphi}|_0)^2}{2}\left(\frac{a_0}{a}\right)^6 + O(\epsilon^2). \tag{87}$$

The above equation yields the equation of state of a stifflike fluid (defined only for background quantities), which is well known to apply for a CSF at the earliest epoch and is independent of the particular potential of the CSF, since at this stage it is the kinetic energy of the scalar field particles [the first term in the quantities defined in Eq. (6)] that dominates the energy density. As pointed out in Ref. [61], this stifflike behavior of the CSF implies that the sound speed associated to it almost reaches the speed of light (the maximum value allowed), which is an analog to the incompressible fluid in Newtonian gas dynamics, where the sound speed is infinite. Substituting the above expression in Eq. (42), we obtain the Hubble parameter H evolution as

$$H^2 = \frac{4\pi(\partial_t|\bar{\varphi}|_0)^2}{3}\left(\frac{a_0}{a}\right)^6 + O(\epsilon^2). \tag{88}$$

Using the above equation in Eq. (86), we obtain an expression for $|\bar{\varphi}|$, namely,

$$|\bar{\varphi}| = |\bar{\varphi}|_0 + \sqrt{\frac{3}{4\pi}}\ln\left(\frac{a}{a_0}\right) + O(\epsilon^2). \tag{89}$$

It is easy to see that in the slow-oscillating regime we will have that $\partial_t\bar{\varphi} = e^{i\theta}\partial_t|\bar{\varphi}|$, so to complete this subsection at the background level we need to calculate the value of the phase of the CSF. We can do this easily with the help of Eq. (62) and the above two expressions as follows:

$$\theta - \theta_0 = \int_0^a \omega \frac{d\tilde{a}}{\tilde{a}H(\tilde{a})} = -\frac{Q}{a_0^3\partial_t|\bar{\varphi}|_0}\frac{1}{|\bar{\varphi}|_0 + \sqrt{3/(4\pi)}\ln(a/a_0)} + O(\epsilon^2), \tag{90}$$

where θ_0 is the global constant phase defined in the paragraph just after Eq. (67). Notice that the above result can be expressed as $\theta - \theta_0 \sim -\omega|\bar{\varphi}|/H$, and, given that $H \sim \partial_t|\bar{\varphi}|$, we have $\theta - \theta_0 \sim -\omega|\bar{\varphi}|/\partial_t|\bar{\varphi}| \ll 1$, since we are in the slow-oscillating regime. This last result then allows us to approximate

$$\bar{\varphi} \simeq |\bar{\varphi}|e^{i\theta_0}, \qquad \partial_t\bar{\varphi} \simeq \partial_t|\bar{\varphi}|e^{i\theta_0}. \tag{91}$$

This is consistent with the physical picture of the slow-oscillating regime, in which the Hubble time is much smaller than the oscillation period, so that the CSF rolls down the potential well, before completing a cycle of spin.

It is worth mentioning that, in the slow-oscillating regime, a RSF would be expected to feature a subsequent attractor solution of an effective cosmological constant [106] (see also Refs. [94,107]). For a CSF, however, such behavior demands specific conditions, so the inflationary models that would arise from a CSF are not as generic as in the case of real fields.

Order $O(\epsilon^2)$.—Before presenting the solutions for this regime at second order, the reader should note that some of the solutions found in the fast-oscillating regime are also valid in the present case. This is because the background solutions are the same up to time derivatives of the field. Such factors are not present in the solutions provided by Eqs. (71)–(77). As for the rest of the solutions, we note that, by using Eqs. (86), (88), and (91), Eq. (77) can be expressed as

$$e^{-i\theta_0}\delta\varphi + e^{i\theta_0}\delta\varphi^* = -\frac{1}{4\sqrt{12\pi}}\left(\frac{a}{a_0}\right)^4 R^{(2)}(x^k) + O(\epsilon^4), \tag{92}$$

where in the above expression we have used that $\bar{\rho}_0 = (\partial_t|\bar{\varphi}|_0)^2/2$. Using the last equation, we can solve Φ by using Eq. (58). In such a case, we obtain

$$\Phi = -\frac{R^{(2)}(x^k)}{(\partial_t|\bar{\varphi}|)^2}\left(\frac{a}{a_0}\right)^6\left[\frac{1}{4\sqrt{12\pi}}\left(\frac{a}{a_0}\right)^4 2V'(|\bar{\varphi}|^2)\left(|\bar{\varphi}|_0 + \sqrt{\frac{3}{4\pi}}\ln\left(\frac{a}{a_0}\right)\right) + \frac{2}{3}\frac{(\partial_t|\bar{\varphi}|_0)^2 a_0^2}{a^2}\right] + O(\epsilon^4). \tag{93}$$

It is interesting to notice that the value of Φ will depend on the particular value of the potential under which the CSF is subject. Then, from Eq. (50), we have

$$\Psi = \int_0^a \Phi\frac{d\tilde{a}}{\tilde{a}} + O(\epsilon^4) = -\frac{R^{(2)}(x^k)}{(\partial_t|\bar{\varphi}|)^2}\int_0^a \frac{\tilde{a}^5}{a_0^6}\left[\frac{1}{4\sqrt{12\pi}}\left(\frac{\tilde{a}}{a_0}\right)^4 2V'(|\bar{\varphi}|^2)\left(|\bar{\varphi}|_0 + \sqrt{\frac{3}{4\pi}}\ln\left(\frac{\tilde{a}}{a_0}\right)\right) + \frac{2}{3}\frac{(\partial_t|\bar{\varphi}|_0)^2 a_0^2}{\tilde{a}^2}\right]d\tilde{a} + O(\epsilon^4). \tag{94}$$

While the slow-oscillating regime is not usually featured in reheating models, it is plausible to consider a stiff-fluid component at early times given its dependence with the scale factor. In such a scenario, the formation of PBHs has been studied in previous papers (e.g., Refs. [108,109]) but with no rigorous criterion for the threshold amplitude for their formation. The present study serves as a first step in the determination of such an amplitude.

To close this section, let us note from Eq. (76) that all inhomogeneities present a spatial dependence related to $R^{(2)}(x^k)$ and its derivatives. This means that a single spatial distribution governs all quantities in the system. This is consistent with the picture in which our solutions take into account only the growing mode, which is the case of structure formation preceded by an inflationary period which erased all decaying modes. In particular, keeping exclusively this mode is crucial in the study of the formation of PBHs, since the incorporation of the decaying mode brings uncertainties to the determination of the threshold amplitude (at horizon crossing), with which an overdensity may collapse gravitationally and form a PBH.

V. CHARACTERIZATION OF TWO EXPLICIT POTENTIALS

We have written the Einstein equations in Sec. IV B in the gradient expansion approximation and presented solutions for a generic canonical potential in Sec. IV C for both the fast- and the slow-oscillating regime. We now proceed to derive explicit solutions for some particular potentials, namely, a quadratic and a quartic potential. Since the fast-oscillating regime is of most interest in cosmology, we will focus on studying examples only in this regime (considering also that solutions in the regime of slow oscillations are immediately recovered by simply substituting the quantities in the general solutions presented above).

Our choice of potentials responds to our focus on cases of interest for reheating, where our solution is suitable to model the evolution of perturbations generated during inflation and that lie outside the cosmological horizon. The idea is that, immediately after the end of inflation, the CSF quickly rolled down to its minimum, where, in the case of large field models, the potential is approximated as $V(|\varphi|^2) \simeq C_n^2|\varphi|^{2n}$, with C_n a suitable constant. With this in mind, we shall analyze the cases $n = 1$ and $n = 2$. Also, for our purposes, we must remember that inflation typically ends when $\epsilon \equiv \frac{M_{\rm pl}}{2}\left(\frac{dV/d\varphi}{V}\right)^2 \simeq 1$. Assuming for simplicity that the transition from the domain of the inflationary behavior to the behavior of oscillations around the minimum is instantaneous, we can use the above potential in the end-of-inflation condition, which means that inflation ends when $\varphi \sim M_{\rm pl}$ and $C_n \sim H$. Subsequently, at some point soon after the end of inflation,

$$C_n \gg H. \tag{95}$$

Then the CSF experiences fast oscillations around the minimum of its potential until interaction with other fields results in its decay to standard model fields.

A. The quadratic potential

We first consider the following simple harmonic potential:

$$V(|\varphi|^2) = \frac{\mu^2}{2}|\varphi|^2, \tag{96}$$

where $C_1 = \mu^2/2$. This can be used to describe a mass term associated to the CSF particles, and this potential is the minimum necessary that is usually used for the CSF to have a dust-like behavior at late times, so it is precisely this potential that is used to consider the CSF as a candidate for dark matter or to describe a type of reheating process. Observe that in this case

$$V(|\bar{\varphi}|^2) = \frac{\mu^2}{2}|\bar{\varphi}|^2 \quad \text{and} \quad V'(|\varphi|^2) = \mu^2 = V'(|\bar{\varphi}|^2). \quad (97)$$

Then, in this example, the fast-oscillating regime is fulfilled when the condition $\mu \gg H$ applies, which is equivalent to the condition in Eq. (95).

1. Order $O(\epsilon^0)$

We can start by finding the solutions that are valid to $O(\epsilon^0)$. Observe that from Eq. (65) we have $\omega = \pm\mu$. From Eq. (66), this yields

$$|\bar{\varphi}|^2 = \pm\frac{Q}{\mu a^3} + O(\epsilon^2). \quad (98)$$

Equation (67) follows immediately:

$$\bar{\varphi} = \frac{C}{\mu^{1/2}a^{3/2}}e^{i\mu t} + O(\epsilon^2). \quad (99)$$

The frequency μ brings naturally an associated characteristic length scale, which appears explicitly in the stability analysis of linear perturbations, namely, the instability scale $l_{\rm inst} \approx a/\mu$, which divides fluctuation sizes into two regimes. For inhomogeneities of size $L \gg l_{\rm inst}$, the background behaves like pressureless dust and perturbations can grow without limit [62,110], while, in the opposite regime, the scalar field fluctuations dilute and are, therefore, irrelevant for structure formation. Note that, since $l_{\rm inst} \ll 1/H$, the configurations in the long-wavelength approximation lie well within the regime relevant for structure formation.[2]

From the above equation, we have

$$\partial_t\bar{\varphi} = \mu\left(i - \frac{3}{2}\frac{H}{\mu}\right)\bar{\varphi} \simeq i\mu\bar{\varphi}. \quad (100)$$

We substitute these expressions in Eqs. (43) and (45) or in Eq. (69), to obtain

$$\bar{\rho} \simeq \frac{\mu Q}{a^3} + O(\epsilon^2) \quad \text{and} \quad \bar{p} \simeq O(\epsilon^2). \quad (101)$$

This implies that the Hubble parameter H evolves from Eq. (42) or (70) as

$$H^2 = \frac{8\pi}{3}\frac{\mu Q}{a^3} + O(\epsilon^2), \quad (102)$$

i.e., as a dustlike component, as expected.

The constants Q and C are determined in each specific example. In our case, a reheating scenario, we assume that reheating started immediately after the end of inflation. Then Eq. (102) sets

$$Q = \frac{3H_0^2}{8\pi\mu}a_0^3, \quad (103)$$

where the subscript "0" refers to quantities evaluated at the end of inflation. Correspondingly, C is given by

$$C = \sqrt{\frac{3H_0^2 a_0^3}{8\pi\mu}}e^{i\theta_0}. \quad (104)$$

2. Order $O(\epsilon^2)$

Now, we proceed by finding the solutions that are valid up to the order of $O(\epsilon^2)$. First of all, by substituting our background solutions in Eq. (72), we obtain

$$\tilde{A}_{ij} = p_{ij}^{(2)}\sqrt{\frac{3}{2\pi\mu Q}}\frac{1}{5a^{1/2}} + O(\epsilon^4). \quad (105)$$

Replacing the above expression into Eq. (73) results in

$$f_{ij} = -p_{ij}^{(2)}\frac{3}{10\pi\mu Q}a + O(\epsilon^4). \quad (106)$$

We can also substitute our background quantities in Eqs. (74) and (75), in which case we have

$$\delta = \frac{a}{a_0}R^{(2)}(x^k) + O(\epsilon^4),$$
$$u_i = \sqrt{\frac{1}{6\pi\mu Q}}\frac{a^{5/2}}{5a_0}\partial_i R^{(2)}(x^k) + O(\epsilon^5), \quad (107)$$

where in the above expression we substituted $\bar{\rho}_0 a_0^2 = \mu Q/a_0$.

In Fig. 1, we show the evolution of the above two solutions in terms of the scale factor, normalized at the end of inflation. Note that, as expected, the density contrast and velocity of the CSF subject to a potential of the form (96) evolve as those of a dustlike component [91].

The next step is to find Φ from Eq. (82). In that case, we found

$$\Phi = -\frac{7}{15}\frac{a}{a_0}R^{(2)}(x^k) + O(\epsilon^4). \quad (108)$$

If we use this result in Eq. (83), it follows that

[2]The linear instability scale is found in the analysis of the Mukhanov-Sasaki equation and is often dubbed the Jeans instability of the scalar field [62,110]. The growth of inhomogeneities above the instability scale is observed in the nonlinear regime both through numerical simulations (with the formation of soliton structures, e.g., [69]) and in the long-wavelength approximation of nonlinear fluctuations in a real scalar field [87].

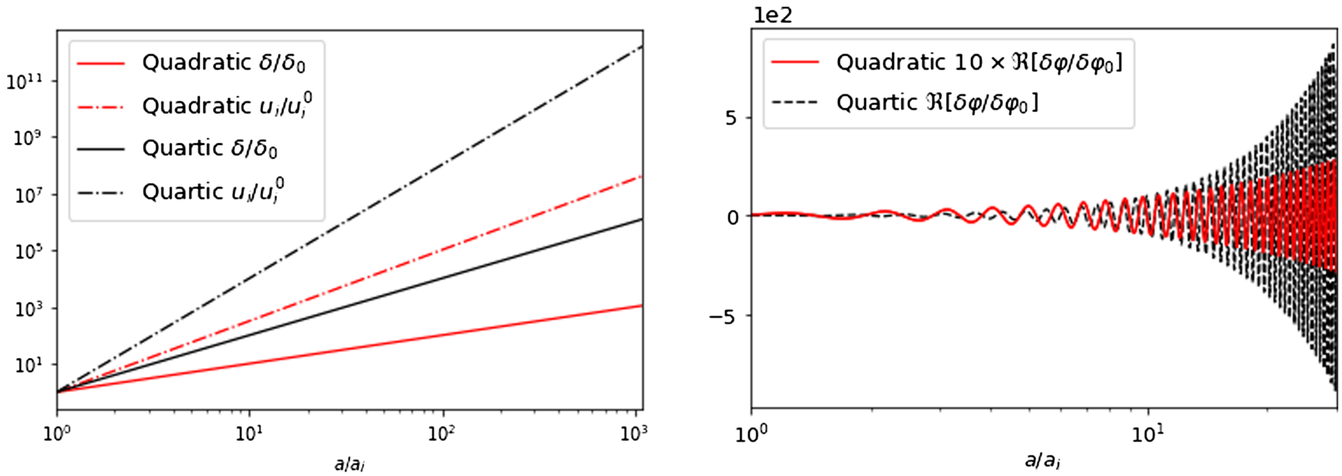

FIG. 1. Left: normalized contrast density and velocity fields for the quadratic and quartic scenarios in terms of the scale factor. Right: evolution of the real parts of $\delta\varphi/\delta\varphi_0$ for the quadratic and quartic scenarios in terms of the scale factor. In the plots δ_0, u_i^0, and $\delta\varphi_0$ are the contrast density, velocity, and CSF inhomogeneity, respectively, measured at the end of inflation.

$$\Psi = -\frac{7}{30}\frac{a}{a_0}R^{(2)}(x^k) + O(\epsilon^4). \tag{109}$$

Finally, we compute the solution of the CSF perturbation (84). In this case, we obtain

$$\delta\varphi = \left[\frac{13}{15}\frac{\sqrt{Q}}{\sqrt{\mu}a_0\sqrt{a}} - i\frac{4}{15}\sqrt{\frac{3}{8\pi}}\frac{a}{a_0}\right]R^{(2)}(x^k)e^{i\mu t+i\theta_0} + O(\epsilon^4). \tag{110}$$

If we use Eqs. (103) and (104), the above expression is reduced to

$$\delta\varphi = \frac{1}{15}\sqrt{\frac{3}{8\pi}}\left[13\frac{H_0}{\mu}\sqrt{\frac{a_0}{a}} - i4\frac{a}{a_0}\right]R^{(2)}(x^k)e^{i\mu t+\theta_0} + O(\epsilon^4). \tag{111}$$

In Fig. 1, we show the evolution of the real part of the field fluctuation as a function of the scale factor, with the numerical value $\mu/H_0 = 10$ (which fulfills the fast-oscillating condition $\mu \gg H$) and $t_0 = 2/(3H_0)$. In the figure, we have multiplied $\delta\varphi$ for this case by a factor of 10 to ease the comparison with the quartic case presented below. As expected, the CSF experiences fast oscillations in both the background and its fluctuation (the reader may, in fact, verify that the oscillation frequency of $\delta\varphi$ coincides with the one for $\bar{\varphi}$). In addition to the oscillating behavior, the value of the amplitude of $\delta\varphi$ grows with the scale factor. This is, of course, a hallmark of the growing mode of the CSF configuration.

B. The quartic potential

We now consider the quartic potential

$$V(|\bar{\varphi}|^2) = \frac{\lambda}{4}|\varphi|^4, \tag{112}$$

with $\lambda > 0$. Then

$$V(|\bar{\varphi}|^2) = \frac{\lambda}{4}|\bar{\varphi}|^4, \qquad V'(|\varphi|^2) = \frac{\lambda}{2}|\varphi|^2, \qquad \text{and} \quad V'(|\bar{\varphi}|^2) = \frac{\lambda}{2}|\bar{\varphi}|^2. \tag{113}$$

The fast-oscillating behavior, in this case, is guaranteed as long as the condition $\sqrt{\lambda}|\bar{\varphi}| \gg H$ holds.

It is well known that this potential undergoes a radiation-like era in its fast-oscillating regime [61,94]. It has been used as an intermediate epoch for the SFDM; that is, a potential term like this could dominate before the mass term does, or it could take place in the reheating process. In this sense, our results may be valid for potentials which contain both terms, Eqs. (96) and (112), in the limit of large values of φ.

1. Order $O(\epsilon^0)$

Following the same procedure as with the quadratic case, we start by finding the solutions that are valid at zeroth order. Observe that from Eq. (65) we have $\omega = \pm\sqrt{\lambda}|\bar{\varphi}|$, and then, from Eq. (66), we find

$$|\bar{\varphi}| = \pm\frac{(Q/\sqrt{\lambda})^{1/3}}{a} + O(\epsilon^2). \tag{114}$$

We use the above expression and Eq. (113) in Eq. (69) to obtain

$$\bar{\rho} \simeq \frac{3}{4}\frac{(\lambda Q^4)^{1/3}}{a^4} + O(\epsilon^2) \quad \text{and} \quad \bar{p} \simeq \frac{1}{4}\frac{(\lambda Q^4)^{1/3}}{a^4} + O(\epsilon^2). \tag{115}$$

Observe that from the above expression $\bar{p} = \bar{\rho}/3$, which thus mimics a radiationlike fluid. This implies that the Hubble parameter H evolves from Eq. (42) or (70) as

$$H^2 = 2\pi \frac{(\lambda Q^4)^{1/3}}{a^4} + O(\epsilon^2). \tag{116}$$

Using the above expressions, we can finally solve for the homogeneous scalar field. If we replace Eqs. (113) and (114) in Eq. (67), we get

$$\bar{\varphi} = \frac{C}{(\lambda Q)^{1/6} a} \exp\left[i(\lambda Q)^{1/3} \int_0^a \frac{d\tilde{a}}{\tilde{a}^2 H(\tilde{a})}\right] + O(\epsilon^2). \tag{117}$$

Substituting Eq. (116) in the above expression, we obtain

$$\bar{\varphi} = \frac{C}{(\lambda Q)^{1/6} a} \exp\left[i \frac{1}{\sqrt{2\pi}} \left(\frac{\sqrt{\lambda}}{Q}\right)^{1/3} a\right] + O(\epsilon^2). \tag{118}$$

Considering a reheating stage, the values of Q and C set the initial conditions at the end of inflation:

$$Q = \left(\frac{a_0^4 H_0^2}{2\pi \lambda^{1/3}}\right)^{3/4}, \qquad C = \left(\frac{a_0^4 H_0^2}{2\pi \lambda^{1/3}}\right)^{3/8} e^{i\theta_0}. \tag{119}$$

2. Order $O(\epsilon^2)$

Let us now find the solutions that are valid up to the order of $O(\epsilon^2)$. By substituting Eq. (116) in Eq. (72), we obtain

$$\tilde{A}_{ij} = p_{ij}^{(2)} \frac{1}{\sqrt{2\pi}(\lambda Q)^{1/6}} + O(\epsilon^4). \tag{120}$$

Equation (73) yields immediately

$$f_{ij} = -p_{ij}^{(2)} \frac{a^2}{\pi (\lambda Q^4)^{1/3}} + O(\epsilon^4). \tag{121}$$

Substituting our background quantities (115) in Eqs. (74) and (75) gives

$$\delta = \frac{a^2}{a_0^2} R^{(2)}(x^k) + O(\epsilon^4),$$
$$u_i = \frac{1}{4\sqrt{2\pi}(\lambda Q^4)^{1/6}} \frac{a^4}{a_0^2} \partial_i R^{(2)}(x^k) + O(\epsilon^5), \tag{122}$$

where we used $\bar{\rho}_0 a_0^2 = 3(\lambda Q^4)^{1/3}/(4a_0^2)$.

In Fig. 1, we show the evolution of the two expressions above as a function of the scale factor, normalized at the end of inflation. As mentioned earlier, the density contrast and velocity of the CSF subject to a quartic potential evolve like a pure radiation fluid [91], which results in a faster growth with respect to the quadratic case. This implies that the size of the perturbations (and, in general, their behavior) when they enter the Hubble horizon will be strongly affected by its potential. In the specific case of our two examples, for a common initial power spectrum, we expect a CSF subject to a quartic potential to have a larger value of the amplitude of the overdensity when reentering the Hubble horizon than for the massive field case.

Let us finally derive Φ from Eq. (82). In this particular case, we find

$$\Phi \simeq -\frac{1}{2}\frac{a^2}{a_0^2} R^{(2)}(x^k) + O(\epsilon^4). \tag{123}$$

If we use this result in Eq. (83), then

$$\Psi = -\frac{1}{8}\frac{a^2}{a_0^2} R^{(2)}(x^k) + O(\epsilon^4). \tag{124}$$

Thus, the solution of the CSF perturbation up to the order of $O(\epsilon^2)$ is, from Eq. (84),

$$\delta\varphi = \left[\frac{1}{2}\left(\frac{Q^2}{\lambda}\right)^{1/6} \frac{1}{a a_0^2} - \frac{i}{2\sqrt{2\pi}} \frac{a^2}{a_0^2}\right] R^{(2)}(x^k) \exp\left[i \frac{1}{\sqrt{2\pi}} \left(\frac{\sqrt{\lambda}}{Q}\right)^{1/3} a + i\theta_0\right] + O(\epsilon^4). \tag{125}$$

Using the parameters in Eq. (119), the above expression is

$$\delta\varphi = \left[\frac{1}{2}\left(\frac{H_0^2}{2\pi\lambda}\right)^{1/4} \frac{a_0}{a} - i \frac{1}{2\sqrt{2\pi}} \left(\frac{a}{a_0}\right)^2\right] \exp\left[i \frac{1}{\sqrt{2\pi}} \left(\frac{2\pi\lambda}{H_0^2}\right)^{1/4} \frac{a}{a_0} + i\theta_0\right] + O(\epsilon^4). \tag{126}$$

In Fig. 1, we show the evolution of the real part of $\delta\varphi$ as a function of the scale factor. The fast-oscillating condition is fulfilled as long as $(2\pi H^2/\lambda)^{1/4} \ll 1$, which turns out to be the condition imposed in Eq. (95). In our plots, we specifically set $(\lambda/2\pi H_0^2)^{1/4} = 10$. We remark that the growth and oscillating frequency differ from the quadratic case. However, just as in the quadratic scenario, the background $\bar{\varphi}$ and the fluctuation $\delta\varphi$ share the same oscillating frequency.

C. Comparison between models

We reproduced the dustlike behavior of the background quadratic potential [Eq. (101)], as well as the radiationlike behavior of the quartic potential in Eq. (115). These are well-known properties of the cosmological evolution of homogeneous scalar fields (zeroth order in the gradient expansion), as the solutions in Eqs. (107) and (122) show (see also Fig. 1). Additionally, the growth rate of inhomogeneities in the superhorizon scale is faster for the density contrast in the quartic case ($\propto t$) than for the quadratic potential ($\propto t^{2/3}$).

We must emphasize that the solutions of this section show that the matter fields present the same time dependence as their equivalents from linear cosmological perturbation theory. In fact, the continuity and Euler equations [Eq. (54)] are equivalent to Eqs. (8.32) and (8.33) in Ref. [111], expressed for an arbitrary gauge. On the other hand, the dominant contribution of the metric fluctuations at superhorizon scales comes from the zeroth-order, time-independent profiles of ψ, expressed in Eq. (71). This term is responsible for the conservation of the curvature perturbation on superhorizon scales at all orders in a perturbative expansion in a (multiple) fluid-dominated universe (see, e.g., [112–115]) and in a scalar-field-dominated universe (see, e.g., [88,116–118]). The zeroth-order metric fluctuation is the source of the second-order matter fields, since they are related through the relativistic version of the Poisson equation (see, e.g., [119–121]).

VI. DISCUSSION

After showing in Sec. II how the energy-momentum tensor of a CSF can be expressed in terms of perfect fluid variables, in Sec. III we presented the EKG system of equations in a $3+1$ formalism. Also, in Sec. III, we showed how to rewrite the system in the cosmological scenario, by reexpressing it in terms of a (cosmological) conformal decomposition. In Sec. IV, we presented the gradient expansion, employed to obtain the system of equations and the solutions valid to zeroth and second order in the ratio H^{-1}/L. The solutions derived here may be used to describe a universe dominated by a CSF at the background with superhorizon inhomogeneities. Finally, in Sec. V, we applied our results to two simple examples, namely, a quadratic and a quartic potential. They both have been proved useful in the description of dark matter models and reheating scenarios. At the background level, we were able to reproduce the results previously obtained in the literature. We also found the solutions of the inhomogeneous variables of the system for each of these potentials, which have not been previously reported in the literature.

The perfect fluid description of the complex scalar field is a common practice [30,61,94,97,99–101,104,122–126]. However, the use of this description is valid as long as there are no nodes in the distribution of the CSF inhomogeneities. Typically, these nodes are expected during the structure formation process, at highly nonlinear stages. The regime of the solutions we obtained in this article pertains to much larger scales, and, therefore, the solutions remain valid.

As mentioned earlier, the solutions here provided are valid for cosmological fluctuations much larger in size than the cosmological horizon. Moreover, the gradient expansion formalism we used has the advantage of not imposing any restriction on the amplitude of the inhomogeneities, as opposed to the standard, linear theory of cosmological perturbations, which is valid only for small amplitudes. In this way, our description allows the study of inhomogeneities of any amplitude, ideal to assess the formation of PBHs. Our solutions are useful as initial conditions of numerical codes solving the EKG system which encodes all the relativistic effects at play. This is crucial in the accurate study of the origin supermassive and/or primordial black holes.

The above results bring important consequences for the evolution of fluctuations in a universe dominated by a canonical CSF. In the reheating scenario, setting initial conditions at the inflationary stage, fluctuations may reach an amplitude at the horizon-crossing time, dependent on the potential of the dominating field at the fast-oscillation period. Typically, a critical amplitude of fluctuations at horizon crossing is defined as a criterion to reach either the formation of structures (for amplitudes above the threshold) or a dilution of fluctuations. The difference in the evolution of matter fluctuations at superhorizon scales implies that the critical amplitude for collapse must strongly depend on the model.

The existence of a critical amplitude can be inferred from the fact that a scalar field modeling dark matter usually shows characteristic dilution scales (the instability scale for each particular model). In the case of an oscillating scalar field, the numerical values for the threshold amplitude are, to the best of our knowledge, still to be determined (for perturbative analytical approximations to the matter density threshold amplitude, in the RSF scenario, see, e.g., [127–129], and for a numerical study see [130]). We shall explore this aspect for the complex scalar field in a follow-up study.

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APPENDIX: THE 4-CURRENT CONSERVATION EQUATION

The 4-current conservation equation (4) can be reexpressed as

$$\nabla_\mu \mathcal{J}^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\mathcal{J}^\mu), \quad \text{where } \mathcal{J}^\mu = -i[\varphi^*\nabla^\nu\varphi - \varphi\nabla^\nu\varphi^*]. \tag{A1}$$

Expanding the above equation, and using that $\sqrt{-g} = \alpha\sqrt{\gamma}$ and the conformal cosmological decomposition $\sqrt{\gamma} = \psi^6 a^3\sqrt{\eta}$, we have

$$-\frac{1}{\alpha\psi^6 a^3\sqrt{\eta}}\left\{\partial_t\left[\alpha\psi^6 a^3\sqrt{\eta}\left(-\frac{1}{\alpha^2}(\varphi^*\partial_t\varphi - \varphi\partial_t\varphi^*) + \frac{\beta^i}{\alpha^2}(\varphi^*\partial_i\varphi - \varphi\partial_i\varphi^*)\right)\right] + \partial_i\left[\alpha\psi^6 a^3\sqrt{\eta}\left(\frac{\beta^i}{\alpha^2}(\varphi^*\partial_t\varphi - \varphi\partial_t\varphi^*) + \left(\gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2}\right)(\varphi^*\partial_j\varphi - \varphi\partial_j\varphi^*)\right)\right]\right\} = 0. \tag{A2}$$

The $O(\epsilon^0)$ of the above equation results in

$$\partial_t[a^3(\bar{\varphi}^*\partial_t\bar{\varphi} - \bar{\varphi}\partial_t\bar{\varphi}^*)] = O(\epsilon^2). \tag{A3}$$

On the other hand, the $O(\epsilon^2)$ of Eq. (A2) is given by

$$\partial_t\{(6\Psi - \Phi)a^3(\bar{\varphi}^*\partial_t\bar{\varphi} - \bar{\varphi}\partial_t\bar{\varphi}^*) + a^3[((\bar{\varphi}^*\partial_t\delta\varphi - \bar{\varphi}\partial_t\delta\varphi^*)) + (\delta\varphi^*\partial_t\bar{\varphi} - \delta\varphi\partial_t\bar{\varphi}^*)]\} = O(\epsilon^4). \tag{A4}$$

Using Eq. (50), we have $6\partial_t\Psi = 3H\Phi$, and then $6\partial_t\Psi - \partial_t\Phi = 3H\Phi - \partial_t\Phi = -a^3\partial_t(\Phi/a^3)$. Then, we can rewrite the above equation as

$$\partial_t\{a^3[\partial_t(\bar{\varphi}^*\delta\varphi - \bar{\varphi}\delta\varphi^*) - 2(\partial_t\bar{\varphi}^*\delta\varphi - \partial_t\bar{\varphi}\delta\varphi^*)]\} - a^3\partial_t\left(\frac{\Phi}{a^3}\right)[a^3(\bar{\varphi}^*\partial_t\bar{\varphi} - \bar{\varphi}\partial_t\bar{\varphi}^*)] = O(\epsilon^4), \tag{A5}$$

where in the above equation we have used Eq. (A3) to simplify the expression.

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