Late-time Universe, H_0-tension, and unparticles

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Lambda cold dark matter (ΛCDM) is increasingly facing challenges in late-time cosmology, notably in the Hubble parameter $H(z)$ at redshift zero known as H_0-tension. Extensions to ΛCDM have been approached mostly by models with an extra parameter. In this work, we provide a framework for evaluating the viability of models of late-time cosmology by investigating unparticle cosmology as a case study. Unparticle cosmology proposes a scale invariant contribution by a dimensionless parameter δ. We focus on H_0-tension in comparison with scale invariant and ΛCDM models. The dynamical behavior of unparticle cosmology with and without a cosmological constant shows that for most values of $\delta \in [-6, 1]$ (corresponding to $d_u \in [-2, 3/2]$), the late-time Universe will be Λ and matter dominated, respectively, with negligible contribution of unparticles. It predicts $H(z)$ in the future to be zero or constant, the latter pointing to a stable de Sitter phase for our Universe. Our data analysis shows $\delta = -2.06 \pm 0.46$ as the best fit to the data. Consequently, the conventional value of $d_u = 3/2 \equiv \delta = 1$ is ruled out by 6.6σ. However, there is a pronounced gap between the models and what is demanded by observational data in the qQ-diagram for *all* δ. We conclude that unparticle cosmology fails to come to the rescue of challenges to late-time ΛCDM, but it provides a pointer to holographic dark energy.

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I. INTRODUCTION

Our Universe is currently experiencing an accelerated expansion phase and the Lambda cold dark matter (ΛCDM) paradigm is a successful model for describing a vast majority of Universe evolution process from early times to the current phase [1]. Therefore, this model is known as the standard cosmological theory, but there are some observational features that cannot be explained by ΛCDM prompting an increasing interest to look beyond ΛCDM or perhaps replacing it with a different model [2,3].

As for ΛCDM challenges, we can mention the original cosmological constant problem arising from quantum field theory as a large gap of 120 orders in magnitude between predictions and observational data [4,5]. This is not strictly a ΛCDM issue, as it comes out of the assumption that vacuum energy scales with classical volume. As the most compact objects, black holes of mass M and Schwarzschild radius $R_s = 2R_g$, $R_g = GM/c^2$ introduce a maximum energy density $\rho_M = 3Mc^2/8\pi R_s^2 G$, where c is the velocity of light and G is the Newton constant. In the holographic interpretation [6–9], this would be universal suggesting the closure density $\rho_c = 3H^2/8\pi$ to be a corresponding bound in cosmological space-time with Hubble parameter H and Hubble radius $R_H = c/H$. By this correspondence, Zel'dovich's integral is immediately ruled out [10,11] rendering *this* cosmological constant problem to be moot.

We further encounter H_0-tension: the discrepancy between measurements of H_0 by the cosmic microwave background and local distant ladder. The first gives $H_0 \cong 68\ \mathrm{kms^{-1}\,Mpc^{-1}}$, while the latter one gives $H_0 \cong 73\ \mathrm{kms^{-1}\,Mpc^{-1}}$, currently distinct at a level of confidence better than 4.4σ [12–17]. Riess *et al.*'s interpretation of type Ia supernova observational data is not without critique, however [18], leaving some ambiguity in this level of confidence.

ΛCDM predictions also give inconsistent results for galaxy properties. A critical challenge for ΛCDM in this regard is the "missing satellite" problem. ΛCDM simulations point to large numbers of subhaloes including their satellite galaxies, but observations indicate that the number of satellite galaxies is much smaller than the predictions. Also, the angular momentum of baryons is calculated to be much smaller than the observed ones, and the actual speed of structure formation seems to be much larger than ΛCDM predictions [2,19].

These challenges to ΛCDM are motivating us to look beyond or to renew the theory. In the past few decades, many attempts have addressed some of the ΛCDM problems [4,20–26] including gravitational theories containing

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dark energy and various approaches to modified gravity. Scale invariant cosmology and unparticle physics are two of many that we aim to investigate in this work.

Scale invariant cosmology suggests that empty space is scale invariant at large scales. This assumption leads to additional terms in the field equations that are supposed to replace the cosmological constant [22]. This model recognizes Minkowski space-time as a solution for Einstein equations, thus predicting the future value of the Hubble parameter to be zero. However, there are some indications that this model is inconsistent with the observational data and it should be ruled out [22,27].

Unparticles, on the other hand, represent nontrivial scale invariant "stuff" at low energy scales. First proposed by Georgi in 2007 [21], such couples with standard model particles at quite high energy scales as Banks-Zaks fields (Sec. II). But at low energy scales (i.e., the IR limit), Banks-Zaks fields are decoupled from standard model particles and their new phase known as the unparticle has some interactions with standard matter. Unparticles are defined by an energy cutoff scale [21], which invites us to study the model using H_0-tension as a novel diagnostic. What they have in common with Maeder's scale invariant model [22] is that unparticles likewise are scale invariant at sufficiently low energy scales.

A main objective of this work is to provide a framework for evaluating the viability of such models. In this way, the viability of unparticle cosmology will be examined by considering the main required features of the theory.

Our starting point is the dynamical properties of unparticle cosmology. As a cosmological model, it must produce the cosmological epochs consecutively. We will see that unparticle cosmology has some challenges in creating the various cosmological epochs. In particular, big bang nucleosythesis (BNS) and Hubble parameter evolution will be investigated.

To study Hubble parameter evolution, we consider, quite generally, three possible scenarios for the future of the Hubble parameter. The value of the Hubble parameter may drop to zero in the distant future ($z = -1$) or decrease to a constant value in the same limit. Another possibility is that the Hubble parameter blows up. Each of these scenarios might be satisfied in some gravitational theories, but our Universe selects just one of them.

A key objective of our approach is the development of an effective diagnostic distinguishing between theses three scenarios based on the data in the present late-time Universe.

Our classification of the Hubble parameter with respect to redshift is presented in Fig. 1. Note that each of these three classes has their distinct footprint on the current value of Hubble parameter H_0. We will identify the future evolution of our models and categorize them accordingly in a comparison with the observational data, both for unparticle and scale invariant cosmology alongside ΛCDM.

While we mostly focus on the late-time behavior of the Universe, one may find exotic scenarios in each model in the early Universe. Yet, all the models should conform to BNS. This common requirement may not always be satisfied, as some of the proposed models do not tend to follow it for all or any of their parameter values.

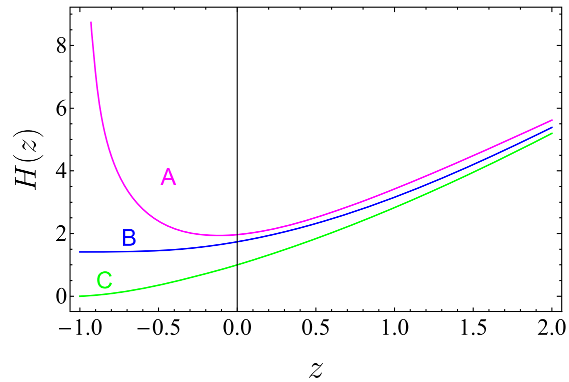

FIG. 1. Different theoretical possibilities for Hubble parameter behavior in the future, $-1 < z < 0$. Based on the model specification, the Hubble parameter may blow up (class A) which implies a dynamical instability on the Hubble timescale, goes to a finite value (class B), the case which proposes a stable de Sitter phase in a distant future, or drops to zero (class C), suggesting asymptotically Minkowski space-time for the future of the Universe. These three classes are anchored by BNS at high z, but distinct futures imply different predictions for $H_0 = H(0)$.

The paper is organized as follows: A brief introduction to unparticles and their main features is presented in Sec. II. In Sec. III, we determine the fixed points of unparticle dynamics and their stability. This provides us an overview to see if the theory can describe the proper ordering of the cosmological epochs. To have a general consideration of its early-time behavior, Sec. IV briefly shows the inconsistency of unparticle physics with BNS. A detailed discussion on the evolution of Hubble parameter for the unparticle and scale invariant model in comparison with ΛCDM is presented in Sec. V. The Hubble parameter turning point and radius of convergence are two key concepts that will be explained to perform a reliable confrontation between the theoretical results and available datasets. These two are presented in Secs. V A and V B. We also discuss the consistency between the common χ^2 analysis method and our use of polynomial fits as a reference for data analysis in Sec. V B 3. The comparison with the data based on these considerations is presented in the rest of the section. The results are summarized in Sec. VI.

II. MAIN FEATURES OF UNPARTICLES

The prototype of unparticles derives from a low energy limit of Banks-Zaks fields [28]. Banks-Zaks fields have a nontrivial conformal IR fixed point at low energy scales. At high energy, above the energy scale $\mathcal{M}_{\mathcal{U}}$, they interact with standard model particles by exchanging particles with mass scale $\mathcal{M}_{\mathcal{U}}$. But below this scale, their interaction is suppressed. Moreover, below the energy scale $\Lambda_{\mathcal{U}}$, the scale

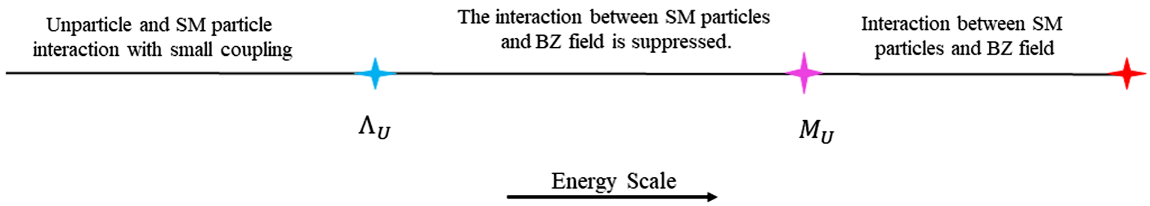

FIG. 2. Schematic view of the energy scales in which the Bank-Zaks (BZ) field, standard model (SM) particles, and unparticles can have interaction with each other. At high energy scales (above $\mathcal{M}_{\mathcal{U}}$), Bank-Zaks fields have interactions with standard model particles, but at low energy scales they are completely decoupled (below $\Lambda_{\mathcal{U}}$). At this energy scale, the interaction exists between standard model particles and unparticles as low energy, scale invariant stuff.

invariance feature of Banks-Zaks field appears. This is where dimensional transmutation of the Banks-Zaks field occurs due to their renormalizable coupling. In fact, at energy scales below $\Lambda_{\mathcal{U}}$ (i.e., the IR regime), Banks-Zaks fields match that of unparticles. At this energy scale, Banks-Zaks fields are completely decoupled from standard model particles with interactions between standard model particles and unparticles (see Fig. 2).

Self-consistency implies $k_B T < \Lambda_{\mathcal{U}}$, where k_B is the Boltzmann constant [29], and as the interactions of unparticles are yet to be discovered, the lower limit on the energy cutoff must satisfy $\Lambda_{\mathcal{U}} \geq 1$ Tev [30]. Unparticles have somewhat unconventional features. They lack a definite mass because a scale transformation multiplies all dimensional quantities of this stuff. Thus, in a scale invariant sector, there is no particle with constant nonzero mass: Unparticles have no definite mass unless this definite mass is equal to zero.

To derive the field equations of a cosmological model which includes unparticles, we begin with a thermodynamic description. For a gauge theory and in a situation where all renormalized masses disappear, the trace anomaly for the energy-momentum tensor is defined as [31]

$$\theta^\mu_\mu = \frac{\beta}{2g} N[F^{\mu\nu}_a F_{a\mu\nu}], \tag{1}$$

where β is the beta function for coupling g, and N represents the normal product [32]. For unparticles, the beta function has a nontrivial IR fixed point at $g = g_* \neq 0$, where the scale invariant feature of these species appears. So at low enough energy scales, one can write the beta function as

$$\beta = a(g - g_*), \qquad (a > 0) \tag{2}$$

comprising a fixed point at $g = g_*$. This implies a running coupling g, which can be defined as

$$g(\mu) = g_* + u\mu^a, \qquad \beta[g(\mu)] = au\mu^a, \tag{3}$$

with μ as the renormalization scale and u an integration constant. Here the lowest order corrections to the conformal limit ($\theta^\mu_\mu = 0$) are demanded, where the system is isotropic, homogeneous, in thermal equilibrium, and has no conserved charge. In this situation, the thermal average of θ^μ_μ ($\langle N[F^{\mu\nu}_a F_{a\mu\nu}]\rangle$) can be taken equal to its conformal value because β vanishes at the conformal limit. With $\mu = T$ at temperature T, one will have

$$\langle N[F^{\mu\nu}_a F_{a\mu\nu}]\rangle = bT^{4+\gamma}, \tag{4}$$

where γ is the anomalous dimension of the unparticle operator. γ can be negative for scalar unparticles, but for vector unparticles, it should be non-negative [33].

Let $\langle \theta^\mu_\mu \rangle$ denote the expectation value of the trace of the energy-momentum tensor for unparticles, and we will then have

$$\langle \theta^\mu_\mu \rangle = \rho_{\mathcal{U}} - 3\mathcal{P}_{\mathcal{U}}, \tag{5}$$

with $\rho_{\mathcal{U}}$ and $\mathcal{P}_{\mathcal{U}}$ the energy density and pressure of unparticles, respectively. The expectation value of Eq. (1) is obtained by using Eq. (5) on the left hand side and Eqs. (3)–(4) on the right hand side of Eq. (1). Therefore, the lowest order correction to the conformal limit ($T = \mu$) of $\langle \theta^\mu_\mu \rangle$ due to the unparticles contribution satisfieses

$$\rho_{\mathcal{U}} - 3\mathcal{P}_{\mathcal{U}} = AT^{4+\delta}, \qquad A = \frac{aub}{2g_*}, \qquad \delta = a + \gamma. \tag{6}$$

Here, δ is a parameter which depends on the scaling dimension of unparticles by the equation

$$\delta = 2(d_u - 1) \tag{7}$$

inferred from [32,34].

Valid values of d_u have been discussed pertaining to different physical features. The most conventional interval is $1 < d_u < 2$, while $0 < d_u < 2$ and $d_u > 1$ have also been considered for the unparticle operator [34–43]. Negative values for d_u have also been considered [33,44,45], although this may be at odds with the unitarity of the model [46,47]. The value of d_u perturbs the entropy of black holes away from the Bekenstein value corresponding to $d_u = 1$ [48,49].

Here, to cover all the proposed values of d_u, the theory will be investigated for $\delta \in [-6, 1]$ corresponding to $d_u \in [-2, \frac{3}{2}]$.

Combining Eq. (6) with the first law of thermodynamics, the expressions for energy density and pressure of the unparticles satisfies

$$\rho_{\mathcal{U}} = \sigma T^4 + A\left(1+\frac{3}{\delta}\right)T^{4+\delta},$$
$$\mathcal{P}_{\mathcal{U}} = \frac{1}{3}\sigma T^4 + \left(\frac{A}{\delta}\right)T^{4+\delta}, \quad (8)$$

where we follow the conventional definition for σ [32,44]. $\delta = 0$ is a special case related to logarithmic correction to standard radiation and should be calculated directly from the first thermodynamic law.

Three other interesting values are $\delta = -2$ which is the holographic limit, where the extra contribution $T^{\delta+4}$ reduces to T^2. We will see that this is the most important one for unparticle cosmology (Sec. V C), $\delta = -3$ is responsible for some apparent singularities, and $\delta = -4$ puts the theory at the boundary of ΛCDM. As the value of δ decreases, the decay rate of unparticles slows down, and this fact secures the role of unparticles at the late-time Universe. Therefore, if unparticles are to govern the late-time Universe, δ must be negative.

The idea of an additional fluid with unknown barotropic index ($w = p/\rho$) is mostly used to describe properties of dark energy [50–57] and only occasionally applied to dark matter [58,59]. Here, however, we use this concept more broadly which extends to unparticles with presumably matterlike properties. According to Eq. (8), the barotropic index of unparticles $w = \mathcal{P}_{\mathcal{U}}/\rho_{\mathcal{U}}$ satisfies

$$w = \frac{\frac{1}{3}\sigma T^4 + (\frac{A}{\delta})T^{4+\delta}}{\sigma T^4 + A(1+\frac{3}{\delta})T^{4+\delta}} = \frac{1}{\delta+3}, \quad (9)$$

where the second equality holds at late-time cosmology when radiation energy density is safely neglected. We note that a constant w in Eq. (9) implies that unparticles carry an adiabatic index 1 associated with infinite degrees of freedom in a classical fluid dynamics perspective.

Furthermore, there might be a concern regarding the ability of the unparticle model to produce positive energy density in the late-time Universe, $A(\delta+3)/\delta > 0$. In the absence of a complete theory that would determine δ and A, we have taken δ and $\Omega_{\mathcal{U}_0}$ as free parameters in our comparison with the data (Sec. V C), the combination of which fixes A.

We assume our Universe to be flat, isotropic, and homogeneous, which is governed by the Friedmann-Robertson-Walker metric

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2) \quad (10)$$

with $a(t)$ the Friedmann scale factor. Using Eqs. (8)–(10), we write the field equations of unparticle cosmology as

$$3H^2 = \rho = \rho_{\mathcal{U}} + \sum_i \rho_i,$$
$$\dot{H} = -\frac{1}{2}(\rho + p)$$
$$= -\frac{1}{2}((\rho_{\mathcal{U}} + \mathcal{P}_{\mathcal{U}}) + \sum_i(\rho_i + p_i)), \quad (11)$$

where the "i" index stands for other species alongside unparticles, e.g., radiation, matter, cosmological constant, etc., and H is the Hubble parameter $H = \dot{a}/a$.

We next investigate different aspects of unparticle cosmology by dynamical behavior analysis and stability of fixed points.

III. DYNAMICAL BEHAVIOR OF UNPARTICLE COSMOLOGY

Our Universe is described by a dynamical system with energy components changing in time. By (10), ultimately the effectiveness of this model derives from the consistency of its dynamical behavior with observational data. In standard cosmology and according to observations, our Universe has experienced different epochs ever since its journey from early times. It passed a radiation-dominated epoch followed by a matter-dominated epoch and, more recently, reached accelerated expansion indicative of dark energy or modified gravitational theories. By now, radiation- and matter-dominated epochs have already faded. A conventional two-dimensional dynamical system analysis can be used to explore if a theory can satisfy the ordering of these epochs [60,61].

In this section, we will study the dynamical properties of unparticle cosmology by fixed points and associated stability. We will employ Eqs. (11) as the main ingredients.

Illustrative for the possible role of unparticles in cosmological evolution is to consider their content as a perturbation to ΛCDM. To this end, we write Eqs. (11) as

$$3H^2 = \rho_{\mathcal{U}} + \rho_r + \rho_m + \rho_\Lambda,$$
$$\dot{H} = -\frac{1}{2}(\rho + p) = -\frac{1}{2}((\rho_{\mathcal{U}} + \mathcal{P}_{\mathcal{U}})$$
$$+ (\rho_r + p_r + \rho_m + p_m + \rho_\Lambda + p_\Lambda)), \quad (12)$$

where indices r and m refer to radiation and matter, respectively. As usual, $p_r = \frac{1}{3}\rho_r$, $p_m = 0$, and $\rho_\Lambda = -p_\Lambda$. Also, the continuity equation holds for all four species

$$\dot{\rho}_r + 3H(\rho_r + p_r) = 0,$$
$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0,$$
$$\dot{\rho}_{\mathcal{U}} + 3H(\rho_{\mathcal{U}} + \mathcal{P}_{\mathcal{U}}) = 0. \quad (13)$$

At energy scales where different species interact, there will be additional interaction terms on the right-hand side of the continuity equations (13), most likely in the form of $\zeta H \rho_i$, where ζ is the coupling constant. Here we have assumed no such interactions between the species.

One of the present challenges of ΛCDM is that it does not describe the late-time Universe properly. It implicitly assumes the Universe to settle down to a stable de Sitter phase with constant Hubble parameter. This model assumption is conceivably challenged by late-time cosmological evolution. Here, we primarily focus on the role of unparticles in late-time cosmology, when radiation can be effectively neglected.

Equations (12) and (13) govern the evolution of a universe which includes unparticles in addition to common species. To proceed, we define some dimensionless parameters for the energy density of each energy component ($\rho = \rho_m + \rho_\Lambda + \rho_{\mathcal{U}}$):

$$\Omega_\Lambda = \frac{\rho_\Lambda}{3H^2}, \qquad \Omega_m = \frac{\rho_m}{3H^2}, \qquad \Omega_{\mathcal{U}} = \frac{\rho_{\mathcal{U}}}{3H^2}. \tag{14}$$

These densities change as time passes, and accordingly the dynamics of the Universe is governed by dimensionless content. The time derivative of the parameters in Eq. (14) can be reexpressed by the number of e-folding (N),

$$\begin{aligned}
\Omega'_m &= \frac{d\Omega_m}{dN} = -3\Omega_m - \left(\frac{2\dot{H}}{H^2}\right)\Omega_m,\\
\Omega'_\Lambda &= \frac{d\Omega_\Lambda}{dN} = -2\Omega_\Lambda\left(\frac{\dot{H}}{H^2}\right),\\
\Omega'_{\mathcal{U}} &= \frac{d\Omega_{\mathcal{U}}}{dN}\\
&= -3\Omega_{\mathcal{U}} - \frac{3}{\delta+3}\Omega_{\mathcal{U}} - \left(\frac{2\dot{H}}{H^2}\right)\Omega_{\mathcal{U}},\\
\frac{\dot{H}}{H^2} &= \frac{-1}{2}\left(3\Omega_m + 3\Omega_{\mathcal{U}}\left(1+\frac{1}{\delta+3}\right)\right),
\end{aligned} \tag{15}$$

where the prime denotes the derivative with respect to N. Although it is not common to use N as a measure of time in the late-time Universe, mathematically it is convenient to employ it for dynamical system analysis. This makes our dynamical system to be expressed purely as a function of energy densities. For a three-flat universe, one of the above three equations depends on the other two

$$1 = \Omega_m + \Omega_\Lambda + \Omega_{\mathcal{U}}. \tag{16}$$

We eliminate Ω_m. The remaining equations are solved jointly ($\Omega'_{\mathcal{U}} = 0, \Omega'_\Lambda = 0$). We thus identify the fixed points to be

$$\begin{aligned}
\text{Matter}\colon &\to (\Omega_\Lambda \to 0, \Omega_{\mathcal{U}} \to 0),\\
\text{Unparticle}\colon &\to (\Omega_\Lambda \to 0, \Omega_{\mathcal{U}} \to 1),\\
\Lambda\colon &\to (\Omega_\Lambda \to 1, \Omega_{\mathcal{U}} \to 0).
\end{aligned} \tag{17}$$

The first fixed point represents a matter-dominated phase of the Universe while in the second, unparticles are dominant. The third refers to a de Sitter state defined by a cosmological constant.

Assuming unparticle cosmology as a perturbation to ΛCDM, one is required to investigate the ordering of these three epochs. A Jacobian matrix for Eqs. (15) will reveal the stability of our fixed points by the associated eigenvalues of

$$J = \begin{pmatrix} \frac{\partial\Omega'_\Lambda}{\partial\Omega_\Lambda} & \frac{\partial\Omega'_\Lambda}{\partial\Omega_{\mathcal{U}}} \\ \frac{\partial\Omega'_{\mathcal{U}}}{\partial\Omega_\Lambda} & \frac{\partial\Omega'_{\mathcal{U}}}{\partial\Omega_{\mathcal{U}}} \end{pmatrix}. \tag{18}$$

The stability of each fixed point is defined by the eigenvalues of Eq. (18). Here we have a two-dimensional dynamical system, so there will be two eigenvalues for each fixed point. In the case where both eigenvalues are positive, the corresponding fixed point is unstable. If both are negative, the fixed point is stable representing an attractor; if one is positive and one is negative, we have a saddle point. In our case, as the expressions for Ω_Λ and $\Omega_{\mathcal{U}}$ depend on the value of δ, naturally the components of the Jacobian matrix will also be the same. We have performed the above procedure for the second and third equations of Eqs. (15) by using Eqs. (17) as fixed points. Our results show some unexpected features for an unparticle universe.

In standard cosmology, the matter-dominated epoch corresponds to a saddle fixed point, while the cosmological constant epoch is responsible for a stable fixed point at late times. Here, unparticles also have their fair share in the dynamics of the Universe.

Figure 3 shows the stability of each fixed point for $\delta \in [-6, 1)$. The results show that for $-3 < \delta < 0$ (and $0 < \delta < 1$), the matter-dominated fixed point is a saddle point, the unparticle fixed point is unstable, and the cosmological constant epoch is stable (as in ΛCDM). In fact, in this situation, the unparticles' role in the Universe is similar to ordinary radiation. This implies that there should be an epoch of unparticle physics followed by a matter epoch, and a cosmological constant governs the late-time Universe. In this case, any contribution from unparticles is sub-dominant at late times.

For $\delta < -3$, the matter-dominated epoch is unstable, the unparticle fixed point represents a saddle point, and again the cosmological constant regulates the late-time Universe. An interesting feature of these results is in the middle of the δ interval, where the unparticle fixed point is stable and the other two are saddle and unstable nodes. In a dynamical system, we often refer to fixed points as nodes when discussing stability. Thus, unparticles can be the ultimate states of our Universe only if δ is naturally chosen to be in that middle interval ($-4 < \delta < -3$).

It is notable that $\delta = -3$ acts as the transition value for stability originating in the definition of unparticle energy density and pressure [Eqs. (8)]. In Fig. 4, the flux lines are presented for $\delta = -2$ and $\delta = -6$. The fixed points and

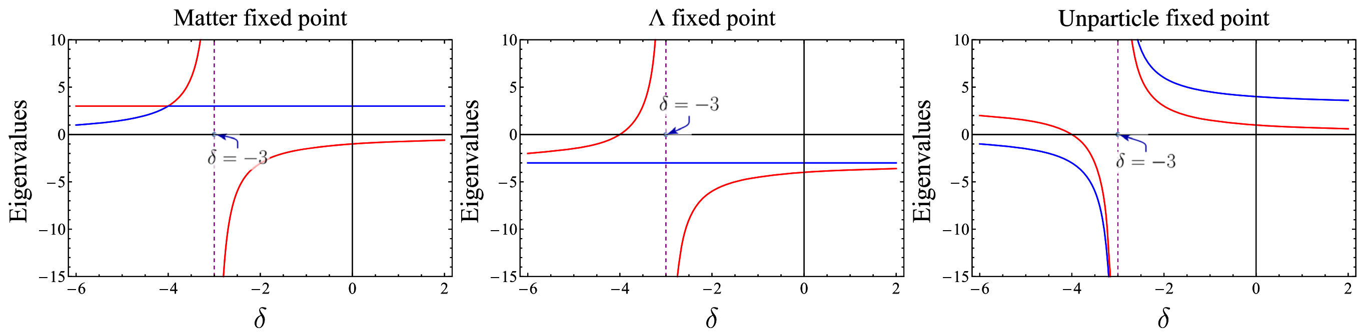

FIG. 3. Stability of each fixed point, Eq. (17). Left panel: matter-dominated epoch is a saddle fixed point for $-3 < \delta < 0$ and an unstable fixed point for $\delta < -3$. Middle panel: cosmological-constant-dominated epoch is a stable fixed point for all the values of δ, except for a narrow interval where it becomes a saddle fixed point. Right panel: unparticle fixed point is unstable for $-3 < \delta < 1$ and is a saddle point for $\delta < -3$. Unparticle epoch is stable only for $-4 < \delta < -3$.

their stability indicate that $\Omega_{\mathcal{U}}$ cannot be dominant at the late-time Universe unless δ takes the value $-4 < \delta < -3$. In all other cases, the cosmological constant governs the late-time Universe.

It is useful for our further purpose to have a glance at the case which considers unparticles as an energy component of the Universe in the *absence* of a cosmological constant. In this case, the field equations are

$$
\begin{aligned}
3H^2 &= \rho_r + \rho_m + \rho_{\mathcal{U}}, \\
\dot{H} &= -\frac{1}{2}(\rho + p) \\
&= -\frac{1}{2}((\rho_{\mathcal{U}} + \mathcal{P}_{\mathcal{U}}) + (\rho_r + p_r + \rho_m + p_m)). \qquad (19)
\end{aligned}
$$

Here also, the contribution of radiation is considered to be quite small at late times, and the continuity equation is satisfied for all three species separately.

Defining dimensionless energy densities as before gives us the fixed points and their stability of this model. We now find for $-3 < \delta < 0$ the radiation-dominated fixed point is a saddle point, the matter-dominated epoch is a stable fixed point, and finally, unparticles are responsible for an unstable fixed point. For $0 < \delta < 1$, the results are the same, but the radiation- and unparticle-dominated epochs will exchange the position. Therefore, unparticles cannot have any contributions at late times in these two intervals, but for $\delta < -3$, these fixed points are unstable, saddle, and stable fixed points, respectively. Thus, if nature demands $\delta < -3$, unparticles contribute to the late-time Universe and can be considered a replacement for Λ.

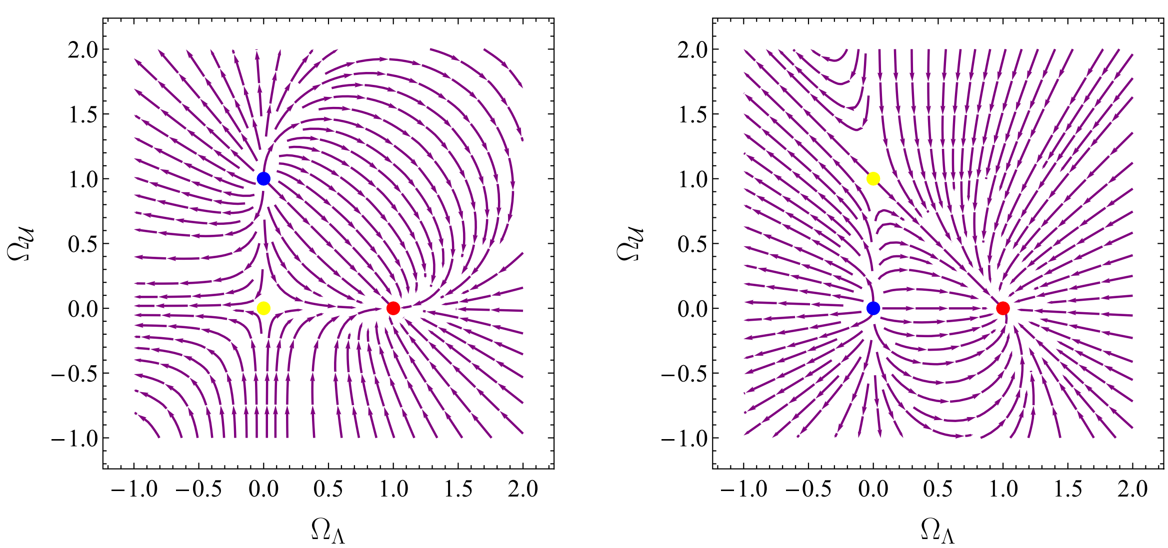

FIG. 4. The phase space diagram for the unparticle cosmology with $\delta = -2$ (left) and $\delta = -6$ (right). The blue, yellow, and red dotes correspond to unstable, saddle, and stable fixed points, respectively. The flux lines are consistent with the explanation of stability of the fixed points in Fig. 3. For $\delta > -3$, unparticle fixed point ($\Omega_{\mathcal{U}} \to 1, \Omega_\Lambda \to 0$) is an unstable point, while it is a saddle point for $\delta < -3$, and vice versa for a matter fixed point ($\Omega_{\mathcal{U}} \to 0, \Omega_\Lambda \to 0$). In both cases, the model will not allow the unparticle to rule the late-time Universe.

IV. UNPARTICLE COSMOLOGY BIG BANG NUCLEOSYNTHESIS

Big bang nucleosynthesis happens at early times of the Universe. Unparticle cosmology should be checked to see if its BNS and the evolution of the scale factor is compatible with the radiation-dominated era. To this end, an analytical solution for the scale factor, which is consistent with BNS, is derived from Eq. (12). At this early epoch, there is no matter component or cosmological constant, so Eq. (12) will be reduced to

$$3H^2 = \rho = \rho_{\mathcal{U}} + \rho_r, \tag{20}$$

which leads to

$$dt = \int \sqrt{H_0^2(\Omega_{r_0} a^{-4} + \Omega_{\mathcal{U}_0} a^{-3(\frac{\delta+4}{\delta+3})})} da. \tag{21}$$

The first term of Eq. (21) shows the presence of radiation at early times, as usual, and the second term is due to the presence of unparticles. For radiation only $a \propto t^{\frac{1}{2}}$, and for combined radiation and unpaticles we can evaluate the above integral explicitly. For the latter, we derive

$$t = \frac{2(\delta+3) a^{\frac{4\delta+15}{\delta+3}} \sqrt{\frac{a^{-\frac{\delta}{\delta+3}}\Omega_{r_0}}{\Omega_{\mathcal{U}_0}} + 1} \sqrt{a^{-\frac{\delta+6}{\delta+3}}\Omega_{\mathcal{U}_0} + \frac{\Omega_{r_0}}{a^2}}\, {}_2F_1\left(\frac{1}{2}, -\frac{3(\delta+4)}{2\delta}; -\frac{1}{2} - \frac{6}{\delta}; -\frac{a^{-\frac{\delta}{\delta+3}}\Omega_{r_0}}{\Omega_{\mathcal{U}_0}}\right)}{3(\delta+4)(a^{\frac{\delta+6}{\delta+3}}\Omega_{r_0} + a^2\Omega_{\mathcal{U}_0})}, \tag{22}$$

where F_1 is the hypergeometric function type [62], and Ω_{r_0}, $\Omega_{\mathcal{U}_0}$ refer to present densities of the unparticle and radiation energy components.

In an early universe containing both radiation and unparticles with no coupling, a scale factor $a(t)$ will deviate from that of a radiation universe depending on the value of δ. Therefore, generally the model cannot be anchored as does ΛCDM at BNS. We will come back to this point in the next section.

V. HUBBLE PARAMETER EVALUATION

Up to now, we have some specifications and challenges of unparticle cosmology. A comparison between the evolution of the Hubble parameter and its properties for the three models will be performed (i.e., ΛCDM, scale invariant, and unparticle).

First, we review the deceleration parameter $q(z)$ in unparticle cosmology and its first derivative with respect to redshift $Q(z)$. The quantities H, q, and Q can give us complete information about cosmological evolution. In addition, these will facilitate our understanding of the model in comparison with the data. The deceleration parameter $q = -\ddot{a}a/\dot{a}^2$ is

$$q(z) = -1 + (1+z)H^{-1}(z)H'(z),$$
$$a = \frac{a_0}{1+z}. \tag{23}$$

Currently, our Universe is in an accelerated expansion phase with $q < 0$. By Eq. (23), one can detect the behavior of $H'(z)$ during this time, which is important when $H(z)$ changes the slope. Q, the first derivative of $q(z)$, captures the curvature of the graph of the $H(z)$ diagram and maybe the inflection point. $q(z)$ and its first derivative hereby effectively characterize the proposed model, here in late-time cosmology.

To apply these functions to our models, we start with the first Friedmann equation. The most general form of this equation for unparticle cosmology is

$$3H^2 = \rho_r + \rho_m + \rho_\Lambda + \rho_{\mathcal{U}}. \tag{24}$$

The corresponding Hubble parameter satisfies

$$H(z) = H_0\sqrt{\Omega_r(z) + \Omega_m(z) + \Omega_\Lambda(z) + \Omega_{\mathcal{U}}(z)}, \tag{25}$$

with

$$\Omega_r(z) = \Omega_{r_0}(z+1)^4,$$
$$\Omega_m(z) = \Omega_{m_0}(z+1)^3,$$
$$\Omega_{\mathcal{U}}(z) = \Omega_{u_0}(z+1)^{3(1+\frac{1}{\delta+3})}. \tag{26}$$

As before, Ω_{r_0}, Ω_{m_0}, and $\Omega_{\mathcal{U}_0}$ represent the densities of radiation, matter, and unparticle energy components, respectively, at $z = 0$. The third term in Eq. (25) is present only if we assume unparticle cosmology as a perturbation to ΛCDM, wherein $\Omega_r(z)$ is ignored. Otherwise, the Universe is governed only by radiation, matter, and unparticles. For ΛCDM and scale invariant cosmology, $H(z)$ satisfies [63]

$$H_\Lambda(z) = H_0\sqrt{(1-\Omega_m) + \Omega_m(1+z)^3},$$
$$H_{sc}(z) = H_0(\Omega_m(1+z)^{\frac{9}{4}} + (1-\Omega_m)(1+z)^{\frac{3}{4}})^{\frac{2}{3}}, \tag{27}$$

where "sc" and Λ subscription stand for the scale invariant model and ΛCDM, respectively. In some sense, these two set boundaries for unparticle cosmology. According to Eq. (27), ΛCDM predicts a constant $H(z)$ for the future of the Universe—a de Sitter universe. But scale invariant cosmology expects the Universe to have *zero* expansion rate in the future—a Minkowski space-time. Roughly speaking, unparticle cosmology as a perturbation to ΛCDM predicts a future with constant Hubble parameter ($z = -1$), while in the absence of a cosmological constant, where unparticles are supposed to be responsible for the late-time Universe, the model predicts the Hubble parameter will vanish in the future. It is in this situation that the local observations measure H_0 larger than the one from the cosmic microwave background (CMB), and there are some arguments indicating that there should be a moment close to the present day for which $H'(0) \simeq 0$ and any proposed model requires this condition to be satisfied.

The exact expressions of the Hubble parameter for unparticles, scale invariant, and ΛCDM in Eqs. (25)–(27) clearly show that at high redshifts where z becomes large, the ΛCDM and scale invariant model obey $H(z) \propto z^{\frac{3}{2}}$, but for unparticle cosmology, the overall behavior of $H(z)$ depends on the value of δ in

$$H(z) \simeq H_0 \sqrt{\Omega_{m_0}(1+z)^3 + \Omega_{\mathcal{U}_0}(1+z)^{3\left(\frac{4+\delta}{3+\delta}\right)}}. \quad (28)$$

Generally, for $\delta < -3$, the first term is dominant, and unparticle cosmology shows $H(z) \propto z^{\frac{3}{2}}$ at the large z limit. For $\delta > -3$, the same limit yields $H(z) \propto z^{\frac{3}{2}\left(\frac{\delta+4}{\delta+3}\right)}$ showing deviation from ΛCDM and scale invariant cosmology, making reconciliation with BNS difficult.

On the other hand, at low redshifts the behavior of these three models will be determined by the first order expansion of Hubble parameters which are

$$H_\Lambda(z) = 1 + \frac{3}{2}\Omega_m z + \mathcal{O}(z^2),$$

$$H_{sc}(z) = 1 + \frac{1}{2}(1 + 2\Omega_m)z + \mathcal{O}(z^2),$$

$$H_{\mathcal{U}}(z) = 1 + \frac{1}{2}\left(3\Omega_m + \frac{3\Omega_{\mathcal{U}}(4+\delta)}{3+\delta}\right)z + \mathcal{O}(z^2). \quad (29)$$

As mentioned before, a divergent $H(z)$ in the future implies $H(z)$ at low redshift to have a turning point, $H'(0) \simeq 0$. Equation (29) gives the following current values of $H'(0)$:

$$H'_\Lambda(0) \simeq 0.5 H_0,$$

$$H'_{sc}(0) \simeq \frac{5}{6} H_0,$$

$$H'_{\mathcal{U}}(0) \simeq \frac{H_0}{2}\left(1 + \frac{2\Omega_{\mathcal{U}}(4+\delta)}{3+\delta}\right). \quad (30)$$

As inferred from Fig. 1, the first two models have no turning points. The same appears for unparticle cosmology

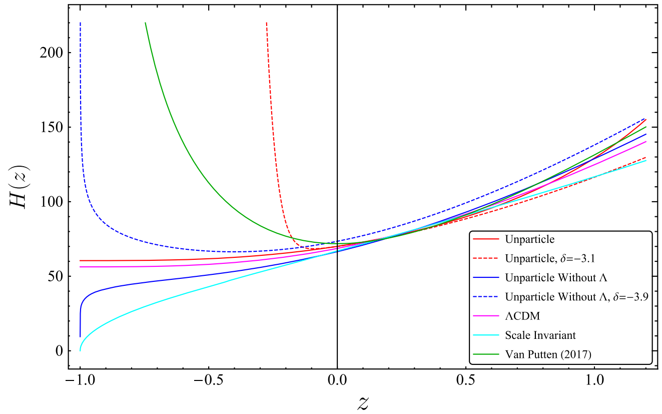

FIG. 5. The evolution of the Hubble parameter as a function of the redshift for $-1 < z < 1$ for unparticle cosmology (with and without a cosmological constant), the scale invariant model, and ΛCDM. We also added the graph for the model proposed in [23] because of its turning point. According to Fig. 1 and the evolution of the Hubble parameter for each model, scale invariant is in class C and ΛCDM is categorized in class B. Unparticle cosmology is in class B for the majority of δ values, while for some $-4 < \delta < -3$, it classifies as class A. Unparticle without Λ might take place either in class A or C, each for some values of δ. This model will be in class A for $\delta = -4$ as it shows a constant Hubble parameter in the future.

except for a special case when $-4 < \delta < -3$. According to Eq. (30), in this case a balance between the values of $\Omega_{\mathcal{U}}$ ($0 < \Omega_{\mathcal{U}} < 1$) and δ may support the existence of a turning point at small z. The future behavior of the Hubble parameter for these models is depicted in Fig. 5. Based on Fig. 1, these models can be classified according to their Hubble evolution in the future. Scale invariant and unparticle cosmology without a cosmological constant for some values of δ sit in class C, while ΛCDM and unparticle cosmology for the majority of δ values take place in class B, and unparticle cosmology for δ values close to -3, unparticle without a cosmological constant for some specific δ values, and the model proposed by van Putten [23] are in class A. Unparticle cosmology without a cosmological constant takes place in class B for $\delta = -4$.

A. The importance of a Hubble turning point

We are in a Universe in an accelerating expansion phase. While it may suggest a future de Sitter phase, the eternity of this de Sitter phase is questionable; it may or may not last forever [64–68]. ΛCDM, the standard cosmological model, implicitly assumes that the future de Sitter phase of the Universe is stable and consequently eternal [69]. At the same time, the existence of plentiful vacua has been challenging to string theory, and the insufficient de Sitter vacua in the theory have given rise to a new conjecture [70]: de Sitter vacua may be inconsistent with low energy effective theories and live in the swampland [70,71]. The swampland conjectures put two criteria on scalar fields and their corresponding potential. The cosmological consequence is that, as a solution, dark energy might be a dynamic instead of a cosmological constant. As a result, swampland conjectures hereby directly challenge ΛCDM due to its constant dark energy and assumed stable de Sitter phase in the future.

Based on the swampland conjectures, de Sitter is unstable and, as a consequence, the Hubble parameter would bear a change of slope at a redshift close to $z = 0$, in the close past or close future. Thus, there will be a turning point in such a way that $H'(0) \simeq 0$ [23,70]. Looking at Fig. 1, the existence of the Hubble turning point is an inevitable consequence of class A whenever the future of the Universe is unstable. The highlighted point is that the presence of a turning point is in agreement with the available observational data, as the datasets indicate that the Universe is in favor of the situation where $w < -1$ [70]. This equation of state violates the null energy condition (and will automatically put aside numerous models, but there will still be some theories that can satisfy this condition [23,72]).

The concept of a turning point for Hubble parameter at a redshift close to zero was first directly introduced in [23], where the author proposed a gravitational theory which is in agreement with observational data and offers the existence of a Hubble turning point. As the theory does not have any additional free parameters, the author concludes that it can be a replacement for ΛCDM, since it shares essentially the same $H(z)$ for $z \gtrsim 1$.

The standard model of cosmology does not have a Hubble turning point as its Hubble parameter goes to a constant value in a distant future (Fig. 1, case B), which comes later in Fig. 5. The scale invariant model also offers a zero Hubble parameter likewise [Eq. (27), Fig. 1 case C]. Thus, a turning point easily rules out the ΛCDM and scale invariant model as they expect a constant Hubble parameter in the future.

Unparticle cosmology has a longer story on this matter. For the vast majority of δ values, the model suggests the $H'(0) > 0$ and Hubble parameter will have a constant future $H(z)$ (when considered as a perturbation to ΛCDM) or zero (when considered in the absence of a cosmological constant), which means it has no turning point. But there is a very narrow interval of δ values around $\delta = -3$, where the Hubble parameter would experience a turning point at a moment close to $z = 0$, or will have $H'(0) < 0$, indicating that there exists a turning point in the close past and might be observable. For some values of δ, the Hubble parameter of unparticle cosmology will blow up in the distant future. The last two are consistent with the results of [73]. They have predicted a phase transition for the future of the Universe within a time on the order of a Hubble time by assuming the swampland conjectures. The presence of a turning point in Hubble evolution is qualitatively similar to this implication of the swampland conjectures.

Although the existence of a turning point is uncertain, it nevertheless serves to put two scenarios on the table; if it occurred in the close past $0 < z \ll 1$, it may perhaps be falsified or identified in the coming years with improved data.

A Hubble turning point, if present, can distinguish different models which are anchored by BNS at high redshifts yet are distinct at late times and in the future. It also opens a new window for alleviating H_0-tension, as it can explicitly explain the increased value of H_0 measured by local observations compared to the CMB measurements.

B. Comparison with the data

During the last decade, there have been numerous efforts to measure the expansion rate of the Universe and the Hubble parameter from low to intermediate redshifts slightly more than 2. Various methods has been employed for these measurements with different results [74–77]. We will use data on $H(z)$ provided by Ryan *et al.* [78], which is the updated version of the data by Farooq *et al.* [76,79]. This dataset includes 31 data points for redshifts from $z = 0$ today, back to $z = 2.36$. But these data points are not entirely independent. There are some correlations in the data which have been taken from Blake *et al.* [80] and Alam *et al.* [81]. Also the Baryon Acoustic Oscillations (BAO) used for observations used for measuring $H(z)$ needed to apply a prior for sound horizon radius [79]. We will use this dataset in the rest of this work without making any correction for these dependences. There are many other $H(z)$ compilations provided in the literature which can be used similarly.

To do analysis using observational data, we first need to know which data points are applicable. The concept of radius of convergence (RC) is what makes restrictions on the data points that we are allowed to use in estimating Taylor series coefficients.

1. Radius of convergence

Any analytic function can be expressed as a Taylor series around a pivot point z_0, and $H(z)$ is no exception. On the other hand, a Taylor series may converge or diverge for certain intervals of the variable. The interval of the variable in which the series converges to a finite value is the RC. In practice, the RC is defined as the shortest distance from the pivot point (z_0) to a singularity or branch point. Mathematically, it can be calculated by using a root test [82], ratio test [83], or by the Cauchy integral formula in the complex plane [84]. This means that the plot of $H(z)$ in the complex plane of z enables us to read the RC from the figure by the distance to the singularity or a branch point closest to the pivot point. Note that the value of the RC depends of the choice of z_0.

The RC is introduced here to fix the domain of data that are allowed in analyzing a particular model by a Taylor series. The results may be compared with an arbitrary (unbiased) polynomial fit where the polynomial serves as a truncated Taylor series. Furthermore, while $H(z)$ is

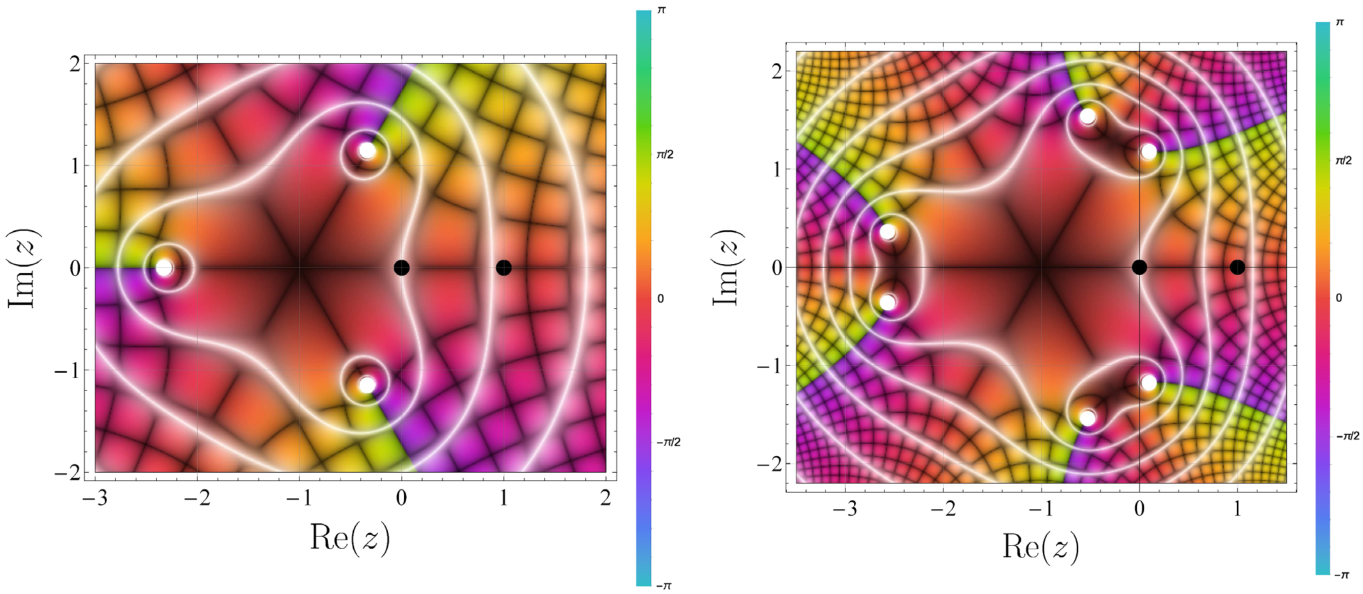

FIG. 6. The branch points of $H(z)$ defined by the zeros $H(z) = 0$ in ΛCDM [Eq. (27) left panel and Eq. (25) right panel]. These points determine the radius of convergence of the Hubble parameter with respect to a pivot point, here $z_0 = 0, 1$ (black dots) for ΛCDM and unparticle cosmology, respectively.

expected to be analytic in $-1 < z < \infty$, its Taylor series about a finite z_0 will have a finite RC: RC_0. RC_0 is positive and cannot be infinity because if $H(z)$ were entire, cosmological space-time would reduce to an eternal de Sitter phase. With $z_0 = 0$, a zero RC_0 represents a singularity for today, and a large RC_0 indicates a de Sitter phase for the current Universe. Therefore, we expect RC_0 to be of order unity around $z_0 = 0$. Ideally, for reliable results, we need to remain inside the RC which is defined by

$$\mathrm{RC}_* = \min(\mathrm{RC}_0, \mathrm{RC}_m), \tag{31}$$

where RC_m represents RC for the model we are investigating. The data points outside RC_* should be generally ignored in a polynomial fit. We note that if RC_* turns out to be small, a very limited number of data points are left for a polynomial fit. Since, as mentioned, RC_0 is unknown but expected to be of order unity, we shall take $\mathrm{RC}_* = \mathrm{RC}_m$ in what follows. Using RC_m for this purpose may lead to high sensitivity on the order of a polynomial fit when $\mathrm{RC}_0 < \mathrm{RC}_m$.

In comparing two or more models alongside the polynomial fit, we would use Eq. (31) for each model independently.

Here, $z_0 = 0$ is a perfect choice because it enables us to have better analysis according to the previous sections. On the other hand, $z_0 = 1$ is beneficial because all the data points will remain within the RC allowing us to look broader. We will apply the unparticles' RC ($\mathrm{RC}_{\mathcal{U}}$) for the polynomial fit to the data.

To find the RC for unparticle cosmology, we take the advantages of illustrating the behavior of $H(z)$ in Eq. (25) in the complex z plane. A small deviation from the current values of Ω_m, Ω_Λ, and $\Omega_{\mathcal{U}}$ and even the vast majority of δ values have no significant impact on $\mathrm{RC}_{\mathcal{U}}$. So it would be safe to use the common values of Ω_m and Ω_Λ with a three-flat condition for calculating the radius of convergence. For $\Omega_\Lambda = 0.7$, $\Omega_m = 0.26$, and $\delta = -2$, the branching points and zeros of Eq. (25) are depicted in Fig. 6.

An interesting result is when $\delta \to -3$. For this case, there is an unavoidable singularity at $z = -1$, and the radius of convergence is smaller than the cases where δ has considerable deviation from $\delta = -3$ (Fig. 7).

To have the analytic value of $\mathrm{RC}_{\mathcal{U}}$ for unparticle cosmology, $H_{\mathcal{U}}(z) = 0$ has been solved for complex z. The minimum distance from the solution to the pivot point ($z_0 = 0$ and $z_0 = 1$) reads as the radius of convergence. It

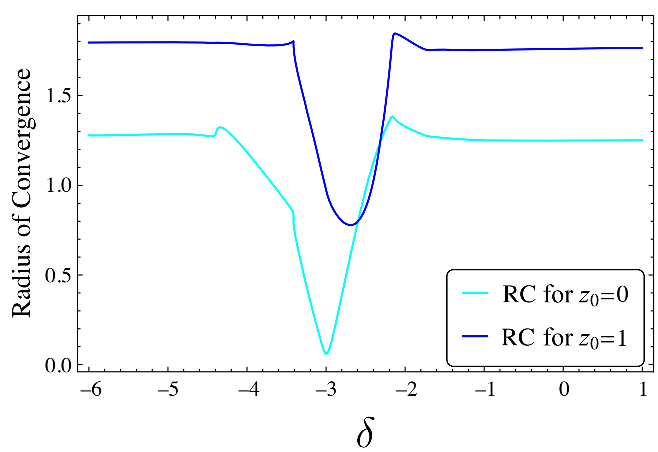

FIG. 7. The value of the RC for unparticle cosmology as a function of δ with $z_0 = 0, 1$. For $z_0 = 1$, the radius of convergence is larger and covers the whole dataset considered in this work.

came out that $\mathrm{RC}_{\mathcal{U}}$ for unparticle cosmology is approximately $RC_{\mathcal{U}} \simeq 1.3$ and 1.8 using $z_0 = 0$ and $z_0 = 1$, respectively. Thus, $z_0 = 0$ allows us to cover the data up to $z \simeq 1.3$ for reliable comparison of the results with a fit to a polynomial, while for $z_0 = 1$ the whole dataset remains inside the RC. We do our analysis for both $z_0 = 0$ and $z_0 = 1$. As $z_0 = 1$ gives us a broader RC and consequently more observational information, we will focus on this case for our final conclusions.

For $z_0 = 1$, the ΛCDM and scale invariant models have $RC_{\Lambda\mathrm{CDM}} \simeq 1.8$ and $RC_{sc} \simeq 2$, respectively, very similar to $\mathrm{RC}_{\mathcal{U}}$.

2. Unparticle cosmology: A four-parameter model

Looking back at Eq. (25), the explicit form of the Hubble parameter for unparticle cosmology as a perturbation to ΛCDM, will be

$$H_{\mathcal{U}} = H_0 \sqrt{\Omega_{\Lambda_0} + \Omega_{m_0}(1+z)^3 + \Omega_{\mathcal{U}_0}(1+z)^{3\frac{4+\delta}{3+\delta}}}, \tag{32}$$

with five free parameters. These five reduce by one by the three-flat assumption ($\Omega_{\mathcal{U}} + \Omega_m + \Omega_\Lambda = 1$). For unparticle cosmology, in the absence of a cosmological constant the theory has three free parameters after reduction. This is while scale invariant and ΛCDM have only two free parameters. In comparing these models with the same dataset, they are supposed to have equal numbers of free parameters. Therefore, we study the unparticle cosmology as a semi-two-parameter model by fixing two (one for unparticle cosmology without Λ) of the parameters and perform the numerical fit for the other two. In this way, δ will be the parameter which is considered a fixed one, and we will perform fitting with running δ in its interval (here, $-6 < \delta < 1$). This gives us an area of solutions for the model. For unparticle cosmology as a perturbation to ΛCDM, an extra fixed parameter is also needed. As we are aiming to investigate the Hubble parameter evolution and unparticle energy component, the extra fixed parameter can be either Ω_{m_0} or Ω_{Λ_0}.

Keeping one of them fixed, unparticles share the remaining energy portion with the other one, satisfying the three-flat condition. Our inspections show that the choice between Ω_{m_0} and Ω_{Λ_0} will not significantly affect the final results.

3. χ^2 vs polynomial fit

Here we employ a polynomial fit of the data as an unbiased reference for our data analysis. On the other hand, χ^2 analysis is the most common method for checking the goodness of fit. Although χ^2 analysis is a powerful method, its result is only a bare number, but polynomial fits can give more information about the data, the information which is directly visual. Thus, it would be useful to examine this method versus the common χ^2 analysis. The common expression for χ^2 is

$$\chi_i^2 = \sum_i \frac{(x_{o_i} - x_{th_i})^2}{\sigma_i^2}, \tag{33}$$

with x_o to be the observed value of the parameter, x_{th_i} to be the theoretical value of x, and σ^2 is the variance of each data point. Basically, the denominator of this expression might also be x_o or x_{th_i} assuming it to be a Pearson or Neyman χ^2, respectively, but the choice of σ^2 is more natural as it will make the data points weighted according to their statistical uncertainty indicated by their error bars.

Calculating a polynomial fit χ^2 [Eq. (33)], we find that the two give very consistent results where applied to each proposed model (here ΛCDM, scale invariant, unparticle, and van Putten). Doing so gives us confidence that the polynomial fit can be applied reliably.

An additional investigation has been done using the best polynomial and models fit within their 1σ band, which turn out to highly overlap each other. This indicates the reliability of the polynomial fit as a reference. It is also possible to calculate χ^2 for the models in comparison with the data that can be manipulated from the polynomial function best fit, which will give the same qualitative result as before.

C. H_0-tension deterioration

H_0-tension can be a powerful diagnostic tool to check the viability of a proposed Hubble evolution $H(z)$, given distinct H_0 values by means of CMB and the one obtained at low redshifts by local distance ladders, namely, $H_0 = 68$ for the first and $H_0 = 73$ for the second [16,17]. The literature present some models that can alleviate H_0-tension mostly with additional fields or interactions [23,85–88]. Here we study $H(z)$ for unparticle cosmology in comparison with the data.

The dataset which we are going to use is a heterogeneous set of tabulated $[z, H(z)]$ data that cover the redshift range $z \in (0, 2)$. As such, it extends well beyond the RC of Hubble parameters of the three models for $z_0 = 0$ mentioned previously. We will keep our analysis inside the RC for reliable results by also considering $z_0 = 1$.

Using the arbitrary polynomial regression over the dataset extracts an unbiased estimate of the parameters. To do so, we performed the analysis using a Taylor series of the Hubble parameter about $z_0 = 0$ satisfying

$$H(z) = H_0 + (1+q_0)z + \frac{1}{2}(Q_0 + q_0(1+q_0))z^2 + b_3 z^3. \tag{34}$$

Fitting this polynomial to the data and with no prior assumptions for the coefficients, the values of H_0, q_0, and Q_0 derived are compared with the model fit to the same data.

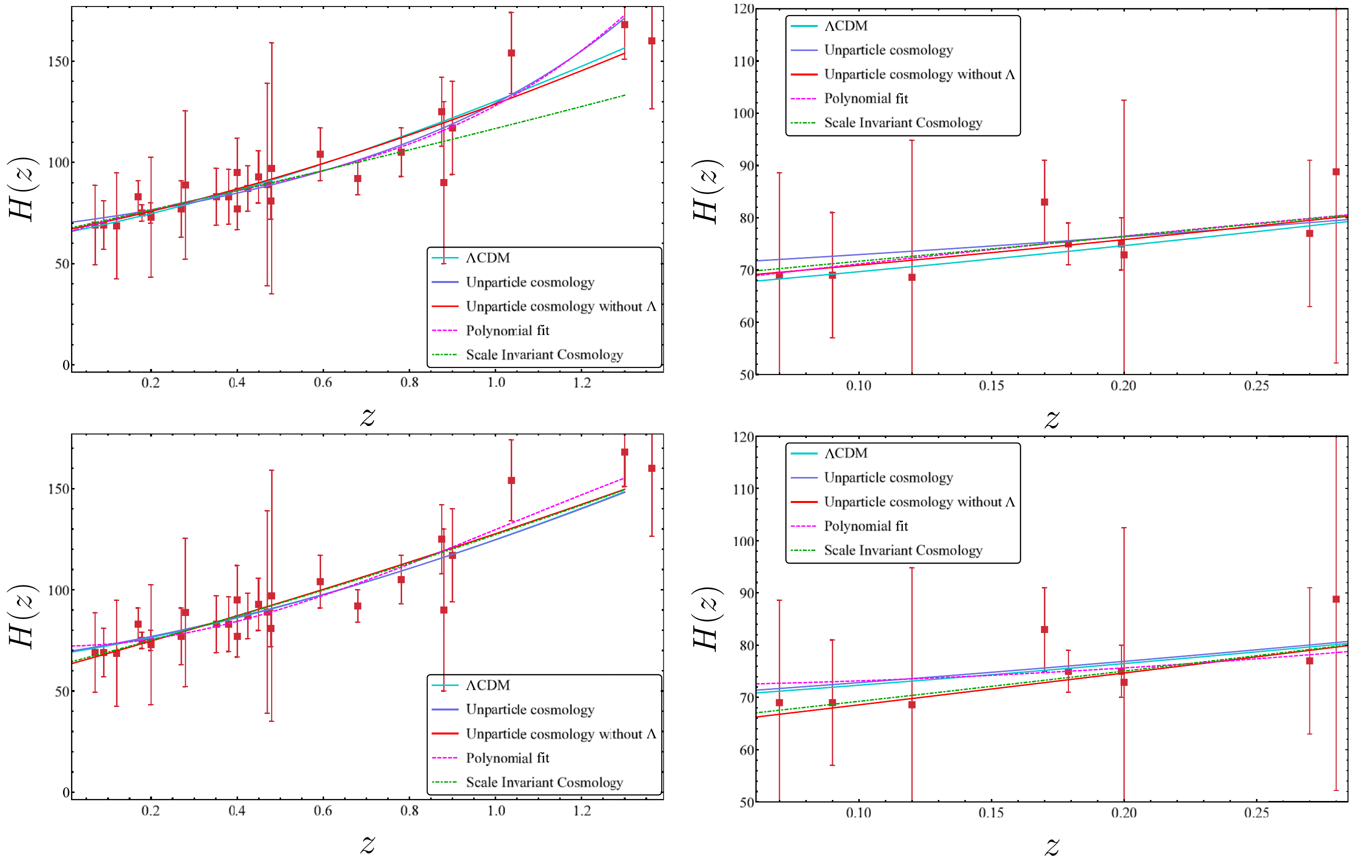

FIG. 8. The evolution of the Hubble parameter for ΛCDM, scale invariant, unparticle cosmology as a perturbation to ΛCDM, and unparticle cosmology in the absence of a cosmological constant, versus the model-independent polynomial fit to the data. Top left panel is for the range of data inside the RC_* for $z_0 = 0$, and the top right panel is an enlarged version of the left one, representing the discrepancy between the polynomial fit and the models more evidently. There is a tendency for the Hubble turning point in the polynomial fit which none of the models are going to follow. The bottom panels represent the results for $z_0 = 1$. The bottom left is the same as the top left with $z_0 = 1$, which covers all the data points and shows a more evident gap between the polynomial fit and the models. The bottom right figure is an enlargement of the left figure.

Figure 8 shows $H(z)$ for the polynomial fit alongside the best fits of ΛCDM, scale invariant, unparticle cosmology as a perturbation to ΛCDM and unparticle in the absence of a cosmological constant, all within their RC. A key observation is that the (unbiased) polynomial fit points to a turning point $[H'(z) = 0]$ near redshift zero. None of these models appear to follow this. Consequently, neither scale invariant cosmology nor unparticle cosmology can relieve H_0-tension, as both suggest $H'(0) > 0$. Though it is notable that for some exceptional values of δ close to -3 in unparticle cosmology, the Hubble parameter diverges in the future. For these values of δ, unparticle cosmology has a turning point and as such might conceivably alleviate H_0-tension somewhat.

Probing the estimated values of H_0 for the range of data with $z \in [0.07, z_{\max}]$ and $0.09 < z_{\max} < 2$ confirms the previous result by depicting that there are deviations between H_0 coming from the models and the value predicted by the data (Fig. 9). The importance of remaining inside the RC is also evident in this figure.

Repeating this analysis with standard errors of H_0 and Ω_{m_0} shows that at lower redshifts, unparticle cosmology is insensitive to the value of Ω_{m_0} in comparison with the dataset. This is a strange feature as we have a three-flat assumption as a prior and unparticle cosmology is *not* expected to be insensitive to the value of Ω_{m_0}.

As δ is the model parameter, it is useful to do the same procedure for the best δ values having $0.09 < z_{\max} < 2$. For unparticle cosmology as a perturbation to ΛCDM, the best fit to $\delta[z_{\max}]$ reveals a preference for

$$\delta \simeq -2.06 \pm 0.46, \tag{35}$$

which is essentially the holographic limit of this model (Fig. 9). We note one outlier with $\delta = -6$ that strongly violates the unitarity [46,47]. Repeating our procedure excluding this data point yields $\delta = -1.7 \pm 0.41$, consistent with Eq. (35). Thus, taking Eq. (8) as an ansatz inherited from unparticles with δ and A to be free parameters, we did a four-parameter fit for unparticle cosmology ($H_0, \delta, \Omega_{\mathcal{U}_0}$, and

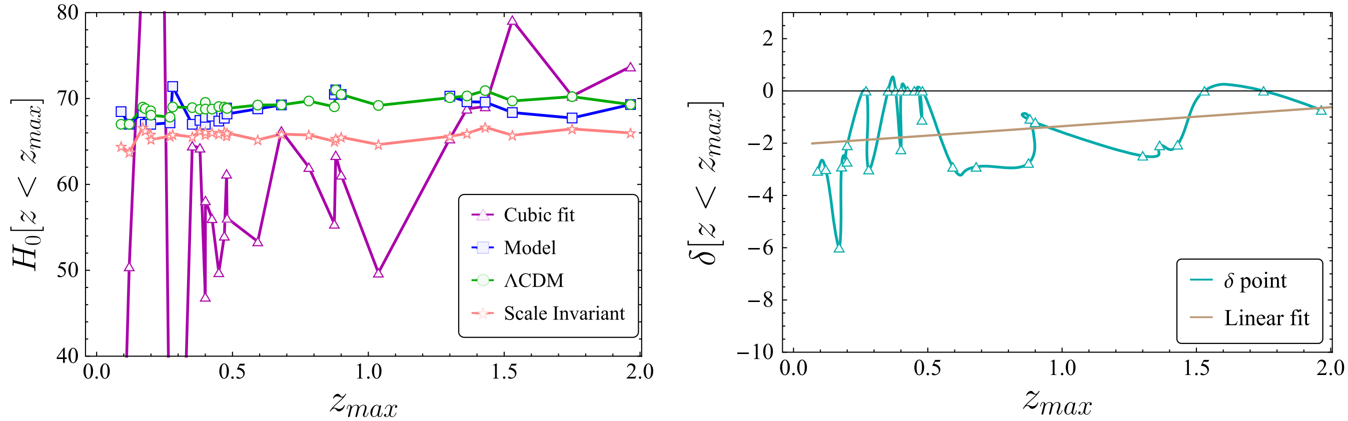

FIG. 9. Estimated values of H_0 (left panel) and δ (right panel) for the best fit of the models with data range $z \in [0.07, z_{\rm max}]$ and $0.09 < z_{\rm max} < 2$ using a model-independent polynomial fit of the data, unparticle cosmology, the scale invariant model, and ΛCDM. The right panel shows the average value of δ with various $z_{\rm max}$ to be $\delta = -2.06 \pm 0.46$, indicating the holographic limit.

Ω_{m_0}). This takes us observationally to $\delta \simeq -2$ in comparison with the data and $\Omega_{\mathcal{U}_0}$ fixes A in Eq. (8). Our results [Eq. (35)] corroborate $c_1 H^2$ in the discussion on entropic gravity (holographic dark energy) with unknown coefficient c_1.

It is different for unparticle cosmology without a cosmological constant giving

$$\delta \simeq -4.8 \pm 0.5 \tag{36}$$

favored by the data, essentially pointing back to ΛCDM. Therefore, it can be inferred that the holographic and ΛCDM limits are two natural boundaries of the theory in our Universe. Note that $\delta = -4$ is equivalent to $d_u = -1$ which is at odds with unitarity mentioned previously.

We summarize our results in a qQ-diagram by data over $z < 2$ (Fig. 10). The figure shows the evolution of $Q(z)$ versus $q(z)$ for the model fits inside their corresponding RC. The models that are included are the unparticle as a perturbation to ΛCDM, the unparticle without Λ, ΛCDM as the standard model, the scale invariant model, and polynomial fits as a reference. We have also added the model which was introduced by van Putten [23] to clarify the effect of the Hubble turning point near $z = 0$. It shows that the models we investigated are quite far from a polynomial fit to the data, while the model with a Hubble turning point appears consistent with the polynomial fit.

In all of the above, unparticle cosmology has been compared with models with only two free parameters, while unparticle cosmology has three free parameters in the absence of a cosmological constant and four if it is considered as a perturbation to ΛCDM. Now let us inspect unparticle cosmology as a semi-two-parameter model by varying the δ value and plot the results in the qQ-diagram. Running $\delta \in [-5, 1]$ results in the qQ-diagram build up of a region that covers the gap between the polynomial fit and the best fit curve of unparticles. So if δ is not really a free parameter, there is a possibility for unparticle cosmology to satisfy the data, but this is in the region which is at odds with unitarity. An interesting feature is that for $\delta = -4$, unparticle cosmology without a cosmological constant falls back to ΛCDM.

Figure 11 illustrates the evolution of the Hubble and deceleration parameter with respect to redshift using different values of δ in the range $-5 < \delta < 1$ for unparticle cosmology both with and without a cosmological constant. For the earlier case, we fixed $\Omega_{\Lambda_0} = 0.744$ to have two free parameters for the model. As $\delta = -3$ has some special features and is the transition value of δ for the stability properties, and because of exotic behaviors in

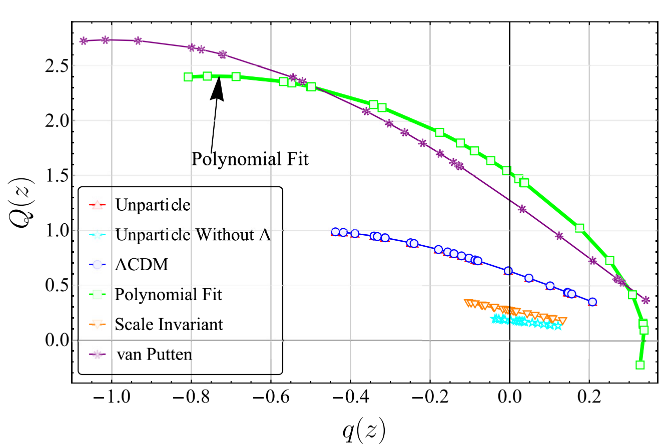

FIG. 10. qQ-diagram for unparticle cosmology, unparticle without Λ, ΛCDM, scale invariant, polynomial fit of the data, and the model proposed by van Putten for using their best fit values. The best fit values for the model parameters applied for this figure have been obtained using the data points inside the radius of convergence for each model with $z_0 = 1$. Apart from the model proposed by van Putten, the other four have obvious distinction with the data demands.

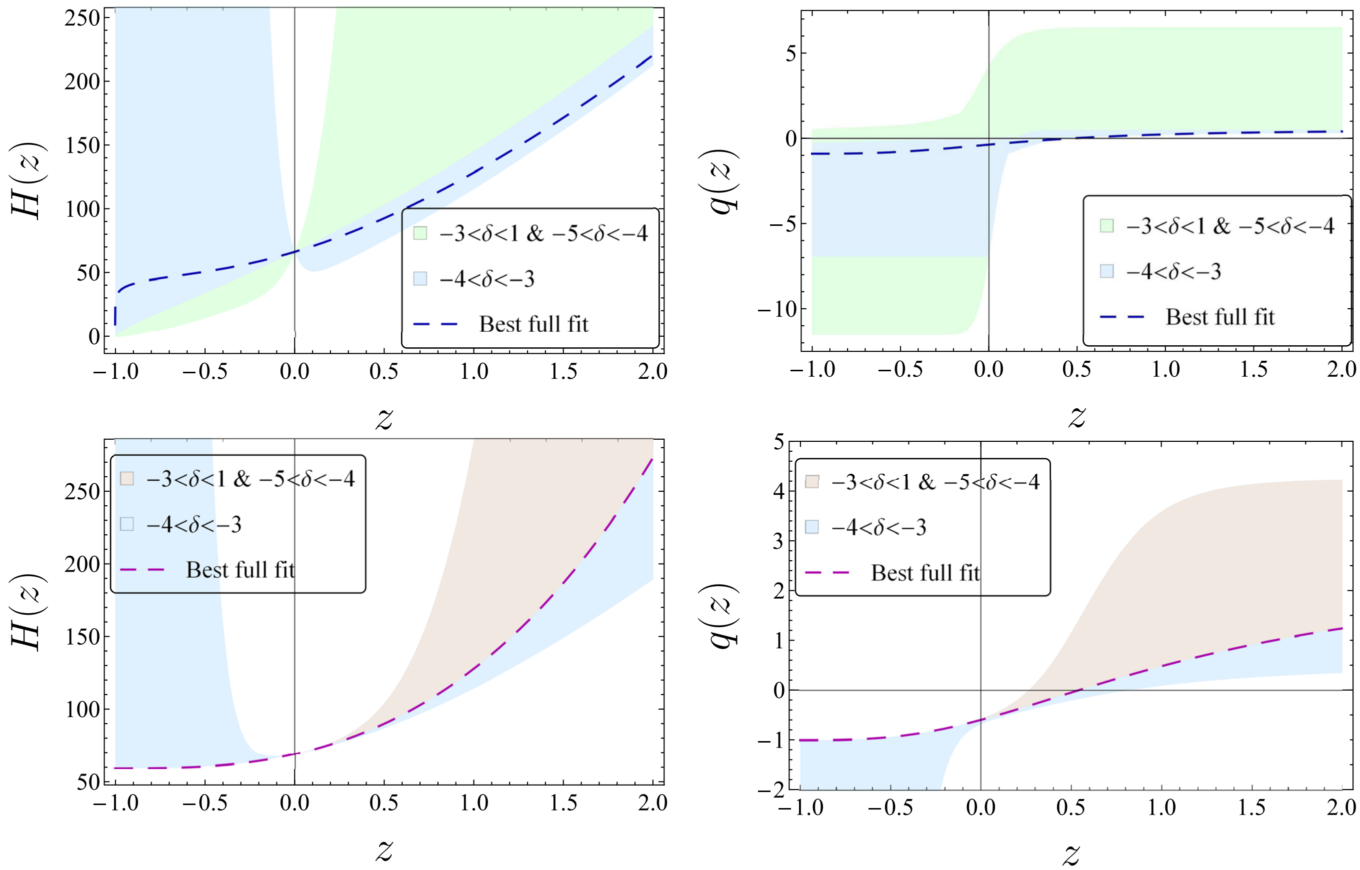

FIG. 11. The evolution of the Hubble parameter $H(z)$ (left column) and deceleration parameter $q(z)$ (right column) with respect to redshift z using $-5 < \delta < 1$, for unparticle cosmology without a cosmological constant (top row) and unparticle cosmology as perturbation to ΛCDM (bottom row). For the latter, we fix $\Omega_{\Lambda_0} = 0.744$. The blue area represents the result for $-5 < \delta < -4$ and $-3 < \delta < 1$, while the brownish (light green) area refers to $-4 < \delta < -3$. The dashed line is the evolution of H_0 and q_0 according to the best fit value of δ when considered to be a free parameter. Although the left panels show turning points in Hubble parameter for $-4 < \delta < -3$, it cannot relax H_0-tension because of violation of unitarity and our observational result on δ shown in Fig. 9.

$-4 < \delta < -3$, the δ interval is divided into $-5 < \delta < -4$, $-4 < \delta < -3$, and $-3 < \delta < 1$. The brownish (light green) area represents the result for $-3 < \delta < 1$ and $-5 < \delta < -4$, while the blue area is to show the results for $-4 < \delta < -3$. Based on this figure, when δ considered to be a running parameter, it will not have a unique value, so there will be an area of solutions for $H(z)$ and $q(z)$. Parts of this solutions may satisfy $H'(0) \simeq 0$, which is consistent with available observational data. On the other hand, if δ is assumed to be a free parameter which must be fitted using the dataset, its best fit values show that the model is not in favor of the observational data (polynomial fit), although it shows a preference for a holographic limit [Eq. (35)] of unparticle cosmology.

VI. CONCLUSION

ΛCDM is best known for its excellent fit to the CMB power spectrum. A key feature that is singled out is Λ, which accounts for accelerated expansion of our Universe observed today. But, with the time passing and observational data improving, serious inconsistencies in the current observations emerged, notably, H_0-tension, the missing satellite problem, the speed of structure formation, etc. These challenges stimulated the development of new ideas to modify or even replace ΛCDM. A number of these ideas seek to modify gravitation, each of them to alleviate at least one of these challenges of ΛCDM.

Here, we provide a mixed theoretical-numerical framework to evaluate a subclass of such models in late-time cosmology according to Fig. 1. As a case study, we apply our framework to unparticle cosmology in comparison with ΛCDM and a scale invariant model of cosmology.

Unparticle and scale invariant models are two of several modified gravity theories satisfying the scale invariant feature at low energies. Scale invariance introduces no cosmological constant, but it can take the role of Λ in late-time evolution. Unparticles, on the other hand, have distinct behavior across an energy scale $\Lambda_{\mathcal{U}}$. The first is supposed to be a candidate to solve the cosmological constant problem, while the second offers a potential explanation for the discrepancy between late- and early-time measurements of

H_0. Here we look at these theories because of their common feature, taking up unparticle cosmology as a special case.

Unparticles were first introduced by Georgi [21] as a low energy phase of the Banks-Zaks field in weak interaction with the standard model particles. Banks-Zaks fields have a nontrivial IR fixed point which causes the scale invariant properties of the unparticle to arise *below* the energy scale $\Lambda_{\mathcal{U}}$. At this scale, the field equations of unparticles can be written by applying Eqs. (8) in Eqs. (12), in which δ is a dimensionless parameter responsible for the scale invariant feature of unparticles and is related to scaling dimension of the unparticles operator by Eq. (7). As negative scaling dimension violates the unitarity of the model, this equation implies that δ cannot be lower than -2. On the other hand, Eq. (8) requires negative δ for unparticles to govern the late-time Universe. Therefore, it would be safer to take $-2 < \delta < 0$. However, as broader range of δ has been considered in the literature, here we include the entire range of $\delta \in [-6, 1]$.

Unparticle cosmology can be considered in two distinct ways: as a perturbation to ΛCDM, in which a small share of total energy density of the Universe can be attributed to unparticles, or as a pure cosmological model devoid of any *ab initio* cosmological constant.

Our investigation of the dynamical behavior of unparticle cosmology shows that the stability of the fixed points of the theory depends on the δ in both cases. Assuming that there is no interaction between unparticles and matter, we find that there are three fixed points for this model (a perturbation to ΛCDM) which are either matter-, unparticle-, or cosmological-constant-dominated ones. For unparticle cosmology without a cosmological constant, the last one is replaced by a radiation-dominated universe. The eigenvalues of the Jacobian matrix at fixed points show that for unparticle cosmology as a perturbation to ΛCDM and for $\delta < -3$, a matter-dominated universe is an unstable fixed point, while the unparticle fixed point is a saddle point, and a cosmological-constant-dominated universe is a stable fixed point. For $\delta > -3$, matter and unparticle fixed points represent exchanges between them. For $-4 < \delta < -3$, the unparticle fixed point is stable at late times and the Λ-dominated universe is a saddle point.

In the case of unparticle cosmology without Λ, for $\delta < -3$, radiation, matter, and unparticle fixed points are unstable, saddle, and stable, respectively. This ordering is in accordance with already known cosmological epochs. But $-3 < \delta < 0$ gives an unparticle-dominated epoch that is unstable, followed by a saddle radiation epoch, and then a stable matter epoch. For positive δ, radiation and unparticle epochs will exchange the position. In this situation, unparticles have no contribution to the late-time Universe. Our results are summarized in Table. I. In both cases, $\delta = -3$ defines a change of stability arising from the definition of unparticle energy density and pressure.

In discussing the stability of fixed points, we note that in all cases the relevant timescale is the Hubble time. Stability and instability of cosmological epochs, therefore, are pertinent to temporal ordering. At the present epoch, however, neither stability nor instability in a Hubble timescale poses a prior constraint. To assess these stability or instabilities precisely, a detailed comparison with the data is required, which is elaborated in the main body of this work beginning from Fig. 1 to obtain the results of Fig. 10.

According to Eq. (21), unparticle cosmology also deviates from standard BNS for all δ. This deviation prevents the theory from being anchored with known theories at BNS (Fig. 5). Although this theory has some exotic features at the early Universe, we defer this to a future study, our focus here being on the late-time Universe.

We compare the model predictions with the dataset provided by Farooq *et al.* [78,79]. We employ a technique different from conventional χ^2 analysis by using an unbiased polynomial fit to indicate a goodness of fit because of its more visual properties. In Sec. V B 3, we confirm that using a polynomial fit gives results in good agreement with that of χ^2 analysis. The RC for the models at hand is also calculated. The RC is important when employing polynomial fits to measure Taylor series coefficients, as the RC fixes the domain of data which is allowed to be taken into account. The pivot point for the RC is considered to be $z_0 = 0$ and $z_0 = 1$ in our calculations, but we present our results mostly based on $z_0 = 1$ as it covers more data points, meaning more observational information.

The number of free parameters is another concern when comparing different models. The ΛCDM, the scale invariant model, and the model proposed in [23] all have two free

TABLE I. Stabilities of fixed points in unparticle cosmology.

Model	δ value	Radiation	Matter	Unparticle	Λ
Unparticle without Λ	$-3 < \delta < 0$	Saddle	Stable	Unstable	⋯
	$0 < \delta < 1$	Unstable	Stable	Saddle	⋯
	$\delta < -3$	Unstable	Saddle	Stable	⋯
Unparticle	$-3 < \delta < 1$	⋯	Saddle	Unstable	Stable
	$-4 < \delta < -3$	⋯	Unstable	Stable	Saddle
	$\delta < -4$	⋯	Unstable	Saddle	Stable

parameters restricted to three-flat cosmologies, but unparticle cosmology is a four-parameter model (three in the absence of a cosmological constant). When comparing these models, it is better if we reduce the number of free parameters to two. Taking δ and Ω_{Λ_0} (Ω_{m_0}) fixed or running (instead of free), it is possible to investigate the theory in two parameter fits. In case of three free parameters, a running δ is sufficient.

Starting with a polynomial fit of the data, one can see that there is a turning point for the Hubble parameter. The existence of this turning point is still uncertain, but if it is confirmed by the future observation, it will be the explanation for H_0-tension [23]. Equations (29) show that the ΛCDM and scale invariant model cannot propose such a turning point, while according to Eqs. (29) and Fig. 11, unparticle cosmology features a turning point for some special values of δ. In contrast, the model proposed in [23] has this feature in good consistency with the data. In Fig. 1, the models that suggest the existence of a turning point is classified in class A.

Figures 8 and 5 represent the evolution of the Hubble parameter of these models in comparison with the data for $0.07 < z < 2$. Based on the results, the turning point of the polynomial fit to the data is estimated to be at $z \cong -0.1$. The scale invariant model has the most weird behavior, as its Hubble parameter is flat at the late-time Universe and drops to zero in the distant future; the limit is a Minkowski space-time. Thus, the scale invariant model does not make a reliable Λ-free gravitational theory. According to Fig. 1, this gravitational model is classified as type C and should be ruled out because of its certain conflicts with the observational data as shown in the qQ-diagram (Fig. 10). ΛCDM is type B with a constant value of H at $z = -1$. Finally, unparticle cosmology behaves differently depending on the value of δ [see Figs. 11(a) and 11(c)]. Unparticle cosmology as a perturbation to ΛCDM is type A or B, while unparticle cosmology without a cosmological constant can take place in each of the three classes. However, this model shows pronounced deviations from the polynomial fit at late time, indicating some inconsistencies with the data.

An interesting feature comes out if we let δ be a free parameter along with H_0 and $\Omega_{\mathcal{U}_0}$. In this case, the holographic limit $\delta \simeq -2$ [Eq. (35)] and the ΛCDM limit $\delta \simeq -4$ [Eq. (36)] will show up as the natural fitted values of δ for unparticles as a perturbation to ΛCDM versus unparticles without Λ, respectively.

Interestingly, the appearance of $\delta \simeq -2$ is persistent for all values of $z_{\max} \in [0.07, 2]$ as shown in the right panel of Fig. 9 [removing the outlier mentioned below Eq. (35) leaves this conclusion essentially unchanged]. However, as δ should be greater than or equal to -2 to avoid violation of unitarity, the holographic limit is the lower bound of unparticle cosmology and any lower value of δ is forbidden. This result implies that the most conventional value of $d_u = 3/2$ is ruled out by a 6.6σ deviation from the best fit value to the data. $\delta \simeq -2$ sets the barotropic index to be $w = 1$, meaning that as presumed, unparticles are matterlike in the best fit to the data used here. Furthermore, $\delta \simeq -4$ is also at odds with unitarity, which brings us back to ΛCDM. The violation of unitarity may, in fact, be the inconsistency anticipated by the swampland conjectures. Therefore, ΛCDM appears to be challenged again.

The unexpected preference of a holographic limit by the data is remarkable, giving credence to the recent studies of alleviating H_0-tension using holographic dark energy. The holographic limit has also been derived for a cosmological constant assuming a stable de Sitter late-time universe [89], but we illustrate that our Universe does not tend to have a stable de Sitter phase. The results of [90] are also remarkable in this regard given the inferred trend of increasing H_0 estimates with bins closer to zero.

An important point is that *ab initio* unparticle theory is local in terms of interactions between various constituents, while holography is a nonlocal theory. Therefore, the appearance of the holographic limit takes us, strictly speaking, out of the realm of unparticle cosmology.

We summarize our findings of our comparison with the data in a qQ-diagram presented in Fig. 10. This figure features a huge gap between the model fits and polynomial fit, representative of the presence or absence of a turning point. It is notable that ΛCDM also represents an obvious discrepancy with the polynomial fit, and this makes the standard model below the gap, in fact, inside the swampland [71].

Current observational data suggest an unstable future for the Universe which, if true, inevitably requires a turning point in Hubble parameter evolution, suggesting class A Hubble evolution. Importantly, ΛCDM implicitly assumes a stable de Sitter phase for the late-time Universe, and this may not hold true. Alternative theories such as unparticle cosmology and scale invariant and many others of types B or C appear inconsistent with the data featuring no turning point. The absence of turning is here interpreted as the origin of H_0-tension. A precise look at Figs. 1 and 10 translates this inconsistency as a significant gap between the qQ-diagram of the models and polynomial fit, the upper boundary of the gap.

Our results show that, although there is a chance for unparticle cosmology to alleviate H_0-tension for special δ values, generally speaking, it cannot be an explanation for H_0-tension. But, as the data show preference for the holographic limit of this model, we conjecture the need to look for a holographic model of dark energy with a turning point indicating an unstable de Sitter phase for the late-time Universe in seeking to alleviate H_0-tension. Furthermore, taking Eq. (8) as an ansatz for the energy density, we find that a comparison with the data shows

$\delta \simeq -2$, equivalent to the common assumption of $c_1 H^2$ for entropic gravity, yet here we identify this by fitting both the coefficient and the exponent. Although the model cannot relax H_0-tension, the appearance of $\delta \simeq -2$, persistent in best fits over $z_{\max} \in [0.07, 2]$ (also $z_{\max} \in [0.07, 1.3]$) is reported here as an unexpected and therefore conceivably powerful hint for holographic dark energy to govern the late-time Universe.

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