Integral representation for three-loop banana graph

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(Received 26 May 2021; accepted 7 September 2021; published 20 October 2021)

It has recently been shown that two-loop kite-type diagrams can be computed analytically in terms of iterated integrals with algebraic kernels. This result was obtained using a new integral representation for two-loop sunset subgraphs. In this paper, we have developed a similar representation for a three-loop banana integral in $d = 2 - 2\varepsilon$ dimensions. This answer can be generalized up to any given order in the ε-expansion and can be calculated numerically both below and above the threshold. We also demonstrate how this result can be used to compute more complex three-loop integrals containing the three-loop banana as a subgraph.

DOI: 10.1103/PhysRevD.104.076017

I. INTRODUCTION

Computing Feynman diagrams, in particular those with masses is one of the most important problems in modern quantum field theory. There are various methods for calculating these integrals,[1] the most effective of which is the differential equations (DE) method [2–8]. The latter is essentially based on the existence of the so-called integration by parts (IBP) relations [9–11], due to which any integral from a given family can be represented as a linear combination of a finite number of master integrals. The number of master integrals is fixed and determined by critical points of the integrand [12] in Feynman or Baikov [13] representation.

Feynman integrals are usually expressed in terms of special functions. The multiple polylogarithms (MPLs) [14,15] are proved to be the most successful here. For MPLs there are many functional dependencies and, which is important, they can be calculated numerically with high precision, see [16,17] and references therein. The system of differential equations for the system of master integral can be solved in terms of MPLs if it can be reduced to the so-called ε-form [18,19], which exists only in certain cases. Here we need to give a little clarification about the ε-form. In this work, we will talk about the ε-form if the kernels of differential equations in $d\log$ form contain only rational arguments. Otherwise, if the arguments are algebraic, then it may not be possible to express the solution in terms of MPLs [20]. However, it is known that not every system of differential equations can be reduced to the ε-form. In these cases, MPLs are no longer sufficient. To solve such equations, it is necessary to involve more complex functions, the simplest functions beyond multiple polylogarithms are the so-called elliptic polylogarithms (eMPLs) [21–43], but wider extensions are also possible [29,44–50].

The purpose of this work is to generalize the functions and methods used to calculate the two-loop sunset graph from the work [50] to the case of the three-loop banana graph with equal masses.[2] The main element of this technique for a two-loop sunset was the use of the Feynman parametrization in order to combine two propagators from three into one. Then a system of differential equations was written for the integrand. Finally, the system of equations is brought into ε-form and solved iteratively. In the three-loop generalization of the technique, we use the Feynman parameterization twice separately for two pairs of propagators. This change was necessary and allows us to reduce the system of differential equations for the obtained integrand to the ε-form, which would not have been possible otherwise. After that, the technique works without any fundamental changes. We restrict ourselves to the case

[1]For a detailed overview of basic methods for computing loop Feynman integrals see [1].

[2]For other methods of calculating banana integrals, see [29,41,46,51,52] and references therein, a similar elliptic integral also occurs when calculating the ρ parameter at three loops, see [53–55] and references therein.

in $d = 2 - 2\varepsilon$ dimensions using the analogy with work [50] where in this case the results were more compact. We also demonstrate with a simple example that the obtained results can be used to compute three-loop diagrams which contain a three-loop banana as a subgraph.

We believe that the results of this work can be of practical use for calculating three-loop corrections to actually measurable processes and quantities. As an example, we think that the developed techniques can be applied to the analytic calculation of the three-loop Higgs-gluon form factor, see [56–58] and references therein, and similarly to the three-loop Higgs-photon form factor in QCD, see [59] and references therein.

The remainder of the paper is organized as follows. In Sec. II, we will explain our notations for iterated integrals that we will frequently use in this paper. Further, in Sec. III we derive a new representation for three nontrivial master integrals from the three-loop banana family. Next, in Sec. IV, we use this representation to compute a simple three-loop integral containing the three-loop banana as a subgraph. Finally, in the last Sec. V, we will draw our conclusions.

II. THE CLASS OF FUNCTIONS

In this section, we shall review the main classes of functions that we will use in subsequent sections.

In this paper we will frequently use functions called MPLs [14,15]. They can be defined recursively:

$$G(a_1, \ldots, a_n; x) = \int_0^x \frac{G(a_2, \ldots, a_n; x')}{x' - a_1} dx', \quad n > 0, \tag{1}$$

where $a_i, x \in \mathbb{C}$, $n \in \mathbb{N}$- is called the weight and the recursion starts with $G(; x) = 1$.

This description is not fully complete because if all a_i are equal to zero then the integral will be divergent. Therefore, it is necessary to add a regularization rule to the definition, it is defined as:

$$G(\underbrace{0, \ldots, 0}_{n}; x) = \frac{\log^n x}{n!}, \tag{2}$$

MPLs are a well-known and widely used class of functions. Among their main advantages is that they obey the Shuffle algebra and the Hopf algebra [60], the latter allows one to find many functional dependencies between them. A more detailed review of MPLs including the related Hopf algebra can be found in [61–63].

We will express the answers for master integrals in a special class of functions similar to the one introduced in the work [50] and called iterated integrals with algebraic kernels. In general, the iterated integrals used in our work will have the form:

$$J(\Psi, \omega_1^s, \ldots, \omega_n^s, \omega_1^x, \ldots, \omega_n^x, \omega_1^\alpha, \ldots, \omega_m^\alpha, \omega_1^{y_1}, \ldots, \omega_l^{y_1}; s) \tag{3}$$

where Ψ is some 2-form in y_1 and y_2 integrated in the limits $y_{1,2} \in [2, \infty]$ the presence of this form is the only noticeable difference from the functions presented in the work [50] where only 1-forms took place. Next, ω^s,ω^x,ω^α, and ω^{y_1}, are some 1-form in s, x, $\alpha = y_2/y_1$, and y_1 respectively and J-functions form iterated integrals in these 1-forms. In this case, the variable x is not independent and depends on the parameters s, y_1 and α as

$$x(s, \alpha, y_1) = \frac{y_1^2(1 + \alpha^2) - s}{2y_1^2\alpha} - \frac{\sqrt{(s - y_1^2(1 - \alpha^2))^2 - 4y_1^4\alpha^2}}{2y_1^2\alpha} \tag{4}$$

Note that since s and x are not independent variables, integrals over s variables can be easily rewritten in terms of integrals over x variables, and actually, we have only 1-forms in three variables x, α, and y_1. The use of 1-forms in s simply makes the results of Sec. IV more compact.

In general, iterated integrals in our results contain the following Ψ and Λ 2-forms ($J_y = \frac{4}{y^2\sqrt{y^2-4}}$):

$$\Psi_{\pm n} = \frac{y_1^2 J_{y_1} J_{y_2} dy_1 dy_2}{(x \mp 1)^n}, \tag{5}$$

$$\Psi_{\pm n}^m = \frac{y_1^2 \alpha^m J_{y_1} J_{y_2} dy_1 dy_2}{(x \mp 1)^n}, \tag{6}$$

$$\Lambda_n = y_1^2 \alpha^n J_{y_1} J_{y_2} dy_1 dy_2 \tag{7}$$

and ω, ζ and η 1-forms:

$$\omega_a^x = \frac{dx}{x - a}, \qquad \omega_b^\alpha = \frac{d\alpha}{\alpha - b}, \tag{8}$$

$$\omega_c^{y_1} = \frac{dy_1}{y_1 - c}, \qquad \omega_a^s = \frac{ds}{s - a}, \tag{9}$$

$$\omega_{a,b}^s = \frac{ds}{(s - a)(x - b)}, \qquad \zeta_{a,b}^s = \frac{ds}{(s - a)(x - b)^2}, \tag{10}$$

$$\eta_{a,b}^s = \frac{ds}{(s - a)(x - b)^3}. \tag{11}$$

Where we tried to choose notations so that they, if possible, coincide with notations from [50].

Here are some examples that explain the structure of these functions.

$$J(\Psi_1,\omega_a^s,\omega_b^x,\omega_c^\alpha,\omega_d^{y_1};s)=\int_2^\infty\int_2^\infty\frac{y_1^2J_{y_1}J_{y_2}dy_1dy_2}{(x-1)}\int_0^s\frac{ds'}{s'-a}\int_0^x\frac{dx'}{x'-b}\int_0^\alpha\frac{d\alpha'}{\alpha'-c}\int_0^{y_1}\frac{dy_1'}{y_1'-d}, \tag{12}$$

$$J(\Psi_1,\zeta^s_{a_1,b_0},\omega^s_{a_2},\omega^x_{b_1},\omega^x_{b_2},\omega^x_{b_3};s)=\int_2^\infty\int_2^\infty\frac{y_1^2J_{y_1}J_{y_2}dy_1dy_2}{(x-1)}\int_0^s\frac{ds'}{(s'-a_1)(x(s')-b_0)^2}\int_0^{s'}\frac{ds''}{s''-a_2}\times\int_0^x\frac{dx'}{x'-b_1}\int_0^{x'}\frac{dx''}{x''-b_2}\int_0^{x''}\frac{dx'''}{x'''-b_3}, \tag{13}$$

$$J(\Psi_1,\omega^s_{a_1},\omega^x_{b_1},\omega^\alpha_{c_1},\omega^\alpha_{c_2};s)=\int_2^\infty\int_2^\infty\frac{y_1^2J_{y_1}J_{y_2}dy_1dy_2}{(x-1)}\int_0^s\frac{ds'}{s'-a_1}\int_0^x\frac{dx'}{x'-b_1}\int_0^\alpha\frac{d\alpha'}{\alpha'-c_1}\int_0^{\alpha'}\frac{d\alpha''}{\alpha''-c_2}, \tag{14}$$

$$J(\Psi_1,\omega^s_{a_1},\omega^s_{a_2},\omega^x_{b_1},\omega^x_{b_2},\omega^x_{b_3};s)=\int_2^\infty\int_2^\infty\frac{y_1^2J_{y_1}J_{y_2}dy_1dy_2}{(x-1)}\int_0^x\frac{dx'}{x'-b_1}\int_0^{x'}\frac{dx''}{x''-b_2}\times\int_0^{y_1}\frac{dy_1'}{y_1'-b_1}\int_0^{y_1'}\frac{dy_1''}{y_1''-b_2}\int_0^{y_1''}\frac{dy_1'''}{y_1'''-b_3}, \tag{15}$$

and regularization occurs in a similar way as in the case of MPLs

$$J(\Psi^m_{\pm n},\underbrace{\omega^x_0,\ldots,\omega^x_0}_{j},\underbrace{\omega^\alpha_0,\ldots,\omega^\alpha_0}_{k},\underbrace{\omega^{y_1}_0,\ldots,\omega^{y_1}_0}_{l};s)=\int_2^\infty\int_2^\infty\frac{y_1^2\alpha^mJ_{y_1}J_{y_2}dy_1dy_2}{(x\mp1)^n}\frac{\log^j(x)\log^k(\alpha)\log^l(y_1)}{j!k!l!}, \tag{16}$$

It is clear from these examples that the presented functions differ from Chen iterated integrals [64] in two ways. First, in our definition, iterated integrations are performed over several variables, and at least three of them are independent. Second, in our definition, the last two integrations are not iterated. Hence, we can conclude that iterated integrals with algebraic kernels are a more complex class of functions than the Chen iterated integrals.

The following considerations can serve as a motivation for using this particular class of functions. First, it was shown in [50] that these functions are more general than eMPLs and can serve as a solution to more complex integrals which cannot be solved in the eMPL class of functions. This was done using the example of a two-loop kite with one massless line and a two-loop kite with all massive lines. Second, as shown in Ref. [50] solutions expressed in terms of this class of functions can be more compact than solutions expressed in terms of eMPLs. The later was demonstrated using the example of a kite-type integral with two massless lines. Of course, this statement applies only to those cases where a solution via eMPLs is possible.

It should also be noted that at the moment the properties of these functions are not fully understood. In our future work we hope to study these functions in more detail.

III. BANANA GRAPH

First let us define the following notation for the master integrals in the elliptic banana family, see Fig. 1.

$$j^{\mathrm{ban}}(a_1,\ldots,a_4)=\frac{e^{3\epsilon\gamma_E}(m^2)^{a-\frac{3}{2}d}}{(i\pi^{d/2})^3}\int\frac{d^dl_1d^dl_2d^dl_3}{D_1^{a_1}D_2^{a_2}D_3^{a_3}D_4^{a_4}}, \tag{17}$$

$$D_1=m^2-l_3^2,\qquad D_2=m^2-(l_2+l_3)^2, \tag{18}$$

$$D_3=m^2-(l_1+l_2)^2,\qquad D_4=m^2-(l_1+p)^2, \tag{19}$$

with $d=2-2\varepsilon$, $s=p^2/m^2$, $a=\sum_{i=1}^4 a_i$ and γ_E is the Euler-Mascheroni constant. Further in this work, to simplify the formulas, we will always assume that $m=1$, bearing in mind that, if necessary, the dependence from m can always be restored based on simple dimensional considerations. The vector of four IBP master integrals obtained as a result of IBP reduction [9–11] can be chosen in the following form:

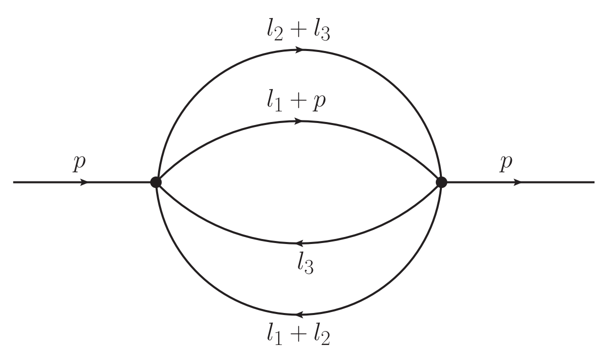

FIG. 1. Banana graph. Thick lines represent massive propagators.

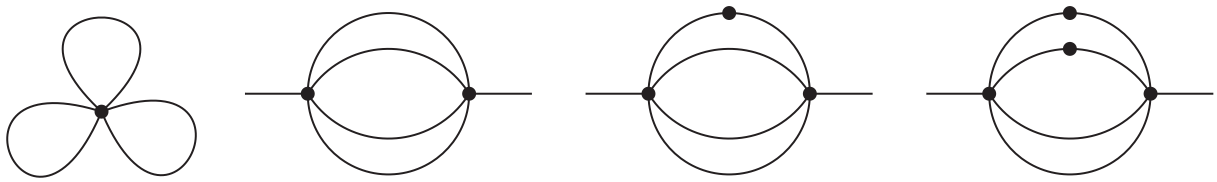

FIG. 2. Set of IBP master integrals for the three-loop banana family. Thick lines represent massive propagators and a dot on a line means that the corresponding propagator is taken in the power two.

$$I_{\mathrm{IBP}} = \{j^{\mathrm{ban}}(1,1,1,0), j^{\mathrm{ban}}(1,1,1,1), j^{\mathrm{ban}}(2,1,1,1), j^{\mathrm{ban}}(2,1,2,1)\}^{\top}, \quad (20)$$

a graphical representation of these master integrals can be found in Fig. 2.

The first master integral is a simple constant and can be expressed analytically in the following form:

$$j^{\mathrm{ban}}(1,1,1,0) = \Gamma(\varepsilon)^3 \quad (21)$$

where $\Gamma(x)$ is the gamma function.

In order to find the other three master integrals we use Feynman parameter trick for pairing two pairs of propagators[3] [50,70–72], and we introduce a new family of integrals defined as ($\bar{t}_{1,2} = 1 - t_{1,2}$)

$$j^{\mathrm{sub}}(b_1, b_2; t_1, t_2) = e^{3\varepsilon\gamma_E} \int \frac{d^d l_1 d^d l_2 d^d l_3}{(i\pi^{d/2})^3} \frac{1}{D_{12}^{b_1} D_{34}^{b_2}}, \quad (22)$$

$$D_{12} = 1 - t_1 l_3^2 - \bar{t}_1 (l_2 + l_3)^2, \quad (23)$$

$$D_{34} = 1 - t_2 (l_1 + l_2)^2 - \bar{t}_2 (l_1 + p)^2, \quad (24)$$

where t_1 and t_2 are Feynman parameters which run through the unit segment ($t_1, t_2 \in [0,1]$), for the convenience of the reader, we will suppress the dependence on parameters t_1 and t_2 in the following and we write instead $j^{\mathrm{sub}}(b_1, b_2; t_1, t_2) = j^{\mathrm{sub}}(b_1, b_2)$. Then the three nontrivial master integrals from (20) can be expressed by integrating the integrals of the family (22) over these parameters, we have

$$j^{\mathrm{ban}}(n,1,m,1) = nm \int_0^1 t_1^{n-1} dt_1 \int_0^1 t_2^{m-1} dt_2 \times j^{\mathrm{sub}}(n+1, m+1). \quad (25)$$

Now we will use the DE method in order to evaluate the necessary integrals from the family (22). The vector of three IBP master integrals can be chosen in such a way that they can be immediately substituted into expression (25) without any IBP reduction

$$I_{\mathrm{IBP}} = \{j^{\mathrm{sub}}(2,2), j^{\mathrm{sub}}(3,2), j^{\mathrm{sub}}(3,3)\}^{\top} \quad (26)$$

To evaluate these master integrals we consider their system of differential equations with respect to the variable x which is associated with the old variable s in the following way

$$s = -\frac{(xy_1 - y_2)(xy_2 - y_1)}{x} \quad (27)$$

where we have introduced new notations

$$y_1 = \frac{1}{\sqrt{t_1(1-t_1)}}, \qquad y_2 = \frac{1}{\sqrt{t_2(1-t_2)}}. \quad (28)$$

With the help of IBP identities the system of differential equations with respect to the variable x can be reduced to the ε-form[4] [18,19] and we have:

$$\frac{d\tilde{I}_{\mathrm{canonical}}}{dx} = \varepsilon \mathcal{M} \tilde{I}_{\mathrm{canonical}} \quad (29)$$

where($\alpha = \frac{y_2}{y_1}$):

$$\mathcal{M} = \frac{1}{x}\mathcal{M}_0 + \frac{1}{x-1}\mathcal{M}_1 + \frac{1}{x+1}\mathcal{M}_{-1} + \frac{1}{x-\alpha}\mathcal{M}_\alpha + \frac{1}{x-1/\alpha}\mathcal{M}_{1/\alpha}, \quad (30)$$

The particular expressions for coefficient matrices $\mathcal{M}_i$ together with transformation matrix to canonical basis T ($I_{\mathrm{IBP}} = T\tilde{I}_{\mathrm{canonical}}$) can be found in the accompanying *Mathematica* notebook [76]. The canonical basis (i.e., the T matrix) after the variable change (27) was obtained with the help of standard Lee algorithm [19,73].

The boundary conditions for (29) at $x = \alpha$ ($s = 0$) can be found by direct integration. Using the Feynman parametrization we find

[3]For an example of using such a trick for nonelliptic cases see [65–69] and references therein.

[4]The subsequent reduction of system of differential equations to ε-form was performed with the use of Libra [73] package and the IBP reduction was performed with the help of LiteRed [74,75] package

$$j^{\mathrm{sub}}(b_1,b_2)|_{s=0} = y_1^{2-2\varepsilon}\alpha^{2(b_2-1+\varepsilon)}\frac{\Gamma(b_1+b_2+3\varepsilon-3)}{\Gamma(b_1)\Gamma(b_2)} \times\int_0^1 \frac{t^{b_1-2+\varepsilon}(1-t)^{b_2-2+\varepsilon}dt}{(t+(1-t)\alpha^2)^{b_1+b_2-3+3\varepsilon}}. \quad (31)$$

For our values of b_1 and b_2 integral (31) is convergent and can be easily calculated in the form of a Laurent series, the results are as follows:

$$j^{\mathrm{sub}}(2,2)|_{s=0} = \frac{y_1^2\alpha^2 G(0;\alpha)}{\alpha^2-1} + \mathcal{O}(\varepsilon), \quad (32)$$

$$j^{\mathrm{sub}}(3,2)|_{s=0} = \frac{y_1^2\alpha^2(\alpha^2-1-2G(0;\alpha))}{2(\alpha^2-1)^2} + \mathcal{O}(\varepsilon), \quad (33)$$

$$j^{\mathrm{sub}}(3,3)|_{s=0} = \frac{y_1^2\alpha^2(\alpha^4-1-4\alpha^2G(0;\alpha))}{4(\alpha^2-1)^3} + \mathcal{O}(\varepsilon). \quad (34)$$

With the boundary conditions available the solution for all master integrals (26) can be found recursively in the regularization parameter ε, after substituting these results into the formula (25) and changing variables from t_1, t_2 to y_1, y_2 the results for nontrivial banana master integrals will be as follows:

$$j^{\mathrm{ban}}(1,1,1,1) = \int_2^\infty\int_2^\infty \frac{4dy_1}{y_1^2\sqrt{y_1^2-4}}\frac{4dy_2}{y_2^2\sqrt{y_2^2-4}}\frac{2\alpha x y_1^2 G(0;x)}{x^2-1} + \mathcal{O}(\varepsilon), \quad (35)$$

$$j^{\mathrm{ban}}(2,1,1,1) = \int_2^\infty\int_2^\infty \frac{4dy_1}{y_1^2\sqrt{y_1^2-4}}\frac{4dy_2}{y_2^2\sqrt{y_2^2-4}}\left[\frac{x^2y_1^2(x^2-2\alpha x+1)G(0;x)}{(x^2-1)^3} + \frac{xy_1^2(\alpha+\alpha x^2-2x)}{2(x^2-1)^2}\right] + \mathcal{O}(\varepsilon), \quad (36)$$

and

$$j^{\mathrm{ban}}(2,1,2,1) = \int_2^\infty\int_2^\infty \frac{4dy_1}{y_1^2\sqrt{y_1^2-4}}\frac{4dy_2}{y_2^2\sqrt{y_2^2-4}}[B_{33}^1+B_{33}^2] + \mathcal{O}(\varepsilon), \quad (37)$$

where

$$B_{33}^1 = \frac{x^3y_1^2(2\alpha+2\alpha x^4-3(\alpha^2+1)x^3+8\alpha x^2-3(\alpha^2+1)x)G(0;x)}{(x^2-1)^5} \quad (38)$$

$$B_{33}^2 = \frac{x^2y_1^2(\alpha^2+(\alpha^2+1)x^4-12\alpha x^3+10(\alpha^2+1)x^2-12\alpha x+1)}{4(x^2-1)^4} \quad (39)$$

or in notations from Sec. II:

$$j^{\mathrm{ban}}(1,1,1,1) = J(\Psi_{-1}^1,\omega_0^x,s) + J(\Psi_1^1,\omega_0^x,s) + \mathcal{O}(\varepsilon), \quad (40)$$

$$\begin{aligned} j^{\mathrm{ban}}(2,1,1,1) = &-\frac{1}{4}J(\Psi_{-3},\omega_0^x,s) + \frac{3}{8}J(\Psi_{-2},\omega_0^x,s) - \frac{1}{8}J(\Psi_{-1},\omega_0^x,s) + \frac{1}{8}J(\Psi_{1},\omega_0^x,s) + \frac{3}{8}J(\Psi_{2},\omega_0^x,s) \\ &+\frac{1}{4}J(\Psi_{3},\omega_0^x,s) - \frac{1}{4}J(\Psi_{-3}^1,\omega_0^x,s) + \frac{3}{8}J(\Psi_{-2}^1,\omega_0^x,s) - \frac{3}{8}J(\Psi_{2}^1,\omega_0^x,s) - \frac{1}{4}J(\Psi_{3}^1,\omega_0^x,s) - \frac{1}{4}J(\Psi_{-2},s) \\ &+\frac{1}{4}J(\Psi_{-1},s) - \frac{1}{4}J(\Psi_{1},s) - \frac{1}{4}J(\Psi_{2},s) - \frac{1}{4}J(\Psi_{-2}^1,s) + \frac{1}{4}J(\Psi_{-1}^1,s) + \frac{1}{4}J(\Psi_{1}^1,s) + \frac{1}{4}J(\Psi_{2}^1,s) + \mathcal{O}(\varepsilon), \end{aligned} \quad (41)$$

$$\begin{aligned} j^{\mathrm{ban}}(2,1,2,1) = &\frac{3}{16}J(\Psi_{-5},\omega_0^x,s) - \frac{15}{32}J(\Psi_{-4},\omega_0^x,s) + \frac{21}{64}J(\Psi_{-3},\omega_0^x,s) - \frac{3}{128}J(\Psi_{-2},\omega_0^x,s) \\ &-\frac{3}{128}J(\Psi_{-1},\omega_0^x,s) + \frac{3}{128}J(\Psi_{1},\omega_0^x,s) - \frac{3}{128}J(\Psi_{2},\omega_0^x,s) - \frac{21}{64}J(\Psi_{3},\omega_0^x,s) - \frac{15}{32}J(\Psi_{4},\omega_0^x,s) \\ &-\frac{3}{16}J(\Psi_{5},\omega_0^x,s) + \frac{3}{8}J(\Psi_{-5}^1,\omega_0^x,s) - \frac{15}{16}J(\Psi_{-4}^1,\omega_0^x,s) + \frac{23}{32}J(\Psi_{-3}^1,\omega_0^x,s) - \frac{9}{64}J(\Psi_{-2}^1,\omega_0^x,s) \end{aligned}$$

$$
\begin{aligned}
&+\frac{9}{64}J(\Psi^1_2,\omega^x_0,s)+\frac{23}{32}J(\Psi^1_3,\omega^x_0,s)+\frac{15}{16}J(\Psi^1_4,\omega^x_0,s)+\frac{3}{8}J(\Psi^1_5,\omega^x_0,s)+\frac{3}{16}J(\Psi^2_{-5},\omega^x_0,s)\\
&-\frac{15}{32}J(\Psi^2_{-4},\omega^x_0,s)+\frac{21}{64}J(\Psi^2_{-3},\omega^x_0,s)-\frac{3}{128}J(\Psi^2_{-2},\omega^x_0,s)-\frac{3}{128}J(\Psi^2_{-1},\omega^x_0,s)+\frac{3}{128}J(\Psi^2_1,\omega^x_0,s)\\
&-\frac{3}{128}J(\Psi^2_2,\omega^x_0,s)-\frac{21}{64}J(\Psi^2_3,\omega^x_0,s)-\frac{15}{32}J(\Psi^2_4,\omega^x_0,s)-\frac{3}{16}J(\Psi^2_5,\omega^x_0,s)+\frac{3}{16}J(\Psi_{-4},s)-\frac{3}{8}J(\Psi_{-3},s)\\
&+\frac{5}{32}J(\Psi_{-2},s)+\frac{1}{32}J(\Psi_{-1},s)-\frac{1}{32}J(\Psi_1,s)+\frac{5}{32}J(\Psi_2,s)+\frac{3}{8}J(\Psi_3,s)+\frac{3}{16}J(\Psi_4,s)+\frac{3}{8}J(\Psi^1_{-4},s)\\
&-\frac{3}{4}J(\Psi^1_{-3},s)+\frac{3}{8}J(\Psi^1_{-2},s)-\frac{3}{8}J(\Psi^1_2,s)-\frac{3}{4}J(\Psi^1_3,s)-\frac{3}{8}J(\Psi^1_4,s)+\frac{3}{16}J(\Psi^2_{-4},s)-\frac{3}{8}J(\Psi^2_{-3},s)\\
&+\frac{5}{32}J(\Psi^2_{-2},s)+\frac{1}{32}J(\Psi^2_{-1},s)-\frac{1}{32}J(\Psi^2_1,s)+\frac{5}{32}J(\Psi^2_2,s)+\frac{3}{8}J(\Psi^2_3,s)+\frac{3}{16}J(\Psi^2_4,s)+\mathcal{O}(\varepsilon).
\end{aligned}
\tag{42}
$$

Results for master integrals (20) up to $\mathcal{O}(\varepsilon^2)$ corrections can be found in the accompanying *Mathematica* file [76].

Integrals in Eqs. (35), (36) and (37) as well as higher ε corrections can be taken numerically, for this it is convenient to change the integration variables $y_{1,2}\to iy_{1,2}$ and change the contour of integration to $y_{1,2}\in[2i,-\infty]$, see Fig. 3.

After this trick, all double integrals can be calculated using standard methods, one can simply use the NIntegrate function that is implemented in the Wolfram *Mathematica* software. To numerically calculate the MPLs that are present in the integrand, we used the handyG package [17]; for greater accuracy, one can also use the GiNaC package [16,77]. We want to note that the numerical calculation goes much faster than with the usage of the sector decomposition method [78–84] as implemented in the FIESTA package [85], for example, the gain in speed reaches almost two orders of magnitude in the calculation of ε^0 corrections.[5] For higher ε-corrections, the speed gain is not so significant, but it also takes place. The example of numerical integration and their comparison with the sector decomposition method for the ε^0 corrections and ε^1 correction for $j^{\mathrm{ban}}(1,1,1,1)$ integral can be found in Figs. 4 and 5. Similar pictures can be drawn for higher ε-corrections, but they would greatly clutter up the text.

At the end of this chapter, we would like to give a more detailed discussion of our results and compare them with other solutions, in particular with the results obtained in [41]. In particular, in [41], there are two fundamental points to which we would like to draw attention. First, the technique presented in [41] allows one to obtain solutions for the integrals in the banana family only up to the order ε^0 in the regularization parameter. To obtain higher corrections with respect to ε an additional nontrivial work is required, if at all possible within the framework of this technique. Nevertheless, for the master integrals, we would like to have a technique that allows us to get an answer to any given order in ε. This is important because we do not know in advance how these master integrals will be included in the expressions for the final measurable quantities. Second, the results obtained in Ref. [41] are applicable only to a three-loop banana and we do not see a way, at least obvious one, how these results can be directly used to calculate more complex three-loop integrals containing a three-loop banana as a subgraph. And such integrals should undoubtedly appear in more complex practical calculations. Therefore, we would like to be able to use solutions for banana subgraphs for calculation of more complex integrals. This was done, for example, with a two-loop sunset graph in [50]. The method we have developed in this article enabled us to overcome the above difficulties. First of all, results (40), (41), and (42) are obtained by solving a system of differential equations (29) which are in canonical form in the sense as understood in [18,19]. This means that we can obtain solutions up to any predetermined order in the regularization parameter ε simply by recursively solving Eq. (29). This does not

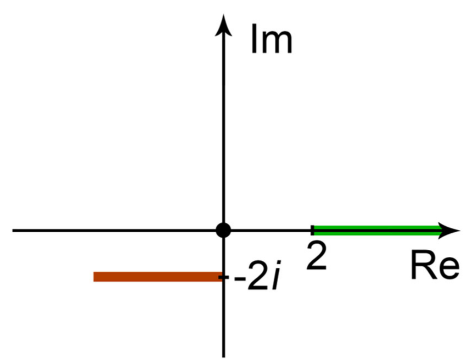

FIG. 3. Change of the integration contour, green—old integration contour, red—new integration contour. Note, that we do not just replace $y_{1,2}\to iy_{1,2}$, we also additionally deform the integration contour itself so that the integration goes to $-\infty$ instead of $-i\infty$, such deformation is possible since the integration contour lying entirely at infinity makes a zero contribution to the integral.

[5]These results may vary depending on the characteristics of the computer.

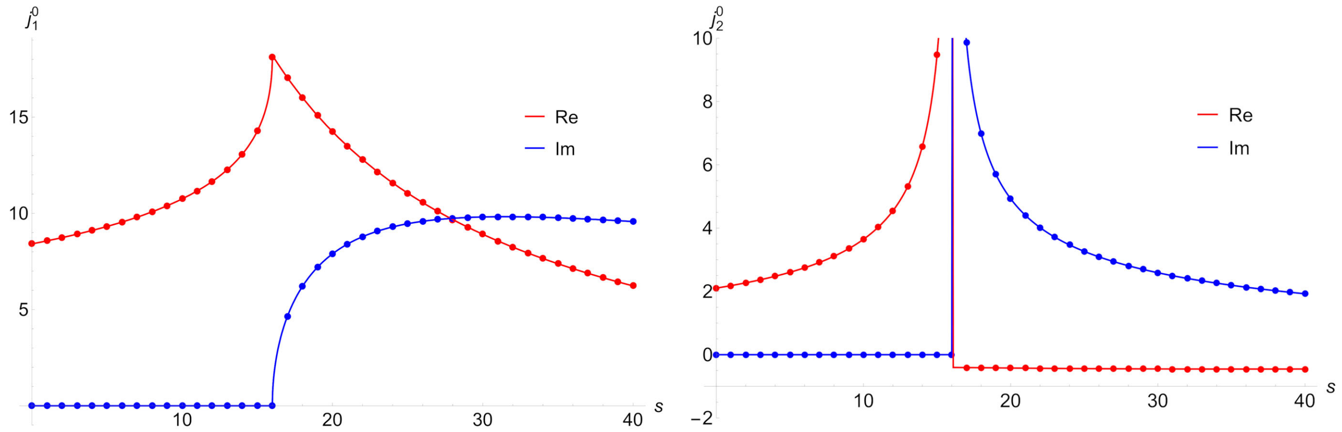

FIG. 4. Plot of the ε^0 correction to the $j^{\mathrm{ban}}(1,1,1,1)$ integral on the left and ε^0 correction to the $j^{\mathrm{ban}}(2,1,1,1)$ integral on the right. The solid points represent values computed numerically with the FIESTA package [85]. Here, for convenience, we have introduced shortened notations $j^{\mathrm{ban}}(1,1,1,1) = j_1^0 + j_1^1\varepsilon + \mathcal{O}(\varepsilon^2)$ and $j^{\mathrm{ban}}(2,1,1,1) = j_2^0 + \mathcal{O}(\varepsilon)$.

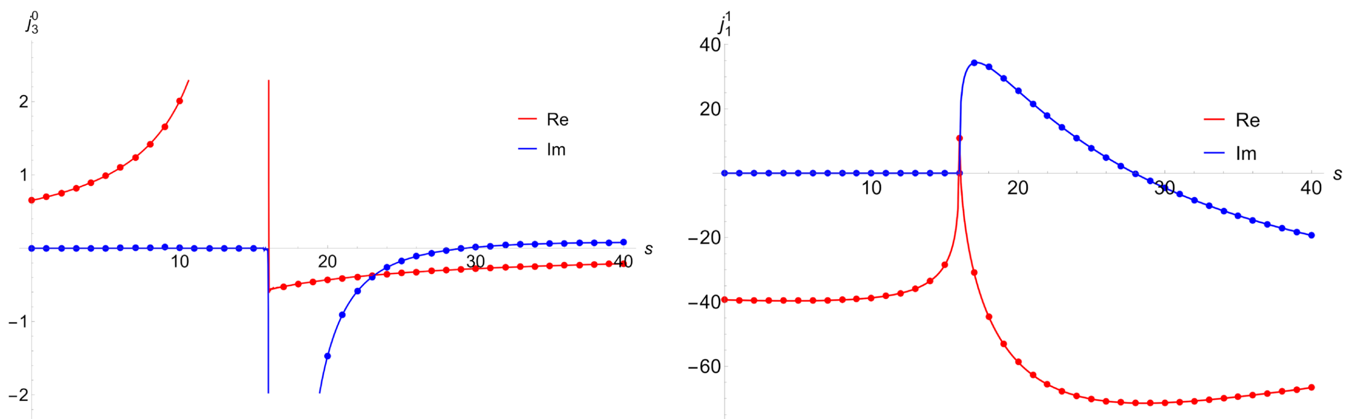

FIG. 5. Plot of the ε^0 correction to the $j^{\mathrm{ban}}(2,1,2,1)$ integral on the left and ε^1 correction to the $j^{\mathrm{ban}}(1,1,1,1)$ integral on the right. The solid points represent values computed numerically with the FIESTA package [85]. Here, for convenience, we have introduced shortened notations $j^{\mathrm{ban}}(1,1,1,1) = j_1^0 + j_1^1\varepsilon + \mathcal{O}(\varepsilon^2)$ and $j^{\mathrm{ban}}(2,1,2,1) = j_3^0 + \mathcal{O}(\varepsilon)$.

require the introduction of any new methods and is quite simple computationally. The only thing we need is the initial conditions, but they can be easily obtained from (31) to any order in ε. Further, results (40), (41), and (42) as well as all higher ε-corrections have a special structure. The latter is related to the fact that the kinematic variable s is contained only at the upper limit of integration and in the Ψ 2-form.[6] This property allows us to use the obtained results to calculate more complex three-loop integrals. An example of such calculation will be given in the next section. It should also be noted that our results, at least for the integrals $j^{\mathrm{ban}}(1,1,1,1)$ and $j^{\mathrm{ban}}(2,1,1,1)$, are written in a relatively compact form. The latter can greatly help in understanding of the results.

[6]The kinematic variable s is hidden in the variable x through relation (27) and does not appear in any other form in the results (40), (41), and (42) as well as in higher ε corrections.

Unfortunately, we cannot say that our results are perfect. The main advantage of [41], in our opinion, is the fact that these results are presented in the form of iterated integrals of modular forms which are a well-established class of functions [36,86–88]. Iterated integrals of modular forms that was used in [41] also can be rewritten in the form of eMPls. The latter are also a well-studied class of functions [36–39]. In contrast, the functions that was used in our work were previously presented only in [50] and require additional study.

IV. TRIANGLE WITH TWO MASSIVE LOOPS

In the previous section, we obtained an integral representation for the three-loop banana family; in this section, we will show how this representation can be used to compute more complex Feynman integrals. For this purpose, we will use the family associated with the triangle with two massive loops defined as (see Fig. 6):

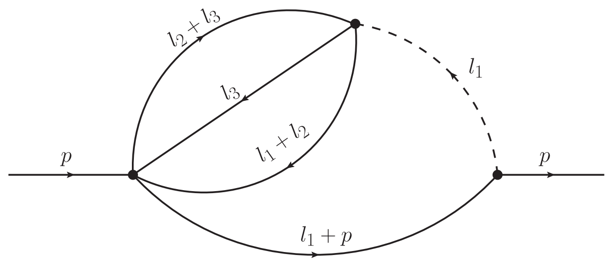

FIG. 6. Triangle with two massive loops. Dashed lines denote massless propagators and thick lines represent massive propagators.

$$j^{\mathrm{tri}}(a_1,\ldots,a_5) = \frac{e^{3\varepsilon\gamma_E}(m^2)^{a-\frac{3}{2}d}}{(i\pi^{d/2})^3}\int\frac{d^dl_1d^dl_2d^dl_3}{D_1^{a_1}D_2^{a_2}D_3^{a_3}D_4^{a_4}D_5^{a_5}} \quad (43)$$

$$D_1 = m^2 - l_3^2, \qquad D_2 = m^2 - (l_2+l_3)^2, \quad (44)$$

$$D_3 = m^2 - (l_1+l_2)^2, \qquad D_4 = m^2 - (l_1+p)^2, \quad (45)$$

$$D_5 = -l_1^2, \quad (46)$$

with $d = 4 - 2\varepsilon$, $s = p^2/m^2$, $a = \sum_{i=1}^5 a_i$ and γ_E is the Euler-Mascheroni constant. Further, we will set $m = 1$ exploiting the same considerations as in the previous section. The vector of seven IBP master integrals obtained as a result of IBP reduction [9–11] together with dimension recurrence relations [89] can be chosen in the following form:

$$I_{\mathrm{IBP}} = \{j^{\mathrm{tri}}(0,2,2,2,0), j^{\mathrm{tri}}(0,2,2,2,1), j^{\mathrm{tri}}(1,1,1,0,1), j^{\mathrm{ban}}(1,1,1,1), j^{\mathrm{ban}}(2,1,1,1), j^{\mathrm{ban}}(2,1,2,1), j^{\mathrm{tri}}(2,2,1,1,1)\}^{\top}, \quad (47)$$

a graphical representation of these master integrals can be found in Fig. 7. Note that we are using the expressions for the three-loop banana graph in *two* dimensions as basis elements for the triangle with two massive loops family in *four* dimensions. This can be done because the master integrals in two and four dimensions are not independent but are related to each other by linear dependencies. These linear dependencies are called dimension recurrence relations [89]. In other words, we choose three linear combinations of four-dimensional integrals as three IBP master integrals so that these combinations are exactly equal to three nontrivial two-dimensional master integrals from the three-loop banana family.

The first and third master integrals are simple constants and can be written as

$$j^{\mathrm{tri}}(0,2,2,2,0) = \Gamma(\varepsilon)^3 \quad (48)$$

and

$$j^{\mathrm{tri}}(1,1,1,0,1) = \frac{1}{\varepsilon^3} + \frac{15}{4\varepsilon^2} + \frac{65+2\pi^2}{8\varepsilon} + \left(\frac{135}{16} - \zeta_3 + \frac{45}{8}\zeta_2 + \frac{81}{4}S_2\right) + \mathcal{O}(\varepsilon) \quad (49)$$

where $S_2 = \frac{4}{9\sqrt{3}}\mathrm{Cl}_2(\frac{\pi}{3})$, $\mathrm{Cl}_2(x) = \mathrm{Im}(\mathrm{Li}_2(e^{ix}))$ and $\mathrm{Li}_2(x)$ is the dilogarithm. And the integrals $j^{\mathrm{ban}}(1,1,1,1)$, $j^{\mathrm{ban}}(2,1,1,1)$, and $j^{\mathrm{ban}}(2,1,2,1)$ were found in the previous section.

To evaluate the remaining master integrals we consider their system of differential equations with respect to $p^2 = s$. Using balance transformations of [19] via the package [73] the latter can be reduced to the following $A + B\varepsilon$ form:

$$\frac{d\tilde{I}_{\mathrm{canonical}}}{ds} = \mathcal{A}\tilde{I}_{\mathrm{canonical}} \quad (50)$$

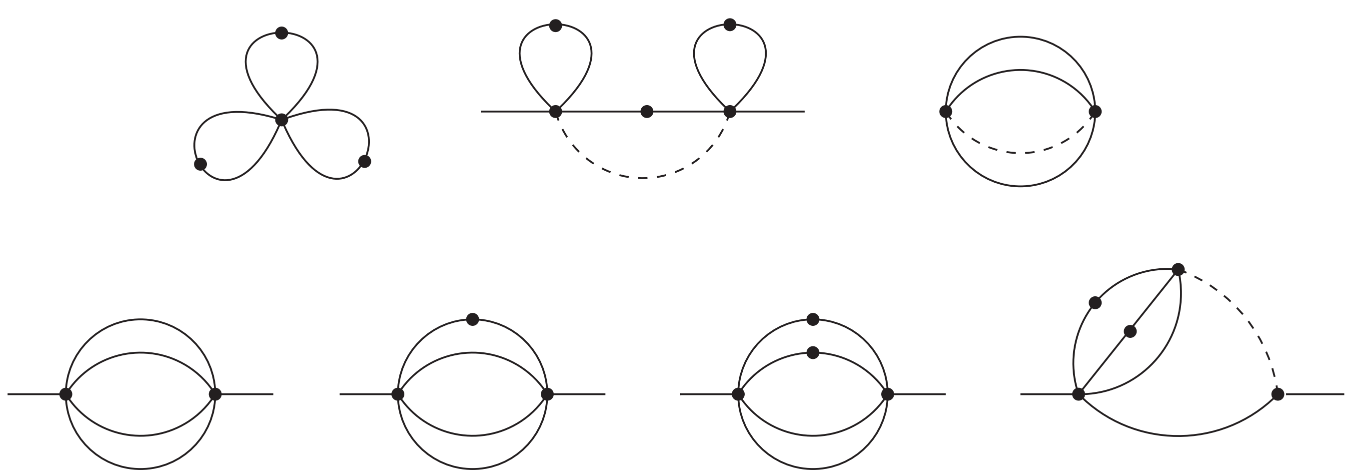

FIG. 7. Set of IBP master integrals for the family (43). Dashed lines denote massless propagators and thick lines represent massive propagators. A dot on a line means that the corresponding propagator is taken in the power two.

with

$$\mathcal{A} = \frac{1}{s}\mathcal{A}_0 + \frac{1}{s-1}\mathcal{A}_1 + \frac{1}{s-4}\mathcal{A}_4 + \frac{1}{s-16}\mathcal{A}_{16}, \quad (51)$$

and

$$\mathcal{A}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3\epsilon-1 & 4(4\epsilon+1) & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4}(-3\epsilon-1) & 4\epsilon+1 & 0 & 0 \\ \frac{\epsilon}{8} & 0 & 0 & -\frac{3}{32}(2\epsilon+1) & \frac{5}{8}(2\epsilon+1) & -\epsilon-1 & 0 \\ 0 & 0 & 0 & \frac{5}{12}(4\epsilon+1) & -\frac{5}{3}(4\epsilon+1) & 0 & \epsilon \end{pmatrix}, \quad (52)$$

$$\mathcal{A}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\epsilon & -2\epsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{6}(-11\epsilon-2) & \frac{5}{3}(4\epsilon+1) & 0 & -2\epsilon \end{pmatrix}, \quad (53)$$

$$\mathcal{A}_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4}(3\epsilon+1) & -2(3\epsilon+1) & 3(3\epsilon+1) & 0 \\ 0 & 0 & 0 & \frac{1}{8}(2\epsilon+1) & -2\epsilon-1 & \frac{3}{2}(2\epsilon+1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (54)$$

$$\mathcal{A}_{16} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\epsilon}{8} & 0 & 0 & \frac{1}{32}(-2\epsilon-1) & \frac{3}{8}(2\epsilon+1) & -\frac{3}{2}(2\epsilon+1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (55)$$

And the elements of the canonical basis $\tilde{I}_{\mathrm{canonical}} = \{I_1, \ldots, I_7\}^\top$ are related to the elements of the IBP basis (7) as

$$I_1 = \varepsilon^2 j^{\mathrm{tri}}(0,2,2,2,0), \quad (56)$$

$$I_2 = s\varepsilon^2 j^{\mathrm{tri}}(0,2,2,2,1), \quad (57)$$

$$I_3 = j^{\mathrm{tri}}(1,1,1,0,1), \quad (58)$$

$$I_4 = (1+3\varepsilon)(1+4\varepsilon) j^{\mathrm{ban}}(1,1,1,1), \quad (59)$$

$$I_5 = (1+3\varepsilon) j^{\mathrm{ban}}(2,1,1,1), \quad (60)$$

$$I_6 = j^{\mathrm{ban}}(2,1,2,1) \quad (61)$$

and

$$I_7 = \frac{(1+3\varepsilon)(1+4\varepsilon)}{s-1}\left(\frac{5}{12} j^{\mathrm{ban}}(1,1,1,1) - s(1-2\varepsilon) j^{\mathrm{tri}}(2,2,1,1,1)\right). \quad (62)$$

Having obtained the differential system in this form it is easy to see, that the solution for required master integrals $j^{\mathrm{tri}}(0,2,2,2,1)$ and $j^{\mathrm{tri}}(2,2,1,1,1)$ can be obtained recursively in the regularization parameter ε similarly to what one typically does for differential systems reduced to ε-form. Of course, of greatest interest is the solution for the integral $j^{\mathrm{tri}}(2,2,1,1,1)$ which can be written through the J-functions from the Appendix

$$\begin{aligned} j^{\mathrm{tri}}(2,2,1,1,1) = \frac{s-1}{12s}\Big[& -5J(\Lambda_0, \zeta^s_{0,-1}, s) - 5J(\Lambda_0, \zeta^s_{0,1}, s) + 5J(\Lambda_0, \zeta^s_{1,-1}, s) + 5J(\Lambda_0, \zeta^s_{1,1}, s) \\ & - 5J(\Lambda_1, \zeta^s_{0,-1}, s) + 5J(\Lambda_1, \zeta^s_{0,1}, s) + 5J(\Lambda_1, \zeta^s_{1,-1}, s) - 5J(\Lambda_1, \zeta^s_{1,1}, s) + 5J(\Lambda_0, \omega^s_{0,-1}, s) \\ & - 5J(\Lambda_0, \omega^s_{0,1}, s) - 5J(\Lambda_0, \omega^s_{1,-1}, s) + 5J(\Lambda_0, \omega^s_{1,1}, s) + 5J(\Lambda_1, \omega^s_{0,-1}, s) + 5J(\Lambda_1, \omega^s_{0,1}, s) \\ & - 5J(\Lambda_1, \omega^s_{1,-1}, s) - 5J(\Lambda_1, \omega^s_{1,1}, s) + \frac{15}{2} J(\Lambda_0, \zeta^s_{0,-1}, \omega^x_0, s) + \frac{15}{2} J(\Lambda_0, \zeta^s_{0,1}, \omega^x_0, s) \\ & - \frac{15}{2} J(\Lambda_0, \zeta^s_{1,-1}, \omega^x_0, s) - \frac{15}{2} J(\Lambda_0, \zeta^s_{1,1}, \omega^x_0, s) + \frac{15}{2} J(\Lambda_1, \zeta^s_{0,-1}, \omega^x_0, s) - \frac{15}{2} J(\Lambda_1, \zeta^s_{0,1}, \omega^x_0, s) \end{aligned}$$

$$
\begin{aligned}
&-\frac{15}{2}J(\Lambda_1,\zeta^s_{1,-1},\omega^x_0,s)+\frac{15}{2}J(\Lambda_1,\zeta^s_{1,1},\omega^x_0,s)-5J(\Lambda_0,\eta^s_{0,-1},\omega^x_0,s)+5J(\Lambda_0,\eta^s_{0,1},\omega^x_0,s)\\
&+5J(\Lambda_0,\eta^s_{1,-1},\omega^x_0,s)-5J(\Lambda_0,\eta^s_{1,1},\omega^x_0,s)-5J(\Lambda_1,\eta^s_{0,-1},\omega^x_0,s)-5J(\Lambda_1,\eta^s_{0,1},\omega^x_0,s)\\
&+5J(\Lambda_1,\eta^s_{1,-1},\omega^x_0,s)+5J(\Lambda_1,\eta^s_{1,1},\omega^x_0,s)-\frac{5}{2}J(\Lambda_0,\omega^s_{0,-1},\omega^x_0,s)+\frac{5}{2}J(\Lambda_0,\omega^s_{0,1},\omega^x_0,s)\\
&+\frac{5}{2}J(\Lambda_0,\omega^s_{1,-1},\omega^x_0,s)-\frac{5}{2}J(\Lambda_0,\omega^s_{1,1},\omega^x_0,s)-5J(\Lambda_1,\omega^s_{0,-1},\omega^x_0,s)-5J(\Lambda_1,\omega^s_{0,1},\omega^x_0,s)\\
&+4J(\Lambda_1,\omega^s_{1,-1},\omega^x_0,s)+4J(\Lambda_1,\omega^s_{1,1},\omega^x_0,s)+\frac{5J(\Psi^1_{-1},\omega^x_0,s)}{s-1}+\frac{5J(\Psi^1_1,\omega^x_0,s)}{s-1}\Bigg]+\mathcal{O}(\varepsilon) \qquad (63)
\end{aligned}
$$

For reference, we also present the result for the second master integral, which can be expressed in terms of usual MPLs:

$$
\begin{aligned}
j^{\mathrm{tri}}(0,2,2,2,1)=&-\frac{G(1,s)}{s\varepsilon^2}+\frac{2G(1,1,s)-G(0,1,s)}{s\varepsilon}\\
&-\frac{\pi^2G(1,s)}{4s}-\frac{G(0,0,1,s)}{s}\\
&+\frac{2G(0,1,1,s)}{s}+\frac{2G(1,0,1,s)}{s}\\
&-\frac{4G(1,1,1,s)}{s}+\mathcal{O}(\varepsilon) \qquad (64)
\end{aligned}
$$

Note, that with the use of the presented procedure we can have as many terms in ε expansion of considered master integrals as required.

Unfortunately, the method of numerical evaluation of these functions that was applied in previous section does not work for the case of a triangle with two massive loops. The reason for this is the appearance of additional singularities, for example, along the line $y_1 = y_2$ and the similar ones. Nevertheless, we were able to verify the results numerically below the threshold using the CUBA package [90]. In the future, we hope to develop a methodology for calculating these functions similar to the one for conventional MPLs [16].

We believe that the technique presented in this chapter can be useful for calculating three-loop integrals that contain a three-loop banana integral as a subgraph. This technique definitely works for the integrals discussed in this paper. As for more complex cases, this issue requires additional study.

V. CONCLUSION

In this paper, we have obtained a new representation for the three-loop equal-mass banana graph in $d = 2 - 2\varepsilon$ dimensions. These results are written in terms of new functions defined as iterated integrals with algebraic kernels. These functions have already been used earlier in [50] to compute the two-loop sunset diagram as well as the massive kite diagram. Our work can be seen as a straightforward generalization of techniques from [50] to the three-loop case. The obtained representation for the three-loop banana graph can be used to calculate some more complex three-loop graphs, we have illustrated the last statement by using the example of the triangle with two massive loops. In our work, we also carried out a comparative analysis of our results with those of [41] and showed that our results have both advantages and disadvantages over these results. The analytical results for the three-loop banana can be numerically calculated with good accuracy both above and below the threshold and are in agreement with the sector decomposition method [78–84] as implemented in [85]. The result for a triangle with two massive loops was verified numerically only below the threshold and its analytical continuation above it will be the subject of our future research. All main results of this work can be found in digital form in the supplemental materials [76].

ACKNOWLEDGMENTS

I would like to thank A. I. Onishchenko for interesting and stimulating discussions as well as for general guidance in writing this work. The work was supported by the Foundation for the Advancement of Theoretical Physics and Mathematics “BASIS”.

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