Hopf algebra structure of the two loop three mass nonplanar Feynman diagram

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The method of using Hopf algebras for calculating Feynman integrals developed by Abreu *et al.* is applied to the two-loop nonplanar on-shell diagram with massless propagators and three external mass scales. We show that the existence of the method of cut Feynman diagrams comprising of the coproduct, the first entry condition and integrability condition that was found to be true for the planar case also holds for the nonplanar case; furthermore, the nonplanar symbol alphabet is the same as for the planar case. This is one of the main results of this work which have been obtained by a systematic analysis of the relevant cuts, using the symbolic manipulation codes HypExp and PolyLogTools. The obtained result for the symbol is cross-checked by an analysis of the known two-loop original Feynman integral result. In addition, we also reconstruct the full result from the symbol. This is the second main result of this paper.

I. INTRODUCTION

In a series of publications [1–4], Abreu *et al.* have developed the study of cut Feynman diagrams with the intention to exploit generalized unitarity, as encoded in the corresponding Hopf algebras, to calculate Feynman integrals in those cases where the final results and the corresponding cuts can be expressed by multiple poly-logarithms. In fact, using the properties of the Hopf algebra, the Feynman diagrams one seeks can be reconstructed from the cut Feynman integrals without the need to perform a tedious dispersion integral, although, in practice, there is still a considerable calculational effort required even for the evaluation of the cut diagrams. In passing, we note that in the case of, say, elliptic polylogarithms (e.g., the two-loop sunset), there is as yet no known (Hopf) algebraic structure that would simplify the dispersion integral. In an important paper, Duhr [5] has argued that for the large class of Feynman that result in multiple polylogarithms, the method of Hopf algebras can be applied. It is not trivial that abstract mathematical objects like Hopf algebras can be found in a real physical system and it is of great interest to further study the extent to which the new method applies in order to discover new calculational tools. The work described in this paper indeed intends to contribute further to the method of using Hopf algebras in the context of Feynman integrals.

While at the one-loop level, the calculations are straightforward, the corresponding diagrams also exhibit important and useful mathematical properties such as the integrability condition and the first-entry condition made evident by the new formalism. These are indeed related to the discovery that Feynman integrals obey Hopf Algebras, which plays an important role in the new method. At two-loop and higher orders, the level of complexity is much higher in terms of the number of integrations for a particular cut integral and the existence of divergences. The appearance of the delta function is what is particular to the cut method (Cutkosky condition). The general advantage of the method is that one effectively computes lower-loop diagrams, and integrations of cut diagrams are simpler.

In order to set up the formalism, the authors introduced and developed a methodology that is encoded in the following concepts: multiple polylogarithms, Hopf algebra, symbol alphabet, integrability condition, first-entry condition, and coproduct. In the next section, each of the relevant terms is explained. In Ref. [1], besides several one-loop examples, the two-loop (planar) ladder diagram is considered. This diagram, which has been studied in the past and has received a lot of attention, is a very useful example for illustrative purposes. It is finite and can be expressed in closed form in 4 dimensions [6]. The calculation using the new method involves several steps: First of all, the total cut diagram for any one channel (of the external momenta), which is a sum of all the possible cuts (the cut propagator momenta add up to the channel momentum), needs to be evaluated for that channel. Then, using this first result, the

maximal iteration of the coproduct or the symbol is evaluated using the first-entry condition and the integrability condition. Finally, the original Feynman integral is reconstructed by finding the class of Feynman integrals, which can give rise to the symbol. This is done by explicitly singling out the possible expressions which can give rise to each of the terms in the symbol and finally adding them up to get the class.

In ref. [1] it is pointed out that no non-planar diagram has been evaluated by the authors using the present Hopf algebra based method. A non-planar "ladder" diagram has been calculated by Ussyukina and Davydychev [7], in addition to the planar topology. The non-planar result is known in closed form, which is finite in 4 dimensions. In view of these considerations, we fell it is a natural choice for study using the Hopf algebra method. We also note that, more recently, nonplanar examples with a related technology involving the symbol alphabet have been studied in [8–11]. However, in addition to the symbol, these authors also involve Mellin-Barnes technology and differential equations to compute the nonplanar integrals. We intend to rely only on the symbols in order to expand their range of use. The diagrams are given by Figs. 1(a) and 1(b).

The property that the result [7] appears as the square of a one-loop triangle is an important aspect of the nonplanar diagram. For a discussion on this property as well as of other (planar) related diagrams, see [12]. The availability of the result [7] offers the possibility to checking whether or not the Hopf algebra results of the planar case also carry over to the nonplanar case. The objective of this paper is to answer this question; it is found to be in the affirmative.

In order to tackle the above question, we have studied the cuts of this nonplanar diagram in the p_2^2 channel. There are two cuts possible for this channel, a two-propagator cut (with two propagators on-shell) and a three-propagator cut (with three propagators on-shell). For evaluating the integrals associated with the first cut and the second cut, we have used the one-loop results of [4] and [13] respectively. The net cut across the p_2^2 channel is the sum of these two cuts. The expression for this total cut is one of the significant result of this paper. (We have chosen the p_2^2 channel because its more simpler than the p_3^2 channel in terms of the number of cuts. The final result is independent of the choice of the channel.)
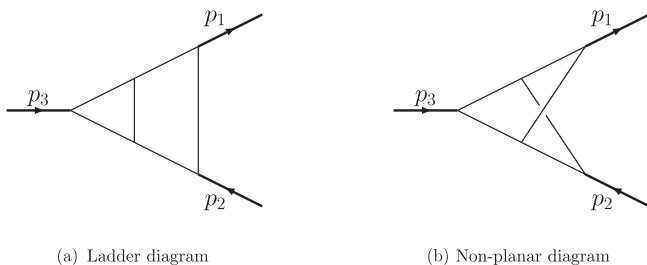
Using this, we have reconstructed the original Feynman diagram. The reason behind choosing this particular nonplanar diagram is the similarity in topology. It is a sufficiently simple extension of the planar ladder, but in practice, far more complicated.

The plan of this paper is the following: In Sec. II, we give a brief overview of the method given in Abreu *et al.* Here first of all we give the definitions of multiple polylogarithms [14,15], their coproduct [15,16] and the symbol [15,17]. After this, we show the equivalence of the coproduct with the symbol and then tell about the integrability condition [17–21]. Then we give the relation between the discontinuity, cut, and the coproduct [5,22–24], and the definition of the first-entry condition [25,26] and finally, the reconstruction of the symbol and the original Feynman integral. In Sec. III, we give the diagram and the kinematics. In Sec. IV, we evaluate the first kind of cut for the p_2^2 channel. We follow the conventions of Abreu *et al.* for parametrizing the momentum variables. The symbol alphabet is the same as discussed in [1] for the three mass triangle and the two-loop ladder diagram owing to the fact that the nonplanar integral falls into the same family of integrals considered in [7]. In Sec. V, we evaluate the second kind of cut for the p_2^2 channel. In Sec. VI, we add the results of the two cuts giving the total cut result across the p_2^2 channel, the result being finite. This checks well with the full result given in [7]. In Sec. VII, we reconstruct the symbol of the original Feynman diagram using the first-entry condition and the integrability condition. To verify it, we also calculate the symbol of [7] and find agreement. The reconstruction of the symbol verifies the Duhr conjecture [5] for the two loop nonplanar triangle diagram and is one of the main results of this paper. In other words, for the nonplanar example, the Hopf Algebra structure is preserved. In Sec. VIII, we reconstruct the full function (final result of the diagram). The reconstruction of the full function involves using a systematic search for all the harmonic polylogarithms (HPL's) [27,28] of relevant weights and imposing the constraints coming from the Cutkosky rules. In Sec. IX, we give our conclusions and a discussion. The Appendix describes our technical calculations including the cancellation of the divergences.

We note here that at the time the work of Abreu *et al.* was performed, the PolyLogTools [29] package was not publicly available. The availability of the code has rendered this evaluation possible by us. In addition, we have used the HypExp [30] package. It is of note that a new work extending the diagrammatic coaction already established at one loop level [4] has now been extended to some two loop examples [31] yet. Applications to nonplanar diagrams is yet to be explored. Also the extension of the method to Feynman integrals with irreducible numerators and higher powers of propagators [32] and a deeper understanding of the underlying mathematical structure [33] is a work for future.

(a) Ladder diagram   (b) Non-planar diagram

FIG. 1. The planar and nonplanar two-loop diagrams discussed in this paper.

II. BRIEF OVERVIEW OF THE METHOD OF CUT FEYNMAN DIAGRAMS

Here we give a brief discussion of the method of cut Feynman diagrams given in [1] worked out for the planar examples. It is a rephrasing of the known information that is given here for completeness. The interested reader is urged to go through the original references in order to have a complete picture.

A. Introduction

A large class of Feynman integrals can be expressed in terms of transcendental functions called multiple polylogarithms, which are defined by certain iterated integrals and include classical polylogarithms as a special case. Multiple polylogarithms form a Hopf algebra which involves a coproduct Δ. The discontinuity across a channel of the Feynman integral is related to the coproduct. The coproduct, in turn, is related to the symbol alphabet of the Feynman integral function, which encodes the information about this function. Thus doing the simpler task of analytically evaluating the discontinuity of the integral can lead us to determine the original Feynman integral.

B. Multiple polylogarithms

Multiple polylogarithms are defined by the iterated integral [14,15]

$$G(a_1, ..., a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, ..., a_n; t), \quad (1)$$

with $a_i, z \in \mathbb{C}$. The number n of integrations is called the *weight* of the multiple polylogarithm. We denote by $\bar{\mathcal{H}}$ the $\mathbb{Q}$-vector space spanned by all multiple polylogarithms, which can be turned into an algebra as well. The algebra of multiple polylogarithms is *graded* by the weight,

$$\bar{\mathcal{H}} = \bigoplus_{n=0}^{\infty} \bar{\mathcal{H}}_n \quad \text{with} \quad \bar{\mathcal{H}}_{n_1} \cdot \bar{\mathcal{H}}_{n_2} \subset \bar{\mathcal{H}}_{n_1 + n_2}, \quad (2)$$

where $\bar{\mathcal{H}}_n$ is the $\mathbb{Q}$-vector space spanned by all multiple polylogarithms of weight n, and we define $\bar{\mathcal{H}}_0 = \mathbb{Q}$.

C. The coproduct

Multiple polylogarithms can be endowed with surprising algebraic structures. They obey the quotient space $\mathcal{H} = \bar{\mathcal{H}}/(\pi\bar{\mathcal{H}})$ (the algebra $\bar{\mathcal{H}}$ modulo π), where $\mathcal{H}$ is a Hopf algebra [15,16]. $\mathcal{H}$ can be equipped with a *coproduct* $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, which is coassociative,

$$(\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta, \quad (3)$$

with the multiplication rule,

$$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b), \quad (4)$$

and the weight rule,

$$\mathcal{H}_n \xrightarrow{\Delta} \bigoplus_{k=0}^{n} \mathcal{H}_k \otimes \mathcal{H}_{n-k}. \quad (5)$$

The coproduct of the multiple polylogarithms is defined by [16]

$$\Delta(G(a_0; a_1, ..., a_n; a_{n+1})) = \sum_{0 = i_1 < i_2 < ... < i_k < i_{k+1} = n} G(a_0; a_{i_1}, ..., a_{i_k}; a_{n+1}) \otimes \left[\prod_{p=0}^{k} G(a_{i_p}; a_{i_p+1}, ..., a_{i_{p+1}-1}; a_{i_{p+1}}) \right]. \quad (6)$$

Using this, the coproduct of the ordinary logarithm and the classical polylogarithms are

$$\Delta(\log z) = 1 \otimes \log z + \log z \otimes 1 \quad \text{and} \quad \Delta(\text{Li}_n(z)) = 1 \otimes \text{Li}_n(z) + \sum_{k=0}^{n-1} \text{Li}_{n-k}(z) \otimes \frac{\log^k z}{k!}. \quad (7)$$

Also

$$\begin{aligned} \Delta(\log x \log y) &= \Delta(\log x)\Delta(\log y) = [1 \otimes \log x + \log x \otimes 1][1 \otimes \log y + \log y \otimes 1] \\ &= 1 \otimes (\log x \log y) + \log x \otimes \log y + \log y \otimes \log x + (\log x \log y) \otimes 1. \end{aligned} \quad (8)$$

If $(n_1, ..., n_k)$ is a partition of n, we define

$$\Delta_{n_1, ..., n_k} : \mathcal{H}_n \to \mathcal{H}_{n_1} \otimes ... \otimes \mathcal{H}_{n_k}. \quad (9)$$

Using Eq. (6) we can then write

$$\Delta = \sum_{p+q=n} \Delta_{p,q}, \quad (10)$$

and it satisfies the recursion

$$\Delta_{n_1, ..., n_k} = (\Delta_{n_1, ..., n_{k-1}} \otimes \text{id})\Delta_{n, n_k}, \quad n = n_1 + \cdots + n_{k-1}. \quad (11)$$

D. The symbol

The symbol of a transcendental function $F_w(x_1, ..., x_n)$ of weight w in the variables $x_1, ..., x_n$ is defined recursively by [17]

$$\mathcal{S}(F_w) = \sum_i \mathcal{S}(F_{i,w-1}) \otimes R_i. \qquad (12)$$

where $F_{i,w-1}$ are transcendental functions of weight $w-1$ and the R_i are rational functions in the variables $x_1, \ldots, x_n$, with the condition that the total differential of F_w can be written in the form

$$dF_w = \sum_i F_{i,w-1} d \log R_i, \qquad (13)$$

Multiple polylogarithms satisfy a differential equation of the type Eq. (13) [15],

$$dG(a_0; a_1, \ldots, a_n; a_{n+1})$$
$$= \sum_{i=1}^{n} G(a_0; a_1, \ldots, \hat{a}_i, \ldots, a_n; a_{n+1}) d \log \left(\frac{a_{i+1} - a_i}{a_{i-1} - a_i} \right), \qquad (14)$$

where the hat indicates that the corresponding element is omitted.

E. Equivalence of the coproduct with the symbol and the integrability condition

The maximal iteration of the coproduct, corresponding to the partition $(1, \ldots, 1)$, is equivalent with the symbol of a transcendental function F [17–21]

$$\mathcal{S}(F) \equiv \Delta_{1,\ldots,1}(F) \in \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_1. \qquad (15)$$

Not every element in $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_1$ corresponds to the symbol of a function in $\mathcal{H}$, it should satisfy the integrability condition: If we take an element

$$s = \sum_{i_1, \ldots, i_n} c_{i_1, \ldots, i_n} \log x_{i_1} \otimes \ldots \otimes \log x_{i_n} \in \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_1, \qquad (16)$$

then there is a function $F \in \mathcal{H}_n$ such that $\mathcal{S}(F) = s$ if and only if s satisfies the *integrability condition*

$$\sum_{i_1, \ldots, i_n} c_{i_1, \ldots, i_n} d \log x_{i_k} \wedge d \log x_{i_{k+1}} \log x_{i_1} \otimes \ldots \otimes \log x_{k-1}$$
$$\otimes \log x_{k+2} \otimes \ldots \otimes \log x_{i_n} = 0, \qquad (17)$$

where $\wedge$ denotes the usual wedge product on differential forms.

F. Disc and cut

The operator $\mathrm{Disc}_s F$ gives the direct value of the discontinuity of F as the variable s crosses the real axis.

$$\mathrm{Disc}_s[F(s \pm i0)] = \lim_{\varepsilon \to 0}[F(s \pm i\varepsilon) - F(s \mp i\varepsilon)], \qquad (18)$$

The operator Cut_s gives the sum of *cut* Feynman integrals, in which some propagators in the integrand of F are replaced by Dirac delta functions. For a massless scalar theory, the rules for cut may be depicted as:

$$\bullet = i \qquad \circ = -i \qquad (19)$$

$$(20)$$

$$(21)$$

G. Relations between disc, cut, and coproduct

Cutkosky's rule is given by [22–24]:

$$\mathrm{Disc}_s F = -\mathrm{Cut}_s F. \qquad (22)$$

The disc and coproduct are related by [5]

$$\mathrm{Disc}\, f_n \cong \mu[(\mathrm{Disc} \otimes \mathrm{id})\Delta_{1,n-1}f_n] \qquad (23)$$

where $\mu : \bar{\mathcal{H}} \otimes \bar{\mathcal{H}} \to \bar{\mathcal{H}}$ denotes the multiplication in $\bar{\mathcal{H}}$, i.e., we simply multiply the two factors in the coproduct, and $\cong$ denotes equivalence modulo π^2. Thus from Eqs. (22) and (23) we have

$$\mathrm{Cut}\, f_n \cong -\mu[(\mathrm{Disc} \otimes \mathrm{id})\Delta_{1,n-1}f_n] \qquad (24)$$

H. The first-entry condition

For massless propagators, the branch points of the integral, seen as a function of the invariants $s_{ij} = (p_i + p_j)^2$ are the points where one of the invariants is zero or infinite [25].

Using Eq. (23) this implies the so-called first entry condition, i.e., the statement that the first entries of the symbol of a Feynman integral with massless propagators can only be logarithms of Mandelstam invariants [26] to give a nonzero value to the discontinuity of the original function. Thus

$$\Delta_{1,n-1}f_n = \log s_{ij} \otimes f_{n-1} \qquad (25)$$

which implies

$$\mathrm{Cut}\, f_n \cong -\mu[(\mathrm{Disc} \otimes \mathrm{id})(\log s_{ij} \otimes f_{n-1})] \qquad (26)$$

and hence

$$\text{Cut} f_n \cong -\mu[\text{Disc}(\log s_{ij}) \otimes f_{n-1}] = \text{Disc}(\log s_{ij}) f_{n-1}$$
(27)

Thus the cut is proportional to the second entry of $\Delta_{1,n-1}$.

I. Reconstructing the symbol from the cut

The symbol can be obtained from the cut using the following three conditions:

(i) Cut is proportional to second entry of $\Delta_{1,n-1}$.
(ii) The first-entry condition.
(iii) The integrabilty condition.

First of all, we take the obtained result for the cut as the second-entry of $\Delta_{1,n-1}$ and the respective channel (the channel which has been cut) as its first-entry. Then using Eqs. (3)–(11), we can find out a possible first guess for the maximal iteration of the coproduct $\Delta_{1,\ldots,1}$ which is equal to the symbol. After this, we need to apply the integrability condition and the first entry condition repeatedly

until both these conditions are satisfied, and finally, we get the symbol.

J. Reconstructing the Feynman integral from the symbol

After getting the symbol, we can guess which kind of function has given rise to each term in the obtained symbol by application of the coproduct $\Delta_{1,\ldots,1}$ on it. After getting all such functions, we add them up to get the family of functions that give the obtained complete symbol. After this, we can use Eq. (22) to get the actual function out of the family of functions.

III. THE TWO-LOOP NONPLANAR DIAGRAM

As we have pointed out, the above method has been checked only for planar topologies. Here we take a topologically related two-loop three mass nonplanar diagram with massless internal propagators. It is illustrated in Fig. 2 and the corresponding Feynman integral is given by

$$C(p_1^2, p_2^2, p_3^2) = e^{2\gamma_E \epsilon} \int \frac{d^D l}{\pi^{D/2}} \frac{d^D k}{\pi^{D/2}} \frac{1}{(l^2)((l-p_2)^2)(k+l-p_1)^2(k+l-p_2)^2(k-p_1)^2(k)^2},$$
(28)

which can be evaluated using direct integration and is given by [7]

$$C(p_1^2, p_2^2, p_3^2) = \left(\frac{i\pi}{p_3^2}\right)^2 \left(\frac{1}{\lambda}\left[2\text{Li}_2(-\rho x) - 2\text{Li}_2(-\rho y) + \log(\rho x)\log(\rho y) + \log\left(\frac{y}{x}\right)\log\left(\frac{1+\rho y}{1+\rho x}\right) + \frac{\pi^2}{3}\right]\right)^2$$
(29)

with

$$\lambda = \sqrt{(1-x-y)^2 - 4xy}, \qquad \rho(x,y) = \frac{2}{1-x-y+\lambda}, \qquad x = \frac{p_1^2}{p_3^2}, \qquad y = \frac{p_2^2}{p_3^2}.$$
(30)

It is worth emphasizing that the result can be expressed in terms of the same variables z and $\bar{z}$ in [1] which is used to express the results of the topologically related one-loop triangle and the two-loop ladder diagram. Now we will try to prove the above method for the p_2^2 channel cut. In a channel cut, the momenta of internal lines which are cut add up to the momentum of the channel. So using this, we have two types of cuts possible for this channel. The first cut, which we are going to consider, is a two-propagator cut and the second one is a three-propagator cut.

In contrast to the original Feynman integral being finite in $D = 4$ dimensions, it turns out that the individual cut diagrams are divergent. Hence, we need to apply the dimensional regularization $D = 4 - 2\epsilon$ in order to separate out the divergences in form of ϵ^{-1} terms. Also, in similarity with the two-loop ladder discussed in [1], the divergent terms from the individual cut diagrams cancel, giving the
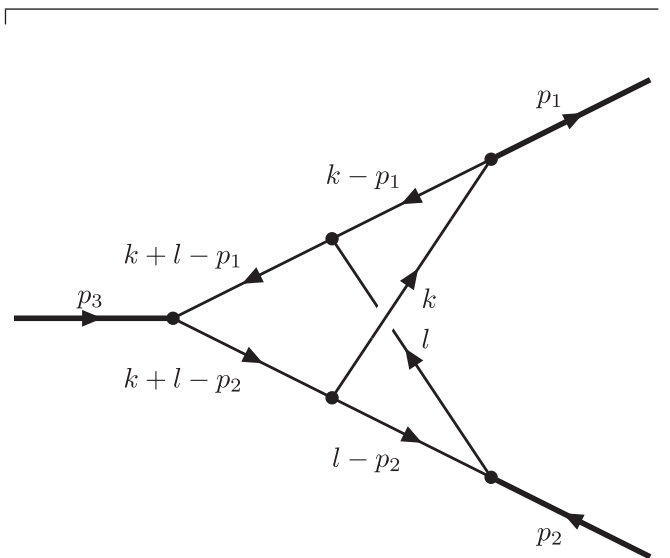

FIG. 2. $C(p_1^2, p_2^2, p_3^2)$.

total cut finite, which is a required criterion as our original Feynman diagram is finite.

Also unlike in [1], here we have not considered a close examination of the kinematic regions explicitly where the kinematic invariants are of different sign for different regions. This is because the close inspection of these regions becomes essential only while considering the double unitary cuts and beyond, whereas, in our paper, we focus only on single unitary cuts for a diagram.

IV. THE FIRST CUT

The Feynman integral for the first cut we are going to consider is given by

$$\text{Cut}[(l - p_2)^2, l^2] = -(2\pi)^2 e^{2\gamma_E \epsilon} \int \frac{d^D l}{\pi^{D/2}} \frac{d^D k}{\pi^{D/2}} \frac{\delta^+(l^2)\delta^+((l - p_2)^2)}{(k + l - p_1)^2(k + l - p_2)^2(k - p_1)^2(k)^2}, \quad (31)$$

and the corresponding diagram is given in Fig. 3(a). Here $\delta^+(l^2) = \delta(l^2)\theta(l_0)$ in accordance with Eq. (21). By close inspection of the diagram we can see that all the uncut propagators form a one loop two mass easy box and hence can be evaluated first independently.

So the integral can be written as

$$\text{Cut}[(l - p_2)^2, l^2] = \frac{e^{\gamma_E \epsilon}}{\pi^{2-\epsilon}} \int d^{4-2\epsilon} l (2\pi)^2 \delta(l^2)\delta((l - p_2)^2) B^{2me}(s, t, p_1^2, p_3^2), \quad (32)$$

with B^{2me} being the two mass easy box integral which is evaluated independently and is given by [4]

$$B^{2me}(s, t, p_1^2, p_3^2) = \frac{2c_\Gamma}{\epsilon^2(st - p_1^2 p_3^2)} \left\{ (-s)^{-\epsilon} + (-t)^{-\epsilon} - (-p_1^2)^{-\epsilon} - (-p_3^2)^{-\epsilon} \right.$$
$$\left. + \sum_{j=0}^{3}(-1)^j \left(\frac{s + t - p_1^2 - p_3^2}{\alpha_j}\right)^\epsilon {}_2F_1\left(\epsilon; \epsilon; 1 + \epsilon; \frac{st - p_1^2 p_3^2}{\alpha_j}\right) \right\} \quad (33)$$

with

$$\begin{aligned}
\alpha_0 &= (p_1^2 - s)(p_1^2 - t) & \alpha_1 &= (p_1^2 - s)(s - p_3^2) \\
\alpha_2 &= (p_3^2 - s)(p_3^2 - t) & \alpha_3 &= (p_1^2 - t)(t - p_3^2)
\end{aligned} \quad (34)$$

and

$$s = (p_3 + l)^2, \qquad t = (p_3 - l + p_2)^2, \qquad c_\Gamma = \frac{\Gamma(1 + \epsilon)\Gamma^2(1 - \epsilon)}{\Gamma(1 - 2\epsilon)}. \quad (35)$$

(a) Cut $\left[(l - p_2)^2, l^2\right]$
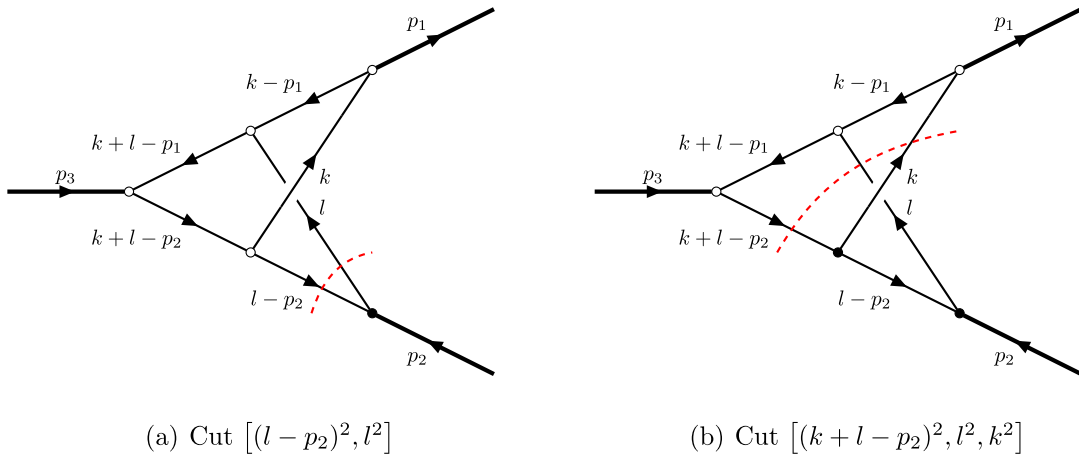
(b) Cut $\left[(k + l - p_2)^2, l^2, k^2\right]$

FIG. 3. Cut$[(k + l - p_2)^2, l^2, k^2]$

Note that here we have taken that expression for the result from literature which is complete for all orders in ϵ, rather than taking the result where it is only up to ϵ^0 order. This is because this result is actually more convenient for doing the subsequent calculations and to match the pre-factors of the two-different type of cuts which is required to do the cancellation of individual cut divergences. Now in similarity with the rules followed in [1], we will parametrize the momenta as

$$p_2 = \sqrt{p_2^2}(1, 0, \mathbf{0}_{D-2}), \qquad p_3 = \sqrt{p_3^2}(\alpha, \sqrt{\alpha^2 - 1}, \mathbf{0}_{D-2}),$$
$$l = (l_0, |l| \cos\theta, |l| \sin\theta \mathbf{1}_{D-2}), \tag{36}$$

where $\theta \in [0, \pi]$ and $|l| > 0$, and $\mathbf{1}_{D-2}$ ranges over unit vectors in the dimensions transverse to p_2 and p_3. Note that this parametrization is possible because of the fact that three vectors can be constrained within a 3-dimensional coordinate space defined accordingly. Again using momentum conservation we have

$$\alpha = \frac{p_1^2 - p_2^2 - p_3^2}{2\sqrt{p_2^2}\sqrt{p_3^2}}. \tag{37}$$

and using the parametrization above we rewrite the momentum integration as

$$\frac{e^{\gamma_E \epsilon}}{\pi^{2-\epsilon}} \int d^{4-2\epsilon} l (2\pi)^2 \delta(l^2) \delta((l - p_2)^2)$$
$$= \frac{4\pi e^{\gamma_E \epsilon}}{\Gamma(1 - \epsilon)} \int dl_0 \int d|\mathbf{l}|^2 |\mathbf{l}|^{1-2\epsilon} \delta(l_0^2 - |\mathbf{l}|^2) \delta\left(p_2^2 - 2l_0\sqrt{p_2^2} + l^2\right) \int_{-1}^{1} d\cos\theta (1 - \cos^2\theta)^{-\epsilon}. \tag{38}$$

The two delta functions allow us to trivially perform the l_0 and $|\mathbf{l}|$ integrations.

The first integration enforces $|\mathbf{l}| = l_0$ and the second one enforces $l_0 = \frac{\sqrt{p_2^2}}{2}$ everywhere in the integral with an additional factor of $2\sqrt{p_2^2}$ overall. For the remaining integration, we will do the following change of variables:

$$\cos\theta = 2x - 1, \qquad x \in [0, 1], \tag{39}$$

$$u_i = \frac{p_i^2}{p_1^2}, \quad i = 2, 3. \tag{40}$$

$$z = \frac{1}{2}(1 + u_2 - u_3 + \sqrt{\lambda}), \qquad \bar{z} = \frac{1}{2}(1 + u_2 - u_3 - \sqrt{\lambda}), \tag{41}$$

$$\lambda \equiv \lambda(1, u_2, u_3), \qquad \lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc, \tag{42}$$

After the first two integrations are performed, with the above change of variables we have

$$s = p_1^2[1 - \bar{z} - x(z - \bar{z})], t = p_1^2[(1 - z) + x(z - \bar{z})],$$
$$p_3^2 = p_1^2[(1 - z)(1 - \bar{z})], p_2^2 = p_1^2[z\bar{z}]$$

and

$$st - p_1^2 p_3^2 = (p_1^2)^2 x(1 - x)(z - \bar{z})^2. \tag{43}$$

Finally Eq. (32) becomes

$$
\text{Cut}[(l - p_2)^2, l^2] = (-1)^\epsilon \frac{4\pi c_\Gamma e^{\gamma_E \epsilon}}{\Gamma(1 - \epsilon)} \frac{(z\bar{z})^{-\epsilon}}{(z - \bar{z})^2} \frac{(p_1^2)^{-2-2\epsilon}}{\epsilon^2} \int_0^1 dx\, x^{-1-\epsilon} (1 - x)^{-1-\epsilon}
$$

$$
\times \left\{ (1 - \bar{z} - x(z - \bar{z}))^{-\epsilon} + (1 - z + x(z - \bar{z}))^{-\epsilon} - (1)^{-\epsilon} - ((1 - z)(1 - \bar{z}))^{-\epsilon} \right.
$$

$$
+ \left(\frac{z\bar{z}}{(\bar{z} + x(z - \bar{z}))(z - x(z - \bar{z}))} \right)^\epsilon {}_2F_1\left(\epsilon; \epsilon; 1 + \epsilon; \frac{x(1 - x)(z - \bar{z})^2}{(\bar{z} + x(z - \bar{z}))(z - x(z - \bar{z}))} \right)
$$

$$
- \left(\frac{z\bar{z}}{(\bar{z} + x(z - \bar{z}))(z(1 - \bar{z}) - x(z - \bar{z}))} \right)^\epsilon {}_2F_1\left(\epsilon; \epsilon; 1 + \epsilon; \frac{x(1 - x)(z - \bar{z})^2}{(\bar{z} + x(z - \bar{z}))(z(1 - \bar{z}) - x(z - \bar{z}))} \right)
$$

$$
- \left(\frac{z\bar{z}}{(\bar{z}(1 - z) + x(z - \bar{z}))(z - x(z - \bar{z}))} \right)^\epsilon {}_2F_1\left(\epsilon; \epsilon; 1 + \epsilon; \frac{x(1 - x)(z - \bar{z})^2}{(\bar{z}(1 - z) + x(z - \bar{z}))(z - x(z - \bar{z}))} \right)
$$

$$
+ \left(\frac{z\bar{z}}{(\bar{z}(1 - z) + x(z - \bar{z}))(z(1 - \bar{z}) - x(z - \bar{z}))} \right)^\epsilon
$$

$$
\left. \times {}_2F_1\left(\epsilon; \epsilon; 1 + \epsilon; \frac{x(1 - x)(z - \bar{z})^2}{(\bar{z}(1 - z) + x(z - \bar{z}))(z(1 - \bar{z}) - x(z - \bar{z}))} \right) \right\}. \tag{44}
$$

So it turns out that we are able to express the cut totally in terms of the momentum invariant p_1^2 and the defined variables z and $\bar{z}$ completely as expected. Now the first four integrations can be done by first expanding them in orders of epsilon and then integrating using PolyLogTools [29]. The last four integrations can be done by first expanding the hypergeometric ${}_2F_1$ functions using HypExp [30] and then integrating using PolyLogTools. The results are as follows:

$$
\text{Cut}\,[(l - p_2)^2, l^2] = (-1)^\epsilon \frac{4\pi c_\Gamma e^{\gamma_E \epsilon}}{\Gamma(1 - \epsilon)} \frac{(z\bar{z})^{-\epsilon}}{(z - \bar{z})^2} (p_1^2)^{-2-2\epsilon} \sum_{k=-1}^\infty \epsilon^k [f^{(k)}(z, \bar{z})] \tag{45}
$$

with

$$
f^{(-1)}(z, \bar{z}) = -G\left(0, \frac{z - 1}{z - \bar{z}}, 1\right) - G\left(0, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right) + G\left(1, \frac{z - 1}{z - \bar{z}}, 1\right) + G\left(1, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right) \tag{46}
$$

$$
f^{(0)}(z, \bar{z}) = G(1, z)\left(G\left(0, \frac{z - 1}{z - \bar{z}}, 1\right) - G\left(0, \frac{z}{z - \bar{z}}, 1\right) + G\left(0, \frac{z - z\bar{z}}{z - \bar{z}}, 1\right) - G\left(1, \frac{z - 1}{z - \bar{z}}, 1\right) \right.
$$

$$
+ G\left(1, \frac{z}{z - \bar{z}}, 1\right) - G\left(1, \frac{z - z\bar{z}}{z - \bar{z}}, 1\right) \right) + G(1, \bar{z})\left(G\left(0, \frac{(z - 1)\bar{z}}{z - \bar{z}}, 1\right) - G\left(0, -\frac{\bar{z}}{z - \bar{z}}, 1\right) \right.
$$

$$
+ G\left(0, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right) - G\left(1, \frac{(z - 1)\bar{z}}{z - \bar{z}}, 1\right) + G\left(1, -\frac{\bar{z}}{z - \bar{z}}, 1\right) - G\left(1, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right) \right)
$$

$$
+ 2G\left(0, 0, \frac{z - 1}{z - \bar{z}}, 1\right) + 2G\left(0, 0, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right) + 2G\left(0, 1, \frac{z - 1}{z - \bar{z}}, 1\right) + 2G\left(0, 1, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right)
$$

$$
+ G\left(0, \frac{z - 1}{z - \bar{z}}, 0, 1\right) + G\left(0, \frac{z - 1}{z - \bar{z}}, 1, 1\right) + G\left(0, \frac{z - 1}{z - \bar{z}}, \frac{z - 1}{z - \bar{z}}, 1\right) - G\left(0, \frac{z}{z - \bar{z}}, \frac{z - 1}{z - \bar{z}}, 1\right)
$$

$$
- G\left(0, \frac{(z - 1)\bar{z}}{z - \bar{z}}, \frac{z - 1}{z - \bar{z}}, 1\right) - G\left(0, \frac{z - z\bar{z}}{z - \bar{z}}, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right) - G\left(0, -\frac{\bar{z}}{z - \bar{z}}, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right)
$$

$$
+ G\left(0, \frac{\bar{z} - 1}{\bar{z} - z}, 0, 1\right) + G\left(0, \frac{\bar{z} - 1}{\bar{z} - z}, 1, 1\right) + G\left(0, \frac{\bar{z} - 1}{\bar{z} - z}, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right) - 2G\left(1, 0, \frac{a - 1}{a - b}, 1\right)
$$

$$
- 2G\left(1, 0, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right) - 2G\left(1, 1, \frac{z - 1}{z - \bar{z}}, 1\right) - 2G\left(1, 1, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right) - G\left(1, \frac{z - 1}{z - \bar{z}}, 0, 1\right)
$$

$$
- G\left(1, \frac{z - 1}{z - \bar{z}}, 1, 1\right) - G\left(1, \frac{z - 1}{z - \bar{z}}, \frac{z - 1}{z - \bar{z}}, 1\right) + G\left(1, \frac{z}{z - \bar{z}}, \frac{z - 1}{z - \bar{z}}, 1\right) + G\left(1, \frac{(z - 1)\bar{z}}{z - \bar{z}}, \frac{z - 1}{\bar{z} - \bar{z}}, 1\right)
$$

$$
+ G\left(1, \frac{z - z\bar{z}}{z - \bar{z}}, \frac{z - 1}{\bar{z} - z}, 1\right) + G\left(1, -\frac{\bar{z}}{z - \bar{z}}, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right) - G\left(1, \frac{\bar{z} - 1}{\bar{z} - z}, 0, 1\right) - G\left(1, \frac{\bar{z} - 1}{\bar{z} - z}, 1, 1\right)
$$

$$
- G\left(1, \frac{\bar{z} - 1}{\bar{z} - z}, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right). \tag{47}
$$

Some of the terms in the above expression are divergent, and it turns out that the divergent parts from such terms exactly cancel each other, making the total contribution divergenceless. Note that this is a required criterion because a finite result shows that there will be no more divergent contributions in the form of more ϵ^{-1} terms. Similarly, the divergent parts of some terms from $f^{(-1)}(z, \bar{z})$ cancels out, showing that there will no divergent contributions in the

form of ϵ^{-2} terms. The process where we show how these divergent parts cancel has been worked out explicitly in the Appendix for the total cut.

V. THE SECOND CUT

The Feynman integral for the second cut (see Fig. 3) is given by

$$\text{Cut}\,[(k+l-p_2)^2, l^2, k^2] = -(2\pi)^3 e^{2\gamma_E \epsilon} \int \frac{d^D l}{\pi^{D/2}} \frac{d^D k}{\pi^{D/2}} \frac{\delta^+(l^2)\delta^+((k+l-p_2)^2)\delta^+(k^2)}{(k+l-p_1)^2(k-p_1)^2(l-p_2)^2}, \tag{48}$$

Again by inspecting the cut diagram we can write the integral as:

$$\text{Cut}\,[(k+l-p_2)^2, l^2, k^2] = \frac{2\pi e^{\gamma_E \epsilon}}{\pi^{2-\epsilon}} \int d^{4-2\epsilon} l \frac{\delta(l^2)}{(l-p_2)^2} \text{Cut}_{[(l-p_2)^2]} B^{3m}(0, p_3^2, (l-p_2)^2, p_1^2, s, t). \tag{49}$$

where $\text{Cut}_{[(l-p_2)^2]} B^{3m}(0, p_3^2, (l-p_2)^2, p_1^2, s, t)$ is the p_3^2 channel cut of a three mass box $B^{3m}(0, p_2^2, p_3^2, p_4^2, s, t)$.

Now this can be evaluated in two ways—one is the direct way where we directly evaluate the integral with delta functions inside it in place of the propagators which are cut. It turns out that the calculations become very difficult using this method. The other way to evaluate is to use the Cutkosky's rule Eq. (22) where we have to use the result for the three mass box from the literature and then find out the cut. The three mass box is given by the integral Eq. (17) of [13]

$$B^{3m}(0, p_2^2, p_3^2, p_4^2, s, t) = i\Gamma(1+\epsilon) \int_0^1 dy dz \frac{1}{z(s-p_2^2) + (1-z)(p_4^2 - t)}$$
$$\times \{[-y(1-y)(zs + (1-z)p_4^2) - z(1-z)y^2 p_3^2]^{-1-\epsilon}$$
$$- [-y(1-y)(zp_2^2 + (1-z)t) - z(1-z)y^2 p_3^2]^{-1-\epsilon}\}. \tag{50}$$

The reason behind writing it in this way is that it becomes easier to evaluate the cut to all orders in ϵ using the above expression which is preferred over evaluating the cut of this box diagram when the result is only available up to ϵ^0 order. Now the cut or the discontinuity across the p_3^2 channel can be evaluated using the following formula [2]

$$\text{Cut}_a[(a-b)^{-\epsilon}] = \text{Disc}_a[(a-b)^{-\epsilon}] = \frac{2\pi i\epsilon}{\Gamma(1-\epsilon)\Gamma(1+\epsilon)}(b-a)^{-\epsilon}\theta\left(\frac{a}{b}-1\right) \tag{51}$$

Using this formula operated inside the integral we have

$$\text{Cut}_{p_3^2} B^{3m}(0, p_2^2, p_3^2, p_4^2, s, t) = -\frac{2\pi}{\Gamma(-\epsilon)} \int_0^1 dz \frac{1}{z(s-p_2^2) + (1-z)(p_4^2 - t)}$$
$$\times \left\{ \int_0^{\frac{zs+(1-z)p_4^2}{zs+(1-z)p_4^2+z(z-1)p_3^2}} dy[y(1-y)(zs+(1-z)p_4^2) + z(1-z)y^2 p_3^2]^{-1-\epsilon} \right.$$
$$\left. - \int_0^{\frac{zp_2^2+(1-z)t}{zp_2^2+(1-z)t+z(z-1)p_3^2}} dy[y(1-y)(zp_2^2+(1-z)t) + z(1-z)y^2 p_3^2]^{-1-\epsilon} \right\} \tag{52}$$

which after some adjustments become

$$\text{Cut}_{p_3^2} B^{3m}(0, p_2^2, p_3^2, p_4^2, s, t) = -\frac{2\pi}{\Gamma(1-\epsilon)} \int_0^1 dz \frac{1}{z(s-p_2^2) + (1-z)(p_4^2 - t)}$$
$$\times \left\{ \frac{(zs+(1-z)p_4^2)^{-2\epsilon-1}}{(zs+(1-z)p_4^2+z(z-1)p_3^2)^{-\epsilon}} {}_2F_1\left(-\epsilon, 1+\epsilon; 1-\epsilon; 1-\frac{2z(z-1)p_3^2}{zs+(1-z)p_4^2+z(z-1)p_3^2}\right) \right.$$
$$\left. - \frac{(zp_2^2+(1-z)t)^{-2\epsilon-1}}{(zp_2^2+(1-z)t+z(z-1)p_3^2)^{-\epsilon}} {}_2F_1\left(-\epsilon, 1+\epsilon; 1-\epsilon; 1-\frac{2z(z-1)p_3^2}{zp_2^2+(1-z)t+z(z-1)p_3^2}\right) \right\}. \tag{53}$$

This result is complete in all orders in ϵ which is actually required to evaluate the total contribution to the ϵ^0 term for the second cut. Now we can use Euler's theorem for hypergeometric functions namely

$$_2F_1(a, b; c; x) = (1 - x)^{c-a-b} {}_2F_1(c - a, c - b; c; x) \tag{54}$$

to rewrite it as

$$\begin{aligned}
\mathrm{Cut}_{p_3^2} B^{3m}(0, p_2^2, p_3^2, p_4^2, s, t) = &-\frac{2\pi}{\Gamma(1-\epsilon)} \int_0^1 dz \frac{1}{z(s - p_2^2) + (1 - z)(p_4^2 - t)} \\
&\times \left\{ \frac{(zs + (1-z)p_4^2)^{-2\epsilon-1}(2z(z-1)p_3^2)^{-\epsilon}}{(zs + (1-z)p_4^2 + z(z-1)p_3^2)^{-2\epsilon}} {}_2F_1\left(1, -2\epsilon; 1 - \epsilon; 1 - \frac{2z(z-1)p_3^2}{zs + (1-z)p_4^2 + z(z-1)p_3^2}\right) \right. \\
&\left. -\frac{(zp_2^2 + (1-z)t)^{-2\epsilon-1}(2z(z-1)p_3^2)^{-\epsilon}}{(zp_2^2 + (1-z)t + z(z-1)p_3^2)^{-2\epsilon}} {}_2F_1\left(1, -2\epsilon; 1 - \epsilon; 1 - \frac{2z(z-1)p_3^2}{zp_2^2 + (1-z)t + z(z-1)p_3^2}\right) \right\}.
\end{aligned} \tag{55}$$

The significance behind this step is that the prefactor term $(2p_3^2)^{-\epsilon}$ in the numerator actually helps us to cancel out the divergences of the remaining integration. We will discuss it in more detail later.

Again we parametrize the variables as in Eqs. (39)–(42), but this time there is only one delta function in the integral after explicit evaluation of the three mass box cut. This enforces $|\mathbf{l}| = l_0$, but l_0 is left as an integration variable which we again parametrize as:

$$l_0 \to \frac{\sqrt{p_2^2}}{2} y \tag{56}$$

After which the momentum invariants become:

$$s = p_1^2[(1 - z)(1 - \bar{z}) + y(z(1 - \bar{z}) - x(z - \bar{z}))], \quad t = p_1^2[1 - y(z - x(z - \bar{z}))],$$
$$p_3^2 = p_1^2[(1 - z)(1 - \bar{z})], \quad p_2^2 = p_1^2[z\bar{z}], \quad (l - p_2^2)^2 = p_1^2(z\bar{z})(1 - y)$$

and

$$st - p_1^2 p_3^2 = (p_1^2)^2 y[z^2(1 - \bar{z}) + x(1 - z)(1 - \bar{z})(z - \bar{z}) - y(z(1 - \bar{z}) - x(z - \bar{z}))(z - x(z - \bar{z}))]. \tag{57}$$

and finally the momentum integration becomes

$$\frac{2\pi e^{\gamma_E \epsilon}}{\pi^{2-\epsilon}} \int d^{4-2\epsilon} l \frac{\delta(l^2)}{(l - p_2)^2} = \frac{2 e^{\gamma_E \epsilon} (z\bar{z})^{-\epsilon}(p_1^2)^{-\epsilon}}{\Gamma(1 - \epsilon)} \int_0^1 dx\, x^{-\epsilon}(1 - x)^{-\epsilon} \int_0^\infty dy\, y^{1-2\epsilon}(1 - y)^{-1}. \tag{58}$$

Note that the y-integration limits are restricted to the positive real axis because of the Heaviside theta function in Eq. (21). Substituting all this Eq. (49) becomes

$$\begin{aligned}
\mathrm{Cut}[(k + l - p_2)^2, l^2, k^2] = &\frac{4\pi c_\Gamma e^{\gamma_E \epsilon}}{\Gamma(1 - \epsilon)} (z\bar{z})^{-\epsilon}(p_1^2)^{-2-2\epsilon} \\
&\times (2z\bar{z})^{-\epsilon} \int_0^1 dx \int_0^1 dy\, x^{-\epsilon}(1 - x)^{-\epsilon} y^{1-2\epsilon}(1 - y)^{-1-\epsilon} \int_0^1 \frac{dz_1}{c_\Gamma \Gamma(1 - \epsilon)} \left[\frac{1}{z_1(z\bar{z}) - z + x(z - \bar{z})} \right. \\
&\times \left\{ \frac{(z_1(z_1 - 1))^{-\epsilon}(z_1((1 - z)(1 - \bar{z}) + y(z(1 - \bar{z}) - x(z - \bar{z})) - 1) + 1)^{-2\epsilon-1}}{(z_1((1 - z)(1 - \bar{z}) + y(z(1 - \bar{z}) - x(z - \bar{z})) - 1) + 1 + z_1(1 - z_1)(z\bar{z})(1 - y))^{-2\epsilon}} \right. \\
&\times {}_2F_1\left(1, -2\epsilon; 1 - \epsilon; \frac{z_1((1 - z)(1 - \bar{z}) + y(z(1 - \bar{z}) - x(z - \bar{z})) - 1) + 1 - z_1(1 - z_1)(z\bar{z})(1 - y)}{z_1((1 - z)(1 - \bar{z}) + y(z(1 - \bar{z}) - x(z - \bar{z})) - 1) + 1 + z_1(1 - z_1)(z\bar{z})(1 - y)}\right) \\
&-\frac{(z_1(z_1 - 1))^{-\epsilon}(z_1((1 - z)(1 - \bar{z}) - 1 + y(z - x(z - \bar{z}))) + 1 - y(z - x(z - \bar{z})))^{-2\epsilon-1}}{(z_1((1 - z)(1 - \bar{z}) - 1 + y(z - x(z - \bar{z}))) + 1 - y(z - x(z - \bar{z})) + z_1(1 - z_1)(z\bar{z})(1 - y))^{-\epsilon}} \\
&\times {}_2F_1(1, -2\epsilon; 1 - \epsilon; \\
&\left.\left. \frac{z_1((1 - z)(1 - \bar{z}) - 1 + y(z - x(z - \bar{z}))) + 1 - y(z - x(z - \bar{z})) - z_1(1 - z_1)(z\bar{z})(1 - y)}{z_1((1 - z)(1 - \bar{z}) - 1 + y(z - x(z - \bar{z}))) + 1 - y(z - x(z - \bar{z})) + z_1(1 - z_1)(z\bar{z})(1 - y)}\right) \right\}.
\end{aligned} \tag{59}$$

Note that the upper limit of the y- integration is changed to 1. This is because after evaluation of the $\text{Cut}_{p_3^2} B^{3m}(0, p_2^2, p_3^2, p_4^2, s, t)$ there is a $\theta(p_3^2)$ multiplied to the final result in Eq. (55) in accordance with Eq. (21). So we have in our case $(l - p_2^2)^2 > 0$ and hence using Eq. (57) we have $y < 1$. Now the hypergeometric $_2F_1$ function can be expanded order by order in ϵ using HypExp and the remaining integration can be done using PolyLogTools.

We can see that the second cut is not as trivial as the first cut. After the hypergeometric $_2F_1$ expansion we have three integrations that need to be done in contrast to a single integration for the first cut. There are many contributions coming out to final answer.

$$\text{Cut}\left[(k + l - p_2)^2, l^2, k^2\right] = \frac{4\pi c_\Gamma e^{\gamma_E \epsilon}}{\Gamma(1 - \epsilon)} \frac{(z\bar{z})^{-\epsilon}}{(z - \bar{z})^2} (p_1^2)^{-2-2\epsilon} \sum_{k=-1}^{\infty} \epsilon^k [g^{(k)}(z, \bar{z})] \tag{60}$$

with

$$g^{(-1)}(z, \bar{z}) = G\left(0, \frac{z-1}{z-\bar{z}}, 1\right) + G\left(0, \frac{\bar{z}-1}{\bar{z}-z}, 1\right) - G\left(1, \frac{z-1}{z-\bar{z}}, 1\right) - G\left(1, \frac{\bar{z}-1}{\bar{z}-z}, 1\right) \tag{61}$$

and

$$g^{(0)}(z, \bar{z}) = g_M^{(0)}(z, \bar{z}) + g_{\epsilon^1}^{(0)}(z, \bar{z}) + g_{\epsilon^{-1}}^{(0)}(z, \bar{z}) + g_P^{(0)}(z, \bar{z}). \tag{62}$$

We will describe each contribution as follows:

(i) The principal contribution:

After expanding the Hypergeometric $_2F_1$ functions in Eq. (59) in orders of epsilon, the obtained ϵ^0 term gives the main contribution. After integration by z_1, y and x, it is given by

$$
\begin{aligned}
g_M^{(0)}(z, \bar{z}) =\ & G\left(\frac{z-1}{z-\bar{z}}, 1\right) \text{HPL}(\{1\}, 1)^2 + G\left(\frac{\bar{z}-1}{\bar{z}-z}, 1\right) \text{HPL}(\{1\}, 1)^2 + G(1, \bar{z}) G\left(\frac{z}{z-\bar{z}}, 1\right) \text{HPL}(\{1\}, 1) \\
& - G(1, \bar{z}) G\left(\frac{z - z\bar{z}}{z-\bar{z}}, 1\right) \text{HPL}(\{1\}, 1) + G\left(\frac{z-1}{z-\bar{z}}, 0, 1\right) \text{HPL}(\{1\}, 1) + G\left(\frac{z-1}{z-\bar{z}}, 1, 1\right) \text{HPL}(\{1\}, 1) \\
& - G\left(\frac{z-1}{z-\bar{z}}, \frac{z-1}{z-\bar{z}}, 1\right) \text{HPL}(\{1\}, 1) + 2G\left(\frac{z}{z-\bar{z}}, \frac{z-1}{z-\bar{z}}, 1\right) \text{HPL}(\{1\}, 1) \\
& + 2G\left(\frac{z - z\bar{z}}{z-\bar{z}}, \frac{\bar{z}-1}{\bar{z}-z}, 1\right) \text{HPL}(\{1\}, 1) + G\left(\frac{\bar{z}-1}{\bar{z}-z}, 0, 1\right) \text{HPL}(\{1\}, 1) + G\left(\frac{\bar{z}-1}{\bar{z}-z}, 1, 1\right) \text{HPL}(\{1\}, 1) \\
& - G\left(\frac{\bar{z}-1}{\bar{z}-z}, \frac{\bar{z}-1}{\bar{z}-z}, 1\right) \text{HPL}(\{1\}, 1) - G(1, \bar{z}) G\left(0, \frac{z}{z-\bar{z}}, 1\right) + G(1, \bar{z}) G\left(0, \frac{z - z\bar{z}}{z-\bar{z}}, 1\right) \\
& + G(1, \bar{z}) G\left(1, \frac{z}{z-\bar{z}}, 1\right) - G(1, \bar{z}) G\left(1, \frac{z - z\bar{z}}{z-\bar{z}}, 1\right) + G\left(0, 0, \frac{z-1}{z-\bar{z}}, 1\right) + G\left(0, 0, \frac{\bar{z}-1}{\bar{z}-z}, 1\right) \\
& + G\left(0, 1, \frac{z-1}{z-\bar{z}}, 1\right) + G\left(0, 1, \frac{\bar{z}-1}{\bar{z}-z}, 1\right) + G\left(0, \frac{z-1}{z-\bar{z}}, \frac{z-1}{z-\bar{z}}, 1\right) - 2G\left(0, \frac{z}{z-\bar{z}}, \frac{z-1}{z-\bar{z}}, 1\right) \\
& - 2G\left(0, \frac{z - z\bar{z}}{z-\bar{z}}, \frac{\bar{z}-1}{\bar{z}-z}, 1\right) + G\left(0, \frac{\bar{z}-1}{\bar{z}-z}, \frac{\bar{z}-1}{\bar{z}-z}, 1\right) - G\left(1, 0, \frac{z-1}{z-\bar{z}}, 1\right) - G\left(1, 0, \frac{\bar{z}-1}{\bar{z}-z}, 1\right) \\
& - G\left(1, 1, \frac{z-1}{z-\bar{z}}, 1\right) - G\left(1, 1, \frac{\bar{z}-1}{\bar{z}-z}, 1\right) - G\left(1, \frac{z-1}{z-\bar{z}}, \frac{z-1}{z-\bar{z}}, 1\right) + 2G\left(1, \frac{z}{z-\bar{z}}, \frac{z-1}{z-\bar{z}}, 1\right) \\
& + 2G\left(1, \frac{z - z\bar{z}}{z-\bar{z}}, \frac{\bar{z}-1}{\bar{z}-z}, 1\right) - G\left(1, \frac{\bar{z}-1}{\bar{z}-z}, \frac{\bar{z}-1}{\bar{z}-z}, 1\right) + G(0, (z-1)(\bar{z}-1)) \left(G\left(0, \frac{z}{z-\bar{z}}, 1\right)\right. \\
& - G\left(0, \frac{z - z\bar{z}}{z-\bar{z}}, 1\right) - G\left(1, \frac{z}{z-\bar{z}}, 1\right) + G\left(1, \frac{z - z\bar{z}}{z-\bar{z}}, 1\right) - G\left(\frac{z}{z-\bar{z}}, 1\right) \text{HPL}(\{1\}, 1) \\
& \left. + G\left(\frac{z - z\bar{z}}{z-\bar{z}}, 1\right) \text{HPL}(\{1\}, 1)\right) + G(1, z) \left(G\left(0, \frac{z-1}{z-\bar{z}}, 1\right) - 3G\left(0, \frac{z}{z-\bar{z}}, 1\right) + 3G\left(0, \frac{z - z\bar{z}}{z-\bar{z}}, 1\right)\right.
\end{aligned}
$$

$$- G\left(0, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right) - G\left(1, \frac{z - 1}{z - \bar{z}}, 1\right) + 3G\left(1, \frac{z}{z - \bar{z}}, 1\right) - 3G\left(1, \frac{z - z\bar{z}}{z - \bar{z}}, 1\right) + G\left(1, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right)$$

$$- G\left(\frac{z - 1}{z - \bar{z}}, 1\right) \mathrm{HPL}(\{1\}, 1) + 3G\left(\frac{z}{z - \bar{z}}, 1\right) \mathrm{HPL}(\{1\}, 1) - 3G\left(\frac{z - z\bar{z}}{z - \bar{z}}, 1\right) \mathrm{HPL}(\{1\}, 1)$$

$$+ G\left(\frac{\bar{z} - 1}{\bar{z} - z}, 1\right) \mathrm{HPL}(\{1\}, 1)). \tag{63}$$

The final result again has many of the terms divergent, which exactly cancels out, giving the final result divergenceless. Here we have used the following formula to get rid of the end point singularities occurring for the y-integration at $y = 1$.

$$\int_0^1 dy \frac{g(y, \epsilon)}{(1 - y)^{1+\epsilon}} = \frac{g(1, \epsilon)}{\epsilon} + \int_0^1 \frac{g(y, \epsilon) - g(1, \epsilon)}{(1 - y)^{1+\epsilon}}. \tag{64}$$

Because of the first term in the right-hand side (rhs) of the above equation, this step produces the ϵ^{-1} term $g^{(-1)}(z, \bar{z})$ and a subsequent contribution to the ϵ^0 term $g_{\epsilon^{-1}}^{(0)}(z, \bar{z})$. Again we use a similar formula for the x-integration to get rid of the endpoint singularities occurring at $x = 0$ and $x = 1$. The similar ϵ^{-1} kind of terms coming out after this step add up to zero and hence giving no contribution at all. Also we are able to use Eq. (64) to cancel the singularities only because of the step in Eqs. (54) and (55).

(ii) The contribution from the ϵ^{-1} term:

The ϵ^{-1} term $g^{(-1)}(z, \bar{z})$ coming after the y-integration when multiplied with $x^{-\epsilon}(1 - x)^{-\epsilon}$ gives a ϵ^0 contribution which after the x-integration is given by

$$g_{\epsilon^{-1}}^{(0)}(z, \bar{z}) = -G\left(0, 0, \frac{z - 1}{z - \bar{z}}, 1\right) - G\left(0, 0, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right) - G\left(0, 1, \frac{z - 1}{z - \bar{z}}, 1\right) - G\left(0, 1, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right) - G\left(0, \frac{z - 1}{z - \bar{z}}, 0, 1\right)$$

$$- G\left(0, \frac{z - 1}{z - \bar{z}}, 1, 1\right) - G\left(0, \frac{\bar{z} - 1}{\bar{z} - z}, 0, 1\right) - G\left(0, \frac{\bar{z} - 1}{\bar{z} - z}, 1, 1\right) + G\left(1, 0, \frac{z - 1}{z - \bar{z}}, 1\right) + G\left(1, 0, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right)$$

$$+ G\left(1, 1, \frac{z - 1}{z - \bar{z}}, 1\right) + G\left(1, 1, \frac{\bar{z} - 1}{\bar{z} - z}, 1\right) + G\left(1, \frac{z - 1}{z - \bar{z}}, 0, 1\right) + G\left(1, \frac{z - 1}{z - \bar{z}}, 1, 1\right) + G\left(1, \frac{\bar{z} - 1}{\bar{z} - z}, 0, 1\right)$$

$$+ G\left(1, \frac{\bar{z} - 1}{\bar{z} - z}, 1, 1\right) \tag{65}$$

Here also, the final result is divergenceless after the cancellation of the divergence arising from individual terms. Note that this is a kind of a feedback contribution that is arising only because of the need to cancel the singularities using Eq. (64) and does not have an existence of its own if there had not been any endpoint singularity due to the y-integration at $y = 1$. The divergences that occur due to the end-point singularities at $x = 0$ and $x = 1$ while doing x-integration add up to zero giving no net contribution.

(iii) The contribution from the ϵ^1 term:

After expanding the hypergeometric $_2F_1$ functions in Eq. (59) in orders of epsilon, the obtained ϵ^1 term gives back a contribution to the ϵ^0 term since we need to get rid of the endpoint singularities arising from the y-integration at $y = 1$. It is given by

$$\begin{aligned}
g_{\epsilon^1}^{(0)}(z,\bar{z}) = -\Big(2G(1,z)\Big(&G\Big(0,\frac{z-1}{z-\bar{z}},1\Big) - G\Big(0,\frac{z}{z-\bar{z}},1\Big) + G\Big(0,\frac{z-z\bar{z}}{z-\bar{z}},1\Big) - G\Big(1,\frac{z-1}{z-\bar{z}},1\Big) \\
&+ G\Big(1,\frac{z}{z-\bar{z}},1\Big) - G\Big(1,\frac{z-z\bar{z}}{z-\bar{z}},1\Big)\Big) + G(1,\bar{z})\Big(G\Big(0,\frac{z-1}{z-\bar{z}},1\Big) + G\Big(0,\frac{\bar{z}-1}{\bar{z}-z},1\Big) \\
&- G\Big(1,\frac{z-1}{z-\bar{z}},1\Big) - G\Big(1,\frac{\bar{z}-1}{\bar{z}-z},1\Big)\Big) + 2G\Big(0,0,\frac{z-1}{z-\bar{z}},1\Big) + 2G\Big(0,0,\frac{\bar{z}-1}{\bar{z}-z},1\Big) \\
&+ 2G\Big(0,1,\frac{z-1}{z-\bar{z}},1\Big) + 2G\Big(0,1,\frac{\bar{z}-1}{\bar{z}-z},1\Big) + G\Big(0,\frac{z-1}{z-\bar{z}},\frac{z-1}{z-\bar{z}},1\Big) - 2G\Big(0,\frac{z}{z-\bar{z}},\frac{z-1}{z-\bar{z}},1\Big) \\
&- 2G\Big(0,\frac{z-z\bar{z}}{z-\bar{z}},\frac{\bar{z}-1}{\bar{z}-z},1\Big) + G\Big(0,\frac{\bar{z}-1}{\bar{z}-z},\frac{\bar{z}-1}{\bar{z}-z},1\Big) - 2G\Big(1,0,\frac{z-1}{z-\bar{z}},1\Big) - 2G\Big(1,0,\frac{\bar{z}-1}{\bar{z}-z},1\Big) \\
&- 2G\Big(1,1,\frac{z-1}{z-\bar{z}},1\Big) - 2G\Big(1,1,\frac{\bar{z}-1}{\bar{z}-z},1\Big) - G\Big(1,\frac{z-1}{z-\bar{z}},\frac{z-1}{z-\bar{z}},1\Big) + 2G\Big(1,\frac{z}{z-\bar{z}},\frac{z-1}{z-\bar{z}},1\Big) \\
&+ 2G\Big(1,\frac{z-z\bar{z}}{z-\bar{z}},\frac{\bar{z}-1}{\bar{z}-z},1\Big) - G\Big(1,\frac{\bar{z}-1}{\bar{z}-z},\frac{\bar{z}-1}{\bar{z}-z},1\Big)\Big).
\end{aligned}$$
(66)

Again the divergent parts cancel here. This term is again a feedback term that is produced only due to the endpoint singularity at $y = 1$. Here also the divergences due to the endpoint singularities of x-integration add up to zero. Note that as discussed earlier, the contribution due to this term could not have been calculated if we had taken the result from literature for Eq. (50) only up to ϵ^0 term.

(iv) The contribution from the prefactor term:

There is an additional prefactor term $(2z\bar{z})^{-\epsilon}$ in Eq. (59) which contributes to the overall ϵ^0 term through its product with the ϵ^{-1} term $g^{(-1)}(z,\bar{z})$. It is given by

$$g_P^{(0)}(z,\bar{z}) = (G(0,\bar{z}) + G(0,z))\Big(G\Big(0,\frac{z-1}{z-\bar{z}},1\Big) + G\Big(0,\frac{\bar{z}-1}{\bar{z}-z},1\Big) - G\Big(1,\frac{z-1}{z-\bar{z}},1\Big) - G\Big(1,\frac{\bar{z}-1}{\bar{z}-z},1\Big)\Big).$$
(67)

Note that for each contribution, the divergences inside them always cancel out. This has been explicitly checked for each contribution. This term is again produced only because of the steps in Eqs. (54) and (55) to cancel the singularities.

The last three contributions are not at all trivial to guess at first in the sense that the prefactor, the lower order (ϵ^{-1}) terms, and higher-order (ϵ^1) terms contribute back to the required result.

VI. SUMMING OF THE CUTS

Now we have all the contributions from both the cuts and hence we can find the net cut across the p_2^2 channel, which is a sum of the two cuts. Since the prefactors of both the cuts match, we can just focus on the multiple polylogarithms part. As expected, since the original Feynman integral is not divergent in 4 dimensions, the divergences (ϵ^{-1} terms) cancel:

$$f^{(-1)}(z,\bar{z}) + g^{(-1)}(z,\bar{z}) = 0.$$
(68)

Again as expected, the ϵ^0 (finite) terms do not cancel, and they give the following result.

$$\begin{aligned}
f^{(0)}&(z,\bar{z}) + g^{(0)}(z,\bar{z}) \\
&= -G(1,\bar{z})G\Big(0,\frac{z-1}{z-\bar{z}},1\Big) + G(0,(z-1)(\bar{z}-1))G\Big(0,\frac{z}{z-\bar{z}},1\Big) - 2G(1,z)G\Big(0,\frac{z}{z-\bar{z}},1\Big) \\
&\quad - G(1,\bar{z})G\Big(0,\frac{z}{z-\bar{z}},1\Big) + G(1,\bar{z})G\Big(0,\frac{(z-1)\bar{z}}{z-\bar{z}},1\Big) - G(0,(z-1)(\bar{z}-1))G\Big(0,\frac{z-z\bar{z}}{z-\bar{z}},1\Big) \\
&\quad + 2G(1,z)G\Big(0,\frac{z-z\bar{z}}{z-\bar{z}},1\Big) + G(1,\bar{z})G\Big(0,\frac{z-z\bar{z}}{z-\bar{z}},1\Big) - G(1,\bar{z})G\Big(0,-\frac{\bar{z}}{z-\bar{z}},1\Big)
\end{aligned}$$

$$- G(1,z)G\left(0,\frac{\bar{z}-1}{\bar{z}-z},1\right) - G(1,\bar{z})G\left(\frac{z-1}{z-\bar{z}},1,1\right) + G(0,(z-1)(\bar{z}-1))G\left(\frac{z}{z-\bar{z}},1,1\right)$$

$$- 2G(1,z)G\left(\frac{z}{z-\bar{z}},1,1\right) - G(1,\bar{z})G\left(\frac{z}{z-\bar{z}},1,1\right) + G(1,\bar{z})G\left(\frac{(z-1)\bar{z}}{z-\bar{z}},1,1\right)$$

$$- G(0,(z-1)(\bar{z}-1))G\left(\frac{z-z\bar{z}}{z-\bar{z}},1,1\right) + 2G(1,z)G\left(\frac{z-z\bar{z}}{z-\bar{z}},1,1\right) + G(1,\bar{z})G\left(\frac{z-z\bar{z}}{z-\bar{z}},1,1\right)$$

$$- G(1,\bar{z})G\left(-\frac{\bar{z}}{z-\bar{z}},1,1\right) - G(1,z)G\left(\frac{\bar{z}-1}{\bar{z}-z},1,1\right) + G(0,z)\left(G\left(0,\frac{z-1}{z-\bar{z}},1\right) + G\left(0,\frac{\bar{z}-1}{\bar{z}-z},1\right)\right.$$

$$\left. + G\left(\frac{z-1}{z-\bar{z}},1,1\right) + G\left(\frac{\bar{z}-1}{\bar{z}-z},1,1\right)\right) + G(0,\bar{z})\left(G\left(0,\frac{z-1}{z-\bar{z}},1\right) + G\left(0,\frac{\bar{z}-1}{\bar{z}-z},1\right)\right.$$

$$\left. + G\left(\frac{z-1}{z-\bar{z}},1,1\right) + G\left(\frac{\bar{z}-1}{\bar{z}-z},1,1\right)\right) + G\left(0,\frac{z-1}{z-\bar{z}},\frac{z-1}{z-\bar{z}},1\right) - G\left(0,\frac{z}{z-\bar{z}},\frac{z-1}{z-\bar{z}},1\right)$$

$$- G\left(0,\frac{(z-1)\bar{z}}{z-\bar{z}},\frac{z-1}{z-\bar{z}},1\right) - G\left(0,\frac{z-z\bar{z}}{z-\bar{z}},\frac{\bar{z}-1}{\bar{z}-z},1\right) - G\left(0,-\frac{\bar{z}}{z-\bar{z}},\frac{\bar{z}-1}{\bar{z}-z},1\right)$$

$$+ G\left(0,\frac{\bar{z}-1}{\bar{z}-z},\frac{\bar{z}-1}{\bar{z}-z},1\right) + G\left(\frac{z-1}{z-\bar{z}},1,\frac{z-1}{z-\bar{z}},1\right) + G\left(\frac{z-1}{z-\bar{z}},\frac{z-1}{z-\bar{z}},1,1\right) - G\left(\frac{z}{z-\bar{z}},1,\frac{z-1}{z-\bar{z}},1\right)$$

$$- G\left(\frac{z}{z-\bar{z}},\frac{z-1}{z-\bar{z}},1,1\right) - G\left(\frac{(z-1)\bar{z}}{z-\bar{z}},1,\frac{z-1}{z-\bar{z}},1\right) - G\left(\frac{(z-1)\bar{z}}{z-\bar{z}},\frac{z-1}{z-\bar{z}},1,1\right)$$

$$- G\left(\frac{z-z\bar{z}}{z-\bar{z}},1,\frac{\bar{z}-1}{\bar{z}-z},1\right) - G\left(\frac{z-z\bar{z}}{z-\bar{z}},\frac{\bar{z}-1}{\bar{z}-z},1,1\right) - G\left(-\frac{\bar{z}}{z-\bar{z}},1,\frac{\bar{z}-1}{\bar{z}-z},1\right)$$

$$- G\left(-\frac{\bar{z}}{z-\bar{z}},\frac{\bar{z}-1}{\bar{z}-z},1,1\right) + G\left(\frac{\bar{z}-1}{\bar{z}-z},1,\frac{\bar{z}-1}{\bar{z}-z},1\right) + G\left(\frac{\bar{z}-1}{\bar{z}-z},\frac{\bar{z}-1}{\bar{z}-z},1,1\right). \tag{69}$$

Here we have used the command ShuffleRegulate of PolyLogTools to cancel the divergent terms within themselves and rewrite the whole result with each term being nondivergent (see Appendix).

VII. RECONSTRUCTION OF THE SYMBOL OF THE ORIGINAL FEYNMAN INTEGRAL

As discussed earlier, the original Feynman integral can be reconstructed using the symbol $\mathcal{S}(F)$ which is equal to the maximal iteration of the coproduct corresponding to the partition $(1,...,1)$ $\Delta_{1,...,1}(F)$. Thus, for getting this, we will first find out the maximal iteration of the coproduct for our obtained result Eq. (69). This can be done using the commands SymbolExpand and SymbolMap of PolyLogTools, and we get the following result:

$$- 2(-(1-z)\otimes(1-z)\otimes\bar{z} - (1-z)\otimes\bar{z}\otimes(1-z) + (1-z)\otimes\bar{z}\otimes(1-\bar{z}) + z\otimes(1-z)\otimes(1-\bar{z})$$

$$+ z\otimes(1-\bar{z})\otimes(1-z) - z\otimes(1-\bar{z})\otimes(1-\bar{z}) + (1-\bar{z})\otimes z\otimes(1-z) - (1-\bar{z})\otimes z\otimes(1-\bar{z})$$

$$- (1-\bar{z})\otimes(1-\bar{z})\otimes z + (1-\bar{z})\otimes(1-\bar{z})\otimes\bar{z} - \bar{z}\otimes(1-z)\otimes(1-z) + \bar{z}\otimes(1-z)\otimes(1-\bar{z})$$

$$+ \bar{z}\otimes(1-\bar{z})\otimes(1-z) - \bar{z}\otimes(1-\bar{z})\otimes(1-\bar{z}) + (1-z)\otimes(1-z)\otimes z - z\otimes(1-z)\otimes(1-z)) \tag{70}$$

Now we have to apply the first entry condition and the integrability condition in order to get the $\Delta_{1,...,1}(F)$. Since it is the p_2^2 channel cut we need to have $z\bar{z}$ as the first entry to the above expression:

$$(z\bar{z})\otimes[-2(-(1-z)\otimes(1-z)\otimes\bar{z} - (1-z)\otimes\bar{z}\otimes(1-z) + (1-z)\otimes\bar{z}\otimes(1-\bar{z})$$

$$+ z\otimes(1-z)\otimes(1-\bar{z}) + z\otimes(1-\bar{z})\otimes(1-z) - z\otimes(1-\bar{z})\otimes(1-\bar{z}) + (1-\bar{z})\otimes z\otimes(1-z)$$

$$- (1-\bar{z})\otimes z\otimes(1-\bar{z}) - (1-\bar{z})\otimes(1-\bar{z})\otimes z + (1-\bar{z})\otimes(1-\bar{z})\otimes\bar{z} - \bar{z}\otimes(1-z)\otimes(1-z)$$

$$+ \bar{z}\otimes(1-z)\otimes(1-\bar{z}) + \bar{z}\otimes(1-\bar{z})\otimes(1-z) - \bar{z}\otimes(1-\bar{z})\otimes(1-\bar{z}) + (1-z)\otimes(1-z)\otimes z$$

$$- z\otimes(1-z)\otimes(1-z))] \tag{71}$$

Now in order to satisfy the integrability condition and the first entry condition, we add the following terms to this expression.

$$
\begin{aligned}
2((1-z) &\otimes (1-z) \otimes z \otimes z - (1-z) \otimes (1-z) \otimes z \otimes \bar{z} - (1-z) \otimes (1-z) \otimes \bar{z} \otimes z \\
&+ (1-z) \otimes (1-z) \otimes \bar{z} \otimes \bar{z} - (1-z) \otimes z \otimes z \otimes (1-z) + (1-z) \otimes z \otimes z \otimes (1-\bar{z}) \\
&+ (1-z) \otimes z \otimes (1-\bar{z}) \otimes z - (1-z) \otimes z \otimes (1-\bar{z}) \otimes \bar{z} + (1-z) \otimes (1-\bar{z}) \otimes z \otimes z \\
&- (1-z) \otimes (1-\bar{z}) \otimes z \otimes \bar{z} - (1-z) \otimes (1-\bar{z}) \otimes \bar{z} \otimes z + (1-z) \otimes (1-\bar{z}) \otimes \bar{z} \otimes \bar{z} \\
&- (1-z) \otimes \bar{z} \otimes (1-z) \otimes z + (1-z) \otimes \bar{z} \otimes (1-z) \otimes \bar{z} + (1-z) \otimes \bar{z} \otimes \bar{z} \otimes (1-z) \\
&- (1-z) \otimes \bar{z} \otimes \bar{z} \otimes (1-\bar{z}) - z \otimes (1-z) \otimes (1-z) \otimes z + z \otimes (1-z) \otimes (1-z) \otimes \bar{z} \\
&+ (1-\bar{z}) \otimes (1-z) \otimes z \otimes \bar{z} - (1-\bar{z}) \otimes (1-z) \otimes \bar{z} \otimes z + (1-\bar{z}) \otimes (1-z) \otimes \bar{z} \otimes \bar{z} \\
&- (1-\bar{z}) \otimes z \otimes z \otimes (1-z) + (1-\bar{z}) \otimes z \otimes z \otimes (1-\bar{z}) + (1-\bar{z}) \otimes z \otimes (1-\bar{z}) \otimes z \\
&- (1-\bar{z}) \otimes z \otimes (1-\bar{z}) \otimes \bar{z} + (1-\bar{z}) \otimes (1-\bar{z}) \otimes z \otimes z - (1-\bar{z}) \otimes (1-\bar{z}) \otimes z \otimes \bar{z} \\
&- (1-\bar{z}) \otimes (1-\bar{z}) \otimes \bar{z} \otimes z + (1-\bar{z}) \otimes (1-\bar{z}) \otimes \bar{z} \otimes \bar{z} - (1-\bar{z}) \otimes \bar{z} \otimes (1-z) \otimes z \\
&+ (1-\bar{z}) \otimes \bar{z} \otimes (1-z) \otimes \bar{z} + (1-\bar{z}) \otimes \bar{z} \otimes \bar{z} \otimes (1-z) - (1-\bar{z}) \otimes \bar{z} \otimes \bar{z} \otimes (1-\bar{z}))
\end{aligned}
\tag{72}
$$

Finally adding the above two expressions we get

$$
\begin{aligned}
2((1-z) &\otimes (1-z) \otimes z \otimes z - (1-z) \otimes (1-z) \otimes z \otimes \bar{z} - (1-z) \otimes (1-z) \otimes \bar{z} \otimes z \\
&+ (1-z) \otimes (1-z) \otimes \bar{z} \otimes \bar{z} - (1-z) \otimes z \otimes z \otimes (1-z) + (1-z) \otimes z \otimes z \otimes (1-\bar{z}) \\
&+ (1-z) \otimes z \otimes (1-\bar{z}) \otimes z - (1-z) \otimes z \otimes (1-\bar{z}) \otimes \bar{z} + (1-z) \otimes (1-\bar{z}) \otimes z \otimes z \\
&- (1-z) \otimes (1-\bar{z}) \otimes z \otimes \bar{z} - (1-z) \otimes (1-\bar{z}) \otimes \bar{z} \otimes z + (1-z) \otimes (1-\bar{z}) \otimes \bar{z} \otimes \bar{z} \\
&- (1-z) \otimes \bar{z} \otimes (1-z) \otimes z + (1-z) \otimes \bar{z} \otimes (1-z) \otimes \bar{z} + (1-z) \otimes \bar{z} \otimes \bar{z} \otimes (1-z) \\
&- (1-z) \otimes \bar{z} \otimes \bar{z} \otimes (1-\bar{z}) - z \otimes (1-z) \otimes (1-z) \otimes z + z \otimes (1-z) \otimes (1-z) \otimes \bar{z} \\
&+ z \otimes (1-z) \otimes \bar{z} \otimes (1-z) - z \otimes (1-z) \otimes \bar{z} \otimes (1-\bar{z}) + z \otimes z \otimes (1-z) \otimes (1-z) \\
&- z \otimes z \otimes (1-z) \otimes (1-\bar{z}) - z \otimes z \otimes (1-\bar{z}) \otimes (1-z) + z \otimes z \otimes (1-\bar{z}) \otimes (1-\bar{z}) \\
&- z \otimes (1-\bar{z}) \otimes z \otimes (1-z) + z \otimes (1-\bar{z}) \otimes z \otimes (1-\bar{z}) + z \otimes (1-\bar{z}) \otimes (1-\bar{z}) \otimes z \\
&- z \otimes (1-\bar{z}) \otimes (1-\bar{z}) \otimes \bar{z} + z \otimes \bar{z} \otimes (1-z) \otimes (1-z) - z \otimes \bar{z} \otimes (1-z) \otimes (1-\bar{z}) \\
&- z \otimes \bar{z} \otimes (1-\bar{z}) \otimes (1-z) + z \otimes \bar{z} \otimes (1-\bar{z}) \otimes (1-\bar{z}) + (1-\bar{z}) \otimes (1-z) \otimes z \otimes z \\
&- (1-\bar{z}) \otimes (1-z) \otimes z \otimes \bar{z} - (1-\bar{z}) \otimes (1-z) \otimes \bar{z} \otimes z + (1-\bar{z}) \otimes (1-z) \otimes \bar{z} \otimes \bar{z} \\
&- (1-\bar{z}) \otimes z \otimes z \otimes (1-z) + (1-\bar{z}) \otimes z \otimes z \otimes (1-\bar{z}) + (1-\bar{z}) \otimes z \otimes (1-\bar{z}) \otimes z \\
&- (1-\bar{z}) \otimes z \otimes (1-\bar{z}) \otimes \bar{z} + (1-\bar{z}) \otimes (1-\bar{z}) \otimes z \otimes z - (1-\bar{z}) \otimes (1-\bar{z}) \otimes z \otimes \bar{z} \\
&- (1-\bar{z}) \otimes (1-\bar{z}) \otimes \bar{z} \otimes z + (1-\bar{z}) \otimes (1-\bar{z}) \otimes \bar{z} \otimes \bar{z} - (1-\bar{z}) \otimes \bar{z} \otimes (1-z) \otimes z \\
&+ (1-\bar{z}) \otimes \bar{z} \otimes (1-z) \otimes \bar{z} + (1-\bar{z}) \otimes \bar{z} \otimes \bar{z} \otimes (1-z) - (1-\bar{z}) \otimes \bar{z} \otimes \bar{z} \otimes (1-\bar{z}) \\
&- \bar{z} \otimes (1-z) \otimes (1-z) \otimes z + \bar{z} \otimes (1-z) \otimes (1-z) \otimes \bar{z} + \bar{z} \otimes (1-z) \otimes \bar{z} \otimes (1-z) \\
&- \bar{z} \otimes (1-z) \otimes \bar{z} \otimes (1-\bar{z}) + \bar{z} \otimes z \otimes (1-z) \otimes (1-z) - \bar{z} \otimes z \otimes (1-z) \otimes (1-\bar{z}) \\
&- \bar{z} \otimes z \otimes (1-\bar{z}) \otimes (1-z) + \bar{z} \otimes z \otimes (1-\bar{z}) \otimes (1-\bar{z}) - \bar{z} \otimes (1-\bar{z}) \otimes z \otimes (1-z) \\
&+ \bar{z} \otimes (1-\bar{z}) \otimes z \otimes (1-\bar{z}) + \bar{z} \otimes (1-\bar{z}) \otimes (1-\bar{z}) \otimes z - \bar{z} \otimes (1-\bar{z}) \otimes (1-\bar{z}) \otimes \bar{z} \\
&+ \bar{z} \otimes \bar{z} \otimes (1-z) \otimes (1-z) - \bar{z} \otimes \bar{z} \otimes (1-z) \otimes (1-\bar{z}) - \bar{z} \otimes \bar{z} \otimes (1-\bar{z}) \otimes (1-z) \\
&+ \bar{z} \otimes \bar{z} \otimes (1-\bar{z}) \otimes (1-\bar{z})).
\end{aligned}
\tag{73}
$$

At this point, it satisfies both the integrability and the first entry condition. Also, this coincides with the coproduct of the original Feynman integral, Eq. (29) when it is written in terms of the variables z and $\bar{z}$.

$$C(p_1^2, p_2^2, p_3^2) = \frac{(2\text{Li}_2(\frac{1}{\bar{z}}) - \log(-z)\log(-\bar{z}) + (\log(z\bar{z}))(\log(\bar{z}) + \log(-\frac{1-z}{1-\bar{z}})) + 2\text{Li}_2(z) + \frac{\pi^2}{3})^2}{(z - \bar{z})^2} \tag{74}$$

We can reconstruct this function using the obtained coproduct by predicting the family of functions that gives this particular coproduct and then extract out that unique function out of the family of the functions by using the rule

$$\text{Disc}_{p_2^2} C(p_1^2, p_2^2, p_3^2) \cong \text{Cut}_{p_2^2} C(p_1^2, p_2^2, p_3^2). \tag{75}$$

This method is well described in Sec. 7 of [1]. Thus in this section we have shown the consistency of the computation of the symbol obtained from the known two loop Feynman integral computation. Equation (73) is the one of the main results of this paper which verifies the Duhr conjecture [5].

VIII. RECONSTRUCTION OF THE FULL FUNCTION

The symbol alphabet for our nonplanar case is same as the massless triangle and massless two-loop ladder. It is given by $\mathcal{A} = \{z, \bar{z}, 1 - z, 1 - \bar{z}\}$. We will use the fact that the most general class of functions giving rise to this symbol alphabet and satisfying the first entry condition are the single-valued harmonic polylogarithms [28].

In order to achieve this goal, we look at each term of the obtained symbol Eq. (73). We need to examine the entire set of HPl's of weight less than 4 which can give rise to the symbol using the commands SymbolExpand and SymbolMap of PolyLogTools. For example the symbol generated by the HPL $H(0, 0, 1, 1, z)$ is the first term $(1 - z) \otimes (1 - z) \otimes z \otimes z$. We can clearly see the pattern of the indices containing 0's and 1's which gives rise to the required symbol. Therefore it may be seen that the index 1 gives rise to terms of $(1 - z)$ kind and index 0 gives rise to terms of (z) kind and the index -1 gives rise to $(1 + z)$ kind which does not appear in the whole symbol and hence the HPL's containing index -1 do not contribute to the final result. This pattern follows from the structure of the HPL's and their associated symbols.

For the terms which contain both the variables z and $\bar{z}$ we can only have product HPL's of lower weight containing the two variables contributing them. The total weight of the product terms is 4 as required. Here also the arguments are such that they give rise to particular kind of symbol. For example, the product HPL $H(0, 1, 1, z)H(0, \bar{z})$ gives rise to the following symbol:

$$(1 - z) \otimes (1 - z) \otimes z \otimes \bar{z} + (1 - z) \otimes (1 - z) \otimes \bar{z} \otimes z + (1 - z) \otimes \bar{z} \otimes (1 - z) \otimes z + \bar{z} \otimes (1 - z) \otimes (1 - z) \otimes z$$

Again here we have only $(\bar{z})$ inside the symbol and no $(1 - \bar{z})$ and hence according to the pattern observed above, we have the index 0 inside the HPL with variable $\bar{z}$. Also all of these four terms are present inside our symbol Eq. (73) and hence the product HPL $H(0, 1, 1, z)H(0, \bar{z})$ contributes to our symbol. Its existence as a contributor to our symbol is predicted by looking at the second term inside the symbol $(1 - z) \otimes (1 - z) \otimes z \otimes \bar{z}$ and comparing with the observed pattern. Likewise we can find out all the HPL's which give rise to the whole symbol by looking at the form of the remaining terms in the symbol which are not yet covered by any HPL's. Using this procedure we get the total sum of all such HPL's which is given by

$$
\begin{aligned}
4(&-H(0, 1, z)H(0, 1, \bar{z}) - H(1, 0, z)H(1, 0, \bar{z}) + H(1, 1, z)H(0, 0, \bar{z}) + H(0, 0, z)H(1, 1, \bar{z}) \\
&+ H(0, 0, 1, z)H(1, \bar{z}) + H(1, z)H(0, 0, 1, \bar{z}) - H(0, 1, 1, z)H(0, \bar{z}) - H(0, z)H(0, 1, 1, \bar{z}) \\
&- H(1, 0, 0, z)H(1, \bar{z}) - H(1, z)H(1, 0, 0, \bar{z}) + H(1, 1, 0, z)H(0, \bar{z}) + H(0, z)H(1, 1, 0, \bar{z}) \\
&+ H(0, 0, 1, 1, \bar{z}) - H(0, 1, 1, 0, \bar{z}) - H(1, 0, 0, 1, \bar{z}) + H(1, 1, 0, 0, \bar{z}) + H(0, 0, 1, 1, z) \\
&- H(0, 1, 1, 0, z) - H(1, 0, 0, 1, z) + H(1, 1, 0, 0, z)) + \sum_i c_i H_i(z, \bar{z}).
\end{aligned}
\tag{76}
$$

Here $H_i(z, \bar{z})$ are the lower weight HPL's that do not contribute to the symbol Eq. (73) but contribute to the final full function and the c_i are coefficients which also carry a weight such that each term in the sum has total weight 4. The total sum $\sum_i c_i H_i(z, \bar{z})$ consists of a basis of weight 3, 2, 1 and 0 HPL's such that any HPL of respective weight

can be represented by a linear combination of the basis HPL's with suitable constants c_i.

Note that for the functions of pure type, i.e., which are not the product of two HPL's there may be the product of two lower weight HPL's giving rise to the same symbol but since using the shuffle product of HPL's we can rewrite the product HPL's in terms of pure functions, therefore

the pure functions form a basis and hence it is enough to represent it in the present form. This function when written in terms of classical polylogarithms by first using the command HToHPL of PolyLogTools and then using HPLConvertToKnownFunctions command of HPL Package is equal to the following expression:

$$\left(-2\mathrm{Li}_2(\bar{z}) - \log^2(\bar{z}) - \log(z)\log(\bar{z}) + (\log(\bar{z}) + \log(z))\left(\log\left(\frac{1-z}{1-\bar{z}}\right) + \log(\bar{z})\right) + 2\mathrm{Li}_2(z)\right)^2$$
$$+ \sum_i c_i H_i(z, \bar{z}) \tag{77}$$

This result has to be further treated in order to get the required correspondence with Eq. (74), which reads after employing the dilogarithm identities as

$$\left(-2\mathrm{Li}_2(\bar{z}) - \log^2(-\bar{z}) - \log(-z)\log(-\bar{z}) + (\log(z\bar{z}))\left(\log\left(-\frac{1-z}{1-\bar{z}}\right) + \log(\bar{z})\right) + 2\mathrm{Li}_2(z)\right)^2.$$

The expression above differs from the known result in the signs of the arguments of the simple logarithms. This is not worrisome, because the symbol method alone does not fix those signs in Eq. (77). This ambiguity is taken care of by the requirement that the result satisfies Cutkosky's rule in Eq. (75) which is achieved by fixing the open constants c_i accordingly.

To do this we first find out the discontinuity of Eq. (77) across the p_2^2 channel. The HPL's with weight less than or equal to 3 can be represented in terms of classical and Nielsen polylogarithms and hence we can easily find out the discontinuity. Then we can numerically evaluate the original integral at certain kinematical points in order to determine the unknown constants c_i. The analytical expression of the rhs of Eq. (75) is already evaluated and is given by Eq. (69). We use the Ginsh command of PolyLogTools to find out the numerical value of rhs at the selected kinematical points. Finally, the constant of the weight 4

term which does not appear in the discontinuity, is calculated by numerically evaluating the Feynman integral at a suitable (integral not divergent) kinematical point (integral not divergent). This is in accordance with the general observations in Sec. 7 of [1] in the event of such a procedure is required. The list of the $H_i(z,\bar{z})'s$ and the corresponding $c_i's$ which actually contribute to our result is given in Table I.

Thus Eq. (76) is the other main result of this paper, and it is on the same footing as Eq. (7.16) of [1]. Thus we have completed the nonplanar extension of the planar results of [1].

IX. DISCUSSION AND CONCLUSIONS

In [1], inspired by the work of Duhr [5], Abreu *et al.* have initiated a new method of calculating Feynman integrals which is based on cut diagrams and Hopf algebras. While the method worked well for planar two-loop diagrams, its extension to nonplanar diagrams and also to more than two loops did not seem clear. In the present paper, therefore, we have considered a nonplanar two-loop diagram to study the new method in this case as well. In particular, we have considered the nonplanar ladder diagram of [7] which is well studied by other methods and can be conveniently compared to the planar ladder. Following the new method, we have calculated the cuts in the p_2^2 channel (see Fig. 1) consisting of a two-propagator cut and three-propagator cut which result in Eqs. (46)–(47) and (60)–(67), respectively. the total cut results are given by Eqs. (68)–(69) and the total symbol by Eq. (72). The kinematics of this nonplanar problem are very much similar to the planar case in the sense that for the first cut (two-propagators), the result was obtained by first evaluating a one-loop diagram and using it

TABLE I. Table of $H_i(z,\bar{z})'s$ and the corresponding $c_i's$.

$H_i(z,\bar{z})$	c_i	$H_i(z,\bar{z})$	c_i
$H(0,0,1,z)$	$-8i\pi$	$H(1,z)H(0,0,\bar{z})$	$16i\pi$
$H(0,0,1,\bar{z})$	$16i\pi$	$H(1,\bar{z})H(0,0,z)$	$-8i\pi$
$H(1,0,0,z)$	$8i\pi$	$H(0,\bar{z})H(0,1,z)$	$-4i\pi$
$H(1,0,0,\bar{z})$	$-16i\pi$	$H(0,z)H(0,1,\bar{z})$	$-4i\pi$
$H(1,0,z)$	$-4\pi^2$	$H(0,\bar{z})H(1,0,z)$	$12i\pi$
$H(1,0,\bar{z})$	$4\pi^2$	$H(0,z)H(1,0,\bar{z})$	$-12i\pi$
$H(0,1,z)$	$4\pi^2$	$H(0,\bar{z})H(1,z)$	$-4\pi^2$
$H(0,1,\bar{z})$	$-4\pi^2$	$H(0,z)H(0,\bar{z})$	$-16\pi^2$
$H(0,0,z)$	$-8\pi^2$	$H(0,z)H(1,\bar{z})$	$4\pi^2$
$H(0,0,\bar{z})$	$-32\pi^2$	$H(0,z)$	$-8i\pi^3$
1	$4\pi^4$	$H(0,\bar{z})$	$-16i\pi^3$

in the subsequent calculation. Also, in the latter cut, the result of the cut of a one-loop diagram is used as input for the ensuing calculations.

One difference between the planar and the nonplanar cases is that in the latter the cut-propagators for the second cut are not coplanar and yet when the ensuing result is added to that coming from the first cut, the total expression is finite. Further, for the first cut, we needed a two-mass easy box rather than a two-mass triangle that was required for the planar case. A noteworthy difference is that we needed to calculate the cut of a three-mass box diagram up to ϵ^1 order, in contrast to only ϵ^0 order for the planar case. Also, there are additional contributions from ϵ^{-1}, ϵ^1 and the prefactor terms for the second cut. It may be noted that the need to go to these orders in ϵ in the nonplanar case could not have been foreseen until the computation was performed.

Compared to the planar case, the nonplanar calculations were more complex because, for the second cut, three integrations after expansion of the hypergeometric $_2F_1$ functions via HypExp were needed. Without using PolyLogTools, it would have been a challenging task. In [2], a specific technique for treating the cuts of the massive internal lines has been introduced. We have adopted the same technique for the massless case and applied it to get the desired results.

It was further discussed in [1], that there is no known algorithm to find the symbol alphabet, which is $\mathcal{A} = \{z, \bar{z}, 1 - z, 1 - \bar{z}\}$ for the nonplanar case. The symbol turns out to be equal to that of the one-loop triangle and the planar two-loop ladder diagram discussed in [1] owing to the fact that they fall in the same family of functions [7].

In [1], it was shown for the planar case, in Eq. (7.16) of that paper, that the generalized result or the class of Feynman integrals which can give rise to the same symbol can be obtained. The search for such a structure in the present case is significantly more complicated and has been carried out and reported in Sec. VIII.

The main points of the present work are to show that (i) the Hopf algebra based method also holds for the nonplanar case, thereby proving the Duhr conjecture [5] for this case, and (ii) that the full result can be reconstructed from the symbol by deploying several tools. The availability of the result (Eq. (30) of [7]) has allowed us to verify this calculation.[1]

The present work also demonstrates that with modern codes such as HypExp and PolyLogTools, one can now access more easily kinematic configurations that were largely out of reach before. It is satisfying to see that a deep understanding of the structure of the amplitudes as a sum of HPLs of suitable weight, and by a systematic use of the properties of the PolyLog tools can lead to the full result for complex calculations. It would of course be interesting to consider other nonplanar diagrams. It is intriguing to think of transformations that relate the planar and nonplanar configurations, which may be possible for the cut versions, thereby providing an answer to many intriguing questions, such as the extension of the Hopf algebra method to nonscalar Feynman integrals with complicated irreducible numerators and higher powers of propagators [32] or the mathematical properties (see for a recent discussion [33]) of the higher orders in ϵ.[2] The present manuscript is a first step in this direction and far from being the last.

It will also be interesting to investigate the usefulness of the results our work for the study of diagrammatic coaction. The diagrammatic coaction is the generalization of the coproduct of multiple polylogarithms to complete Feynman graphs. Recently Abreu *et al.* [31] have defined the diagrammatic coaction starting from one loop and extending it to certain class of (planar) two loop diagrams. The diagrammatic coaction possibly offers a very interesting way of studying Feynman integrals and their cuts, paving the way for an easier calculation, as compared to working to higher orders in the ϵ expansion involving multiple polylogarithms. Basically it sums up to all orders in ϵ, and the relation with cuts is upgraded in terms of a representation involving only diagrams. Thus an extension to nonplanar diagrams could be a project for the future.

APPENDIX: TECHNICAL CALCULATIONS

1. Cancellation of the divergences

Here we show explicitly that the divergences of the obtained total cut cancel out. A similar procedure is done for the individual contributions, which shows each of the contributions is independently divergenceless. As discussed earlier, the verification of the cancellation of the divergences is crucial as it is the basis of all types of contributions we have got in the results.

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[1]For the planar case, in [1], it has been stated that the result can be obtained by suitable manipulations of HPLs. Using the procedure detailed in this paper for the nonplanar case, we have also reproduced their results for the planar case ourselves.

[2]Even at the lowest relevant order in ϵ, the algebraic complexity is considerable and could not be done by software tools alone. Going to higher orders would require a whole new effort.

First of all, we use the command ShuffleRegulate on the sum of the first and the second cut to separate out the divergences, which gives us the following divergent terms, which are now separated from the finite terms.

$$
\begin{aligned}
\mathrm{HPL}(\{1\},1)&\Bigg(G\Big(\frac{z-1}{z-\bar{z}},1\Big)\mathrm{HPL}(\{1\},1)+G\Big(\frac{\bar{z}-1}{\bar{z}-z},1\Big)\mathrm{HPL}(\{1\},1)-G(1,z)G\Big(\frac{z-1}{z-\bar{z}},1\Big)\\
&-G(1,\bar{z})G\Big(\frac{z-1}{z-\bar{z}},1\Big)+G(1,z)G\Big(\frac{z}{z-\bar{z}},1\Big)+G(1,\bar{z})G\Big(\frac{(z-1)\bar{z}}{z-\bar{z}},1\Big)-G(1,z)G\Big(\frac{z-z\bar{z}}{z-\bar{z}},1\Big)\\
&-G(1,\bar{z})G\Big(-\frac{\bar{z}}{z-\bar{z}},1\Big)+G(0,z)\Big(G\Big(\frac{z-1}{z-\bar{z}},1\Big)+G\Big(\frac{\bar{z}-1}{\bar{z}-z},1\Big)\Big)+G\Big(\frac{z-1}{z-\bar{z}},0,1\Big)\\
&+G(0,\bar{z})\Big(G\Big(\frac{z-1}{z-\bar{z}},1\Big)+G\Big(\frac{\bar{z}-1}{\bar{z}-z},1\Big)\Big)+G\Big(\frac{z-1}{z-\bar{z}},1,1\Big)+G\Big(\frac{z}{z-\bar{z}},\frac{z-1}{z-\bar{z}},1\Big)+G\Big(\frac{\bar{z}-1}{\bar{z}-z},0,1\Big)\\
&-G\Big(\frac{(z-1)\bar{z}}{z-\bar{z}},\frac{z-1}{z-\bar{z}},1\Big)+G\Big(\frac{z-z\bar{z}}{z-\bar{z}},\frac{\bar{z}-1}{\bar{z}-z},1\Big)-G\Big(-\frac{\bar{z}}{z-\bar{z}},\frac{\bar{z}-1}{\bar{z}-z},1\Big)+G\Big(\frac{\bar{z}-1}{\bar{z}-z},1,1\Big)\Bigg)
\end{aligned}
\tag{A1}
$$

Here the term $\mathrm{HPL}(\{1\},1)$ is actually divergent, the rest of the expressions are finite. So we just need to check that whether these finite terms are canceling.

The first two terms are nothing but simple logarithms multiplied with $\mathrm{HPL}(\{1\},1)$, and it is very trivial to show that they cancel each other.

For the rest of the terms, we use the command Ginsh of PolyLogTools to check the result for different values of z and $\bar{z}$. It turns out that the result is always zero irrespective of what values of z and $\bar{z}$ we take, which shows that these finite terms cancel and hence also the divergences. We can also rewrite them in terms of normal polylogarithms using the definitions of the multiple polylogarithms and check they add up to zero using dilogarithm identities.

2. Realization of symbol of the Feynman integral

As discussed earlier after the total cut for a particular channel is obtained we use the first-entry condition and the integrability condition repeatedly in order to get the symbol. By analysing the final form of the symbol it turns

out that at least for the studied examples the full symbol is actually given by the following framework:

$$
\mathrm{Symbol}(F) = \Delta_{1,\dots,1}(F) = \sum_i p_i^2 \otimes \mathrm{Cut}_{p_i^2} F, \qquad i \neq 1
\tag{A2}
$$

This means that using the first-entry condition, the first entries of all the terms will always be a particular channel, and the next entry will always be the cut of that particular channel. This is confirmed by the fact that for our example, even if we have started with the other p_3^2 channel, we will end with the same symbol. Though we do not have a rigorous proof of this, it is true for other examples studied in [1].

This also suggests that using the cut for a particular channel, the first-entry condition, and the integrability condition, we can find out the cuts for the other channels. Also, if we know the cuts for all the channels except for anyone channel, we can find out the original symbol without using the first-entry condition and the integrability condition explicitly.

[1] S. Abreu, R. Britto, C. Duhr, and E. Gardi, From multiple unitarity cuts to the coproduct of Feynman integrals, J. High Energy Phys. 10 (2014) 125.

[2] S. Abreu, R. Britto, and H. Grönqvist, Cuts and coproducts of massive triangle diagrams, J. High Energy Phys. 07 (2015) 111.

[3] S. Abreu, R. Britto, C. Duhr, and E. Gardi, Cuts from residues: The one-loop case, J. High Energy Phys. 06 (2017) 114.

[4] S. Abreu, R. Britto, C. Duhr, and E. Gardi, Diagrammatic Hopf algebra of cut Feynman integrals: The one-loop case, J. High Energy Phys. 12 (2017) 090.

[5] C. Duhr, Hopf algebras, coproducts and symbols: An application to Higgs boson amplitudes, J. High Energy Phys. 08 (2012) 043.

[6] N. I. Usyukina and A. I. Davydychev, An approach to the evaluation of three and four point ladder diagrams, Phys. Lett. B **298**, 363 (1993).

[7] N. I. Ussyukina and A. I. Davydychev, Some exact results for two-loop diagrams with three and four external lines, Phys. At. Nucl. **56,** 1553 (1993), https://inspirehep.net/literature/345697.

[8] D. Chicherin, J. Henn, and V. Mitev, Bootstrapping pentagon functions, J. High Energy Phys. 05 (2018) 164.

[9] D. Chicherin, T. Gehrmann, J. M. Henn, N. A. Lo Presti, V. Mitev, and P. Wasser, Analytic result for the non-planar hexa-box integrals, J. High Energy Phys. 03 (2019) 042.

[10] S. Abreu, B. Page, and M. Zeng, Differential equations from unitarity cuts: Non-planar hexa-box integrals, J. High Energy Phys. 01 (2019) 006.

[11] S. Abreu, L. J. Dixon, E. Herrmann, B. Page, and M. Zeng, Two-Loop Five-Point Amplitude in $N = 4$ Super-Yang-Mills Theory, Phys. Rev. Lett. **122,** 121603 (2019).

[12] N. I. Usyukina and A. I. Davydychev, New results for two-loop off-shell three-point diagrams, Phys. Lett. B **332,** 159 (1994).

[13] G. Duplančić and B. Nižić, Dimensionally regulated one-loop box scalar integrals with massless internal lines, Eur. Phys. J. C **20,** 357 (2001).

[14] A. B. Goncharov, Multiple polylogarithms, cyclotomy and modular complexes, Math. Res. Lett. **5,** 497 (1998).

[15] A. B. Goncharov, Multiple polylogarithms and mixed Tate motives, arXiv:math/0103059.

[16] A. B. Goncharov, Galois symmetries of fundamental groupoids and noncommutative geometry, Duke Math. J. **128,** 209 (2005).

[17] A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich, Classical Polylogarithms for Amplitudes and Wilson Loops, Phys. Rev. Lett. **105,** 151605 (2010).

[18] K. T. Chen, Iterated path integrals, Bull. Am. Math. Soc. **83,** 831 (1977).

[19] A. B. Goncharov, A simple construction of Grassmannian polylogarithms, Adv. Math. **241,** 79 (2013).

[20] F. C. Brown, Multiple zeta values and periods of moduli spaces $\mathfrak{M}_{0,n}$, Ann. Sci. Ecole Norm. Sup. **42,** 371 (2009), http://www.numdam.org/item/ASENS_2009_4_42_3_371_0/.

[21] C. Duhr, H. Gangl, and J. R. Rhodes, From polygons and symbols to polylogarithmic functions, J. High Energy Phys. 10 (2012) 075.

[22] R. Cutkosky, Singularities and discontinuities of Feynman amplitudes, J. Math. Phys. (N.Y.) **1,** 429 (1960).

[23] G. 't Hooft and M. Veltman, Diagrammar, NATO Adv. Study Inst. Ser. B Phys. **4,** 177 (1974).

[24] M. Veltman, Diagrammatica: The path to Feynman rules, Cambridge Lect. Notes Phys. **4,** 1 (1994), https://www.cambridge.org/in/academic/subjects/physics/theoretical-physics-and-mathematical-physics/diagrammatica-path-feynman-diagrams?format=PB.

[25] L. Landau, On analytic properties of vertex parts in quantum field theory, Nucl. Phys. **13,** 181 (1959).

[26] D. Gaiotto, J. Maldacena, A. Sever, and P. Vieira, Pulling the straps of polygons, J. High Energy Phys. 12 (2011) 011.

[27] E. Remiddi and J. A. M. Vermaseren, Harmonic polylogarithms, Int. J. Mod. Phys. A **15,** 725 (2000).

[28] F. C. Brown, Single-valued multiple polylogarithms in one variable, C. R. Acad. Sci. Paris, Ser. I **338,** 527 (2004), https://www.cambridge.org/core/services/aop-cambridge-core/content/view/0CAE3B4DCE4B54AA9CA09ACBFA558111/S2050509414000188a.pdf/single-valued-motivic-periods-and-multiple-zeta-values.pdf.

[29] C. Duhr and F. Dulat, PolyLogTools—polylogs for the masses, J. High Energy Phys. 08 (2019) 135.

[30] T. Huber and D. Maître, HypExp 2, expanding hypergeometric functions about half-integer parameters, Comput. Phys. Commun. **178,** 755 (2008).

[31] S. Abreu, R. Britto, C. Duhr, E. Gardi, and J. Matthew, The diagrammatic coaction beyond one loop, arXiv:2106.01280.

[32] N. I. Usyukina and A. I. Davydychev, Two-loop three-point diagrams with irreducible numerators, Phys. Lett. B **348,** 503 (1995).

[33] J. Broedel, C. Duhr, F. Dulat, B. Penante, and L. Tancredi, Elliptic polylogarithms and Feynman parameter integrals, J. High Energy Phys. 05 (2019) 120.