Reconstructing propagators of confined particles in the presence of complex singularities

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Propagators of confined particles, especially the Landau-gauge gluon propagator, may have complex singularities as suggested by recent numerical works as well as several theoretical models, e.g., motivated by the Gribov problem. In this paper, we study formal aspects of propagators with complex singularities in reconstructing Minkowski propagators starting from Euclidean propagators by the analytic continuation. We derive the following properties rigorously for propagators with arbitrary complex singularities satisfying some boundedness condition. The two-point Schwinger function with complex singularities violates the reflection positivity. In the presence of complex singularities, while the holomorphy in the usual tube is maintained, the reconstructed Wightman function on the Minkowski spacetime becomes a nontempered distribution and violates the positivity condition. On the other hand, the Lorentz symmetry and locality are kept intact under this reconstruction. Finally, we argue that complex singularities can be realized in a state space with an indefinite metric and correspond to confined states. We also discuss consequences of complex singularities in the Becchi-Rouet-Stora-Tyutin formalism. Our results could open up a new way of understanding a confinement mechanism, mainly in the Landau-gauge Yang-Mills theory.

I. INTRODUCTION

One of the most fundamental properties of strong interactions is color confinement, the absence of colored degrees of freedom from the physical spectrum. Understanding this property in the framework of relativistic quantum field theory (QFT) is a long-standing problem and of crucial importance for particle and nuclear physics. Analytic structures of the correlation functions enable us to extract valuable information on the state-space structure through, e.g., the Källén-Lehmann spectral representation [1], which will be useful toward understanding a confinement mechanism. Therefore, investigating analytic structures of confined propagators, e.g., the gluon propagator, and considering their implications are of great interest.

In the last decades, the gluon, ghost, and quark propagators in the Landau gauge have been extensively studied by both lattice numerical simulations and semianalytical methods (e.g., Dyson-Schwinger equation and functional

renormalization group), for reviews see [2–4], and also by models motivated by the massivelike gluon propagator of these results [5–7]. Based on these advances, in recent years, there has been an increasing interest in the analytic structures of the gluon, ghost, and quark propagators [8–26]. In particular, unusual singularities invalidating the Källén-Lehmann spectral representation, which we call *complex singularities*, receive much attention. A pair of complex conjugate poles of the gluon propagator, which is a typical example of such singularities, was predicted in old literature [27–32], e.g., by improving the gauge fixing procedure. The most remarkable point of the recent studies without assuming the Källén-Lehmann representation is that the independent approaches represented by numerical reconstruction techniques from Euclidean data [21,25], models of massivelike gluons [12,13,18,23], and the ray technique of the Dyson-Schwinger equation [9,24] consistently suggest the existence of complex singularities of the gluon propagator. Moreover, some results support complex poles of the quark propagator [23].

There are also studies of complex singularities on other models [33–35]. A relation between complex poles of a fermion propagator and confinement in the three-dimensional QED was suggested in [35].

Since complex singularities cannot appear in propagators for observable particles, we expect that the complex singularities are related to color confinement. However, while the

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analytic structures have been investigated in many works, implications of complex singularities for QFTs have been much less studied. Theoretical consequences of complex singularities are of crucial importance since such considerations on complex singularities could play a pivotal role in obtaining a clear description of a confinement mechanism. Thus, we study theoretical aspects of complex singularities in this paper.

For this purpose, the reconstruction of the two-point Wightman function, or the vacuum expectation value of the product of field operators, from the two-point Schwinger function, or the Euclidean propagator, has to be carefully investigated. Thus, we reconstruct the Wightman function based on the holomorphy of the Wightman function in "the tube" [36] following the Osterwalder-Schrader (OS) reconstruction [37,38]. This is the standard method to relate Euclidean field theories to QFTs in axiomatic quantum field theory.

Some argue that the appearance of complex singularities might indicate nonlocality, e.g., [29–31]. Nevertheless, this argument relying on the naive inverse Wick rotation is not fully convincing. Actually, as we briefly remark in this paper, the naive inverse Wick rotation differs from the reconstruction based on the holomorphy of the Wightman function in the presence of complex singularities. Since the relation between complex singularities and locality is thus in a confusing situation, we also address this topic carefully.

In this paper, we study formal aspects of complex singularities, namely, analytic properties of the reconstructed two-point Wightman function and implications of complex singularities for the state-space structure. The standard reconstruction procedure and contents of this paper are illustrated in Fig. 1. Because of the somewhat confusing situation on this subject as mentioned above, it is essential to clarify consequences of complex singularities that can be stated unambiguously. Thus, we derive these analytic properties with rigorous proofs. Moreover, since it is very important to investigate states related to the confined particles for understanding a confinement mechanism, we consider state-space structures yielding complex singularities.

The main results of this paper are listed as follows, as announced in [39]. Suppose that the Euclidean propagator, or the two-point Schwinger function, has complex singularities in the complex squared momentum plane, as defined in Sec. III A. Then, the following claims are derived:

(A) The *reflection positivity* is violated for the Schwinger function (Theorem 6).

(B) The *holomorphy* of the Wightman function $W(\xi - i\eta)$ in the tube (Theorem 3) and the *existence of the boundary value* as a distribution (Theorem 4) are still valid. Thus, we can reconstruct the Wightman function from the Schwinger function.
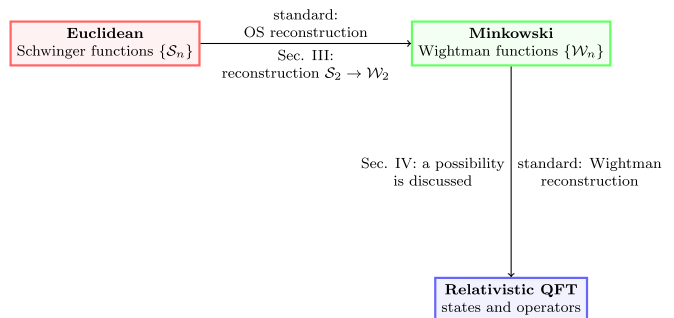

FIG. 1. The reconstruction procedure and contents of this paper. In the standard reconstruction procedure, we start from a family of Schwinger functions satisfying OS axioms and finally reconstruct a QFT by the OS theorem [37,38] and Wightman's reconstruction theorem [[36], Theorems 2–6]. We re-examine this reconstruction procedure when a propagator has complex singularities. In Sec. II, it is pointed out that we should begin with a Schwinger function with complex singularities. In Sec. III, we reconstruct a Wightman function from the Schwinger function in the same way as the OS reconstruction based on the holomorphy in the tube. In Sec. IV, we discuss a possibility in the reconstruction procedure from the Wightman functions to a QFT.

(C) The *temperedness* (Theorem 5) and the *positivity condition* in $\mathscr{D}(\mathbb{R}^4)$ (Theorem 7) are violated for the reconstructed Wightman function. The *spectral condition* is never satisfied since it requires the temperedness as a prerequisite.

(D) The *Lorentz symmetry* (Theorem 8 and Theorem 9) and *spacelike commutativity* (Theorem 10) are kept intact.

(E) A quantum mechanical observation (Claim 3) suggests, together with an example of QFT (Sec. IV B), that complex singularities correspond to pairs of zero-norm eigenstates of complex eigenvalues.

This paper is organized as follows. In Sec. II, we emphasize the difference between complex singularities of a Euclidean propagator and a (real-)time-ordered one in the momentum space and take a glimpse of some properties to be generally derived in Sec. III B. In Sec. III, we give a definition of complex singularities (Sec. III A) and derive the properties (Sec. III B) listed above with a mathematical rigor except for the last one (E). In Sec. IV, based on the results of Sec. III, we consider quantum-theoretical aspects, namely, what complex singularities imply on the state-space structure. We also discuss implications of complex singularities in the Becchi-Rouet-Stora-Tyutin (BRST) formalism. A summary is given in Sec. V, and Sec. VI is devoted to discussion on related topics and future prospects. The mathematical notations and standard axioms are summarized in Appendix A. Appendix B contains a detailed proof of the violation of the reflection positivity (Theorem 6). Appendix C summarizes violated axioms of the OS axioms for Schwinger functions and the Wightman axioms for Wightman functions.

II. PRELIMINARY DISCUSSION

In this section, we sketch out the main properties of complex singularities and emphasize the difference between complex singularities of a Euclidean propagator and those of a (real-)time-ordered propagator in the momentum space. For simplicity, we consider $(0 + 1)$-dimensional field theories in this section. This nonrigorous discussion helps us to determine a point of departure toward the rigorous discussion in Sec. III.

A. Difference between complex singularities of Euclidean propagator and (real-)time-ordered one

We consider complex singularities of Euclidean and real-time propagators on the complex squared momentum plane. We point out that the conventional Wick rotation in the squared momentum plane $p^2 \to -p_E^2$ is *not applicable* in the presence of complex singularities. Thus, we emphasize that complex singularities in the propagators that appear in many works should be regarded as Euclidean ones and that the reconstruction procedure must be carefully considered.

We define the "Wightman functions" $D^>(t)$ and $D^<(t)$ and the real-time propagator $D(t)$ by

$$D^>(t) := \langle 0|\phi(t)\phi(0)|0\rangle,$$
$$D^<(t) := \langle 0|\phi(0)\phi(t)|0\rangle,$$
$$D(t) := \theta(t)D^>(t) + \theta(-t)D^<(t). \quad (1)$$

Usually, we can analytically continue $D^>(t)$ and $D^<(t)$ to the lower and upper half planes of the complex t plane, respectively. In particular, $D^>(-i\tau)$ can be defined for $\tau > 0$, while $D^<(-i\tau)$ can be defined for $\tau < 0$.

Thus, we introduce the Euclidean propagator $\Delta(\tau)$, which is identified with the "two-point Schwinger function", as

$$\Delta^>(\tau) := D^>(-i\tau) \quad (\text{for } \tau > 0),$$
$$\Delta^<(\tau) := D^<(-i\tau) \quad (\text{for } \tau < 0),$$
$$\Delta(\tau) := \theta(\tau)\Delta^>(\tau) + \theta(-\tau)\Delta^<(\tau). \quad (2)$$

This connection between the Wightman and Schwinger functions is consistent with the standard reconstruction method given in (A23) and (A25), where the Schwinger function is regarded as the "values" of the Wightman function at pure imaginary times. We denote the Fourier transforms of $D(t)$ and $\Delta(\tau)$ by $\tilde{D}(p_0)$ and $\tilde{\Delta}(p_E)$, respectively.

We emphasize that the connection between Euclidean correlation functions and vacuum expectation values of the product of field operators should be implemented in the complex time plane rather than in the complex squared momentum plane. Here, with the connection (2), we

demonstrate that the reconstructed propagator $D(t)$ cannot have a well-defined Fourier transform if $\tilde{\Delta}(p_E)$ has complex poles. This indicates that a real-time propagator with complex poles [where $\tilde{D}(p_0)$ has complex poles] is not the reconstructed propagator from a Euclidean propagator with complex poles [where $\tilde{\Delta}(p_E)$ has complex poles].

1. Physical case

First, we observe the physical case for a comparison. Let us assume as a definition of the "physical case",
  (i) completeness: $1 = \sum_n |n\rangle\langle n|$, where $|n\rangle$ is an eigenstate of the Hamiltonian H with an eigenvalue E_n: $H|n\rangle = E_n|n\rangle$,
 (ii) translational covariance: $\phi(t) = e^{iHt}\phi(0)e^{-iHt}$,
(iii) spectral condition: positivity of H, namely, $E_n \geq 0$.
Then, one can relate Euclidean and real-time propagators $\tilde{\Delta}(p_E)$ and $\tilde{D}(p_0)$ by the conventional Wick rotation $p_0^2 \to -p_E^2$. Indeed, these three conditions yield the spectral representations for the Wightman functions and the real-time propagator,

$$D^>(t) = \int_0^\infty d\sigma\, e^{-i\sigma t}\rho(\sigma),$$
$$D^<(t) = \int_0^\infty d\sigma\, e^{i\sigma t}\rho(\sigma),$$
$$\tilde{D}(p_0) = i\int d\sigma \frac{2\sigma\rho(\sigma)}{p_0^2 - \sigma^2 + i\epsilon}, \quad (3)$$

where we have defined the spectral function $\rho(\sigma)$ by

$$\rho(\sigma) := \sum_n \delta(\sigma - E_n)|\langle n|\phi(0)|0\rangle|^2. \quad (4)$$

Consequently, from (2), the Euclidean propagator has the spectral representation given by

$$\Delta^>(\tau) = D^>(-i\tau) = \int_0^\infty d\sigma\, e^{-\sigma\tau}\rho(\sigma),$$
$$\Delta^<(\tau) = D^<(-i\tau) = \int_0^\infty d\sigma\, e^{\sigma\tau}\rho(\sigma),$$
$$\tilde{\Delta}(p_E) = \int d\sigma \frac{2\sigma\rho(\sigma)}{p_E^2 + \sigma^2}. \quad (5)$$

Therefore, in the physical case, the Euclidean propagator $\tilde{\Delta}(p_E)$ and the real-time propagator $\tilde{D}(p_0)$ are related by the analytic continuation on the complex squared momentum plane: $p_0^2 \to -p_E^2$. The spectral representation guarantees this consequence, which does not hold in the presence of complex singularities as is shown below.

2. With complex poles

For example, let us take the Gribov-type propagator with complex poles,

$$\tilde{\Delta}(p_E) := \frac{p_E^2}{p_E^4 + \gamma^4}. \qquad (6)$$

This gives the following Euclidean propagator in the Euclidean time:

$$\Delta(\tau) = \frac{1}{2\gamma} e^{-\frac{\gamma|\tau|}{\sqrt{2}}} \sin\left(-\frac{\gamma|\tau|}{\sqrt{2}} + \frac{\pi}{4}\right), \qquad (7)$$

Although a complete reconstruction method from Euclidean to Minkowski in the presence of complex singularities has not been established, we here assume the connection introduced in (2), which is consistent with the standard reconstruction method, even in the presence of complex singularities. With this connection, we have the Wightman functions,

$$D^>(t) = D^<(-t) = \frac{i}{2\gamma} e^{i\frac{\gamma t}{\sqrt{2}}} \sinh\left(\frac{\gamma t}{\sqrt{2}} - \frac{i\pi}{4}\right). \qquad (8)$$

Then, both $D^>(t)$ and $D^<(t)$ increase exponentially as $t \to \pm\infty$.

Therefore, starting with the Gribov-type Euclidean propagator, we have the Wightman functions $D^>(t)$ and $D^<(t)$ of exponential growth. Such Wightman functions $D^>(t)$ and $D^<(t)$ cannot be regarded as tempered distributions, and therefore, they do not have well-defined Fourier transforms. This is also the case for $D(t)$. Thus, the Minkowski propagator *cannot* be reconstructed from the Euclidean propagator with complex poles by using the simple "inverse Wick rotation" $p_E^2 \to -p_0^2$ in the complex squared momentum plane, since the "reconstructed" real-time propagator has no Fourier transform. In other words, a Euclidean propagator with complex poles [where $\tilde{\Delta}(p_E)$ has complex poles] is different from a real-time propagator with complex poles [where $\tilde{D}(p_0)$ has complex poles]. In particular, one has to take care of the definition of complex singularities.

Again, one should reconstruct the propagator not by the simple inverse Wick rotation on the complex squared momentum plane, $p_E^2 \to -p_0^2$, but by the standard method explained in (A23) and (A25). The former reconstruction is often discussed in some literature, e.g., in [29–31], which is *different* from the latter one. As more is discussed in Sec. VI A, we argue that the latter one should be adopted because of the fundamental relation (A23) and some advantages.

B. Properties

Let us briefly summarize properties of complex poles. Here, we suppose that the Euclidean propagator $\tilde{\Delta}(p_E)$ has complex poles:

(a) The Wightman functions $D^>(t)$ and $D^<(t)$ reconstructed from the Euclidean propagator $\Delta(\tau)$ cannot be regarded as tempered distributions because they grow exponentially as $t \to \pm\infty$.

(b) A Euclidean propagator with only complex poles violates the reflection positivity (A20) because $\Delta(\tau)$ violates the necessary condition for the reflection positivity (A22): $\Delta(\tau) \geq 0$ for all $\tau > 0$.

(c) The positivity in the sector $\{\phi(t)|0\rangle\}_{t\in\mathbb{R}}$ is violated due to the nontemperedness. Indeed, suppose that the sector $\{\phi(t)|0\rangle\}_{t\in\mathbb{R}}$ had a positive metric. From the translational invariance of the two-point function, the time-translation operator defined on this sector, $U(s)\phi(t)|0\rangle := \phi(t+s)|0\rangle$, is unitary, i.e., $\langle 0|\phi(t) U(s)^\dagger U(s)\phi(t')|0\rangle = \langle 0|\phi(t)\phi(t')|0\rangle$. Since the modulus of a matrix element of a unitary operator is not more than one in a space with a positive metric, we would have an upper bound $|\langle 0|\phi(0)U(s)\phi(0)|0\rangle| \leq \langle 0|\phi(0)\phi(0)|0\rangle$ or $|D^<(s)| \leq |D^<(0)|$, which contradicts the nontemperedness.

In the next section, we see that these properties always hold rigorously if $\tilde{\Delta}(p_E)$ has complex singularities (Theorems 5, 6, and 7).

III. COMPLEX SINGULARITIES: DEFINITION AND PROPERTIES

In this section, we give a definition of complex singularities and rigorous proofs of some properties for propagators. These "complex singularities" should be regarded as complex singularities on the complex squared momentum plane of an analytically continued Euclidean propagator. Indeed, in many studies, propagators with complex poles are compared with numerical results on Euclidean ones. Therefore, we start with a two-point Schwinger function. For details of mathematical notations, see Appendix A.

For simplicity, we work in four-dimensional Euclidean space $D = 4$. However, our main results can be easily generalized to arbitrary dimensions $D \geq 2$ except for Theorem 9 and Sec. III B 8, where the Bargmann-Hall-Wightman theorem is used for the proof.

A. Definition

1. Preliminary assumptions

For simplicity, we consider a two-point function for a scalar field. Throughout this paper, we assume the following conditions for a two-point Schwinger function $S_2(x_1, x_2)$, which follow from the OS axioms [37,38] (see Appendix A).

(i) [OS0] Temperedness $S_2(x_1, x_2) \in {}^0\mathscr{S}'(\mathbb{R}^{4\cdot2})$: $S_2(x_1, x_2)$ is a tempered distribution defined on the space of test functions vanishing at coincident points $x_1 = x_2$.

(ii) [OS1] Euclidean (translational and rotational) invariance: $S_2(Rx_1 + a, Rx_2 + a) = S_2(x_1, x_2)$, for all $a \in \mathbb{R}^4, R \in SO(4)$.

From (i) temperedness and (ii) translational invariance, there exists a distribution $S_1(\xi) \in \mathscr{S}'(\mathbb{R}^4_+)$ such that $\mathcal{S}_2(x_1, x_2) = S_1(x_2 - x_1)$ for $x_1^4 < x_2^4$. We can also regard $S_1(\xi) \in \mathscr{S}'(\mathbb{R}^4_{\neq 0})$, where $\mathscr{S}'(\mathbb{R}^4_{\neq 0})$ is the dual space of

$$\mathscr{S}(\mathbb{R}^4_{\neq 0}) := \left\{ f(\xi) \in \mathscr{S}(\mathbb{R}^4); \begin{array}{l} D^\alpha f(\xi)|_{\xi=0} = 0 \\ \text{for any } \alpha \in \mathbb{Z}^4_{\geq 0} \end{array} \right\}. \quad (9)$$

Moreover, (ii) Euclidean rotational invariance implies $S_1(R\xi) = S_1(\xi)$ for all $R \in SO(4)$.

Let us comment on the other conditions of the standard OS axioms [37,38] (see Appendix A). They are [OS2] reflection positivity, [OS3] permutation symmetry, [OS4] cluster property, and [OS0'] Laplace transform condition. Intuitively, [OS2] reflection positivity corresponds to the positivity of the metric of the state space. If we consider gauge theories in Lorentz covariant gauges including confined degrees of freedom, we must allow violation of the reflection positivity. Thus, we do not require the reflection positivity, which is actually broken in the presence of complex singularities (Theorem 6). For a two-point function of a single scalar field, [OS3] permutation symmetry is a consequence of [OS1] Euclidean rotational invariance. For general cases, we assume [OS3] permutation symmetry, see Sec. III B 8. For generality, we do not impose [OS4] the cluster property, which corresponds to the uniqueness of the vacuum and could be violated by a severe infrared singularity of a propagator. In the view of the reconstruction from Euclidean field theories, [OS0'] the Laplace transform condition is introduced for the purpose of controlling higher point functions. Since we focus on the two-point function in this paper, we do not take a further look into this condition. Incidentally, the Laplace transform condition itself is violated if the two-point function has complex singularities due to the nontemperedness of the Wightman functions (Theorem 5).

In addition to the assumptions taken from the standard OS axiom, we further require that the two-point Schwinger function $S_1(\xi) \in \mathscr{S}'(\mathbb{R}^4_{\neq 0})$ has a well-defined Fourier transform $S_1(k)$. Simply, this can be realized by the following assumption:

(iii) The Schwinger function $S_1(\xi)$ can be regarded as an element of $\mathscr{S}'(\mathbb{R}^4)$: $S_1(\xi) \in \mathscr{S}'(\mathbb{R}^4)$.

This assumption allows the well-defined Fourier transform,

$$S_1(k) = \int d^4\xi \, e^{-ik\xi} S_1(\xi). \quad (10)$$

From the rotational invariance, we can write[1]

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[1]Note the difference of conventions with our previous papers [18,20,23], where we took $S_1(k) = D(-k^2)$. In particular, the timelike axis is the negative real axis in this paper unlike the previous ones (see Fig. 2). Note also that the Minkowski notations in [18,20,23] were misleading; they should be Euclidean.

$$S_1(k) = D(k^2). \quad (11)$$

A few remarks are in order.

(a) While the condition $S_1(\xi) \in \mathscr{S}'(\mathbb{R}^4_{\neq 0})$ allows any singularity at $\xi = 0$, the new condition (iii) $S_1(\xi) \in \mathscr{S}'(\mathbb{R}^4)$ imposes that such a singularity is at most derivatives of a delta function $D^\alpha \delta(\xi)$. We do not expect appearance of singularities beyond usual distributions at least in an ultraviolet asymptotic free theory.

(b) For real-valued fields, namely, real-valued $S_1(\xi)$, $S_1(k) = D(k^2)$ is a real distribution from the rotational symmetry (or the permutation symmetry) $S_1(-\xi) = S_1(\xi)$.

(c) There is a constraint on the massless singularities. For example, this formulation excludes the "dipole ghost pole": $D(k^2) \sim 1/k^4$ without a regularization since $D(k^2) = S_1(k) \in \mathscr{S}'(\mathbb{R}^4)$. This constraint depends on the spacetime dimension. The massless pole (without a regularization) is prohibited in the two-dimensional space.

2. Definition of complex singularities

Now, let us define complex singularities of a two-point Schwinger function. We call the positive real axis of the complex k^2 plane the *Euclidean (spacelike) axis* and call the negative real axis of the complex k^2 plane the *timelike axis*. In addition to (i)–(iii), we assume the following for $D(k^2)$:

(iv) $D(k^2) = S_1(k)$ is holomorphic except singularities on the timelike axis $\{k^2; k^2 < 0\}$ and a finite number of poles and branch cuts of finite length satisfying:

(iva) The singularities on the timelike axis can be represented as a tempered distribution on $[-\infty, 0]$, namely,

$$D(-\sigma^2 - i\epsilon) - D(-\sigma^2 + i\epsilon)$$
$$\xrightarrow{\epsilon \to +0} \mathrm{Disc}D(-\sigma^2) \in \mathscr{S}'([0, \infty]), \quad (12)$$

where $\mathscr{S}'([0, \infty])$ is the dual space of $\mathscr{S}([0, \infty]) := \{f(\lambda) = g(-(1 + \lambda)^{-1}); g$ is a C^∞ function on $[-1, 0]\}$. For details, see Appendix A or [[40], Sec. A. 3.].

(ivb) $D(k^2) = S_1(k)$ is holomorphic at least in neighborhoods of all points of the Euclidean axis $\{k^2; k^2 > 0\}$ in the sense that there is no singularity on the Euclidean axis.

(ivc) The complex branch cuts are not located across the real axis.

(v) The analytically continued $D(z)$ on the complex plane $z = k^2$ tends to vanish as $|z| \to \infty$.

With these assumptions (i)–(v), we call singularities except on the negative real axis *complex singularities*.

The first assumption (iva) is imposed for a practical purpose. Without this condition, the spectral function

would generally be a hyperfunction, which makes an analytical treatment difficult. Due to this condition, the "spectral" integral, $\int_0^\infty \frac{\text{Disc}D(-\sigma^2)}{k^2+\sigma^2}$ (see Theorem 1), is well defined. The second assumption (ivb) excludes "tachyonic singularities", which could make $S_1(\xi)$ ill defined. The third one (ivc) claims that, except for the timelike singularities, there are no singularities in the vicinity of the real axis. This is a technical assumption for defining the spectral function and also for simplifying the proof of Theorem 5.

Although assumption (v) is a technical one,[2] we expect that the gluon, ghost, and quark propagators satisfy this property due to the ultraviolet asymptotic freedom. Indeed, in the Landau gauge, the QCD propagators have the asymptotic form of $D(k^2) \sim \frac{1}{k^2(\ln|k^2|)^{\gamma_0/\beta_0}}$, where γ_0 and β_0 are, respectively, the first coefficients of the anomalous dimension and the beta function [41].

The finiteness of branch cuts is required for the reconstruction of the Wightman function. One could allow infinitely long branch cuts whose discontinuities are suppressed faster than any exponential decay as $|z| \to \infty$ and those which approach asymptotically to the negative real axis sufficiently fast. We make a further comment on this point below. For simplicity, we restrict ourselves to the case without branch cuts of infinite length in this paper.

Although we have restricted ourselves to poles and cuts at the assumption (iv), we note that one can easily generalize theorems in Sec. III B, i.e., Theorems 2–11, to arbitrary complex singularities if the following conditions are satisfied: boundedness of locations in $|k^2|$, (iva) regularity of the timelike singularities, (ivb, ivc) holomorphy in a neighborhood of the real axis except for the timelike axis, and (v) $D(k^2) \to 0$ as $k^2 \to \infty$. With these conditions, contributions from complex singularities can be represented as integrals along contours surrounding these singularities according to the Cauchy integral theorem. Then, we can use the same proofs described in Sec. III B for this generalization.

3. Generalized spectral representation

As an immediate consequence following from the complex singularities, we derive the generalized spectral representation for $D(k^2)$.

Here, we consider the setup illustrated in Fig. 2, which is characterized by the following:

(1) $\{z_\ell\}_{\ell=1}^{N_p}$: positions of the complex poles
(2) $\{n_\ell\}_{\ell=1}^{N_p}$: their orders
(3) γ_ℓ: a small contour surrounding z_ℓ clockwise
(4) $\{\mathcal{C}_k\}_{k=1}^{N_c}$: the complex branch cuts
(5) Γ_k: a contour wrapping around $\mathcal{C}_k$ clockwise
(6) $\mathcal{C}_0$: the negative real axis
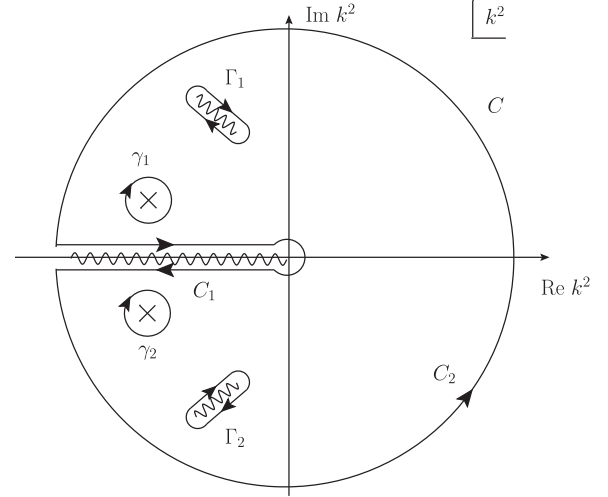
FIG. 2.   The contours γ_ℓ and Γ_k surround the pole z_ℓ and the branch cut $\mathcal{C}_k$ clockwise, respectively. The contour C consists of the path C_1 winding the negative real axis and the large circle C_2: $C = C_1 \cup C_2$. The orientation of the contour C is taken counterclockwise. The propagator $D(k^2)$ is holomorphic in the region bounded by $C \cup \{\gamma_\ell\}_{\ell=1}^{N_p} \cup \{\Gamma_k\}_{k=1}^{N_c}$.

(7) $C = C_1 \cup C_2$: the contour consisting of the path C_1 encompassing $\mathcal{C}_0$ and the large circle C_2 counterclockwise.

The discontinuity of $D(\zeta)$ for a cut $\mathcal{C}_k$ ($k = 0, 1, \ldots, N_c$) is denoted by $\text{Disc}_{\mathcal{C}_k} D(\zeta)$. On a cut with an orientation, $\text{Disc}_{\mathcal{C}_k} D(\zeta) := D(\zeta + id\zeta) - D(\zeta - id\zeta)$, where $d\zeta$ is an infinitesimal along the given orientation of $\mathcal{C}_k$. For example, for the negative real axis $\mathcal{C}_0$ with the orientation from 0 to $-\infty$, $\text{Disc}_{\mathcal{C}_0} D(-\sigma^2) = D(-\sigma^2 - i\epsilon) - D(-\sigma^2 + i\epsilon)$ ($\epsilon \to +0$).

Theorem 1.—Let $D(k^2) = S_1(k)$ be a propagator satisfying (i)–(v). In the above notation, the generalized spectral representation follows for k^2 which is not on singularities of $D(k^2)$,

$$D(k^2) = \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 + k^2} + \sum_{\ell=1}^{N_p} \sum_{m=1}^{n_\ell} \frac{Z_\ell^{(m)}}{(k^2 - z_\ell)^m} + \sum_{k=1}^{N_c} \int_{\mathcal{C}_k} d\zeta \frac{\rho_k(\zeta)}{k^2 - \zeta}, \tag{13}$$

where

$$\rho(\sigma^2) := \frac{1}{2\pi i} \text{Disc}_{\mathcal{C}_0} D(-\sigma^2), \tag{14}$$

$$Z_\ell^{(m)} := -\frac{1}{2\pi i} \oint_{\gamma_\ell} dk^2 D(k^2)(k^2 - z_\ell)^{m-1}$$

$$(\ell = 1, \ldots, N_p; m = 1, \ldots, n_\ell), \tag{15}$$

$$\rho_k(\zeta) := \frac{1}{2\pi i} \text{Disc}_{\mathcal{C}_k} D(\zeta) \quad \text{for } \zeta \in \mathcal{C}_k$$
$$(k = 1, ..., N_c). \tag{16}$$

We have taken the orientation of $\mathcal{C}_k$ ($k = 1, ..., N_c$) in the discontinuities $\text{Disc}_{\mathcal{C}_k} D(\zeta)$ to coincide with the orientation of the integral in (13) and the orientation of $\mathcal{C}_0$ in $\text{Disc}_{\mathcal{C}_0} D(\zeta)$ to be from the origin to negative infinity.

Before proceeding to the proof, let us add several remarks.

(a) If there exists no complex singularity ($N_p = N_c = 0$), this theorem provides the Källén-Lehmann spectral representation

$$D(k^2) = \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 + k^2}, \tag{17}$$

except for the non-negativity $\rho(\sigma^2) \geq 0$. In this sense, (13) is a generalization of the Källén-Lehmann spectral representation.

(b) For real-valued fields, $D(k^2)$ is real for $k^2 > 0$ as noted above. Then, from the Schwarz reflection principle $D(z^*) = [D(z)]^*$, the spectral function can be written in the form

$$\rho(\sigma^2) = \frac{1}{\pi} \text{Im} D(-\sigma^2 - i\epsilon) \quad (\epsilon \to +0), \tag{18}$$

which is the usual dispersion relation.

(c) Similarly, for real-valued fields, the Schwarz reflection principle $D(z^*) = [D(z)]^*$ implies that the complex singularities must appear as complex conjugate pairs (up to arbitrariness of the branch cuts).

(d) $\text{Disc}_{\mathcal{C}_k} D(\zeta)$ is in general a hyperfunction, which is not very convenient for careful analyses. Thus, although Theorem 1 is itself important, we utilize an equation (19) appearing in the proof given below rather than (13) in order to prove subsequent theorems. Only for the timelike part, namely, the first term of (13), we use the spectral representation in the following subsections, since the assumption (iva) makes $\rho(\sigma^2)$ somewhat easy to treat.

(e) Note that the domains of the integrals only represent that $\rho(\sigma^2) \in \mathscr{S}'([0, \infty])$ and that $\text{supp } \rho_k$ lies in the closure of the cut $\mathcal{C}_k$. In particular, we allow a massless pole, namely, a pole at the origin $k^2 = 0$, as long as assumption (iii) is maintained.

Proof.—For any point k^2 not on the singularities, the Cauchy integral formula yields

$$D(k^2) = \oint_C \frac{d\zeta}{2\pi i} \frac{D(\zeta)}{\zeta - k^2} + \sum_{\ell=1}^{N_p} \oint_{\gamma_\ell} \frac{d\zeta}{2\pi i} \frac{D(\zeta)}{\zeta - k^2}$$
$$+ \sum_{k=1}^{N_c} \oint_{\Gamma_k} \frac{d\zeta}{2\pi i} \frac{D(\zeta)}{\zeta - k^2}, \tag{19}$$

where we have chosen the contours $(C_1, \gamma_\ell, \Gamma_k)$ sufficiently close to the singularities.

Assumption (v) guarantees that the integration along the large circle C_2 vanishes. Thus, the first term reads

$$\oint_C \frac{d\zeta}{2\pi i} \frac{D(\zeta)}{\zeta - k^2} = \int_{C_1} \frac{d\zeta}{2\pi i} \frac{D(\zeta)}{\zeta - k^2}, \tag{20}$$

where C_1 surrounds the negative real axis.

For the second term, a calculation yields

$$\sum_{\ell=1}^{N_p} \oint_{\gamma_\ell} \frac{d\zeta}{2\pi i} \frac{D(\zeta)}{\zeta - k^2} = \sum_{\ell=1}^{N_p} \sum_{m=1}^{n_\ell} \frac{Z_\ell^{(m)}}{(k^2 - z_\ell)^m}. \tag{21}$$

Therefore, we have

$$D(k^2) = \int_{C_1^{-1}} \frac{d\zeta}{2\pi i} \frac{D(\zeta)}{k^2 - \zeta} + \sum_{\ell=1}^{N_p} \sum_{m=1}^{n_\ell} \frac{Z_\ell^{(m)}}{(k^2 - z_\ell)^m}$$
$$+ \sum_{k=1}^{N_c} \oint_{\Gamma_k^{-1}} \frac{d\zeta}{2\pi i} \frac{D(\zeta)}{k^2 - \zeta}, \tag{22}$$

where C_1^{-1} and Γ_k^{-1} denote C_1 and Γ_k with inverse directions, respectively. Note that C_1^{-1} and Γ_k^{-1} are roughly "contours surrounding the cuts counterclockwise". By taking a limit shrinking these contours (C_1, Γ_k), the right-hand side of (22) is represented as (13). ∎

B. Some properties of complex singularities

Here, we derive analytic properties of propagators with complex singularities. As a first step, we consider (Sec. III B 1) an example of one pair of complex conjugate simple poles. After that, we prove the properties of general complex singularities: (Sec. III B 2) Holomorphy in the tube, (Sec. III B 3) Violation of temperedness of the reconstructed Wightman function, (Sec. III B 4) Violation of reflection positivity, (Sec. III B 5) Violation of (Wightman) positivity, (Sec. III B 6) Lorentz symmetry, and (Sec. III B 7) Locality. The organization of this section is illustrated in Fig. 3. Some remarks on generalization to arbitrary spinor fields are made in Sec. III B 8.

1. Example: One pair of complex conjugate simple poles

Let us first consider the propagator $D(k^2)$ with one pair of complex conjugate simple poles, which is decomposed
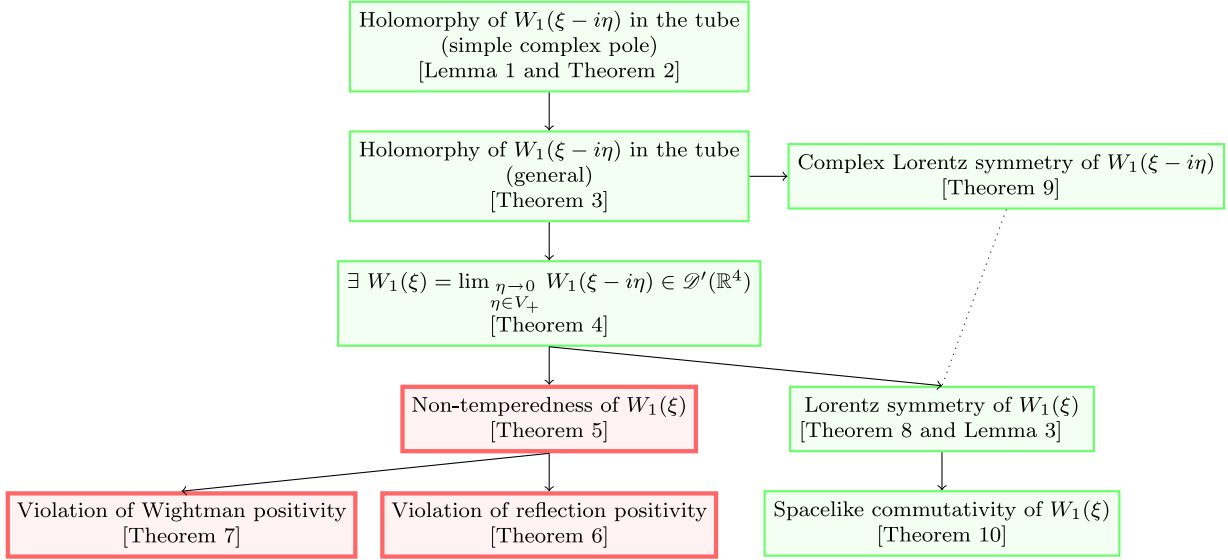
FIG. 3.   Flow chart summarizing Sec. III B. In our proofs, a theorem at a destination of an arrow requires theorems in its upstream. Figure 5 shows the detailed relation on the dotted line between Theorems 9 and 8. The green blocks are consistent with the usual QFT, while the red blocks with thick boxes contradict that.

into the "timelike part" $D_{tl}(k^2)$ and "complex-pole part" $D_{cp}(k^2)$,

$$D(k^2) = D_{tl}(k^2) + D_{cp}(k^2),$$

$$D_{tl}(k^2) = \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 + k^2},$$

$$D_{cp}(k^2) = \frac{Z}{M^2 + k^2} + \frac{Z^*}{(M^*)^2 + k^2}. \qquad (23)$$

Without loss of generality, we can assume $\text{Im}\, M^2 > 0$. Accordingly, the Schwinger function is decomposed as

$$S_1(\xi) = S_{tl}(\xi) + S_{cp}(\xi),$$

$$S_{tl}(\xi) = \int \frac{d^4 k}{(2\pi)^4} e^{ik\xi} D_{tl}(k^2),$$

$$S_{cp}(\xi) = \int \frac{d^4 k}{(2\pi)^4} e^{ik\xi} D_{cp}(k^2). \qquad (24)$$

Our aim here is to demonstrate the reconstruction procedure $S_1(\vec{\xi}, \xi_4) \to W_1(\xi^0, \vec{\xi})$ according to the definition of the reconstruction (A23) and (A25). We can reconstruct each part of the Wightman function separately, as $S_{tl} \to W_{tl}$ and $S_{cp} \to W_{cp}$,

$$W_1(\xi) = W_{tl}(\xi) + W_{cp}(\xi). \qquad (25)$$

We first consider the timelike part $S_{tl} \to W_{tl}$. Since the timelike part is not a main subject of this paper, let us describe the reconstruction procedure of this part only briefly. This reconstruction procedure consists of the following steps:

Step 1. Regarding $S_{tl}(\xi)$ as an ordinary function $\hat{S}_{tl}(\xi)$ on $\{(\vec{\xi}, \xi_4); \xi_4 > 0\}$

Step 2. Performing analytic continuation from $W_{tl}(-i\xi_4, \vec{\xi}) = \hat{S}_{tl}(\vec{\xi}, \xi_4)$ to $W_{tl}(\xi - i\eta)$ defined on the tube $\mathbb{R}^4 - iV_+$

Step 3. Taking the boundary value as a tempered distribution $W_{tl}(\xi) = \lim_{\substack{\eta \to 0 \\ \eta \in V_+}} W_{tl}(\xi - i\eta) \in \mathscr{S}'(\mathbb{R}^4)$,

where V_+ denotes the (open) forward light cone

$$V_+ := \{(\eta^0, \vec{\eta}) \in \mathbb{R}^4; \eta^0 > |\vec{\eta}|\}. \qquad (26)$$

Let us take a closer look into each step. Main properties of the spectral function that we use in these steps are $\rho(\sigma^2) \in \mathscr{S}'([0, \infty])$ and its regularization $\frac{1}{2\pi i}(D(-\sigma^2 - i\epsilon) - D(-\sigma^2 + i\epsilon))$ $(\epsilon \to +0)$.

Step 1. This step claims that there exists a function $\hat{S}_{tl}(\xi)$ such that,[3] for any test function $f(\xi) \in \mathscr{S}(\mathbb{R}_+^4)$,

$$\int \frac{d^4 k}{(2\pi)^4} D_{tl}(k^2) \tilde{f}(k) = \int d^4 \xi f(\xi) \hat{S}_{tl}(\xi), \qquad (27)$$

where $\tilde{f}(k) := \int d^4 \xi f(\xi) e^{ik\xi}$. Noting the properties of $\rho(\sigma^2)$, we have the desired function $\hat{S}_{tl}(\xi)$

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[3]Recall that the Fourier transform of a tempered distribution is defined by the Fourier transform of its test function.

$$\hat{S}_{tl}(\xi) := \int_0^\infty d\sigma^2\, \rho(\sigma^2) \hat{S}_{\sigma^2}(\xi),$$

$$\hat{S}_{\sigma^2}(\xi) := \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{\xi}} \frac{e^{-\sqrt{\sigma^2+\vec{k}^2}|\xi_4|}}{2\sqrt{\vec{k}^2+\sigma^2}}. \tag{28}$$

Step 2. We can confirm that the Cauchy-Riemann equation holds in the tube $\xi - i\eta \in \mathbb{R}^4 - iV_+$ for the following function $W_{tl}(\xi - i\eta)$:

$$W_{tl}(\xi - i\eta) := \int_0^\infty d\sigma^2\, \rho(\sigma^2) W_{\sigma^2}(\xi - i\eta),$$

$$W_{\sigma^2}(\xi - i\eta) := \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{\xi}-i\vec{\eta})} \frac{e^{-i\sqrt{\sigma^2+\vec{k}^2}(\xi^0-i\eta^0)}}{2\sqrt{\vec{k}^2+\sigma^2}}, \tag{29}$$

which satisfies $W_{tl}(-i\xi_4, \vec{\xi}) = \hat{S}_{tl}(\vec{\xi},\xi_4)$. Thus, $W_{tl}(\xi - i\eta)$ is the desired analytic continuation.

Step 3. We can take the limit $\eta \to 0$ ($\eta \in V_+$) of $W_{tl}(\xi - i\eta)$ as a functional of $\mathscr{S}(\mathbb{R}^4)$. For each $f \in \mathscr{S}(\mathbb{R}^4)$, we define

$$W_{tl}(f) := \lim_{\substack{\eta\to 0 \\ \eta\in V_+}} \int d^4\xi\, f(\xi) W_{tl}(\xi - i\eta)$$

$$= \int_0^\infty d\sigma^2\, \rho(\sigma^2) i\Delta^+(f,\sigma^2), \tag{30}$$

where $i\Delta^+(f,\sigma^2)$ is the free Wightman function of mass σ^2,

$$i\Delta^+(f,\sigma^2) := \int d^4\xi\, f(\xi) i\Delta^+(\xi,\sigma^2)$$

$$:= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\sqrt{\vec{k}^2+\sigma^2}}$$

$$\times \left[\int d^4\xi f(\xi) e^{-i\sqrt{\sigma^2+\vec{k}^2}\xi^0 + i\vec{k}\cdot\vec{\xi}} \right],$$

$$i\Delta^+(\xi,\sigma^2) = (2\pi) \int \frac{d^4 k}{(2\pi)^4} e^{-ik\xi} \theta(k_0)\delta(k^2-\sigma^2), \tag{31}$$

with the Loretzian vectors $\xi = (\xi^0,\vec{\xi}), k = (k^0,\vec{k})$. We can check that this linear functional $W_{tl}(f)$ is continuous in $f \in \mathscr{S}(\mathbb{R}^4)$. Hence, we obtain the timelike part of the reconstructed Wightman function which is a tempered distribution.

Let us next reconstruct the complex-pole part $S_{cp} \to W_{cp}$ in a similar way. The complex-pole part $S_{cp}(\xi)$ can be expressed as

$$S_{cp}(\vec{\xi},\xi_4) = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{\xi}} \left[\frac{Z}{2E_{\vec{k}}} e^{-E_{\vec{k}}|\xi_4|} + \frac{Z^*}{2E_{\vec{k}}^*} e^{-E_{\vec{k}}^*|\xi_4|} \right], \tag{32}$$

where $E_{\vec{k}} = \sqrt{\vec{k}^2 + M^2}$ is a branch of $\mathrm{Re}\, E_{\vec{k}} > 0$. We chose $\mathrm{Im}\, M^2 > 0$, so that $\mathrm{Im}\, E_{\vec{k}} > 0$ holds. Note that $S_{cp}(\vec{\xi},\xi_4)$ can be regarded as a function for $\xi_4 > 0$.

For a later purpose, we state this derivation as a lemma.

Lemma 1.—The following equation holds for $\zeta \in \mathbb{C} - (-\infty, 0]$:

$$S_\zeta(\xi) := \int \frac{d^4 k}{(2\pi)^4} e^{ik\xi} \frac{1}{k^2+\zeta} = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{\xi}} \left[\frac{e^{-\sqrt{\vec{k}^2+\zeta}|\xi_4|}}{2\sqrt{\vec{k}^2+\zeta}} \right], \tag{33}$$

where we have chosen $\mathrm{Re}\sqrt{\vec{k}^2+\zeta^2} > 0$, and these Fourier transforms are understood in $\mathscr{S}'(\mathbb{R}^4)$ and $\mathscr{S}'(\mathbb{R}^3)$, respectively. Moreover, the right-hand side is an ordinary function for $\xi_4 > 0$,

$$S_\zeta(\xi) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{\xi}-\sqrt{\vec{k}^2+\zeta}\xi_4}}{2\sqrt{\vec{k}^2+\zeta}} \quad (\text{in } \mathscr{S}'(\mathbb{R}_+^4)), \tag{34}$$

where this integral over $\vec{k}$ is the ordinary integral (namely, not necessarily understood as the Fourier transform of a tempered distribution).

Proof.—For the former assertion (33), it is sufficient to prove that, for any test function $f \in \mathscr{S}(\mathbb{R}^4)$,

$$\int \frac{d^4 k}{(2\pi)^4} \left(\frac{1}{k^2+\zeta} \right) \tilde{f}(k)$$

$$= \int d\xi_4 \int \frac{d^3\vec{k}}{(2\pi)^3} \left(\frac{e^{-\sqrt{\vec{k}^2+\zeta}|\xi_4|}}{2\sqrt{\vec{k}^2+\zeta}} \right) \int d^3\vec{\xi} e^{i\vec{k}\cdot\vec{\xi}} f(\xi), \tag{35}$$

where $\tilde{f}(k) := \int d^4\xi f(\xi) e^{ik\xi}$. Since both $\tilde{f}(k)$ and $f(\xi)$ are of rapid decrease, Fubini's theorem (for $\int d^4 k \to \int d^3\vec{k} \int dk_4$ and $\int dk_4 \int d^4\xi \to \int d^4\xi \int dk_4$) yields

$$\int \frac{d^4 k}{(2\pi)^4} \left(\frac{1}{k^2+\zeta} \right) \tilde{f}(k)$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \int \frac{dk_4}{(2\pi)} \frac{1}{k^2+\zeta} \int d^4\xi f(\xi) e^{ik\xi}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \int d^4\xi \int \frac{dk_4}{(2\pi)} f(\xi) \frac{e^{ik\xi}}{k^2+\zeta}. \tag{36}$$

Therefore, a simple residue calculation gives

$$\int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 + \zeta}\right) \tilde{f}(k)$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \int d^4\xi \, e^{i\vec{k}\cdot\vec{\xi}} \left(\frac{e^{-\sqrt{\vec{k}^2 + \zeta}|\xi_4|}}{2\sqrt{\vec{k}^2 + \zeta}}\right) f(\xi). \quad (37)$$

Since both $f(\xi)$ and $\int d^3\vec{\xi} \, e^{i\vec{k}\cdot\vec{\xi}} f(\xi)$ are of rapid decrease, we can change the order of the integrals to obtain the right-hand side of (35). This establishes the former assertion (35).

For the latter assertion (34), it is enough to prove that, for any test function $f(\xi) \in \mathscr{S}(\mathbb{R}^4_+)$,

$$\int d\xi_4 \int \frac{d^3\vec{k}}{(2\pi)^3} \left(\frac{e^{-\sqrt{\vec{k}^2 + \zeta}|\xi_4|}}{2\sqrt{\vec{k}^2 + \zeta}}\right) \int d^3\vec{\xi} \, e^{i\vec{k}\cdot\vec{\xi}} f(\xi)$$

$$= \int d^4\xi \left[\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{\xi} - \sqrt{\vec{k}^2 + \zeta}\xi_4}}{2\sqrt{\vec{k}^2 + \zeta}}\right] f(\xi). \quad (38)$$

This follows from Fubini's theorem and integrability[4] of $\left|\frac{e^{-\sqrt{\vec{k}^2 + \zeta}\xi_4}}{2\sqrt{\vec{k}^2 + \zeta}} f(\xi)\right|$ for $f \in \mathscr{S}(\mathbb{R}^4_+)$. ∎

Note that $E_{\vec{k}} = |\vec{k}| + O(1/|\vec{k}|)$ strongly suggests that $\text{Im} M^2$ does not affect the convergence. Then, the convergence and holomorphy of the analytically continued Schwinger function is valid in the usual tube $(-i\xi_4, \vec{\xi}) \in \mathbb{R}^4 - iV_+$. This holomorphy is an important step. We prove this claim carefully.

Theorem 2.—The complex-pole part of the Wightman function,

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[4]The integrability can be verified by the following estimation: for $f \in \mathscr{S}(\mathbb{R}^4_+)$,

$$\left|\frac{e^{-\sqrt{\vec{k}^2 + \zeta}\xi_4}}{2\sqrt{\vec{k}^2 + \zeta}} f(\xi)\right|$$

$$\leq \frac{|f(\xi)|\max_{X\geq 0}|e^{-X\xi_4}X^3|}{2|\sqrt{\vec{k}^2 + \zeta}|(\text{Re}\sqrt{\vec{k}^2 + \zeta})^3}$$

$$\leq \left(2|\sqrt{\vec{k}^2 + \zeta}|(\text{Re}\sqrt{\vec{k}^2 + \zeta})^3 (1 + (\xi)^2)^3\right)^{-1}$$

$$\times \sup_{\eta \in \mathbb{R}^4}\left(|f(\eta)(1 + (\eta)^2)^3| \max\left(1, e^{-3}\left(\frac{3}{\eta^4}\right)^3\right)\right),$$

which is integrable in $\vec{k}$ and ξ. Note that the supremum is finite due to $\partial^n_{\xi_4} f(\xi)\big|_{\xi_4=0} = 0$ for any $n \in \mathbb{Z}_{\geq 0}$.

$$W_{cp}(\xi - i\eta) = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{\xi} - i\vec{\eta})}$$

$$\times \left[\frac{Z}{2E_{\vec{k}}} e^{-iE_{\vec{k}}(\xi^0 - i\eta^0)} + \frac{Z^*}{2E^*_{\vec{k}}} e^{-iE^*_{\vec{k}}(\xi^0 - i\eta^0)}\right], \quad (39)$$

is holomorphic in the tube $\xi - i\eta = (\xi^0 - i\eta^0, \vec{\xi} - i\vec{\eta}) \in \mathbb{R}^4 - iV_+$.

Proof.—The first and second terms of the integrand in (39) decreases rapidly as $|\vec{k}| \to \infty$. Indeed, we find

$$\left|\frac{Z}{2E_{\vec{k}}} e^{-iE_{\vec{k}}(\xi^0 - i\eta^0) + i\vec{k}\cdot(\vec{\xi} - i\vec{\eta})}\right|$$

$$= \frac{|Z|}{2|E_{\vec{k}}|} e^{-\eta^0 \text{Re} E_{\vec{k}} + \xi^0 \text{Im} E_{\vec{k}} + \vec{k}\cdot\vec{\eta}}$$

$$= \frac{|Z|}{2|E_{\vec{k}}|} e^{\xi^0 \text{Im} E_{\vec{k}}} e^{-\eta^0(\text{Re} E_{\vec{k}} - |\vec{k}|)} e^{-\eta^0|\vec{k}| + \vec{k}\cdot\vec{\eta}}$$

$$\leq \frac{|Z|}{2|E_{\vec{k}}|} e^{\xi^0 \text{Im} E_{\vec{k}}} e^{-\eta^0(\text{Re} E_{\vec{k}} - |\vec{k}|)} e^{-(\eta^0 - |\vec{\eta}|)|\vec{k}|}. \quad (40)$$

For $\eta \in V_+$, we have, as $|\vec{k}| \to \infty$,
(a) $\text{Im} E_{\vec{k}} \to 0$ and $(\text{Re} E_{\vec{k}} - |\vec{k}|) \to 0$ from $E_{\vec{k}} = |\vec{k}| + O(1/|\vec{k}|)$,
(b) exponential decreasing of $e^{-(\eta^0 - |\vec{\eta}|)|\vec{k}|}$ in $|\vec{k}|$,
from which the first term decreases rapidly: $\frac{Z}{2E_{\vec{k}}} e^{-iE_{\vec{k}}(\xi^0 - i\eta^0) + i\vec{k}\cdot(\vec{\xi} - i\vec{\eta})} \in \mathscr{S}(\mathbb{R}^3)$ for fixed $\xi \in \mathbb{R}^4$ and $\eta \in V_+$. Similarly for the second term, we have $\frac{Z^*}{2E^*_{\vec{k}}} e^{-iE^*_{\vec{k}}(\xi^0 - i\eta^0) + i\vec{k}\cdot(\vec{\xi} - i\vec{\eta})} \in \mathscr{S}(\mathbb{R}^3)$ for fixed $\xi \in \mathbb{R}^4$ and $\eta \in V_+$.

Since the integrand in (39) decreases rapidly as $|\vec{k}| \to \infty$, we can change the order of the integration and differentiations with respect to ξ and η. Therefore, the Cauchy-Riemann equations with respect to (several complex variables) $\xi - i\eta$ hold in the tube $\xi - i\eta \in \mathbb{R}^4 - iV_+$, which guarantees the holomorphy of $W_{cp}(\xi - i\eta)$ in the tube. ∎

Note that, usually, it is the spectral condition that guarantees the holomorphy of the Wightman function in the tube. Without the spectral condition, it is, in general, difficult to establish the analytic arguments based on the holomorphy of the Wightman functions. However, Theorem 2 (and more generally Theorem 3) suggests that such analytic arguments are still valid even in the presence of complex singularities, while complex singularities violate a prerequisite of the spectral condition, namely, the temperedness (see the discussion below or Theorem 5).

Let us regard the Fourier transform in (39) as a tempered distribution in $\vec{\xi}$ with a smooth parameter ξ^0. Then, we can

take the limit $\eta \to 0$ with $\eta \in V_+$ to obtain the reconstructed Wightman function (A25),

$$W_{cp}(\xi^0, \vec{\xi}) = \int \frac{d^3 \vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{\xi}} \left[\frac{Z}{2E_{\vec{k}}} e^{-iE_{\vec{k}}\xi^0} + \frac{Z^*}{2E_{\vec{k}}^*} e^{-iE_{\vec{k}}^* \xi^0} \right]. \tag{41}$$

The first term in the bracket exponentially increases as $\xi^0 \to +\infty$ and so does the second one as $\xi^0 \to -\infty$, with the choice $\mathrm{Im}\, M^2 > 0$. Therefore, complex poles invalidate temperedness of the Wightman function.[5] The nontemperedness is proved more generally in Sec. III B 3.

2. Holomorphy in the tube and boundary value

We have seen the holomorphy of the Wightman function in the usual tube in the presence of the simple complex poles (Theorem 2). Here, we generalize this theorem to the cases with arbitrary complex singularities.

Theorem 3.—Let $S_1(p) = D(p^2)$ be a two-point Schwinger function with complex singularities satisfying (i)–(v). Then, $W_1(-i\xi_4, \vec{\xi}) = S_1(\vec{\xi}, \xi_4)$ $(\xi_4 > 0)$ has an analytic continuation $W_1(\xi - i\eta)$ to the tube $\mathbb{R}^4 - iV_+$.

Proof.—We first recall that

$$S_1(\xi) = \int \frac{d^4 k}{(2\pi)^4} e^{ik\xi} D(k^2), \tag{42}$$

and $D(k^2)$ can be represented as Theorem 1. We know that the timelike part can be analytically continued to the tube. Therefore, we prove the holomorphy for the part coming from complex singularities.

From (19) in the proof of Theorem 1, the contributions of complex singularities can be expressed as[6]

$$S_{\mathrm{complex}}(\xi) = \int \frac{d^4 k}{(2\pi)^4} e^{ik\xi} \left\{ \sum_{\ell=1}^{N_p} \oint_{\gamma_\ell} \frac{d\zeta}{2\pi i} \frac{-D(\zeta)}{k^2 + (-\zeta)} \right.$$
$$\left. + \sum_{k=1}^{N_c} \oint_{\Gamma_k} \frac{d\zeta}{2\pi i} \frac{-D(\zeta)}{k^2 + (-\zeta)} \right\}. \tag{43}$$

Thus, it is sufficient to prove that

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[5]Indeed, suppose that $W_{cp}(\xi^0, \vec{\xi})$ were a tempered distribution. Then, the Fourier transform of $W_{cp}(\xi^0, \vec{\xi})$ in $\vec{\xi}$: $\frac{Z}{2E_{\vec{k}}} e^{-iE_{\vec{k}}\xi^0} + \frac{Z^*}{2E_{\vec{k}}^*} e^{-iE_{\vec{k}}^* \xi^0}$ would be in $\mathscr{S}'(\mathbb{R}^4)$ (by the Schwartz nuclear theorem). This contradicts with the exponential growth in ξ^0.

[6]For this proof, it is enough to take γ_ℓ and Γ_k so close to their singularities that they do not intersect with the positive real axis.

$$\int \frac{d^4 k}{(2\pi)^4} e^{ik\xi} \int_C \frac{d\zeta}{2\pi i} \frac{D(\zeta)}{k^2 + \zeta} \tag{44}$$

can be analytically continued to the tube for any smooth path C of finite length and any smooth function $D(\zeta)$ on C.

To this end, let us proceed with the following steps:

Step 1. Interpreting (44) as an ordinary function on $(\vec{\xi}, \xi_4) \in \mathbb{R}^3 \times (0, \infty)$, that is to say, proving that there exists an analytic function $S_C(\xi)$ on $\mathbb{R}^3 \times (0, \infty)$ such that for any test function $f(\xi) \in \mathscr{S}(\mathbb{R}_+^4)$,

$$\int \frac{d^4 k}{(2\pi)^4} \left(\int_C \frac{d\zeta}{2\pi i} \frac{D(\zeta)}{k^2 + \zeta} \right) \left(\int d^4\xi\, f(\xi) e^{ik\xi} \right)$$
$$= \int d^4\xi\, S_C(\xi) f(\xi), \tag{45}$$

Step 2. Constructing a holomorphic function $W_C(\xi - i\eta)$ in the tube $\mathbb{R}^4 - iV_+$ satisfying $W_C(-i\eta^0, \vec{\xi}) = S_C(\vec{\xi}, \eta^0)$ for $\eta^0 > 0$.

Step 1: Interpreting (44) as a function. We prove that

$$S_C(\xi) := \int_C \frac{d\zeta}{2\pi i} D(\zeta) S_\zeta(\xi) \tag{46}$$

has the desired properties of Step 1, where $S_\zeta(\xi)$ is a function defined by (34) for $\xi_4 > 0$.

(a) $S_C(\xi)$ is an analytic function in $\mathbb{R}^3 \times (0, \infty)$. Indeed, as shown in Theorem 2, $S_\zeta(\xi)$ is an analytic function for $\xi_4 > 0$. Since C is a finite smooth path and $D(\zeta)$ is a smooth function on C, $S_C(\xi)$ defined by (46) is also analytic for $\xi_4 > 0$.

(b) Let us verify that (46) satisfies (45). For any test function $f(\xi) \in \mathscr{S}(\mathbb{R}_+^4)$,

$$\int \frac{d^4 k}{(2\pi)^4} \left(\int_C \frac{d\zeta}{2\pi i} \frac{D(\zeta)}{k^2 + \zeta} \right) \left(\int d^4\xi\, f(\xi) e^{ik\xi} \right)$$
$$= \int_C \frac{d\zeta}{2\pi i} D(\zeta) \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + \zeta} \int d^4\xi\, f(\xi) e^{ik\xi}$$
$$= \int_C \frac{d\zeta}{2\pi i} D(\zeta) \int d^4\xi\, f(\xi) S_\zeta(\xi), \tag{47}$$

where we have used Lemma 1 in the last equality. Since the integrand $D(\zeta) f(\xi) S_\zeta(\xi)$ is integrable in $(\xi, \zeta) \in \mathbb{R}^4 \times C$, we can change the order of the integrals to obtain (45).

Hence, $S_C(\xi)$ given in (46) is the analytic function on $\mathbb{R}^3 \times (0, \infty)$ satisfying (45). This completes the step 1.

Step 2: Analytic continuation of $S_C(\xi)$. We prove that

$$W_C(\xi - i\eta) := \int_C \frac{d\zeta}{2\pi i} D(\zeta) W_\zeta(\xi - i\eta), \tag{48}$$

where

$$W_\zeta(\xi - i\eta) := \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{\xi}-i\vec{\eta})} \left[\frac{1}{2E_{\vec{k}}} e^{-iE_{\vec{k}}(\xi^0-i\eta^0)} \right], \quad (49)$$

is the desired function. Indeed, $W_C(\xi - i\eta)$ satisfies the following properties:

(a) Holomorphy of $W_C(\xi - i\eta)$: From Theorem 2, $W_C(\xi - i\eta)$ is holomorphic in the tube $\mathbb{R}^4 - iV_+$ due to the finiteness of C and smoothness of $D(\zeta)$.

(b) $W_C(-i\eta^0, \vec{\xi}) = S_C(\vec{\xi}, \eta^0)$ for $\eta^0 > 0$. Indeed, we find

$$W_C(-i\eta^0, \vec{\xi}) = \int_C \frac{d\zeta}{2\pi i} D(\zeta) W_\zeta(-i\eta^0, \vec{\xi})$$

$$= \int_C \frac{d\zeta}{2\pi i} D(\zeta) S_\zeta(\vec{\xi}, \eta^0)$$

$$= S_C(\vec{\xi}, \eta^0). \quad (50)$$

Therefore, $W_C(\xi - i\eta)$ provides the analytic continuation of (44) to the tube. This completes the proof of Theorem 3. ∎

Note that the finiteness of branch cuts is essential in this proof. If there existed a branch cut of infinite length with an asymptotic line $\{re^{i\theta}; r > 0\}$, the holomorphic Wightman function would be

$$W_C(\xi - i\eta) = \int_C \frac{d\zeta}{2\pi i} \left[\frac{1}{2\pi} \text{Disc}\, D(\zeta) \right] W_\zeta(\xi - i\eta), \quad (51)$$

and an estimate for large $|\zeta|$ contribution would be

$$W_C(\xi - i\eta) \sim \int dr\, e^{-i\sqrt{r}e^{i\theta/2}(\xi^0-i\eta^0)}$$

$$\sim \int dr\, e^{\sqrt{r}(\xi^0 \sin\theta/2 - \eta^0 \cos\theta/2)}. \quad (52)$$

Unless $\text{Disc}D(\zeta)$ is strongly suppressed faster than any exponential decay as $|\zeta| \to \infty$ or the asymptotic line is the positive real axis ($\theta = 0$), the holomorphy would not be guaranteed at least by this integral representation. Therefore, the finiteness in (iv) plays an important role to reconstruct the Wightman function.

With the finiteness of complex singularities, we can take safely the limit $\eta \to 0$ ($\eta \in V_+$) as a distribution in $\mathscr{D}'(\mathbb{R}^4)$, which is the dual space of $\mathscr{D}(\mathbb{R}^4) = \{f(\xi); f(\xi) \text{ is a } C^\infty \text{function with a compact support}\}$.

Theorem 4.—Let $S_1(p) = D(p^2)$ be a two-point Schwinger function with complex singularities satisfying (i)–(v). By Theorem 3, $W_1(-i\xi_4, \vec{\xi}) = S_1(\vec{\xi}, \xi_4)(\xi_4 > 0)$ has the analytic continuation $W_1(\xi - i\eta)$ to the tube $\mathbb{R}^4 - iV_+$. Then, there exists the limit $\lim_{\substack{\eta \to 0 \\ \eta \in V_+}} W_1(\xi - i\eta) \in \mathscr{D}'(\mathbb{R}^4)$. Moreover, while the part reconstructed from timelike singularities is a tempered distribution in

$\mathscr{S}'(\mathbb{R}^4)$, the part from complex singularities is a tempered distribution in $\vec{\xi}$ with a smooth parameter ξ^0.

Proof.—By Theorem 3, $W_1(-i\xi_4, \vec{\xi}) = S_1(\vec{\xi}, \xi_4)(\xi_4 > 0)$ has an analytic continuation $W_1(\xi - i\eta)$ to the tube $\mathbb{R}^4 - iV_+$.

From the proof of Theorem 3, we can write $W_1(\xi - i\eta)$ corresponding to the representation of Theorem 1 as

$$W_1(\xi - i\eta) = W_{tl}(\xi - i\eta) + W_{\text{complex}}(\xi - i\eta)$$

$$W_{tl}(\xi - i\eta) = \int_0^\infty d\sigma^2 \rho(\sigma^2) W_{\sigma^2}(\xi - i\eta)$$

$$W_{\text{complex}}(\xi - i\eta) = -\sum_{\ell=1}^{N_p} \oint_{\gamma_\ell} \frac{d\zeta}{2\pi i} W_\zeta(\xi - i\eta) D(\zeta)$$

$$- \sum_{k=1}^{N_c} \oint_{\Gamma_k} \frac{d\zeta}{2\pi i} W_\zeta(\xi - i\eta) D(\zeta), \quad (53)$$

where $W_{\sigma^2}(\xi - i\eta)$ and $W_\zeta(\xi - i\eta)$ are given by (29) and (49), respectively.

As seen in Sec. III B 1, the boundary value of the timelike part is a tempered distribution, represented as (30), $W_{tl}(\xi) = \lim_{\substack{\eta \to 0 \\ \eta \in V_+}} W_{tl}(\xi - i\eta) \in \mathscr{S}'(\mathbb{R}^4) \subset \mathscr{D}'(\mathbb{R}^4)$.

Next, we consider the complex part $W_{\text{complex}}(\xi - i\eta)$. As discussed in (41), $W_\zeta(\xi - i\eta)$ has a boundary value that is a tempered distribution in $\vec{\xi}$ with a smooth parameter ξ^0. Indeed, by smearing it with any test function $f(\vec{\xi}) \in \mathscr{S}(\mathbb{R}^3)$,

$$\int d^3\vec{\xi} f(\vec{\xi}) W_\zeta(\xi - i\eta)$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3} e^{\vec{k}\cdot\vec{\eta}} \left[\frac{1}{2E_{\vec{k}}} e^{-iE_{\vec{k}}(\xi^0-i\eta^0)} \right] \left(\int d^3\vec{\xi} f(\vec{\xi}) e^{i\vec{k}\cdot\vec{\xi}} \right)$$

$$(54)$$

converges to, as $\eta \to 0$ ($\eta \in V_+$),

$$\int d^3\vec{\xi} f(\vec{\xi}) W_\zeta(\xi - i\eta)$$

$$\to \int \frac{d^3\vec{k}}{(2\pi)^3} \left[\frac{1}{2E_{\vec{k}}} e^{-iE_{\vec{k}}\xi^0} \right] \left(\int d^3\vec{\xi} f(\vec{\xi}) e^{i\vec{k}\cdot\vec{\xi}} \right), \quad (55)$$

which is a C^∞ function of ξ^0.

Let us show that the boundary value of $W_{\text{complex}}(\xi - i\eta)$ is also a tempered distribution in $\vec{\xi}$ with a smooth parameter ξ^0. It suffices to prove that, for any test function $f(\vec{\xi}) \in \mathscr{S}(\mathbb{R}^3)$ and any finite smooth path C,

$$\int d^3\vec{\xi} f(\vec{\xi}) \left[\int_C \frac{d\zeta}{2\pi i} D(\zeta) W_\zeta(\xi - i\eta) \right] \qquad (56)$$

has a limit that is a C^∞ function of ξ^0 as $\eta \to 0$ ($\eta \in V_+$).

This can be proved as follows. Due to the finiteness of C and the rapid decrease of $f(\vec{\xi})$, we have

$$\int d^3\vec{\xi} f(\vec{\xi}) \left[\int_C \frac{d\zeta}{2\pi i} D(\zeta) W_\zeta(\xi - i\eta) \right]$$
$$= \int_C \frac{d\zeta}{2\pi i} D(\zeta) \left[\int d^3\vec{\xi} f(\vec{\xi}) W_\zeta(\xi - i\eta) \right]. \qquad (57)$$

We have already shown that $\int d^3\vec{\xi} f(\vec{\xi}) W_\zeta(\xi - i\eta)$ has a limit that is a C^∞ function of ξ^0 as $\eta \to 0$ ($\eta \in V_+$). From the finiteness of C, (56) also has such a desired limit.

Therefore, $W_{\text{complex}}(\xi - i\eta)$ has the limit $\lim_{\substack{\eta \to 0 \\ \eta \in V_+}} W_{\text{complex}}(\xi - i\eta)$ that is a tempered distribution in $\vec{\xi}$ with a smooth parameter ξ^0. Since any smooth function can be regarded as a distribution, we have $\lim_{\substack{\eta \to 0 \\ \eta \in V_+}} W_1(\xi - i\eta) \in \mathscr{D}'(\mathbb{R}^4)$. This completes the proof of Theorem 4. ∎

So far, we have seen that, even in the presence of complex singularities, we can analytically continue a Schwinger function to the tube and define its Wightman function $W_1(\xi)$ on the real space as a distribution. However, the existence of complex singularities always violates the temperedness of a Wightman function as a boundary value, which is proved in the next section.

3. Violation of temperedness of Wightman functions and ill-defined asymptotic states

Theorem 5.—Let $S_1(p) = D(p^2)$ be a two-point Schwinger function with complex singularities satisfying (i)–(v). By Theorems 3 and 4, $W_1(-i\xi_4, \vec{\xi}) = S_1(\vec{\xi}, \xi_4)$ ($\xi_4 > 0$) has the analytic continuation $W_1(\xi - i\eta)$ to the tube $\mathbb{R}^4 - iV_+$, and there exists the boundary value as a distribution $W_1(\xi) := \lim_{\substack{\eta \to 0 \\ \eta \in V_+}} W_1(\xi - i\eta) \in \mathscr{D}'(\mathbb{R}^4)$. Then, the boundary value cannot be regarded as a tempered distribution $W_1(\xi) \notin \mathscr{S}'(\mathbb{R}^4)$.

Note that this theorem can be intuitively understood as follows. Readers who can accept the following reasoning can skip the (somewhat technical) proof.
 (a) For simple complex poles, the nontemperedness follows from (41).
 (b) The higher-order poles $\frac{1}{(k^2 - z_\ell)^m}$ can be formally represented as the $(m-1)$th order derivative of the simple pole $\frac{1}{k^2 - z_\ell}$ with respect to z_ℓ. Since the derivative with respect to z_ℓ cannot suppress the exponential growth of $W_{cp}(\xi^0, \vec{\xi})$ given in (41), higher-order complex poles also break temperedness.

(c) The contribution of a complex branch cut $\int_{C_k} d\zeta \frac{\rho_k(\zeta)}{k^2 - \zeta}$ is a superposition of $W_{-\zeta}(\xi^0, \vec{\xi})$ with the weight $\rho_k(\zeta)$. Therefore, the exponential growth of the Wightman function in ξ^0 would be unchanged.
(d) Finally, let us comment on a possibility of cancellation between contributions from different complex singularities. For such cancellations to occur, they must have the same exponentially growing factor $e^{\xi^0 \operatorname{Im} E_{\vec{k}}}$ and oscillating factor $e^{-i\xi^0 \operatorname{Re} E_{\vec{k}}}$. This indicates that this possibility occurs only if singularities are located in the same position in complex k_4 plane. Therefore, we would exclude this possibility.

We prove this theorem rigorously as follows. This proof is based on an intuition that the holomorphy in the tube would essentially imply the spectral condition for the Wightman function in momentum representation, which leads to the usual spectral representation against complex singularities as in Sec. II, if the Wightman function were a tempered distribution.

Proof.—As a preparation, we define a holomorphic function $F_h(\xi^0 - i\eta^0)$ as

$$F_h(\xi^0 - i\eta^0) := \int d^3\vec{\xi} W_1(\xi^0 - i\eta^0, \vec{\xi}) h(\vec{\xi}), \qquad (58)$$

where $h(\vec{\xi})$ is a test function on the spatial directions $h(\vec{\xi}) \in \mathscr{S}(\mathbb{R}^3)$. We require that its Fourier transform has a compact support,

$$\tilde{h}(\vec{k}) := \int d^3\vec{\xi} e^{i\vec{k}\cdot\vec{\xi}} h(\xi) \in \mathscr{D}(\mathbb{R}^3). \qquad (59)$$

This function $F_h(\xi^0 - i\eta^0)$ satisfies the following properties:
(a) $F_h(\xi^0 - i\eta^0)$ is holomorphic in the lower-half plane $\eta^0 > 0$.
(b) In all directions of the limit $|\xi^0 - i\eta^0| \to \infty$ in the lower-half plane ($\eta^0 > 0$), $F_h(\xi^0 - i\eta^0)$ grows at most exponentially as can be seen from representation (53).
(c) For $\xi_4 \neq 0$, $F_h(-i|\xi_4|)$ coincides with the Schwinger function smeared by $h(\vec{\xi})$,

$$S_h(\xi_4) := \int d^3\vec{\xi} S_1(\vec{\xi}, \xi_4) h(\vec{\xi}). \qquad (60)$$

(d) We define, for $\epsilon > 0$,

$$\tilde{S}_h^{(\epsilon)}(k_4^2) := \int d\xi_4 S_h(|\xi_4| + \epsilon) e^{-ik_4\xi_4}$$
$$= \int d\xi_4 F_h(-i(|\xi_4| + \epsilon)) e^{-ik_4\xi_4}. \qquad (61)$$

Representations (13) and (19), together with (28) and (46), yield[7]

$$\tilde{S}_h^{(\epsilon)}(k_4^2) = \int d\xi_4 S_h(|\xi_4| + \epsilon)e^{-ik_4\xi_4}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3}\tilde{h}(\vec{k})\left[\int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 + \vec{k}^2 + k_4^2}e^{-\epsilon\sqrt{\sigma^2+\vec{k}^2}}\right.$$

$$-\sum_{\ell=1}^{N_p}\oint_{\gamma_\ell}\frac{d\zeta}{2\pi i}\frac{D(\zeta)}{(-\zeta) + \vec{k}^2 + k_4^2}e^{-\epsilon\sqrt{\vec{k}^2-\zeta}}$$

$$\left.-\sum_{k=1}^{N_c}\oint_{\Gamma_k}\frac{d\zeta}{2\pi i}\frac{D(\zeta)}{(-\zeta) + \vec{k}^2 + k_4^2}e^{-\epsilon\sqrt{\vec{k}^2-\zeta}}\right],$$

$$(62)$$

from which $\tilde{S}_h^{(\epsilon)}(k_4^2)$ has some singularities in $\mathbb{C} - (-\infty, 0]$ for some $\epsilon > 0$ and some $\tilde{h}(\vec{k}) \in \mathscr{D}(\mathbb{R}^3)$.

Indeed, otherwise, $\tilde{S}_h^{(\epsilon)}(k_4^2)$ would be holomorphic in $\mathbb{C} - (-\infty, 0]$ for all $\epsilon > 0$ and $\tilde{h}(\vec{k}) \in \mathscr{D}(\mathbb{R}^3)$. This implies that the last two terms (except for the first term) of (62) would vanish for all $\epsilon > 0$.[8] Then, $\lim_{\epsilon\downarrow 0}\tilde{S}_h^{(\epsilon)}(k_4^2) = \int d^3\vec{k}\,\tilde{h}(\vec{k})D(k_4^2 + \vec{k}^2)$ would be also holomorphic in $\mathbb{C} - (-\infty, 0]$ for any $\tilde{h}(\vec{k}) \in \mathscr{D}(\mathbb{R}^3)$. By taking the limit of the mollifiers, "approximations" to the delta function, $\tilde{h}(\vec{k}) \to \delta(|\vec{k}| - x_0)(x_0 > 0)$, this leads to holomorphy in $\mathbb{C} - (-\infty, 0]$ of $D(k^2)$.[9] This contradicts with the existence of complex singularities. The above properties follow from the prerequisites of theorems (i)–(v). We prove the theorem by contradiction. Suppose that the boundary value of the Wightman function were a tempered distribution: $\lim_{\substack{\eta\to 0 \\ \eta\in V_+}} W_1(\xi - i\eta) \in \mathscr{S}'(\mathbb{R}^4)$.

(e) Then, the boundary value of $F_h(\xi^0 - i\eta^0)$ would be a tempered distribution $F_h(\xi^0) := \lim_{\eta^0\downarrow 0} F_h(\xi^0 - i\eta^0) \in \mathscr{S}'(\mathbb{R})$.

Let us find a contradiction under the circumstance characterized by (a)–(e).

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[7]Note that the limit $\epsilon \to 0$ gives the smeared Schwinger function $\tilde{S}_h(k_4) := \int d\xi_4 S_h(\xi_4)e^{-ik_4\xi_4}$. In other words, representation (13) enables us to "complete" the point $\xi_4 = 0$ from $F_h(-i|\xi_4|)$ defined on $\xi_4 \neq 0$.

[8]Since the last two terms of (62) are holomorphic at least on the negative real axis [where we have used the third assumption of (iv)], it would be an entire function. Furthermore, it tends to vanish as $|k_4^2| \to \infty$ and therefore would vanish.

[9]Indeed, let $h_\epsilon(\vec{k})$ denote such a mollifier: $h_\epsilon(\vec{k}) \to \delta(|\vec{k}| - x_0), \epsilon \to +0$. Then, $\lim_{\epsilon\downarrow 0}\int d^3\vec{k}\tilde{h}_\epsilon(\vec{k})D(k_4^2 + \vec{k}^2) = C'D(k_4^2 + x_0^2)$ for some $C' > 0$. The last two terms of (62) would not contribute to the left-hand side due to the same argument as the previous footnote. This leads to the holomorphy in $\mathbb{C} - (-\infty, 0]$ of $D(k_4^2 + x_0^2)$ for $x_0 > 0$.

We first decompose $F_h(\xi^0)$ as

$$F_h(\xi^0) = F_+(\xi^0) + F_-(\xi^0)$$

$$F_\pm(\xi^0) = \int \frac{d\omega}{2\pi}e^{-i\omega\xi^0}\tilde{F}_\pm(\omega),$$

$$\text{supp}\,\tilde{F}_+ \subset [0, \infty), \quad \text{supp}\,\tilde{F}_- \subset (-\infty, 0]. \quad (63)$$

Since $F_h(\xi^0)$ is not a function but a tempered distribution, there is a delicate point here. We can prove this decomposition with the following manipulation. We recall (see Appendix A) that $\mathscr{S}(\bar{\mathbb{R}}_+) := \mathscr{S}(\mathbb{R})/\mathscr{S}_-(\mathbb{R})$ and its dual space $\mathscr{S}'(\bar{\mathbb{R}}_+) \simeq \{F \in \mathscr{S}'(\mathbb{R}); \text{supp}\,F \subset [0, \infty)\}$. We similarly define $\mathscr{S}(\bar{\mathbb{R}}_-) := \mathscr{S}(\mathbb{R})/\mathscr{S}_+(\mathbb{R})$. We also define $\mathscr{X} := \{([f]_+, [f]_-) \in \mathscr{S}(\bar{\mathbb{R}}_+) \oplus \mathscr{S}(\bar{\mathbb{R}}_-); f \in \mathscr{S}(\mathbb{R})\}$ and its dual $\mathscr{X}'$. Note the homeomorphism $\mathscr{X} \simeq \mathscr{S}(\mathbb{R})$. By the Hahn-Banach theorem, an element of $\mathscr{X}'$ can be extended to the dual space of $\mathscr{S}(\bar{\mathbb{R}}_+) \oplus \mathscr{S}(\bar{\mathbb{R}}_-)$, which is isomorphic to $\mathscr{S}'(\bar{\mathbb{R}}_+) \oplus \mathscr{S}'(\bar{\mathbb{R}}_-) \simeq \{F \in \mathscr{S}'(\mathbb{R}); \text{supp}\,F \subset [0, \infty)\} \oplus \{F \in \mathscr{S}'(\mathbb{R}); \text{supp}\,F \subset (-\infty, 0]\}$. Therefore, for any $\tilde{F} \in \mathscr{S}'(\mathbb{R})$, there exist $\tilde{F}_+, \tilde{F}_- \in \mathscr{S}'(\mathbb{R})$ such that $\tilde{F} = \tilde{F}_+ + \tilde{F}_-$ with $\text{supp}\,\tilde{F}_+ \subset [0, \infty)$ and $\text{supp}\,\tilde{F}_- \subset (-\infty, 0]$. This justifies (63). For a more general description on this decomposition, see Proposition A.3 of [40].

Next, we list several properties of $F_-(\xi^0)$ as follows:

(a') $F_-(\xi^0)$ can be analytically continued to the whole complex plane. To show this, we consider the holomorphy in the (1) lower and (2) upper half planes separately and (3) glue them.

(1) For the lower-half plane, we define $F_-(\xi^0 - i\eta^0) := F_h(\xi^0 - i\eta^0) - F_+(\xi^0 - i\eta^0)$, where $F_+(\xi^0 - i\eta^0)$ is the Laplace transform of $\tilde{F}_+(\omega)$. This is the desired holomorphic function. Indeed, because of the support property $\text{supp}\,\tilde{F}_+ \subset [0, \infty)$, $F_+(\xi^0 - i\eta^0)$ is holomorphic in the lower-half plane ($\eta^0 > 0$). The holomorphy of $F_+(\xi^0 - i\eta^0)$ and $F_h(\xi^0 - i\eta^0)$ from (a) yields that $F_-(\xi^0 - i\eta^0)$ defined above is holomorphic in the lower-half plane. The boundary values are $F_h(\xi^0 - i\eta^0) \to F_h(\xi^0)$ from (e) and $F_+(\xi^0 - i\eta^0) \to F_+(\xi^0)$ as is well known,[10] from which $F_-(\xi^0 - i\eta^0)$ has the boundary value $F_-(\xi^0)$. Therefore, $F_-(\xi^0 - i\eta^0) = F_h(\xi^0 - i\eta^0) - F_+(\xi^0 - i\eta^0)$ provides the analytic continuation to the lower-half plane.

(2) For the upper-half plane, the Laplace transform of $\tilde{F}_-(\omega)$ provides the analytic continuation due to $\text{supp}\,\tilde{F}_- \subset (-\infty, 0]$.

(3) We have two analytic continuations in the upper- and lower-half planes that have the coincident boundary value on the real axis. By the one-variable version of the edge of the wedge theorem, one can find an

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[10]For example, see Theorems 2–9 in [36].

entire function which is the analytic continuation from both half planes.

(b1′) In all directions of the limit $|\xi^0 - i\eta^0| \to \infty$ in the lower-half plane ($\eta^0 > 0$), $F_-(\xi^0 - i\eta^0)$ grows at most exponentially. Indeed, both $F_+(\xi^0 - i\eta^0)$ and $F_h(\xi^0 - i\eta^0)$ satisfy this condition due to (b) and supp$\tilde{F}_+ \subset [0, \infty)$.

(b2′) In all directions of the limit $|\xi^0 - i\eta^0| \to \infty$ in the upper-half plane ($\eta^0 < 0$), $F_-(\xi^0 - i\eta^0)$ grows at most polynomially because of supp $\tilde{F}_- \subset (-\infty, 0]$.

(c′) $F_-(-i\xi_4)$ is of at most polynomial growth in $\xi_4 > 0$ due to (c) and supp $\tilde{F}_+ \subset [0, \infty)$.

From (a′), (b1′), (c′), and the temperedness of $F_-(\xi^0)$, a variant of the Paley-Wiener-Schwartz theorem for one-sided support (see, e.g., Theorem A of [42]) implies that $F_-(\xi^0 - i\eta^0)$ in the lower-half plane can be written as the Laplace transformation of a tempered distribution $\tilde{F}'_-(\omega)$ of supp$\tilde{F}'_- \subset [0, \infty)$ (which actually coincides with $\tilde{F}_-(\omega)$). Thus, in all directions of the limit $|\xi^0 - i\eta^0| \to \infty$ in the lower-half plane, $F_-(\xi^0 - i\eta^0)$ grows at most polynomially. Together with (b2'), we conclude that the entire function $F_-(\xi^0 - i\eta^0)$ is a polynomial, whose Fourier transform is a point-supported distribution.

Because of the support properties supp$\tilde{F}_- = \{0\}$ and supp$\tilde{F}_+ \subset [0, \infty)$, $\tilde{F}_+(\omega)$ can absorb $\tilde{F}_-(\omega)$ in the decomposition (63). From here on, we assume $\tilde{F}_- = 0$ without loss of generality.

Finally, let us construct $\tilde{S}_h^{(\epsilon)}(k_4^2)$ defined in (d) from $F_h(\xi^0) = F_+(\xi^0)$. Due to supp $\tilde{F}_+ \subset [0, \infty)$, the analytic continuation of $F_h(\xi^0)$ to the lower-half plane is given by the Laplace transform of $\tilde{F}_+$,

$$F_h(\xi^0 - i\eta^0) = \int \frac{d\omega}{2\pi} e^{-i\omega\xi^0} e^{-\omega\eta^0} \tilde{F}_+(\omega), \qquad (64)$$

which is a holomorphic function for $\eta^0 > 0$.

Therefore, using (c) and (d), we have

$$\tilde{S}_h^{(\epsilon)}(k_4^2) = \int d\xi_4 \, F_h(-i|\xi_4| - i\epsilon) e^{-ik_4\xi_4}$$

$$= \int d\xi_4 e^{-ik_4\xi_4} \int \frac{d\omega}{2\pi} \tilde{F}_+(\omega) e^{-\epsilon\omega} e^{-\omega|\xi_4|}. \quad (65)$$

Since a tempered distribution is a sum of derivatives of continuous functions (of at most polynomial growth): $\tilde{F}_+(\omega) = \sum_{n=1}^{M} (-\frac{\partial}{\partial\omega})^{\alpha_n} \tilde{f}_n(\omega)$, we can rewrite

$$\tilde{S}_h^{(\epsilon)}(k_4^2) = \sum_{n=1}^{M} \int d\xi_4 e^{-ik_4\xi_4} \int_0^\infty \frac{d\omega}{2\pi} \tilde{f}_n(\omega) \frac{\partial^{\alpha_n}}{\partial\omega^{\alpha_n}} e^{-\epsilon\omega} e^{-\omega|\xi_4|}$$

$$= \sum_{n=1}^{M} \int_0^\infty \frac{d\omega}{2\pi} \tilde{f}_n(\omega) \frac{\partial^{\alpha_n}}{\partial\omega^{\alpha_n}} \frac{2\omega e^{-\epsilon\omega}}{k_4^2 + \omega^2}, \qquad (66)$$

where α_n is a non-negative integer, and $\tilde{f}_n(\omega)$ is a continuous function of supp$\tilde{f}_n \subset [0, \infty)$. Note that the last line of (66) can be regarded as an ordinary function, although we have calculated the Fourier transformation in (65) and (66) as a tempered distribution in the above equations.

Representation (66) shows the holomorphy of $\tilde{S}_h^{(\epsilon)}(k_4^2)$ on $\mathbb{C} - (-\infty, 0]$ for all $\epsilon > 0$ and $\tilde{h}(\vec{k}) \in \mathscr{D}(\mathbb{R}^3)$, which contradicts with the existence of singularity explained in (d). This completes the proof of Theorem 5. ∎

Let us comment on some implications of the non-temperedness. As seen from (41), a typical nontempered behavior is the exponential growth in ξ^0. The exponential growth of the Wightman function largely affects asymptotic states, which correspond to "$\xi^0 \to \pm\infty$ limit". This indicates that asymptotic states of the field are ill defined without some artificial manipulations.[11] Since such states in the "full" state space are far from being identified with asymptotic particle states and should be eliminated from the physical state space, the complex singularities could be considered as a signal of confinement.

Finally, let us comment on the spectral condition. The spectral condition for the two-point Wightman function states supp $\tilde{W}_1(q) \subset V_+$, where $\tilde{W}_1(q) = \int d^4\xi e^{iq\xi} W_1(\xi)$ with Lorentzian vectors ξ, q. Since the existence of $\tilde{W}_1(q)$ is assumed in the spectral condition, this condition requires the temperedness as a prerequisite. Therefore, Theorem 5 implies that the spectral condition is never satisfied in the presence of complex singularities.

4. Violation of reflection positivity

As a consequence of the nontemperedness, we can prove that the reflection positivity [OS2] is always violated in the presence of complex singularities. Since complex singularities invalidate the Källén-Lehmann spectral representation, some conditions of the standard axiom should be violated. Therefore, the violation of the reflection positivity is in some sense trivial. However, for this paper to be self-contained and because of importance of this claim, we describe the proof in detail in Appendix B. Moreover, to the

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[11] For Lee-Wick theory, which is the simplest model providing complex poles considered below, some manipulations on the S matrix were discussed in old literature, e.g., see [[43], Sec. 16] for a review. However, these manipulations can cause Lorentz noninvariance and acausality. We insist that such states corresponding to complex singularities should be eliminated from the physical state space before taking the asymptotic limit (rather than causing Lorentz noninvariance).

best of our knowledge, an explicit proof on this claim is new.

Theorem 6.—If $S_1(p) = D(p^2)$ is a two-point Schwinger function with complex singularities satisfying (i)–(v), then the reflection positivity [OS2] is violated.

Proof.—The reflection positivity for the two-point function (A20) is a necessary condition of the reflection positivity [OS2].

In Appendix B, it is proved that the reflection positivity for the two-point function (A20) yields temperedness of a reconstructed two-point Wightman function (Theorem 11). Therefore, the nontemperedness (Theorem 5) implies the violation of the reflection positivity. ∎

The reflection positivity, especially (A20) for the two-point function, is often checked by a necessary condition: the positivity of $\underline{S}_1(\vec{k}, \xi_4) := \int d^3\vec{\xi} e^{i\vec{k}\cdot\vec{\xi}} S_1(\vec{\xi}, \xi_4)$ (A22), e.g., [20]. Using this check, one can easily show that a propagator with only simple complex conjugate poles violates the reflection positivity. Indeed, from (32), we have, for $\xi_4 > 0$,

$$\underline{S}_1(\vec{k}, \xi_4) = \frac{Z}{2E_{\vec{k}}} e^{-E_{\vec{k}}\xi_4} + \frac{Z^*}{2E_{\vec{k}}^*} e^{-E_{\vec{k}}^*\xi_4}$$
$$= \frac{|Z|}{|E_{\vec{k}}|} e^{-\xi_4 \operatorname{Re} E_{\vec{k}}} \cos\left(\xi_4 \operatorname{Im} E_{\vec{k}} - \arg\left(\frac{Z}{E_{\vec{k}}}\right)\right),$$
(67)

which is negative for some $\xi_4 > 0$. However, this check is not useful to prove the violation of the reflection positivity for general propagators with complex singularities. For example, in the case seen in Sec. III B 1, we have, by assuming some regularity of the spectral function $\rho(\sigma^2)$,

$$\underline{S}_1(\vec{k}, \xi_4) = \int_{\sqrt{\vec{k}^2}}^{\infty} d\sigma \, e^{-\sigma\xi_4} \rho(\sigma^2 - \vec{k}^2)$$
$$+ \frac{|Z|}{|E_{\vec{k}}|} e^{-\xi_4 \operatorname{Re} E_{\vec{k}}} \cos\left(\xi_4 \operatorname{Im} E_{\vec{k}} - \arg\left(\frac{Z}{E_{\vec{k}}}\right)\right),$$
(68)

which could be positive if the spectral function $\rho(\sigma^2)$ is positive and large. Theorem 6 indicates that the existence of complex singularities always invalidates the reflection positivity irrespective of the timelike singularities. It is redundant to check the positivity of (A22) numerically for a propagator with complex singularities.

5. Violation of (Wightman) positivity

Let us consider the positivity condition of the Wightman function. First of all, the standard positivity condition,

$$\int d^4x d^4y W_1(y-x) f^*(x) f(y) \geq 0 \quad \text{for any } f \in \mathscr{S}(\mathbb{R}^4),$$
(69)

makes no sense for a nontempered distribution $W_1(y-x)$. It is natural to examine a positivity condition in a weak sense using $\mathscr{D}(\mathbb{R}^4)$, instead of $\mathscr{S}(\mathbb{R}^4)$, which we call Wightman positivity in $\mathscr{D}(\mathbb{R}^4)$ (for the two-point function),

$$\int d^4x d^4y W_1(y-x) f^*(x) f(y) \geq 0 \quad \text{for any } f \in \mathscr{D}(\mathbb{R}^4).$$
(70)

Here, we examine this positivity condition. As can be inferred from the violation of the reflection positivity, this condition is also violated in the presence of complex singularities. We prove the following theorem in a way similar to Theorem 6.

Theorem 7.—Let $S_1(p) = D(p^2)$ be a two-point Schwinger function with complex singularities satisfying (i)–(v). By Theorems 3 and 4, $W_1(-i\xi_4, \vec{\xi}) = S_1(\vec{\xi}, \xi_4)$ ($\xi_4 > 0$) has the analytic continuation $W_1(\xi - i\eta)$ to the tube $\mathbb{R}^4 - iV_+$, and there exists the boundary value as a distribution $W_1(\xi) := \lim_{\substack{\eta \to 0 \\ \eta \in V_+}} W_1(\xi - i\eta) \in \mathscr{D}'(\mathbb{R}^4)$. Then, the Wightman positivity in $\mathscr{D}(\mathbb{R}^4)$ for $W_1(\xi)$ is violated.

Proof.—In the next lemma (Lemma 2), we prove that the Wightman positivity implies the temperedness of W_1. Therefore, the Wightman positivity is violated due to the nontemperedness (Theorem 5). ∎

Lemma 2.—Let $W_1(\xi) \in \mathscr{D}'(\mathbb{R}^4)$ be a distribution satisfying the Wightman positivity in $\mathscr{D}(\mathbb{R}^4)$. Then, $W_1(\xi)$ can be regarded as a tempered distribution: $W_1(\xi) \in \mathscr{S}'(\mathbb{R}^4)$.

The following proof of Lemma 2 is based on an intuition that $W_1(\xi)$ is roughly a matrix element of a unitary operator and is therefore bounded above in a positive-definite state space as shown in Sec. II B.

Proof.—We define a sesquilinear form on $\mathscr{D}(\mathbb{R}^4)$: for $f, g \in \mathscr{D}(\mathbb{R}^4)$,

$$(f, g)_W := \int d^4x d^4y \, W_1(y-x) f^*(x) g(y), \quad (71)$$

which is positive semidefinite due to the Wightman positivity (70). For $a \in \mathbb{R}^4$, $\hat{U}(a)$ denotes an operator on $\mathscr{D}(\mathbb{R}^4)$ defined by

$$(\hat{U}(a)f)(x) := f(x-a), \quad (72)$$

which satisfies $(\hat{U}(a)f, \hat{U}(a)f)_W = (f, f)_W$.

Since $(\cdot, \cdot)_W$ is positive semidefinite, the Cauchy-Schwarz inequality yields

$$|(f, \hat{U}(a)g)_W| \le \sqrt{(f,f)_W (g,g)_W}. \tag{73}$$

Thus, for all $f, g \in \mathscr{D}(\mathbb{R}^4)$,

$$(f * (g * W_1))(a) = (f^*, \hat{U}(a)\hat{g})_W \tag{74}$$

is bounded in $a \in \mathbb{R}^4$, where $\hat{g}(x) := g(-x)$ and $(f * g)(x) := \int d^4\xi f(x - \xi)g(\xi)$.

Note that there exists a convenient necessary and sufficient condition for a distribution $T \in \mathscr{D}'(\mathbb{R}^4)$ to be a tempered distribution [[44], Theorem 5, Chapter 7]:

$$T \in \mathscr{S}'(\mathbb{R}^4) \Leftrightarrow$$

$\alpha * T$ is a smooth function of at most

polynomial growth for any $\alpha \in \mathscr{D}(\mathbb{R}^4)$.   (75)

Now, let us fix an arbitrary $g \in \mathscr{D}(\mathbb{R}^4)$. Then, $(f * (g * W_1))(a)$ is a smooth function bounded above for all $f \in \mathscr{D}(\mathbb{R}^4)$. The condition for temperedness (75) implies that we can regard $(g * W_1) \in \mathscr{S}'(\mathbb{R}^4)$, from which $(g * W_1)(x)$ is a smooth function of at most polynomial growth.

Therefore, from arbitrariness of $g \in \mathscr{D}(\mathbb{R}^4)$ and (75), we obtain $W_1 \in \mathscr{S}'(\mathbb{R}^4)$. This completes the proof of Lemma 2.   ∎

6. Lorentz symmetry

Since the Lorentz invariance is itself an important nature and also an essential step to the locality, let us carefully prove the Lorentz invariance of the reconstructed Wightman function.

Theorem 8.—Let $S_1(p) = D(p^2)$ be a two-point Schwinger function with complex singularities satisfying (i)–(v). By Theorems 3 and 4, $W_1(-i\xi_4, \vec{\xi}) = S_1(\vec{\xi}, \xi_4)$ $(\xi_4 > 0)$ has the analytic continuation $W_1(\xi - i\eta)$ to the tube $\mathbb{R}^4 - iV_+$, and there exists the boundary value as a distribution $\lim_{\substack{\eta \to 0 \\ \eta \in V_+}} W_1(\xi - i\eta) \in \mathscr{D}'(\mathbb{R}^4)$. Then, both the holomorphic Wightman function and its boundary value are (restricted) Lorentz invariant. More precisely, for all proper orthochronous Lorentz transformations $\Lambda \in SO(3,1)^+$,

$$W_1(\Lambda(\xi - i\eta)) = W_1(\xi - i\eta), \quad \text{for } \xi - i\eta \in \mathbb{R}^4 - iV_+, \tag{76}$$

and for any $f \in \mathscr{D}(\mathbb{R}^4)$,

$$W_1(f) = W_1(f_\Lambda), \quad \text{with} \quad f_\Lambda(\xi) := f(\Lambda^{-1}\xi). \tag{77}$$

Proof.—Let us first consider the holomorphic Wightman function (76). This can be decomposed as (53): $W_1(\xi - i\eta) = W_{tl}(\xi - i\eta) + W_{\text{complex}}(\xi - i\eta)$. Therefore, the Lorentz invariance of $W_1(\xi - i\eta)$ follows from that of the respective parts.

The timelike part $W_{tl}(\xi - i\eta)$ is expressed as (29). Since the free Wightman function $W_{\sigma^2}(\xi - i\eta)$ is a Lorentz invariant function as is well known, $W_{tl}(\xi - i\eta)$ is also Lorentz invariant.

For the Lorentz invariance of the complex part $W_{\text{complex}}(\xi - i\eta)$, similarly from representation (53), it is sufficient to prove that $W_\zeta(\Lambda(\xi - i\eta)) = W_\zeta(\xi - i\eta)$ in $\xi - i\eta \in \mathbb{R}^4 - iV_+$ for all $\Lambda \in SO(3,1)^+$. We prove this claim in Lemma 3 to be given below. This established the invariance (76).

The latter assertion (77) immediately follows from the former one (76).   ∎

Lemma 3.—The Wightman function $W_\zeta(\xi - i\eta)$, (49), of a simple complex pole defined on $\xi - i\eta \in \mathbb{R}^4 - iV_+$ satisfies, for all $\Lambda \in SO(3,1)^+$,

$$W_\zeta(\Lambda(\xi - i\eta)) = W_\zeta(\xi - i\eta). \tag{78}$$

Proof.—The spatial rotational symmetry is manifest by the expression (49). Therefore, it suffices to prove the invariance under the boost along ξ^3,

$$\xi = (\xi^0, \xi^1, \xi^2, \xi^3)$$
$$\to \xi' := \Lambda\xi = (\gamma(\xi^0 - \beta\xi^3), \xi^1, \xi^2, \gamma(\xi^3 - \beta\xi^0)). \tag{79}$$

As mentioned in [45], one can show the invariance under the boost by a contour deformation.

Under this transformation, $W_\zeta(\xi - i\eta)$ reads

$$W_\zeta(\Lambda(\xi - i\eta)) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2E_{\vec{k}}} e^{i\vec{k}\cdot(\vec{\xi}' - i\vec{\eta}') - iE_{\vec{k}}(\xi^{0'} - i\eta^{0'})}$$
$$= \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2E_{\vec{k}}} e^{i\vec{k}'\cdot(\vec{\xi} - i\vec{\eta}) - iE'_{\vec{k}}(\xi^0 - i\eta^0)}, \tag{80}$$

where we have defined $E_{\vec{k}} := \sqrt{\vec{k}^2 + \zeta}$ of the principal branch $(\text{Re}\, E_{\vec{p}} > 0)$, and

$$E'_{\vec{k}} := \gamma(E_{\vec{k}} + \beta k_3),$$
$$k'_3 := \gamma(k_3 + \beta E_{\vec{k}}), \vec{k}' := (k_1, k_2, k'_3). \tag{81}$$

Note that a simple computation and $\text{Re}\, E_{\vec{k}} > 0$ yield

$$E_{\vec{k}'} = \sqrt{\gamma^2(E_{\vec{k}} + \beta k_3)^2} = \gamma(E_{\vec{k}} + \beta k_3), \tag{82}$$

from which we have

$$\frac{dk_3'}{dk_3} E_{\vec{k}} = E'_{\vec{k}} = E_{\vec{k}'}. \tag{83}$$

By changing the variable from $\vec{k}$ to $\vec{k}'$, we obtain

$$W_\zeta(\Lambda(\xi - i\eta)) = \int_{\mathbb{R}^2 \times C_\beta} \frac{d^3\vec{k}'}{(2\pi)^3} \frac{1}{2E_{\vec{k}'}} e^{i\vec{k}'\cdot(\vec{\xi}-i\vec{\eta})-iE_{\vec{k}'}(\xi^0-i\eta^0)}, \tag{84}$$

where the contour C_β is defined by

$$C_\beta := \{k_3' = \gamma(k_3 + \beta E_{\vec{k}}); k_3 \in \mathbf{R}\}, \tag{85}$$

(see Fig. 4). Note that, for all $|\beta| < 1$, $E_{\vec{k}'} = \gamma(E_{\vec{k}} + \beta k_3)$ does not vanish on the contour $k_3' \in C_\beta$, namely, $k_3 \in \mathbf{R}$. Since the family of the contours $\{C_{\beta'}\}_{0 < \beta' < \beta}$ scans the region bounded by $C_{\beta=0}$ and C_β, the integrand $\frac{1}{2E_{\vec{k}'}} e^{i\vec{k}'\cdot(\vec{\xi}-i\vec{\eta})-iE_{\vec{k}'}(\xi^0-i\eta^0)}$ is holomorphic in the region bounded by $C_{\beta=0}$ and C_β. Therefore, the holomorphy allows us to deform the contour C_β into $C_{\beta=0}$, i.e., the real axis, and finally,
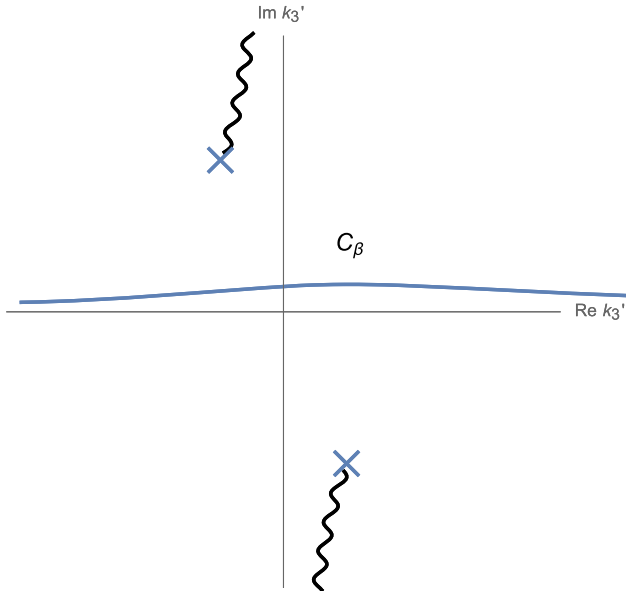

FIG. 4. Schematic picture of the contour C_β in the k_3' complex plane. The cross symbols represent the points at $E_{\vec{k}} = 0$. The integrand $\frac{1}{2E_{\vec{k}}} e^{i\vec{k}'\cdot(\vec{\xi}-i\vec{\eta})-iE'_{\vec{k}}(\xi^0-i\eta^0)}$ has singularities: branch points at these points and cuts represented as wavy lines. This integrand is holomorphic in the region bounded by the real axis $C_{\beta=0}$ and C_β.

$$W_\zeta(\Lambda(\xi - i\eta)) = \int_{\mathbb{R}^2 \times C_\beta} \frac{d^3\vec{k}'}{(2\pi)^3} \frac{1}{2E_{\vec{k}'}} e^{i\vec{k}'\cdot(\vec{\xi}-i\vec{\eta})-iE_{\vec{k}'}(\xi^0-i\eta^0)}$$

$$= \int \frac{d^3\vec{k}'}{(2\pi)^3} \frac{1}{2E_{\vec{k}'}} e^{i\vec{k}'\cdot(\vec{\xi}-i\vec{\eta})-iE_{\vec{k}'}(\xi^0-i\eta^0)}$$

$$= W_\zeta(\xi - i\eta), \tag{86}$$

which establishes the Lorentz invariance. ∎

So far, we have verified the Lorentz invariance explicitly. Because of importance of this assertion, we prove it from another point of view. The Lorentz invariance follows from a stronger symmetry, the proper complex Lorentz symmetry.

Theorem 9.—Let $W_1(\xi - i\eta)$ be a holomorphic function in the tube $\mathbb{R}^4 - iV_+$ and invariant under the Euclidean rotation group $SO(4)$ (within the domain of definition of $W_1(\xi - i\eta))$.[12] Then, $W_1(\xi - i\eta)$ is invariant under the proper complex Lorentz group $L_+(\mathbb{C})$, including the restricted Lorentz group, namely, for any $\Lambda \in L_+(\mathbb{C})$,

$$z, \Lambda z \in \mathbb{R}^4 - iV_+ \Rightarrow W_1(\Lambda z) = W_1(z), \tag{87}$$

where $L_+(\mathbb{C}) := \{\Lambda \in \mathbb{C}^{4\times4}; \Lambda^T G\Lambda = G, \det\Lambda = 1\}$ with the metric $G = \mathrm{diag}(1, -1, -1, -1)$. In particular, the holomorphic Wightman function of Theorem 3 satisfies (87).

Proof.—Since the Euclidean rotation gives a real environment of the complex Lorentz group, the assumption of the theorem and the identity theorem for holomorphic functions guarantee that, for every $z \in \mathbb{R}^4 - iV_+$, there exists a complex neighborhood of the identity element of the complex Lorentz group $L_+(\mathbb{C})$ under which the holomorphic function $W_1(z)$ is invariant.

The same argument for proving the Bargmann-Hall-Wightman theorem (Theorems 2–11 and its lemma of [36]) ensures the existence of a curve in $L_+(\mathbb{C})$, $\{\Lambda(s) \in L_+(\mathbb{C}); s \in [0, 1], \Lambda(0) = 1, \Lambda(1) = \Lambda\}$, such that $\Lambda(s)z \in \mathbb{R}^4 - iV_+$. Therefore, by using repeatedly the invariance under the neighborhood of the identity element for the function on the curve $W_1(\Lambda(s)z)$, the former assertion (87) holds.

An analytic continuation of a $SO(4)$ invariant function is invariant under $SO(4)$ within its domain of definition, since $\hat{M}_{\mu\nu} W_1(z)$ vanishes in the domain due to the identity theorem, where $\hat{M}_{\mu\nu}$ is the $SO(4)$ symmetry generators. Thus, the latter assertion follows from the former one. ∎

Let us add some remarks.

(a) Unlike the other theorems, a generalization of this argument to $D \neq 4$ is nontrivial because of the usage of the same argument as the Bargmann-Hall-Wightman theorem.

---

[12]Note that the action of $R \in SO(4)$ upon $(\xi - i\eta)$ is represented as $(\eta^0 + i\xi^0, \vec{\xi} - i\vec{\eta}) \mapsto R(\eta^0 + i\xi^0, \vec{\xi} - i\vec{\eta})$.
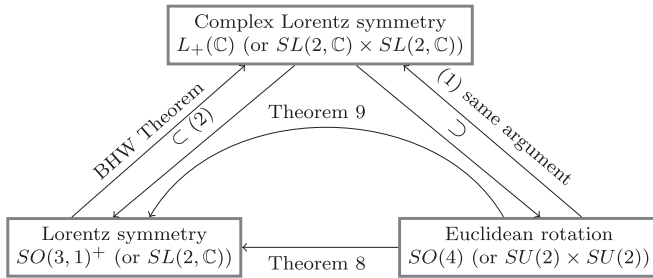
FIG. 5. A sketch of relations among the well-known Bargmann-Hall-Wightman (BHW) theorem, Theorem 8, and Theorem 9. Theorem 9 is a composition of (1) the same argument as the BHW theorem from Euclidean to complex Lorentz symmtry and (2) the restriction from complex Lorentz symmetry to Lorentz symmetry.

(b) Using the Bargmann-Hall-Wightman theorem, we can prove the complex Lorentz invariance also from Theorem 8.

(c) Relations among the well-known Bargmann-Hall-Wightman theorem, Theorem 8, and Theorem 9 are depicted in Fig. 5.

(d) As is well known, this theorem guarantees a single-valued analytic continuation of the Wightman function to the extended tube, $\mathscr{T}' := L_+(\mathbb{C})(\mathbb{R}^4 - iV_+) = \{\Lambda z \in \mathbb{C}^4; \exists (z, \Lambda) \in (\mathbb{R}^4 - iV_+) \times L_+(\mathbb{C})\}$, which includes the Jost points $\mathbb{R}^4 \cap \mathscr{T}'$. Here, the *Jost points* are just spacelike points: $\mathbb{R}^4 \cap \mathscr{T}' = \{(\xi^0, \vec{\xi}) \in \mathbb{R}^4; (\xi^0)^2 - \vec{\xi}^2 < 0\}$. Note that the proper complex Lorentz group includes $-1 \in L_+(\mathbb{C})$, from which the equality $W_1(z) = W_1(-z)$ follows.

(e) The reconstruction is based on the identification of (A23): $W_1(-i\xi_4, \vec{\xi}) = S_1(\vec{\xi}, \xi_4)$. However, we have reconstructed the Wightman function using only the Schwinger function with positive imaginary time $\xi_4 > 0$. It should be possible to use the Schwinger function with negative imaginary time $\xi_4 < 0$ for the reconstruction. The holomorphy in the extended tube together with the invariance under the proper complex Lorentz group, especially $-1 \in L_+(\mathbb{C})$, guarantees the consistency that the reconstruction from $\xi_4 < 0$ would give the same holomorphic Wightman function as that from $\xi_4 > 0$.

7. Locality

Finally, let us comment on locality. Some argue that complex singularities are associated with nonlocality. One might claim that the nonlocality of the Yang-Mills theory in a gauge-fixed picture is rather "natural" due to the Gribov-Singer obstruction, see [4,27,46] and [28,47]. However, we argue that complex singularities themselves do not necessarily lead to nonlocality.

For example, the problem of locality has been discussed in [29–31] (see also Sec. VI A), in which they assert that complex poles describe short-lived excitations and that the

locality is broken in short range at the level of propagators, but the corresponding S matrix remains causal. However, as we have mentioned above, this interpretation is different from our results.

To the best of our knowledge, the only axiomatic way to impose locality is the spacelike commutativity. To argue that complex singularities themselves do not necessarily yield nonlocality, it suffices to prove the spacelike commutativity at the level of two-point functions, because existence of complex singularities is a property of propagators.

Theorem 10.—Let $S_1(p) = D(p^2)$ be a two-point Schwinger function with complex singularities satisfying (i)–(v). By Theorems 3 and 4, $W_1(-i\xi_4, \vec{\xi}) = S_1(\vec{\xi}, \xi_4)(\xi_4 > 0)$ has the analytic continuation $W_1(\xi - i\eta)$ to the tube $\mathbb{R}^4 - iV_+$, and there exists the boundary value as a distribution $W_1(\xi) = \lim_{\substack{\eta \to 0 \\ \eta \in V_+}} W_1(\xi - i\eta) \in \mathscr{D}'(\mathbb{R}^4)$. Then, the boundary value $W_1(\xi)$ satisfies the spacelike commutativity: $W_1(\xi) = W_1(-\xi)$ for spacelike ξ.

Proof.—For a spacelike point ξ, there exists an element of the restricted Lorentz group Λ such that $\Lambda\xi = -\xi$. Therefore, the spacelike commutativity $W_1(\xi) = W_1(-\xi)$ immediately follows from Theorem 8. ∎

Note that the spacelike commutativity at this level is also an immediate consequence of the holomorphy in the extended tube and the invariance under the (proper) complex Lorentz group [see Remark (d) of Theorem 9].

One might argue that, e.g., from the Jost-Lehmann-Dyson (JLD) representation [48], complex singularities could lead to violation of the local spacelike commutativity. Nevertheless, the Wightman function with complex singularities breaks temperedness (Theorem 5). This nontemperedness enables a theory to evade the restriction of the theorems like the JLD representation that assumes existence of Fourier transform of Wightman functions. Hence, there is no contradiction here.

In conclusion, even in the presence of complex singularities, the spacelike commutativity at the level of two-point functions remains intact. Therefore, complex singularities themselves not necessarily lead to nonlocality.

8. Generalization to arbitrary spinor fields

So far, we have restricted our arguments to the case of a single scalar field for simplicity. Since the gluon fields are vector and the quark fields are spinor, it is important to generalize the above arguments to include vector and spinor fields. Here, remarks on this generalization are made. This section is applicable only for four-dimensional cases, since the complex Lorentz symmetry $L_+(\mathbb{C})$ plays a pivotal role.

Except for a single scalar field, the permutation symmetry [OS3] is not an immediate consequence of the Euclidean rotational symmetry. Therefore, for arbitrary spinor fields including vector and (Dirac-)spinor ones, we assume the following (ii'a) and (ii'b) instead of (ii) in Sec. III A.

(ii′a) [OS1] Euclidean (translational and rotational) invariance: for all translation $a \in \mathbb{R}^4$, Euclidean rotation $(A, B) \in SU(2) \times SU(2)$, labels of fields (k_1, k_2), and spinor (dotted and undotted) indices (ν_{k_1}, ν_{k_2}),

$$
\begin{aligned}
\mathcal{S}^{(k_1,k_2)}_{2,\nu_{k_1}\nu_{k_2}} & (\Lambda(A,B)x_1 + a, \Lambda(A,B)x_2 + a) \\
&= \sum_{\mu_{k_1},\mu_{k_2}} S^{(k_1)}(A,B)^{\mu_{k_1}}_{\nu_{k_1}} S^{(k_2)}(A,B)^{\mu_{k_2}}_{\nu_{k_2}} \mathcal{S}^{(k_1,k_2)}_{2,\mu_{k_1}\mu_{k_2}}(x_1, x_2),
\end{aligned}
\tag{88}
$$

where $\Lambda(A,B) \in SO(4)$ is the vector representation, and $S^{(k_1)}(A,B)^{\mu_{k_1}}_{\nu_{k_1}}$ and $S^{(k_2)}(A,B)^{\mu_{k_2}}_{\nu_{k_2}}$ are, respectively, finite-dimensional representations of $SU(2) \times SU(2)$ of the fields labeled by k_1 and k_2.

(ii′b) [OS3] Permutation symmetry:

$$
\mathcal{S}^{(k_1,k_2)}_{2,\nu_{k_1}\nu_{k_2}}(x_1, x_2) = (-1)^\sigma \mathcal{S}^{(k_2,k_1)}_{2,\nu_{k_2}\nu_{k_1}}(x_2, x_1), \tag{89}
$$

where $(-1)^\sigma = 1$ for bosonic fields and $(-1)^\sigma = -1$ for fermionic fields.

For assumptions (iv)–(v), $D(k^2)$ is understood as each component of the tensor decomposition of the Schwinger function $S^{(k_1,k_2)}_{1,\nu_{k_1}\nu_{k_2}}(k)$. For example, for a vector field, the tensor decomposition is

$$
D_{\mu\nu}(k) = D_1(k^2)\delta_{\mu\nu} + D_2(k^2)\frac{k_\mu k_\nu}{k^2}, \tag{90}
$$

and therefore, we assume (iv) and (v) for $D_1(k^2)$ and $D_2(k^2)$. In the case of the Landau gauge, the assumptions apply $D_T(k^2) = D_1(k^2) = -D_2(k^2)$.

By repeating the same arguments for each $(k_1, k_2; \nu_{k_1}\nu_{k_2})$, we can generalize the holomorphy in the tube, boundary value as a distribution, the nontemperedness, and the violation of the (reflection-)positivity. Their proofs do not essentially depend on the Lorentz (or Euclidean) symmetry. Nontrivial points are Lorentz symmetry and locality.

For Lorentz symmetry, Theorem 9 based on the complex Lorentz symmetry can be easily generalized,[13] while this is not the case for Lemma 3 and Theorem 8. The Bargmann-Hall-Wightman argument holds for arbitrary spinor fields.

For locality, note that the proof of Theorem 10 is applicable only for a single scalar field. However, the alternative proof remarked below the proof of Theorem 10 can be generalized.

Let us prove here the spacelike commutativity: for spacelike $\xi \in \mathbb{R}^4$,

$$
W^{(k_1,k_2)}_{1,\nu_{k_1}\nu_{k_2}}(\xi) = (-1)^\sigma W^{(k_2,k_1)}_{1,\nu_{k_2}\nu_{k_1}}(-\xi). \tag{91}
$$

The proof is as follows. The permutation symmetry and the Euclidean rotation $(A, B) = (1, -1) \in SU(2) \times SU(2)$ implies

$$
\begin{aligned}
S^{(k_1,k_2)}_{1,\nu_{k_1}\nu_{k_2}}(\xi) &= (-1)^\sigma S^{(k_2,k_1)}_{1,\nu_{k_2}\nu_{k_1}}(-\xi) \\
&= (-1)^\sigma (-1)^{J(\nu_{k_2},\nu_{k_1})} S^{(k_2,k_1)}_{1,\nu_{k_2}\nu_{k_1}}(\xi), \tag{92}
\end{aligned}
$$

where $J(\nu_{k_2}, \nu_{k_1})$ denotes the number of dotted indices in ν_{k_1} and ν_{k_2}. Therefore, the holomorphic Wightman function satisfies

$$
W^{(k_1,k_2)}_{1,\nu_{k_1}\nu_{k_2}}(\xi - i\eta) = (-1)^\sigma (-1)^{J(\nu_{k_2},\nu_{k_1})} W^{(k_2,k_1)}_{1,\nu_{k_2}\nu_{k_1}}(\xi - i\eta). \tag{93}
$$

Using the complex Lorentz symmetry $(A, B) = (1, -1) \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ again, we have

$$
W^{(k_1,k_2)}_{1,\nu_{k_1}\nu_{k_2}}(\xi - i\eta) = (-1)^\sigma W^{(k_2,k_1)}_{1,\nu_{k_2}\nu_{k_1}}(-\xi + i\eta), \tag{94}
$$

where the right-hand side is understood as a single-valued analytic continuation to the extended tube $\mathcal{T}' = L_+(\mathbb{C})\mathcal{T}$ by the Bargmann-Hall-Wightman theorem. Let us take the limit $\eta \to 0 (\eta \in V_+)$. The left-hand side gives the Wightman function $W^{(k_1,k_2)}_{1,\nu_{k_1}\nu_{k_2}}(\xi)$ as usual. On the other hand, at spacelike ξ, i.e., at the Jost point $\xi \in \mathbb{R}^4 \cap \mathcal{T}'$, the limit $W^{(k_2,k_1)}_{1,\nu_{k_2}\nu_{k_1}}(-\xi - i\eta) \to W^{(k_2,k_1)}_{1,\nu_{k_2}\nu_{k_1}}(-\xi)$ is independent of the direction of the limit $\eta \to 0$. Therefore, the right-hand side of (94) tends to $W^{(k_2,k_1)}_{1,\nu_{k_2}\nu_{k_1}}(-\xi)$. This establishes the spacelike (anti-)commutativity (91).

Therefore, the main conclusions (A)–(D) hold regardless of spins.

IV. INTERPRETATION IN A STATE SPACE WITH AN INDEFINITE METRIC

We have discussed analytic aspects of complex singularities. In this section, we consider a possible kinematic structure yielding complex singularities, i.e., a realization of complex singularities in a quantum theory. Since abandoning the positivity of the state-space metric is very common in Lorentz covariant gauge-fixed descriptions of gauge theories, we consider a quantum theory in a state space with an indefinite metric.

In Sec. IV A, we argue that the natural candidates providing complex singularities in an indefinite-metric state space are zero-norm pairs of eigenstates with complex eigenvalues. In Sec IV B, we present the Lee-Wick model as an example of QFT with complex poles. In Sec. IV C, we discuss complex poles in the BRST formalism in a heuristic

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[13]Note that every finite-dimensional representation of $SU(2) \times SU(2)$ can be analytically continued to that of the complex Lorentz symmetry $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$.

way. Finally, some remarks are made on gauge-invariant propagators in Sec. IV D.

A. Complex singularities and complex spectra

An important observation is that a complex energy spectrum can appear in an indefinite metric state space even if the Hamiltonian is (pseudo-)Hermitian. For a review on indefinite-metric quantum field theories, see, e.g., [43].

Beforehand, let us introduce some notions on an indefinite-metric state space. Note that the completeness of eigenstates of a Hermitian operator does not always hold even in a finite dimensional state space with an indefinite metric. Instead of simple eigenstates, the set of "*generalized eigenstates*" $\{|E^0\rangle, |E^1\rangle, ..., |E^{n-1}\rangle\}$ that are defined to be elements of sequences: $(H - E)|E^0\rangle = E|E^1\rangle$, $(H - E)|E^1\rangle = E|E^2\rangle, ..., (H - E)|E^{n-1}\rangle = 0$ spans the full state space, in general, where H is a Hermitian operator, and the value E of such a sequence $\{|E^0\rangle, |E^1\rangle, ..., |E^{n-1}\rangle\}$ is called the *generalized eigenvalue*. This follows from the standard Jordan decomposition. A generalized eigenstate $|n\rangle$ is said to be of *order M* if and only if both $(H - E_n)^M|n\rangle = 0$ and $(H - E_n)^{M-1}|n\rangle \neq 0$ hold. For example, $|E^0\rangle$ of a sequence $\{|E^0\rangle, |E^1\rangle, ..., |E^{n-1}\rangle\}$ is a generalized eigenstate of order n.

For a while, we consider the $0 + 1$ dimensional case in which a field $\phi(t)$ is regarded as an operator-valued function whose domain contains at least the vacuum $|0\rangle$, for simplicity. Alternatively, one could consider a situation in which field operators are smeared in spatial directions.

We begin with the necessity of complex spectra for existence of complex singularities.

Claim 1.—Let us assume the following:

(i) Completeness of (denumerable) generalized eigenstates $|n\rangle$ of the Hamiltonian H: $1 = \sum_{n,n'} \eta_{n,n'}^{-1}|n\rangle\langle n'|$, where $\eta_{n,n'} = \langle n|n'\rangle$ is the nondegenerate metric

(ii) Translational covariance: $\phi(t) = e^{iHt}\phi(0)e^{-iHt}$

(iii) Real-valuedness of generalized eigenvalues E_n of the Hamiltonian H

Moreover, as technical assumptions, we assume the following:

(iv) Existence of an upper bound M on the orders of generalized eigenstates,[14] finiteness of a sum $\sum_{n'} \eta_{n,n'}^{-1}$ for any $|n\rangle$ in the complete system, and the absolute convergence of the sum,

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[14]Note that all states that are not generalized eigenstates of finite order can be seen as "generalized eigenstates of infinite order". The notion "generalized eigenstates of infinite order" is thus irrelevant to the spectral decomposition. Therefore, it would be appropriate to assume the upper bound.

$$\sum_{n,n'} \eta_{n,n'}^{-1} \sum_{k=0}^{M(n)-1} e^{-iE_n t} \frac{(-it)^k}{k!}$$
$$\times \langle 0|\phi(0)(H - E_n)^k|n\rangle\langle n'|\phi(0)|0\rangle, \quad (95)$$

which actually equals $\langle 0|\phi(t)\phi(0)|0\rangle$, where E_n is the generalized eigenvalue of $|n\rangle$, $M(n)$ is the order of $|n\rangle$, and $|0\rangle$ is the vacuum state satisfying $H|0\rangle = 0$.

Then, the Wightman function $\langle 0|\phi(t)\phi(0)|0\rangle$ can be regarded as a tempered distribution.

Derivation.—Since $|n\rangle$ is a generalized eigenstate of order $M(n)$, $(H - E_n)^{M(n)}|n\rangle = 0$ and $(H - E_n)^{M(n)-1}|n\rangle \neq 0$ hold, which implies

$$e^{-i(H-E_n)t}|n\rangle = \sum_{k=0}^{M(n)-1} \frac{(-it)^k}{k!}(H - E_n)^k|n\rangle. \quad (96)$$

By assumptions (i) and (ii), we have

$$\langle 0|\phi(t)\phi(0)|0\rangle = \sum_{n,n'} \eta_{n,n'}^{-1} e^{-iE_n t}$$
$$\times \langle 0|\phi(0)e^{-i(H-E_n)t}|n\rangle\langle n'|\phi(0)|0\rangle$$
$$= \sum_{n,n'} \eta_{n,n'}^{-1} \sum_{k=0}^{M(n)-1} e^{-iE_n t} \frac{(-it)^k}{k!}$$
$$\times \langle 0|\phi(0)(H - E_n)^k|n\rangle\langle n'|\phi(0)|0\rangle.$$
$$(97)$$

Note that the generalized eigenvalue E_n is real by assumption (iii).

For any test function $f(t) \in \mathscr{S}(\mathbb{R})$, we obtain

$$\left|\int dt f(t)\langle 0|\phi(t)\phi(0)|0\rangle\right|$$
$$= \left|\sum_{n,n'} \eta_{n,n'}^{-1} \sum_{k=0}^{M(n)-1} \frac{1}{k!}\frac{\partial^k \tilde{f}}{\partial\omega^k}\bigg|_{\omega=E_n}\right.$$
$$\left.\times \langle 0|\phi(0)(H - E_n)^k|n\rangle\langle n'|\phi(0)|0\rangle\right|$$
$$\leq \left(\sum_{n,n'} |\eta_{n,n'}^{-1}| \sum_{k=0}^{M(n)-1} \left|\frac{1}{k!}\langle 0|\phi(0)(H - E_n)^k|n\rangle\langle n'|\phi(0)|0\rangle\right|\right)$$
$$\times \left(\sup_{\omega,k<M}\left|\frac{\partial^k \tilde{f}}{\partial\omega^k}\right|\right) \leq \text{const.} \times \left(\sup_{\omega,k<M}\left|\frac{\partial^k \tilde{f}}{\partial\omega^k}\right|\right), \quad (98)$$

where $\tilde{f}(\omega) = \int dt e^{-i\omega t}f(t)$ is the Fourier transform of $f(t)$, and we have used assumptions (iv). This inequality proves $\langle 0|\phi(t)\phi(0)|0\rangle \in \mathscr{S}'(\mathbb{R})$. ∎

From this claim, the nontemperedness (Theorem 5) is incompatible with the reality of the spectrum. Thus,

complex spectra should be allowed for complex singularities to appear. We call eigenvalues that are not real *complex eigenvalues*. Note that eigenstates of complex eigenvalues of a Hermitian operator appear as pairs of zero-norm states. As an introduction to the state-space structure with complex eigenvalues, we prove the following claim.

Claim 2.—Let H be a Hermitian operator and have a complex eigenvalue: $H|\alpha\rangle = E_\alpha|\alpha\rangle, E_\alpha \neq E_\alpha^*$. Suppose that its generalized eigenstates form a complete system. Then,
 (1) $|\alpha\rangle$ is a zero-norm state.
 (2) There exists a partner state $|\beta\rangle$ such that $\langle\beta|\alpha\rangle \neq 0$, $\langle\beta|\beta\rangle = 0$, and $(H - E_\alpha^*)^k|\beta\rangle = 0$ for some integer k.[15]

Derivation.
 (1) Since $E_\alpha \neq E_\alpha^*$, the equation $E_\alpha\langle\alpha|\alpha\rangle = \langle\alpha|H|\alpha\rangle = E_\alpha^*\langle\alpha|\alpha\rangle$ implies that $|\alpha\rangle$ is a zero-norm state: $\langle\alpha|\alpha\rangle = 0$.
 (2) Because of the nondegeneracy of the metric, $|\alpha\rangle$ has a partner state, namely, $|\beta\rangle$ such that $\langle\beta|\alpha\rangle \neq 0$. One can take a generalized eigenstate of H as this state $|\beta\rangle$. Indeed, otherwise, the completeness would imply that $|\alpha\rangle$ is orthogonal to all states, which contradicts with the nondegeneracy. Therefore, $|\beta\rangle$ satisfies: for some integer k,

$$\langle\alpha|\beta\rangle \neq 0, \qquad (H - E_\beta)^k|\beta\rangle = 0,$$
$$(H - E_\beta)^{k-1}|\beta\rangle \neq 0. \qquad (99)$$

From the second and first equations, we have $(E_\alpha^* - E_\beta)^k\langle\alpha|\beta\rangle = 0$, and therefore, $E_\beta = E_\alpha^*$. Similarly to $|\alpha\rangle$, $|\beta\rangle$ is also a zero-norm state: $\langle\beta|\beta\rangle = 0$ since E_β is not real, $E_\beta^* \neq E_\beta$.   ∎

The simplest possibility to provide complex singularities is a pair of the zero-norm states $\{|\alpha\rangle, |\beta\rangle\}$. Let us consider a consequence from such minimal complex spectra.

Claim 3.—Suppose, in addition to (i), (ii), (iv) of Claim 1, the following:
 (iii′) Besides real eigenvalues, the Hermitian Hamiltonian H has one pair of eigenstates $\{|\alpha\rangle, |\beta\rangle\}$ of complex conjugate eigenvalues $E_\alpha, E_\beta = E_\alpha^*$ with a positive real part $\text{Re}E_\alpha > 0$.
 (v) The field operator $\phi(t)$ is Hermitian.
Then, the following statements hold:
 (1) If $\langle\beta|\phi(0)|0\rangle = 0$ or $\langle\alpha|\phi(0)|0\rangle = 0$, then the Wightman function is in $\mathscr{S}'(\mathbb{R})$. In particular, the Schwinger function has no complex singularity.
 (2) If $\langle\beta|\phi(0)|0\rangle \neq 0$ and $\langle\alpha|\phi(0)|0\rangle \neq 0$, then the Schwinger function has a pair of simple complex conjugate poles besides the real singularities.

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[15]One can prove the one-to-one correspondence between a sequence of generalized eigenstates of E_α: $\{|\alpha\rangle, (H - E_\alpha)|\alpha\rangle, (H - E_\alpha)^2|\alpha\rangle, \cdots\}$ and that of E_α^* in finite dimensional cases. For an example, see Sec. 7 of [43].

Derivation.—First, let us examine the metric structure of the state space. The eigenstates of complex eigenvalues, $H|\alpha\rangle = E_\alpha|\alpha\rangle, H|\beta\rangle = E_\alpha^*|\beta\rangle$, are orthogonal to the generalized eigenstates with real eigenvalues $|n\rangle$. Indeed, for every $|n\rangle$ satisfying $(H - E_n)^{M(n)}|n\rangle = 0$ and $(H - E_n)^{M(n)-1}|n\rangle \neq 0$ with real E_n, $(E_\alpha^* - E_n)^{M(n)}\langle\alpha|n\rangle = 0$ and $(E_\alpha - E_n)^{M(n)}\langle\beta|n\rangle = 0$ hold, from which $\langle\alpha|n\rangle = \langle\beta|n\rangle = 0$. The metric $\eta_{n,m} = \langle n|m\rangle$ is "block diagonalized" to the sectors of real energies and of complex energies; we can decompose the completeness relation as

$$\sum_{n,n'}\eta_{n,n'}^{-1} = \sum_{n,n':\text{real}}\eta_{n,n'}^{-1} + \sum_{n,n':\text{complex}}\eta_{n,n'}^{-1}. \qquad (100)$$

The metric $\eta_{n,n'}^{-1}$ in the second term is a two-by-two matrix and can be written as $\eta_{\alpha,\alpha}^{-1} = \eta_{\beta,\beta}^{-1} = 0$, $\eta_{\alpha,\beta}^{-1} = (\langle\beta|\alpha\rangle)^{-1}$, and $\eta_{\beta,\alpha}^{-1} = (\langle\alpha|\beta\rangle)^{-1}$.

Now, we have

$$\langle 0|\phi(t)\phi(0)|0\rangle = \sum_{n,n':\text{real}}\eta_{n,n'}^{-1}e^{-iE_n t}\langle 0|\phi(0)|n\rangle\langle n'|\phi(0)|0\rangle$$
$$+ \sum_{n,n':\text{complex}}\eta_{n,n'}^{-1}e^{-iE_n t}$$
$$\times \langle 0|\phi(0)|n\rangle\langle n'|\phi(0)|0\rangle. \qquad (101)$$

The first term is characterized by Claim 1, which provides singularities only on the negative real axis in the Schwinger function. On the other hand, the second term reads

$$W_{\text{complex}}(t) := \sum_{n,n':\text{complex}}\eta_{n,n'}^{-1}e^{-iE_n t}\langle 0|\phi(0)|n\rangle\langle n'|\phi(0)|0\rangle$$
$$= (\langle\beta|\alpha\rangle)^{-1}e^{-iE_\alpha t}\langle 0|\phi(0)|\alpha\rangle\langle\beta|\phi(0)|0\rangle$$
$$+ (\langle\alpha|\beta\rangle)^{-1}e^{-iE_\alpha^* t}\langle 0|\phi(0)|\beta\rangle\langle\alpha|\phi(0)|0\rangle. \qquad (102)$$

Let us evaluate $W_{\text{complex}}(t)$ in the following cases:
 (1) $\langle\beta|\phi(0)|0\rangle = 0$ or $\langle\alpha|\phi(0)|0\rangle = 0$. The Hermiticity of ϕ yields

$$\langle\alpha|\phi(0)|0\rangle = \langle 0|\phi(0)|\alpha\rangle^*,$$
$$\langle\beta|\phi(0)|0\rangle = \langle 0|\phi(0)|\beta\rangle^*, \qquad (103)$$

from which $W_{\text{complex}}(t) = 0$ in this case. Thus, the Wightman function can be regarded as a tempered distribution.
 (2) $\langle\beta|\phi(0)|0\rangle \neq 0$ and $\langle\alpha|\phi(0)|0\rangle \neq 0$. We define

$$Z := \frac{2E_\alpha\langle 0|\phi(0)|\alpha\rangle\langle\beta|\phi(0)|0\rangle}{\langle\beta|\alpha\rangle}, \qquad (104)$$

which does not vanish in this case. The Schwinger function of this part $S_{\text{complex}}(\tau)$ for $\tau \neq 0$ is given by

$$S_{\text{complex}}(\tau) = W_{\text{complex}}(-i|\tau|)$$

$$= \frac{Z}{2E_\alpha} e^{-E_\alpha|\tau|} + \frac{Z^*}{2E_\alpha^*} e^{-E_\alpha^*|\tau|}. \quad (105)$$

This function can be represented as

$$S_{\text{complex}}(\tau) = \int \frac{dk}{2\pi} e^{ik\tau} \tilde{S}_{\text{complex}}(k),$$

$$\tilde{S}_{\text{complex}}(k) = \frac{Z}{k^2 + E_\alpha^2} + \frac{Z^*}{k^2 + (E_\alpha^*)^2}, \quad (106)$$

which is indeed a pair of simple complex conjugate poles.

Therefore, the pair of eigenstates $\{|\alpha\rangle, |\beta\rangle\}$ leads to either (1) the Wightman function is in $\mathscr{S}'(\mathbb{R})$, or (2) the Schwinger function has a pair of simple complex conjugate poles. ∎

Therefore, complex singularities defined in the previous section can appear in a state space with an indefinite metric, when the Hamiltonian H has complex spectra. This claim suggests *a correspondence between complex singularities and zero-norm pairs of eigenstates of complex eigenvalues.* Finally, let us add remarks on this claim.

(a) The necessity of an indefinite metric for complex singularities is consistent with Theorem 7, the violation of the Wightman positivity.

(b) Claim 3 also implies that, under the assumption of the Hermiticity of the Hamiltonian and field operators, complex singularities should appear as complex conjugate pairs. This statement can be also understood by the (intuitive) representation of the Schwinger function $S(\tau)$: for $\tau > 0$, $S(\tau) = \langle 0|\phi(0)e^{-H\tau}\phi(0)|0\rangle$. The Hermiticity of the Hamiltonian and the field operator yields $S(\tau) \in \mathbb{R}$, from which $D(z)^* = D(z^*)$. This complex-conjugate pairing is consistent with Remark (d) of Theorem 1.

The discussion above is restricted to quantum mechanics, or the $(0+1)$ dimension. In the next subsection, we see an example of QFT with complex poles.

B. Example: Lee-Wick model

A simple possible QFT yielding complex poles is the Lee-Wick model of complex ghosts [49], which has been studied for many years. Here, we briefly review its kinematic structure following its covariant operator formulation given in Ref. [45] and see that there indeed exists a Hermitian field whose propagator has complex poles.

Let us start with the Lagrangian density of the Lee-Wick model of complex scalar field ϕ with complex mass $M^2 \in \mathbb{C}$,

$$\mathscr{L} := \frac{1}{2}[(\partial_\mu \phi)(\partial^\mu \phi) + (\partial_\mu \phi)^\dagger (\partial^\mu \phi)^\dagger$$
$$- M^2 \phi^2 - (M^*)^2 (\phi^\dagger)^2]. \quad (107)$$

We expand the field operator ϕ as

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x),$$

$$\phi^{(+)}(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \alpha(\vec{p}) e^{i\vec{p}\cdot\vec{x} - iE_{\vec{p}}t},$$

$$\phi^{(-)}(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \beta^\dagger(\vec{p}) e^{-i\vec{p}\cdot\vec{x} + iE_{\vec{p}}t}, \quad (108)$$

where $E_{\vec{p}} := \sqrt{M^2 + \vec{p}^2}$, and we chose $\mathrm{Re}\, E_{\vec{p}} \geq 0$ and $\mathrm{Re}\sqrt{E_{\vec{p}}} \geq 0$. The canonical commutation relation implies $[\alpha(\vec{p}), \beta^\dagger(\vec{q})] = [\beta(\vec{p}), \alpha^\dagger(\vec{q})] = (2\pi)^3 \delta(\vec{p} - \vec{q})$. We define the vacuum $|0\rangle$ by $\alpha(\vec{p})|0\rangle = \beta(\vec{p})|0\rangle = 0$, or $\phi^{(+)}(x)|0\rangle = [\phi^{(-)}(x)]^\dagger|0\rangle = 0$. Note that the field operator $\phi(x)$ together with its parts $\phi^{(+)}(x)$ and $\phi^{(-)}(x)$ is a Lorentz scalar, and therefore, the vacuum $|0\rangle$ is a Lorentz invariant state, see [45] for details. Note that the Lorentz symmetry is manifest in this formulation until one (artificially) considers asymptotic states. The Hamiltonian reads

$$H = \int \frac{d^3 p}{(2\pi)^3} [E_{\vec{p}} \beta^\dagger(\vec{p})\alpha(\vec{p}) + E_{\vec{p}}^* \alpha^\dagger(\vec{p})\beta(\vec{p})], \quad (109)$$

ignoring some constant. Notice that the complex-energy states $\alpha^\dagger(\vec{p})|0\rangle$ and $\beta^\dagger(\vec{p})|0\rangle$ form a pair of zero-norm states $(|\vec{p}, \alpha\rangle := \alpha^\dagger(\vec{p})|0\rangle, |\vec{p}, \beta\rangle := \beta^\dagger(\vec{p})|0\rangle)$ for every $\vec{p} \in \mathbb{R}^3$,

$$\langle \vec{p}, \alpha|\vec{q}, \alpha\rangle = \langle \vec{p}, \beta|\vec{q}, \beta\rangle = 0,$$
$$\langle \vec{p}, \alpha|\vec{q}, \beta\rangle = \langle \vec{p}, \beta|\vec{q}, \alpha\rangle = (2\pi)^3 \delta(\vec{p} - \vec{q}). \quad (110)$$

The commutators of the fields are given by

$$[\phi(x), \phi(y)] = i\Delta(x - y, M^2),$$
$$[\phi(x), \phi^\dagger(y)] = 0, \quad (111)$$

where

$$\Delta(x, M^2) := \int \frac{d^3 p}{(2\pi)^3} \frac{1}{E_{\vec{p}}} \sin(\vec{p} \cdot \vec{x} - E_{\vec{p}}t). \quad (112)$$

Note that $\Delta(x, M^2)$ is a Lorentz-invariant function as shown in Lemma 3 as expected from the invariance of the field operator and the vacuum state. This theory is thus spacelike commutative at least in the level of elementary fields, since $\Delta(x - y, M^2)$ vanishes for spacelike $x - y$.

Next, let us show that the Euclidean propagator of a Hermitian combination with a constant $Z \in \mathbb{C}$,

$$\Phi := \sqrt{Z}\phi + \sqrt{Z^*}\phi^\dagger, \quad (113)$$

has indeed complex poles. In this sense, the complex fields ϕ and $\phi^\dagger$ are the counterparts in the covariant operator formalism of so-called i particles [47].

Using the following correlators:

$$\langle 0|\phi(x)\phi(0)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{i\vec{p}\cdot\vec{x}-iE_{\vec{p}}t},$$

$$\langle 0|\phi(x)\phi^\dagger(0)|0\rangle = 0,$$

$$\langle 0|\phi^\dagger(x)\phi^\dagger(0)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}^*}e^{i\vec{p}\cdot\vec{x}-iE_{\vec{p}}^*t}, \quad (114)$$

we find

$$D_\Phi^>(t,\vec{x}) := \langle 0|\Phi(x)\Phi(0)|0\rangle$$
$$= [Z\langle 0|\phi(x)\phi(0)|0\rangle + Z^*\langle 0|\phi^\dagger(x)\phi^\dagger(0)|0\rangle]$$
$$= \int \frac{d^3p}{(2\pi)^3}\left[\frac{Z}{2E_{\vec{p}}}e^{i\vec{p}\cdot\vec{x}-iE_{\vec{p}}t} + \frac{Z^*}{2E_{\vec{p}}^*}e^{i\vec{p}\cdot\vec{x}-iE_{\vec{p}}^*t}\right], \quad (115)$$

which is exactly the same as the Wightman function (41) reconstructed from the Schwinger function (23). From the relation (2), we obtain the Euclidean propagator $\Delta_\Phi(\tau,\vec{x})$ for $\tau \neq 0$,

$$\Delta_\Phi(\tau,\vec{x}) := \theta(-\tau)D_\Phi^>(i\tau,\vec{x}) + \theta(\tau)D_\Phi^<(i\tau,\vec{x})$$
$$= \int \frac{d^3p}{(2\pi)^3}\left[\frac{Z}{2E_{\vec{p}}}e^{i\vec{p}\cdot\vec{x}-E_{\vec{p}}|\tau|} + \frac{Z^*}{2E_{\vec{p}}^*}e^{i\vec{p}\cdot\vec{x}-E_{\vec{p}}^*|\tau|}\right]$$
$$= \int \frac{d^3p}{(2\pi)^3}\int \frac{dp_4}{2\pi}e^{i\vec{p}\cdot\vec{x}+ip_4\tau}$$
$$\times \left[\frac{Z}{p_4^2+E_{\vec{p}}^2} + \frac{Z^*}{p_4^2+(E_{\vec{p}}^*)^2}\right]. \quad (116)$$

The Euclidean propagator in the momentum space is given by

$$\Delta_\Phi(p_E) = \frac{Z}{p_E^2+M^2} + \frac{Z^*}{p_E^2+(M^*)^2}, \quad (117)$$

which indeed exhibits a pair of complex conjugate poles.

Therefore, a kinematic structure of the covariant operator formalism of the Lee-Wick model yields simple complex poles. The simple complex poles correspond to the one-particle-like zero-norm states with complex masses.

Finally, let us comment on a construction of a composite operator whose propagator obeys the Källén-Lehmann representation [47].

As mentioned above, the field $\phi(x)$ corresponds to the so-called i-particle. According to the toy model [47], we define

$$\mathcal{O}(x) := \phi(x)\phi^\dagger(x). \quad (118)$$

This propagator can be expressed as

$$D_\mathcal{O}^>(y-x) := \langle 0|\mathcal{O}(y)\mathcal{O}(x)|0\rangle$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{4E_pE_q^*}$$
$$\times e^{-i(E_p+E_q^*)(y^0-x^0)+i(\vec{p}+\vec{q})\cdot(\vec{y}-\vec{x})}, \quad (119)$$

which seems not tempered since $(E_p + E_q^*)$ is complex, in general. However, the following reasoning indicates that this composite-field propagator involves only real spectra.[16] The Euclidean propagator $\underline{\Delta}_\mathcal{O}(\tau,\vec{k})$ in the imaginary time τ and spatial momentum $\vec{k}$ is given by

$$\underline{\Delta}_\mathcal{O}(\tau,\vec{k}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{4E_pE_{k-p}^*}e^{-(E_p+E_{k-p}^*)|\tau|}, \quad (120)$$

which reads in the momentum space,

$$\Delta_\mathcal{O}(\vec{k},k_4) = \int \frac{d^3p}{(2\pi)^3}\frac{E_p+E_{k-p}^*}{2E_pE_{k-p}^*}\frac{1}{p_4^2+(E_p+E_{k-p}^*)^2}$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\left[\frac{1}{E_p}\frac{1}{(k_4-iE_p)^2+(E_{k-p}^*)^2}\right.$$
$$\left. + \frac{1}{E_{k-p}^*}\frac{1}{(k_4+iE_{k-p}^*)^2+E_p^2}\right]$$
$$= \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2+M^2}\frac{1}{(k-p)^2+(M^*)^2}. \quad (121)$$

This is what is calculated in [47] and takes a form of the Källén-Lehmann spectral representation with a positive spectral density. Back to the real-time propagator, this implies $\langle 0|\mathcal{O}(y)\mathcal{O}(x)|0\rangle$ has only real spectra. Thus, the composite operator $\mathcal{O}(x)$ could be regarded to be "physical".

C. Complex singularities in a BRST quartet

Here, we discuss implications from the interpretation of complex singularities in an indefinite-metric state space in light of confinement. As discussed above, complex singularities correspond to zero-norm states. Such states, which

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[16]This phenomenon corresponds to nonuniqueness of a Cauchy integral. For example, if $D(k^2)$ has singularities only on the negative real axis, one can represent $D(k^2) = \int_C \frac{d\zeta}{2\pi i}\frac{D(\zeta)}{\zeta-k^2}$, where C is an arbitrary contour which separates the positive and negative real axes. In this representation, $D(k^2)$, which has no complex singularities, appears to have complex singularities on the contour C.

are not physical, should be confined according to some confinement mechanism.

It is worthwhile considering implications in the Kugo-Ojima BRST quartet mechanism [50]. Here, we assume existence of a Hermitian nilpotent BRST operator Q_B: $Q_B^2 = 0, Q_B^\dagger = Q_B$. Some issues on this existence are mentioned in Sec. VI C. In this scenario, confined states should belong to BRST quartets, i.e., BRST exact (BRST daughter) or BRST noninvariant (BRST parent) states. Thus, complex energy states, which lead to complex singularities of the propagators, should belong to BRST quartets.

In this section, we provide only a sketch of the argument. Suppose that the gluon propagator has complex singularities. Then, "one-gluon state" has complex energy states, which is schematically expressed as

$$A_\mu(0)|0\rangle = |E\rangle + |E^*\rangle + \cdots, \qquad (122)$$

where $|E\rangle$ and $|E^*\rangle$ stand for a pair of complex energy states, $\langle E^*|E\rangle \neq 0$. Since $|E\rangle$ and $|E^*\rangle$ should be excluded from the physical state space constructed from the BRST cohomology $\mathrm{Ker}\,Q_B/\mathrm{Im}\,Q_B$ to make the theory physical, we require that $|E\rangle$ and $|E^*\rangle$ are either BRST exact or BRST noninvariant states.[17]

We can easily exclude a possibility that both $|E\rangle$ and $|E^*\rangle$ are BRST exact. Indeed, if they were BRST exact, $|E\rangle = Q_B|\gamma\rangle$ and $|E^*\rangle = Q_B|\gamma^*\rangle$, then the nonorthogonality $\langle E^*|E\rangle \neq 0$ contradicts with the nilpotency of the BRST charge Q_B, $Q_B^2 = 0$. Therefore, at least either $|E\rangle \notin \mathrm{Ker}\,Q_B$ or $|E^*\rangle \notin \mathrm{Ker}\,Q_B$ holds.

We assume further that a *CPT* (antiunitary) operator Θ exists and satisfies

$$\Theta^2 = 1, \qquad \Theta|0\rangle = |0\rangle, \qquad \Theta Q_B \Theta = Q_B,$$
$$\Theta H \Theta = H, \qquad \Theta A_\mu(0)\Theta = -A_\mu(0). \qquad (123)$$

$\Theta A_\mu(0)\Theta = -A_\mu(0)$ and $\Theta|0\rangle = |0\rangle$ implies

$$\Theta|E\rangle = -|E^*\rangle, \; \Theta|E^*\rangle = -|E\rangle. \qquad (124)$$

When either $|E\rangle \notin \mathrm{Ker}\,Q_B$ or $|E^*\rangle \notin \mathrm{Ker}\,Q_B$ holds, the possibilities are (i) $|E\rangle \notin \mathrm{Ker}\,Q_B$ and $|E^*\rangle \in \mathrm{Im}\,Q_B$, (ii) $|E\rangle \in \mathrm{Im}\,Q_B$ and $|E^*\rangle \notin \mathrm{Ker}\,Q_B$, and (iii) $|E\rangle \notin \mathrm{Ker}\,Q_B$ and $|E^*\rangle \notin \mathrm{Ker}\,Q_B$. The first two possibilities (i) and (ii) can be excluded by (124) and $Q_B\Theta = \Theta Q_B$, namely, $|E\rangle \in \mathrm{Ker}\,Q_B \Leftrightarrow |E^*\rangle \in \mathrm{Ker}\,Q_B$. Thus, the only possibility is (iii); both complex energy states are BRST noninvariant.

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[17]Notice that, if the complex energy states are confined correctly, asymptotic states in the physical state space are expected to be well defined. Therefore, if such a confinement mechanism works well, the nontemperedness of the Wightman function and the ill definedness of the asymptotic states would not provide any physical issue.

Hence, existence of a *CPT* operator and nonexistence of complex energy states in the physical state space implies that both $|E\rangle$ and $|E^*\rangle$ should contain BRST parent states. In the simplest possibility, complex energy states form a double-BRST quartet.

As a consequence, since $Q_B|E\rangle = |E, c\rangle \neq 0$ and $Q_B|E^*\rangle = |E^*, c\rangle \neq 0$, we have

$$(D_\mu C)(0)|0\rangle = Q_B A_\mu(0)|0\rangle = |E, c\rangle + |E^*, c\rangle + \cdots. \qquad (125)$$

Since the ghost propagator seems to have no complex singularity according to recent analyses, e.g., [9,12,13, 18,21,24,25], this implies that the gluon-ghost bound state should contain complex energy states whose energies are equal to those of the gluon. Therefore, a propagator of the gluon-ghost bound state should have complex singularities at the same position as the gluon propagator.

Let us summarize the discussion above. Complex energy states should be "eliminated" from the physical state space by some confinement mechanism. In the Kugo-Ojima scenario, they should be in BRST quartets. For complex singularities in the gluon propagator, the "one-gluon state" should have complex conjugate energy states (122), $|E\rangle$ and $|E^*\rangle$. The other discussion in this section can be summarized as the following claim.

Claim 4.—Suppose that $|E\rangle$ and $|E^*\rangle$ of the "one-gluon state" with $\langle E|E^*\rangle \neq 0$ are in BRST quartets. Then, either $|E\rangle$ or $|E^*\rangle$ is not a BRST daughter state. Moreover, with the additional assumption of the existence of the *CPT* operator, both $|E\rangle$ and $|E^*\rangle$ contain BRST parent states.

This claim predicts that a propagator of the gluon-ghost bound state should have complex singularities at the same positions as those of the gluon propagator.

D. Complex singularities and gauge-invariant propagators

Regarding complex singularities of the gluon propagator, a natural question is how complex singularities are "canceled out" in the physical gauge-invariant propagators, e.g., the glueball propagators. Here, we make some comments on this issue.

First, we can see that gauge-invariant propagators have no complex singularity in the BRST formalism if all complex spectra are in BRST quartets. To this end, let us consider the following two-point function of a gauge-invariant operator $\mathcal{O}(x)$, e.g., $\mathcal{O}(x) = F_{\mu\nu}^a F^{a\mu\nu}$ for the 0^{++} glueball,

$$\langle 0|\mathcal{O}(x)\mathcal{O}(0)|0\rangle = \langle 0|\mathcal{O}(0)e^{-iP\cdot x}\mathcal{O}(0)|0\rangle. \qquad (126)$$

Since $\mathcal{O}(x)$ is gauge invariant, it is BRST invariant; therefore $Q_B\mathcal{O}(x)|0\rangle = 0$. Assuming the completeness relation, we write

$$\langle 0|\mathcal{O}(x)\mathcal{O}(0)|0\rangle = \sum_{n,n'}\eta_{n,n'}^{-1}e^{-ip_n\cdot x}\langle 0|\mathcal{O}(0)|n\rangle\langle n'|\mathcal{O}(0)|0\rangle.$$

$$(127)$$

As we have seen, if some states with complex energies $(p_n)^0$ contribute, this two-point function has complex singularities. In order to prevent complex-energy states to appear in the physical state space $\mathrm{Ker}Q_B/\mathrm{Im}Q_B$, we assume that such complex-energy states are in BRST quartets. Note that a BRST parent-parent pair does not appear in the completeness relation due to the nilpotency,

$$0 = Q_B^2 = \sum_{n,n'\in\{\text{parents}\}}\eta_{n,n'}^{-1}Q_B|n\rangle\langle n'|Q_B, \quad (128)$$

which yields $\eta_{n,n'}^{-1} = 0$ for $n, n' \in \{\text{parents}\}$, where $\{\text{parents}\}$ denotes a basis of a complementary space of $\mathrm{Ker}Q_B$. Thus, every quartet in the completeness relation satisfies $|n\rangle \in \mathrm{Im}Q_B$ or $|n'\rangle \in \mathrm{Im}Q_B$. This implies $\langle 0|\mathcal{O}(0)|n\rangle = 0$ or $\langle n'|\mathcal{O}(0)|0\rangle = 0$, which is case (1) of Claim 3. Therefore, with those assumptions, the gauge-invariant propagator has no complex singularity.

The above discussion indicates that complex-energy intermediate states are "canceled out" due to the BRST invariance of the operator $\mathcal{O}$. For example, this suggests that possible complex singularities of the $(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2$ propagator should disappear in the full $(F_{\mu\nu}^a)^2$ propagator.

Second, although checking this scenario by a concrete calculation is an interesting topic, it is a highly difficult task. Indeed, the connection between the full gluon propagator and the full glueball propagator is very complicated. For example, one has to maintain all of the composite operators $\partial A\partial A, AA\partial A, AAAA$ to preserve the BRST symmetry.

Third, let us mention some calculations on the glueball propagator from the (refined-)Gribov-Zwanziger propagator. The lowest-order calculation leads to complex branch cuts [28,51]. Note that, because of the soft breaking of the BRST symmetry of the Gribov-Zwanziger model, the composite operator $(F_{\mu\nu}^a)^2$ can mix with BRST noninvariant operators. One can hope that such mixing would somehow eliminate unphysical complex singularities of the glueball propagator, which is expected in the i particle scenario [47]. In terms of the i particle, it is possible to construct a composite operator whose propagator has only timelike cut at the lowest order. However, a systematic mechanism behind this scenario in the Gribov-Zwanziger model is still far from clear.

Finally, in relation to this topic, it is worthwhile noting that the reflection positivity for gauge-invariant quantities was proven in the lattice gauge theory [52]. This is of extreme importance from a viewpoint of the reconstruction from Euclidean field theories. This could lead to the spectral condition and positivity in the physical state space $\mathrm{Ker}Q_B/\mathrm{Im}Q_B$, which have been just assumed.

V. SUMMARY

Let us summarize our findings. In Sec. II, we have presented a sketch of the discussion emphasizing that complex singularities of propagators on the complex squared momentum plane differ depending on whether the propagator is Euclidean one or Minkowski one. This is an important remark for determining a starting point toward considering the reconstruction. We have to regard "complex singularities" as those of Euclidean propagator and consider the reconstruction carefully.

The main part of this paper consists of general properties of Wightman functions in Sec. III and implications on state spaces in Sec. IV.

In Sec. III, we have defined complex singularities and reconstructed Wightman functions from Schwinger functions with complex singularities. We have obtained the following general properties on this reconstruction as stated in the Introduction:

(A) Violation of the reflection positivity of the Schwinger functions (Theorem 6)
(B) Holomorphy in the tube (Theorem 3) and existence of the boundary value as a distribution (Theorem 4)
(C) Violation of the temperedness (Theorem 5) and the positivity condition in $\mathscr{D}(\mathbb{R}^4)$ (Theorem 7)
(D) Validity of Lorentz symmetry (Theorem 8 and Theorem 9) and spacelike commutativity (Theorem 10)

The organization of our proofs of these theorems is depicted in Fig. 3. See Appendix C for a summary of violated axioms.

In Sec. IV, we have considered a possible state-space structure in the presence of complex singularities. Consequently, a quantum mechanical observation (Sec. IV A) suggests the following:

(E) Complex singularities correspond to zero-norm states with complex energy eigenvalues

Indeed, we have first argued the necessity of nonreal spectra by proving Claim 1. Secondly, Claim 2 implies that the complex-energy states have zero-norm and form complex conjugate pairs. Third, Claim 3, which asserts that a pair of zero-norm eigenstates of complex conjugate energies yield a pair of complex conjugate poles in $(0 + 1)$-dimensional theory, indicates that complex singularities correspond to pairs of zero-norm eigenstates of complex conjugate energies.

Moreover, we have discussed an example of a relativistic QFT having propagators with complex poles which is called the Lee-Wick model. This model also supports the correspondence between complex singularities and pairs of zero-norm states. Incidentally, we have argued that the field operator of the Lee-Wick model can be understood as a

counterpart in the covariant operator formalism of the so-called i particle [47].

Finally, we have discussed implications of complex singularities in the BRST formalism. Under assumptions that the Kugo-Ojima quartet mechanism works well and that the CPT operator exists, we have argued that both complex conjugate energy states of the "one-gluon state" $A_\mu(0)|0\rangle$ contain BRST parent states. This predicts that complex singularities of a propagator of the gluon-ghost composite operator should appear at the same locations as those of the gluon propagator.

VI. DISCUSSION

In this section, some remarks are made on related topics.

A. On other interpretation of complex singularities

Let us make comments on another interpretation of complex singularities. We have reconstructed Wightman functions from Schwinger functions based on (A23) and (A25). As remarked in Sec. II, this is different from a naive inverse Wick rotation on the complex momentum plane. An interpretation using the inverse Wick rotation is often discussed, e.g., in [29–31]. In these references, it is claimed that complex poles lead to (a) short-lived gluonic particles, (b) no free limits, (c) violation of causality (in short range), (d) violation of reflection positivity, (e) asymptotic incompleteness, and (f) violation of unitarity (in short range).

In our reconstruction method, there are some differences on (a) short-lived particle, (c) violation of causality, and (f) unitarity: (a) Instead of finite lifetime, the reconstructed Wightman function grows exponentially. (c) The causality as the spacelike commutativity is kept as mentioned in Sec. III B 7. (f) The Hermiticity of Hamiltonian can be consistent with complex poles in an indefinite metric state space as discussed in Sec. IV.

B. On the superconvergence relation

Oehme and Zimmermann derived the superconvergence relation for the gluon spectral function $\rho(\sigma^2)$ [41],

$$\int_0^\infty d\sigma^2 \rho(\sigma^2) = 0. \qquad (129)$$

The core idea of the Oehme-Zimmermann argument is to obtain the exact asymptotic form of the gluon propagator in the complex plane by using the asymptotic freedom and renormalization group. In particular, if the gluon anomalous dimension is negative, the gluon propagator $D(k^2)$ tends to vanish faster than $1/k^2$. If the Källén-Lehmann spectral representation is assumed, the asymptotic form yields the well-known superconvergence relation.

In the presence of complex singularities, the Källén-Lehmann spectral representation is invalidated. Therefore, the superconvergence relation (129) does not hold. In order

to obtain the correct asymptotic form determined by the perturbative renormalization group, the superconvergence relation is modified by complex singularities [18].

Here, we see how this relation is modified with the following assumptions:

 (i) The generalized spectral representation (13) for $D(k^2)$ holds.
(ii) $D(k^2)$ has the asymptotic behavior $|k^2||D(k^2)| \rightarrow 0 (|k^2| \rightarrow \infty)$. For the gluon propagator, the perturbative renormalization group and asymptotic freedom imply $D(k^2) \sim \frac{1}{k^2(\ln|k^2|)^{\gamma_0/\beta_0}}$, where γ_0 and β_0 are, respectively, the first coefficients of the gluon anomalous dimension and the beta function. This assumption is satisfied when both γ_0 and β_0 are negative.

First, let us consider a simple example with one pair of complex conjugate poles,

$$D(k^2) = \int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 + k^2}$$
$$+ \frac{Z}{M^2 + k^2} + \frac{Z^*}{(M^*)^2 + k^2}. \qquad (130)$$

From assumption (ii), by taking the limit $|k^2| \rightarrow \infty$ (or evaluating a contour integral along a semicircle [[18], Sec. II C]^{18}), we have

$$2\mathrm{Re}Z + \int_0^\infty d\sigma^2 \rho(\sigma^2) = 0. \qquad (131)$$

For a more general case with assumption (i), we obtain the modified superconvergence relation by the same evaluation,

$$\sum_{\ell=1}^{N_p} Z_\ell^{(1)} + \sum_{k=1}^{N_c} \int_{C_k} d\zeta \rho_k(\zeta) + \int_0^\infty d\sigma^2 \rho(\sigma^2) = 0. \qquad (132)$$

C. BRST symmetry, confinement, and complex singularities

Finally, let us add some comments on BRST symmetry and confinement in relation to complex singularities.

First, we have assumed a nilpotent BRST charge in Sec. IV C. Since the Kugo-Ojima quartet mechanism is a promising way to construct the physical state space in gauge-fixed pictures, it would be natural to hope the existence of a nilpotent BRST charge. However, part of

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^{18}Again, note that our previous works [18,20,23] have misleading Minkowskian notations, which should be Euclidean. Incidentally, in the presence of complex singularities, note also that the Oehme-Zimmermann renormalization group analysis should be implemented in Euclidean theories, since the renormalization condition for the real-time propagator cannot be imposed due to the nontemperedness.

the evidence for complex singularities in the Landau-gauge gluon propagator relies on numerical lattice calculations in the minimal Landau gauge, where the usual BRST symmetry is not guaranteed. At the present situation, the "best-case scenario" is that the gluon propagator of the minimal Landau gauge would be a good approximation of some gauge with a nilpotent BRST symmetry. Developing the lattice Landau gauge preserving the standard BRST symmetry in the continuum limit overcoming the Neuberger zero [53–55] would be an important future prospect.

Second, since complex singularities cause a problem on the asymptotic completeness as mentioned in Sec. III B 3 in the "full" state space, the Kugo-Ojima arguments could be modified. It would be interesting to explore this possibility.

Third, there are few theoretical developments of the axiomatic method without the spectral condition and positivity to our knowledge. Such studies could yield some constraints on complex singularities and are therefore interesting.

Fourth, Claim 4 predicts complex gluon-ghost bound states with the same energy as that of the gluon. Conversely, the appearance of complex singularities in a propagator of the gluon-ghost composite operator would be a necessary condition for the BRST formalism to "work well" if the gluon propagator has complex singularities. Thus, seeking such a complex gluon-ghost bound state would be interesting. Remarkably, the Bethe-Salpeter equation for the gluon-ghost bound state has been studied in light of BRST quartets in [56].

Fifth, while one can expect that complex singularities of field correlators have something to do with a confinement mechanism, we ought to note that complex singularities could be trivial gauge artifacts. Although the complex singularities yield a violation of (reflection) positivity, this violation is itself neither necessary nor sufficient for the confinement of a particle corresponding to the field, e.g., the gluon confinement. Indeed, this is not sufficient because this violation only indicates that the field involves some negative-norm states and does not deny the existence of asymptotic physical states. This violation is not a necessary condition because BRST parent states can be positive norm, for example. Similarly, although complex singularities correspond to confined states, their existence is neither necessary nor sufficient for the confinement of the corresponding particle. Moreover, such "confined states" corresponding to complex singularities could only be members of BRST quartets that are irrelevant to the confinement mechanism like the timelike photon. There are still many possibilities because understanding a

confining theory as a quantum theory is far from being achieved. Further studies are needed for clarification of relations between complex singularities and confinement mechanism.

APPENDIX A: NOTATIONS AND AXIOMS

In this section, we introduce notations required for mathematical discussions and review the standard Osterwalder-Schrader axiom for Euclidean field theories [37].

1. Notations and conventions

We use the notation $x = (\vec{x}, x^4) = (x^1, x^2, x^3, x^4)$ for a four-vector and Euclidean inner product $xy = x^\mu y^\mu$ (and Lorentzian inner product only when explicitly mentioned). When only one four vector is relevant as in the main text, we also use the lower indices $x = (x_1, x_2, x_3, x_4)$. We call the direction of $e_4 := (\vec{0}, 1)$ "(imaginary-)time direction". We also use the multi-index notation: for a multi-index $\alpha = (\alpha_{1,1}, \alpha_{1,2}, ..., \alpha_{n,4}) \in \mathbb{Z}_{\geq 0}^{4n}$, D_α denotes

$$D_\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1^1)^{\alpha_{1,1}} (\partial x_1^2)^{\alpha_{1,2}} \cdots (\partial x_n^4)^{\alpha_{n,4}}}, \quad \text{(A1)}$$

where $|\alpha| = \alpha_{1,1} + \cdots + \alpha_{n,4}$.

The Schwartz's space on $\mathbb{R}^n$ is denoted by $\mathscr{S}(\mathbb{R}^n)$. Its dual space, the space of tempered distributions, is denoted by $\mathscr{S}'(\mathbb{R}^n)$. We also define $\mathscr{D}(\mathbb{R}^n) := \{f(\xi); f(\xi) \text{ is a } C^\infty \text{ function with a compact support}\}$ and its dual space $\mathscr{D}'(\mathbb{R}^n)$. We can regard $\mathscr{S}'(\mathbb{R}^4) \subset \mathscr{D}'(\mathbb{R}^4)$. An element of $\mathscr{D}'(\mathbb{R}^4)$ can be beyond polynomial growth unlike $\mathscr{S}'(\mathbb{R}^4)$. An element of $\mathscr{D}'(\mathbb{R}^4)$ that cannot be regarded as a tempered distribution is called a nontempered distribution.

The important test function spaces are listed as follows. These spaces are equipped with the topologies in the same way as [[37], Sec. 2].

(i) Space of test functions on noncoincident points,

$$^0\mathscr{S}(\mathbb{R}^{4n}) := \left\{ f \in \mathscr{S}(\mathbb{R}^{4n}); \begin{array}{l} D^\alpha f(x_1, ..., x_n) = 0 \text{ for any } \alpha \in \mathbb{Z}_{\geq 0}^{4n} \\ \text{if } x_i = x_j \text{ for some } 1 \leq i < j \leq n \end{array} \right\}. \quad \text{(A2)}$$

(ii) Space of test functions with (imaginary-)time-ordered supports,

$$\mathscr{S}_+(\mathbb{R}^{4n}) = \left\{ f \in \mathscr{S}(\mathbb{R}^{4n}); \ \begin{array}{l} D^\alpha f(x_1, ..., x_n) = 0 \text{ for any } \alpha \in \mathbb{Z}_{\geq 0}^{4n} \\ \text{unless } 0 < x_1^4 < x_2^4 < \cdots < x_n^4 \end{array} \right\}, \tag{A3}$$

$$\mathscr{S}_<(\mathbb{R}^{4n}) = \left\{ f \in \mathscr{S}(\mathbb{R}^{4n}); \ \begin{array}{l} D^\alpha f(x_1, ..., x_n) = 0 \text{ for any } \alpha \in \mathbb{Z}_{\geq 0}^{4n} \\ \text{unless } x_1^4 < x_2^4 < \cdots < x_n^4 \end{array} \right\}. \tag{A4}$$

(iii) Space of test functions with supports of positive (imaginary-)time.
For functions of one variable, $\mathscr{S}_+(\mathbb{R}) := \{f(s) \in \mathscr{S}(\mathbb{R}); \text{ supp } f \subset [0, \infty)\}$ and also $\mathscr{S}_-(\mathbb{R}) := \{f(s) \in \mathscr{S}(\mathbb{R}); \text{ supp } f \subset (-\infty, 0]\}$. We define

$$\mathscr{S}(\mathbb{R}_+^4) := \mathscr{S}(\mathbb{R}^3) \hat{\otimes} \mathscr{S}_+(\mathbb{R}), \qquad \mathscr{S}(\mathbb{R}_+^{4n}) := \hat{\otimes}^n \mathscr{S}(\mathbb{R}_+^4), \tag{A5}$$

where $\hat{\otimes}$ denotes the completed topological tensor product and $\hat{\otimes}^n$ the n-fold one.
(iv) Space of test functions on "non-negative (imaginary-)time".
$\mathscr{S}(\bar{\mathbb{R}}_+)$ denotes the space of test functions on the non-negative real half line: $\mathscr{S}(\bar{\mathbb{R}}_+) := \mathscr{S}(\mathbb{R})/\mathscr{S}_-(\mathbb{R})$. Note that its dual space can be identified as $\mathscr{S}'(\bar{\mathbb{R}}_+) \simeq \{F \in \mathscr{S}'(\mathbb{R}); \text{ supp } F \subset [0, \infty)\}$. We define as above

$$\mathscr{S}(\bar{\mathbb{R}}_+^4) := \mathscr{S}(\mathbb{R}^3) \hat{\otimes} \mathscr{S}(\bar{\mathbb{R}}_+), \qquad \mathscr{S}(\bar{\mathbb{R}}_+^{4n}) := \hat{\otimes}^n \mathscr{S}(\bar{\mathbb{R}}_+^4). \tag{A6}$$

We introduce the sets of terminating sequences $\underline{\mathscr{L}}, \underline{\mathscr{L}}_+, \underline{\mathscr{L}}_<$, and $\underline{\mathscr{L}}(\bar{\mathbb{R}}_+^4)$ over the spaces $\mathscr{S}(\mathbb{R}^{4n}), \mathscr{S}_+(\mathbb{R}^{4n})$, $\mathscr{S}_<(\mathbb{R}^{4n})$, and $\mathscr{S}(\bar{\mathbb{R}}_+^{4n})$, respectively. An element $\underline{f}$ of one of the spaces $\underline{\mathscr{L}}_*(= \underline{\mathscr{L}}, \underline{\mathscr{L}}_+, \underline{\mathscr{L}}_<, \underline{\mathscr{L}}(\bar{\mathbb{R}}_+^4))$ over $\mathscr{S}_*^n(= \mathscr{S}(\mathbb{R}^{4n}), \mathscr{S}_+(\mathbb{R}^{4n}), \mathscr{S}_<(\mathbb{R}^{4n}), \mathscr{S}(\bar{\mathbb{R}}_+^{4n}))$ is a terminating sequence $\underline{f} := (f_0, f_1, \cdots)$ with $f_0 \in \mathbb{C}, f_n \in \mathscr{S}_*^n(n = 1, 2, ...)$, i.e.,

$$\underline{\mathscr{L}}_* := \bigoplus_{n=1}^\infty \mathscr{S}_*^n, \tag{A7}$$

with $\mathscr{S}_*^0 := \mathbb{C}$.
Next, we define some operations $\times, \cdot^\star$, and Θ on these spaces.
(a) For $\underline{f} = (f_0, f_1, \cdots), \underline{g} = (g_0, g_1, \cdots) \in \underline{\mathscr{L}}_*$, $\underline{f} \times \underline{g}$ is defined as

$$\underline{f} \times \underline{g} = ((\underline{f} \times \underline{g})_0, (\underline{f} \times \underline{g})_1, (\underline{f} \times \underline{g})_2, \cdots),$$

$$(\underline{f} \times \underline{g})_n(x_1, x_2, ..., x_n) = \sum_{k=0}^n (f_{n-k} \times g_k)(x_1, x_2, ..., x_n)$$

$$= \sum_{k=0}^n f_{n-k}(x_1, x_2, ..., x_{n-k}) g_k(x_{n-k+1}, x_{n-k+2}, ..., x_n), \tag{A8}$$

(b) For $\underline{f} = (f_0, f_1, \cdots) \in \underline{\mathscr{L}}_*$,

$$\underline{f}^\star = (f_0^\star, f_1^\star, \cdots), \qquad f_n^\star(x_1, x_2, ..., x_n) := \bar{f}_n(x_n, x_{n-1}, ..., x_1), \tag{A9}$$

$$\Theta \underline{f} = ((\Theta \underline{f})_0, (\Theta \underline{f})_1, \cdots), \qquad (\Theta \underline{f})_n(x_1, x_2, ..., x_n) := f_n(\vartheta x_1, \vartheta x_2, ..., \vartheta x_n), \tag{A10}$$

where $\bar{\cdot}$ is complex conjugation in this appendix (to distinguish from $\cdot^\star$) and $\vartheta x = (\vec{x}, -x^4)$. In the main text, the complex conjugation is denoted by $\cdot^*$.

(c) For an element of the Euclidean group $(a, R) \in \mathbb{R}^4 \rtimes SO(4)$ and $f \in \mathscr{S}_*^n$,

$$f_{(a,R)}(x_1, \ldots, x_n) := f(Rx_1 + a, \ldots, Rx_n + a). \quad \text{(A11)}$$

For the spectral function, we define tempered distribution on a compactified set $[0, \infty]$ [[40], Sec. A. 3.]. We introduce the space of test functions on $[0, \infty]$ as

$$\mathscr{S}([0, \infty]) := \{f(\lambda) = g(-(1 + \lambda)^{-1});$$
$$g \text{ is a } C^\infty \text{ function on } [-1, 0]\}, \quad \text{(A12)}$$

equipped with the topology characterized by the countable norm family[19] $\|f(\lambda)\|_n^{[0,\infty]} := \max_{k \in \{0,1,\ldots,n\}} \sup_{\lambda \geq 0} |((1 + \lambda)^2 \frac{\partial}{\partial \lambda})^k f(\lambda)|$ for $n \in \mathbb{Z}_{\geq 0}$. Its dual space, namely, the space of continuous linear functions of $\mathscr{S}([0, \infty])$, is denoted by $\mathscr{S}'([0, \infty])$. Elements of this space are called *tempered distributions on* $[0, \infty]$. With this definition, for $\rho(\sigma^2) \in \mathscr{S}'([0, \infty])$, the "integral" $\int_0^\infty d\sigma^2 \frac{\rho(\sigma^2)}{k^2 + \sigma^2}$ is formally well defined.

2. Osterwalder-Schrader Axioms

Using the above notations, we state the standard Osterwalder-Schrader axioms, for simplicity, for the scalar field. $\{\mathcal{S}_n\}_{n=0}^\infty$ is a sequence of distributions $\mathcal{S}_n(x_1, \ldots, x_n)$, called Schwinger functions, satisfying

[OS0] (Temperedness):

$$\mathcal{S}_0 = 1, \qquad \mathcal{S}_n \in {}^0\mathscr{S}'(\mathbb{R}^{4n}). \quad \text{(A13)}$$

[OS1] (Euclidean Invariance): for all $(a, R) \in \mathbb{R}^4 \rtimes SO(4)$ and $f \in {}^0\mathscr{S}(\mathbb{R}^{4n})$,

$$\mathcal{S}_n(f) = \mathcal{S}_n(f_{(a,R)}). \quad \text{(A14)}$$

[OS2] (Reflection Positivity): for all $\underline{f} = (f_0, f_1 \cdots) \in \underline{\mathscr{L}}_+$,

$$\sum_{n,m} \mathcal{S}_{n+m}(\Theta f_n^\star \times f_m) \geq 0. \quad \text{(A15)}$$

[OS2] (Symmetry):

$$\mathcal{S}_n(x_1, \ldots, x_n) = \mathcal{S}_n(x_{\pi(1)}, \ldots, x_{\pi(n)}), \quad \text{(A16)}$$

for any pertumutation $\pi(\cdot)$ of n items

[OS4] (Cluster Property): for all $\underline{f} = (f_0, f_1 \cdots), \underline{g} = (g_0, g_1 \cdots) \in \underline{\mathscr{L}}_+$ and $a = (\vec{a}, 0)$,

$$\lim_{\lambda \to \infty} \sum_{n,m} [\mathcal{S}_{n+m}(\Theta f_n^\star \times g_{m,(\lambda a, 1)})$$
$$- \mathcal{S}_n(\Theta f_n^\star) \mathcal{S}_m(g_m)] = 0. \quad \text{(A17)}$$

[OS0'] (Laplace Transform Condition)[20]: From the translational invariance [OS1], we can write $\mathcal{S}_n(x_1, \ldots, x_n)$ as $S_{n-1}(\xi_1, \ldots, \xi_{n-1}) \in \mathscr{S}'(\mathbb{R}_+^{4(n-1)})$, i.e., $\mathcal{S}_n(x_1, \ldots, x_n) =: S_{n-1}(x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1})$ for $x_1^4 < x_2^4 < \cdots < x_n^4$. This condition means that there exists, for every n, a Schwarz seminorm $\| \cdot \|_{\mathscr{S}}$ on $\mathscr{S}(\mathbb{R}_+^{4(n-1)})$ so that

$$|S_{n-1}(f)| \leq \|f^L\|_{\mathscr{S}} \quad \text{for all } f \in \mathscr{S}(\mathbb{R}_+^{4(n-1)}), \quad \text{(A18)}$$

where f^L denotes the Laplace-Fourier transform defined by

$$f^L(q_1, \ldots, q_{n-1})$$
$$:= \int d^{n-1}\xi f(\xi_1, \ldots, \xi_{n-1}) e^{\sum_{k=1}^n (-q_k^4 x_k^4 + i\vec{q}_k \cdot \vec{x}_k)} \bigg|_{q_k^4 \geq 0}$$
$$\in \mathscr{S}(\mathbb{R}_+^{4(n-1)}). \quad \text{(A19)}$$

Let us comment on the standard axiom of Euclidean field theories.

(a) For $\underline{f}, \underline{g} \in \underline{\mathscr{L}}_+$, $\Theta \underline{f}^\star \times \underline{g} \in \underline{\mathscr{L}}_<$, which appears in [OS2] and [OS4].

(b) As a special case of [OS2], $\underline{f} = (0, f, 0, 0, \ldots) \in \underline{\mathscr{L}}_+$, we have the reflection positivity for the two-point function. For any $f(x) \in \mathscr{S}_+(\mathbb{R}^4) = \{f \in \mathscr{S}(\mathbb{R}^4); \text{supp} f \subset \mathbb{R}^3 \times [0, \infty)\}$,

$$\int d^4x d^4y \bar{f}(\vartheta x) f(y) \mathcal{S}_2(x, y) \geq 0, \quad \text{(A20)}$$

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[19]Note that this norm can be written in terms of $g(u)$ on $[-1, 0]$ by identifying $u = -\frac{1}{1+\lambda}$ as $\|f(\lambda)\|_n^{[0,\infty]} = \max_{k \in \{0,1,\ldots,n\} u \in [-1,0]} |\frac{\partial^k g}{\partial u^k}|$, which is clearly finite for $f \in \mathscr{S}([0, \infty])$.

[20]Contrary to the original expectation, temperedness of the (higher-point) Wightman functions would not be guaranteed by [OS0]–[OS4] [38,57]. For the two-point sector, this condition is irrelevant since the temperedness of the Wightman function can be derived by the other conditions. However, it should be noted that complex singularities also violate this condition. Note also that this condition can be replaced by, e.g., a slight stronger condition, "linear-growth condition", which controls growth of $\mathcal{S}_n$ in n [38]. Since the aim of imposing this condition is to control behavior of the higher-point functions, we do not go far into this condition in this paper.

which is equivalent to, in terms of $S_1(y - x) := \mathcal{S}_2(x, y)$ and Fourier transforms of S_1 and f in the spatial directions,

$$\int dx^4 dy^4 \int \frac{d^3\vec{p}}{(2\pi)^3} \bar{f}(\vec{p}, x^4) f(\vec{p}, y^4) S_1(\vec{p}, -x^4 - y^4)$$
$$\geq 0. \tag{A21}$$

(c) Usually, $\underline{S}_1(\vec{p}, \tau)$ is an ordinary function. In this case, the reflection positivity is often checked by a necessary condition,

$$\underline{S}_1(\vec{p}, \tau) \geq 0 \quad \text{for all } \tau > 0, \vec{p}. \tag{A22}$$

If not, there exists $(\vec{p}_*, \tau_*)$ such that $\underline{S}_1(\vec{p}_*, \tau_*) < 0$, and we can choose a test function $f(\vec{p}, \tau)$ with its support sufficiently close to $(\vec{p}_*, \tau_*/2)$ so that the left-hand side of (A21) is negative. Observing the sign of $\underline{S}_1(\vec{p}, \tau)$ is relatively easy but is not enough to test the reflection positivity completely even in the two-point sector. For example, a propagator with complex poles and largely positive spectral function will not show the negativity of $\underline{S}_1(\vec{p}, \tau)$, while the reflection positivity itself is violated as proven in Theorem 6.

(d) The Schwinger function $S_{n-1}(\xi_1, ..., \xi_{n-1}) \in \mathscr{S}'(\mathbb{R}_+^{4(n-1)})$, i.e., $\mathcal{S}_n(x_1, ..., x_n) =: S_{n-1}(x_2 - x_1, x_3 - x_2, ..., x_n - x_{n-1})$ for $x_1^4 < x_2^4 < \cdots < x_n^4$ is regarded as the "values" of the Wightman function at pure imaginary times or Euclidean points,[21]

$$W_{n-1}((-i\tau_1, \vec{\xi}_1), (-i\tau_2, \vec{\xi}_2), ..., (-i\tau_{n-1}, \vec{\xi}_{n-1}))$$
$$= S_{n-1}((\vec{\xi}_1, \tau_1), (\vec{\xi}_2, \tau_2), ..., (\vec{\xi}_{n-1}, \tau_{n-1})). \tag{A23}$$

One expects that the Wightman function is holomorphic in the (extended) holomorphic tube and that the holomorphic Wightman function provides the vacuum expectation value of the fields as its boundary value as the usual case [36],

$$W_{n-1}(\xi_1, ..., \xi_{n-1})$$
$$= \lim_{\substack{\eta_1, \cdots \eta_{n-1} \to 0 \\ \eta_1, \cdots \eta_{n-1} \in V_+}} W_{n-1}(\xi_1 - i\eta_1, ..., \xi_{n-1} - i\eta_{n-1}), \tag{A24}$$

where V_+ denotes the forward light cone. Therefore, the Wightman function is reconstructed from the Schwinger function analytically continued to $\text{Re}\tau_k > 0$,

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$$W_{n-1}((t_1, \vec{\xi}_1), (t_2, \vec{\xi}_2), ..., (t_{n-1}, \vec{\xi}_{n-1}))$$
$$= \lim_{\tau_1, ..., \tau_{n-1} \to +0} S_{n-1}((\vec{\xi}_1, \tau_1 + it_1),$$
$$(\vec{\xi}_2, \tau_2 + it_2), ..., (\vec{\xi}_{n-1}, \tau_{n-1} + it_{n-1})), \tag{A25}$$

since $((-i(\tau_1 + it_1), \vec{\xi}_1), (-i(\tau_2 + it_2), \vec{\xi}_2), ..., (-i(\tau_{n-1} + it_{n-1}), \vec{\xi}_{n-1})$ for $\tau_1, ..., \tau_{n-1} > 0$ is in the tube $\mathbb{R}^{4(n-1)} - iV_+^{n-1}$. Note that the spacelike commutativity of the Wightman functions follows from the permutation symmetry [OS3] and expected holomorphy of the analytically continued Schwinger functions S_{n-1} in the extended tube.

(e) A generalization of the Osterwalder-Schrader axiom without the reflection positivity was proposed in [58]. However, the new axiom ("Hilbert space structure condition" with "$\mathscr{L}$ continuity", where the latter is introduced for a convenient purpose) is strong enough to derive the Laplace transform condition and prohibit complex singularities.

APPENDIX B: PROOF OF VIOLATION OF THE REFLECTION POSITIVITY IN THE PRESENCE OF COMPLEX SINGULARITIES

For the proof of violation of the reflection positivity (Theorem 6), the goal of this section is to prove Theorem 11 that the reflection positivity leads to temperedness of a reconstructed two-point Wightman function. Consequently, violation of the reflection positivity in the presence of complex singularities follows from the nontemperedness (Theorem 5).

This proof is essentially a simplified version of steps (a) and (b) of the Osterwalder-Schrader Theorem [37].

Lemma 4.—Suppose that the two-point Schwinger function $\mathcal{S}_2$ satisfies

(i) temperedness, $\mathcal{S}_2 \in {}^0\mathscr{S}'(\mathbb{R}^{4\cdot2})$,
(ii) translational invariance, $\mathcal{S}_2(x_1 + a, x_2 + a) = \mathcal{S}_2(x_1, x_2)$ for all $a \in \mathbb{R}^4$,
(iii) the reflection positivity for the two-point sector (A20),

which follow from [OS0] temperedness, [OS1] Euclidean invariance, and [OS2] reflection positivity, respectively.

Then, $S_1(x_2 - x_1) := \mathcal{S}_n(x_1, x_2)$ (after smearing the spatial directions) can be analytically continued to the right-half plane ($\text{Re}(x_2^4 - x_1^4) > 0$), and its analytic continuation can be regarded as a tempered distribution on the half-plane and the spatial directions. More precisely, for any $h(\vec{\xi}) \in \mathscr{S}(\mathbb{R}^3)$, there exists a holomorphic function $S_1(\tau + is|h)$ on the right-half plane ($\tau > 0$) such that

(a) On the real axis, $S_1(\tau|h) = \int d^3\vec{\xi} S_1(\vec{\xi}, \tau) h(\vec{\xi})$.
(b) $S_1(\tau + is|h)$ can be regarded as an element of $[\mathscr{S}(\mathbb{R}_+) \hat{\otimes} \mathscr{S}(\mathbb{R})]'$, the dual space of $\mathscr{S}(\mathbb{R}_+) \hat{\otimes} \mathscr{S}(\mathbb{R})$.
(c) $S_1(\tau + is|h)$ is continuous in $h(\vec{\xi}) \in \mathscr{S}(\mathbb{R}^3)$.

Proof.—First of all, using (i) temperedness and (ii) translational invariance, there exists $S_1(\xi) \in \mathscr{S}'(\mathbb{R}^4_+)$ such that $\mathcal{S}_2(x_1, x_2) = S_1(x_2 - x_1)$ for $x_1^4 < x_2^4$.

We prove the claim with the following steps:

Step 1. Constructing a Hilbert space with the form (A20)

Step 2. Defining spatial and (imaginary-)temporal translation operator and "Hamiltonian"

Step 3. Analytic continuation using the holomorphic semigroup generated by the "Hamiltonian"

Step 1. Defining a Hilbert space with the form (A20)

Let us first begin with constructing a Hilbert space. For $f, g \in \mathscr{S}_+(\mathbb{R}^4)$, we define a sesquilinear form on $\mathscr{S}_+(\mathbb{R}^4) \times \mathscr{S}_+(\mathbb{R}^4)$ by

$$(f, g) := \mathcal{S}_2(\Theta f^\star \times g) = \int d^4x d^4y \bar{f}(\vartheta x)g(y)S_1(y - x), \tag{B1}$$

which is positive semidefinite: $\|f\|^2 := (f, f) \geq 0$ from (A20). We introduce $\mathscr{N}$ as the space of all zero norm vectors, i.e.,

$$\mathscr{N} := \{f \in \mathscr{S}_+(\mathbb{R}^4); \|f\|^2 = 0\}. \tag{B2}$$

We then obtain a pre-Hilbert space $\mathscr{S}_+(\mathbb{R}^4)/\mathscr{N}$ and denote its completion by $\mathscr{K}$. Therefore, $\mathscr{K}$ is a Hilbert space and contains $\mathscr{S}_+(\mathbb{R}^4)/\mathscr{N}$ as a dense subset $\mathscr{D}_0$, namely, $\mathscr{S}_+(\mathbb{R}^4)/\mathscr{N} \simeq \mathscr{D}_0 \subset \mathscr{K}$.

We denote the (continuous) natural map by $v: \mathscr{S}_+(\mathbb{R}^4) \to \mathscr{K}$, whose image is $\mathscr{D}_0$, and the inner product on $\mathscr{K}$ by $(\cdot, \cdot)_{\mathscr{K}}$. It follows that, for $f, g \in \mathscr{S}_+(\mathbb{R}^4)$,

$$(v(f), v(g))_{\mathscr{K}} = (f, g). \tag{B3}$$

Step 2. Constructing translation operators and "Hamiltonian"

Next, we define translational operators on $\mathscr{K}$.

For spatial directions, we define $\hat{U}_s(\vec{a})$ on $\mathscr{S}_+(\mathbb{R}^4)$ by

$$(\hat{U}_s(\vec{a})f)(x) := f(x - a), \tag{B4}$$

for $a = (\vec{a}, 0)$ and $f \in \mathscr{S}_+(\mathbb{R}^4)$. Note that $(\hat{U}_s(\vec{a})f, \hat{U}_s(\vec{a})g) = (f, g)$. We then define the unitary operator $U_s(\vec{a})$ on $\mathscr{K}$ by a continuous extension of $U_s(\vec{a})$ defined on $\mathscr{D}_0$,

$$U_s(\vec{a})v(f) := v(\hat{U}_s(\vec{a})f). \tag{B5}$$

Similarly, for $\tau > 0$, we define $\hat{T}^\tau$ on $\mathscr{S}_+(\mathbb{R}^4)$ by

$$(\hat{T}^\tau f)(x) := f(\vec{x}, x^4 - \tau). \tag{B6}$$

Note that $\tau > 0$ is necessary to guarantee supp $(\hat{T}^\tau f) \subset \mathbb{R}^3 \times [0, \infty)$. Recalling (B1), we have

$$(\hat{T}^\tau f, g) = (f, \hat{T}^\tau g), \tag{B7}$$

for $f, g \in \mathscr{S}_+(\mathbb{R}^4)$ and $\tau \geq 0$.

Next, let us derive the following bound: for any $\tau > 0$, $f \in \mathscr{S}_+(\mathbb{R}^4)$,

$$(f, \hat{T}^\tau f) \leq (f, f) = \|f\|^2. \tag{B8}$$

Beforehand, we see that $(f, \hat{T}^\tau f)$ grows at most polynomially in τ. Indeed, by the definition (B1),

$$(f, \hat{T}^\tau f) = \int d^4x d^4y \bar{f}(x)f(y)S_1(\vec{y} - \vec{x}, \tau + x^4 + y^4), \tag{B9}$$

which shows $(f, \hat{T}^\tau f)$ increases at most polynomially as $\tau \to \infty$ because of the temperedness of $S_1(\xi) \in \mathscr{S}'(\mathbb{R}^4_+)$. Then, by using the Cauchy-Schwarz inequality and (B7) recursively, we have

$$\begin{aligned}
(f, \hat{T}^\tau f) &\leq (f, f)^{1/2}(\hat{T}^\tau f, \hat{T}^\tau f)^{1/2} \\
&= (f, f)^{1/2}(f, \hat{T}^{2\tau} f)^{1/2} \\
&\leq (f, f)^{1/2+1/4}(\hat{T}^{2\tau} f, \hat{T}^{2\tau} f)^{1/4} \\
&= (f, f)^{1/2+1/4}(f, \hat{T}^{4\tau} f)^{1/4} \leq \cdots \\
&\leq (f, f)^{\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}} \exp[2^{-n} \ln(f, \hat{T}^{2^n\tau} f)], \tag{B10}
\end{aligned}$$

for all positive integer n, positive real $\tau > 0$, and $f \in \mathscr{S}(\mathbb{R}^4_+)$. As $n \to \infty$, $2^{-n} \ln(f, \hat{T}^{2^n\tau} f) \to 0$ due to (at most) linear growth of $\ln(f, \hat{T}^{2^n\tau} f)$ in n. Therefore, the $n \to \infty$ limit of (B10) gives the bound (B8).

In particular, for any $f \in \mathscr{N}$, $\hat{T}^\tau f$ is also zero-norm $\hat{T}^\tau f \in \mathscr{N}$, since

$$(\hat{T}^\tau f, \hat{T}^\tau f) = (f, \hat{T}^{2\tau} f) \leq (f, f) = 0. \tag{B11}$$

Thus, the natural map of $\hat{T}^\tau$ on $\mathscr{S}(\mathbb{R}^4_+)/\mathscr{N}$ is well defined. We define T_0^τ to be the operator defined on $\mathscr{D}_0$,

$$T_0^\tau v(f) := v(\hat{T}^\tau f). \tag{B12}$$

So far, T_0^τ is defined on the dense domain $\mathscr{D}_0$, symmetric from (B7), and bounded from (B8). Then, we can extend T_0^τ to be defined on the whole $\mathscr{K}$ by continuity and have a self-adjoint contraction[22] T^τ on $\mathscr{K}$. Note that the semigroup $\{T^\tau\}_{\tau \geq 0}$ is strongly continuous due to (1) the

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[22] From (B8), the operator norm of T^τ is less than or equal to 1: $\|T^\tau\|_{\text{op}} \leq 1$. A bounded operator with this property is called a contraction.

boundedness $\|T^\tau\|_{\mathrm{op}} \leq 1$ from (B8) and (2) continuity for points in $\mathscr{D}_0$, $\lim_{\tau \downarrow 0} \|T^\tau v(f) - v(f)\| = 0$ from the definition (B1).

Let us define the infinitesimal generator of the semigroup $\{T^\tau\}_{\tau \geq 0}$, "Hamiltonian", by H. Formally,[23]

$$H := \lim_{\tau \downarrow 0} \frac{1}{\tau}(1 - T^\tau). \tag{B14}$$

Since the family of self-adjoint operators $\{T^\tau\}_{\tau \geq 0}$ satisfies (i) $\|T^\tau\|_{\mathrm{op}} \leq 1$, (ii) $\{T^\tau\}_{\tau \geq 0}$ form a semigroup, and (iii) this semigroup is strongly continuous (iv) $T^0 = 1$, a variant of Stone's theorem yields that the infinitesimal generator H is a self-adjoint operator, e.g., [[59], Theorem VIII. 8 and page 315]. Therefore, we can define a strongly continuous one-parameter group of unitary operators on $\mathscr{K}$ generated by H, $\{T^{is} := e^{-iHs}\}_{s \in \mathbb{R}}$ by Stone's theorem. T^{is} corresponds to the real-time translation operator.

Finally, we define a "holomorphic semigroup",

$$\{T^{\tau + is} := T^\tau T^{is}; \tau > 0, s \in \mathbb{R}\}. \tag{B15}$$

Step 3. Analytic continuation using the holomorphic semigroup generated by the "Hamiltonian"

First, let us consider

$$(v(f), T^{is}v(g))_{\mathscr{K}}, \tag{B16}$$

which is a continuous bilinear functional on $(\bar{f}(\vartheta x), g(y)) \in \mathscr{S}_-(\mathbb{R}^4) \times \mathscr{S}_+(\mathbb{R}^4)$, where $\mathscr{S}_-(\mathbb{R}^4) := \{f(\vartheta x); f(x) \in \mathscr{S}_+(\mathbb{R}^4)\}$. From the Schwartz nuclear theorem, we can write this as a continuous linear functional of $\Theta f^\star \times g$,

$$(v(f), T^{is}v(g))_{\mathscr{K}} = \int dx dy (\Theta f^\star \times g)(x, y)\mathcal{S}_2(x, y|s), \tag{B17}$$

where $\mathcal{S}_2(x, y|s)$ is a distribution over the space $\mathscr{S}_-(\mathbb{R}^4)\hat{\otimes}\mathscr{S}_+(\mathbb{R}^4) \simeq \{f(x, y) \in \mathscr{S}(\mathbb{R}^{4\cdot2}); f = 0 \text{ unless } x^4 < 0 < y^4\}$. Due to the translational invariance arising from $[U_s(\vec{a}), T^{is}] = 0$ and $[T^{a^4}, T^{is}] = 0$, $\mathcal{S}_2(x + a, y + a|s) = \mathcal{S}_2(x, y|s)$ for $a \in \mathbb{R}^3 \times [0, \infty)$, from which $\mathcal{S}_2(x, y|s) = S_1(y - x|s)$ with $S_1(y - x|s) \in \mathscr{S}'(\mathbb{R}^4_+)$.

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[23]In terms of the original space $\mathscr{S}(\mathbb{R}^4_+)$, H can be regarded as $-\partial_4 = -\frac{\partial}{\partial x^4}$. Note that the reflection ϑ in (B1) makes $-\frac{\partial}{\partial x^4}$ Hermitian. More precisely, H can be defined on the dense domain $\mathscr{D}_0$, and

$$Hv(f) = v(-\partial_4 f), \tag{B13}$$

for $f \in \mathscr{S}(\mathbb{R}^4_+)$.

Note that $S_1(\xi|s)$ satisfies

$$S_1(\xi|0) = S_1(\xi). \tag{B18}$$

Moreover, the unitarity of T^{is} provides the upper bound on $(v(f), T^{is}v(g))_{\mathscr{K}}$ in s. We can thus regard $S_1(y - x|s) \in [\mathscr{S}(\mathbb{R}^4_+)\hat{\otimes}\mathscr{S}(\mathbb{R})]'$.

Using $S_1(\xi|s)$, we also have

$$(v(f), T^{\tau + is}v(g))_{\mathscr{K}}$$
$$= \int dx dy (\Theta f^\star \times g)(x, y)S_1(\vec{y} - \vec{x}, y^4 - x^4 + \tau|s). \tag{B19}$$

From the construction of $T^{\tau + is}$, the left-hand side is holomorphic in $\tau + is$ for $\tau > 0$. Therefore, by using the uniqueness of the Schwartz nuclear theorem, $S_1(\vec{\xi}, \tau|s)$ satisfies the Cauchy-Riemann equation in the sense of a distribution.

Now, we consider one smeared in the spatial directions,

$$S_1(\tau, s|h) := \int d^3\vec{\xi}S_1(\vec{\xi}, \tau|s)h(\vec{\xi}) \in [\mathscr{S}(\mathbb{R}_+)\hat{\otimes}\mathscr{S}(\mathbb{R})]', \tag{B20}$$

for $h(\vec{\xi}) \in \mathscr{S}(\mathbb{R}^3)$. The Cauchy-Riemann equation of $S_1(\tau, s|h)$ holds for $\tau > 0$ (still in the sense of a distribution). From [60] [p. 31], $S_1(\tau, s|h)$ is a holomorphic *function* in $\tau + is$ for $\tau > 0$.

$S_1(\tau, s|h)$ also satisfies the following:
(a) On the real axis $s = 0$, $S_1(\tau|h) = \int d^3\vec{\xi}S_1(\vec{\xi}, \tau)h(\vec{\xi})$ from (B18).
(b) $S_1(\tau + is|h)$ can be regarded as an element of $[\mathscr{S}(\mathbb{R}_+)\hat{\otimes}\mathscr{S}(\mathbb{R})]'$ from the definition (B20).
(c) $S_1(\tau + is|h)$ is continuous in $h(\vec{\xi}) \in \mathscr{S}(\mathbb{R}^3)$ from $S_1(y - x|s) \in [\mathscr{S}(\mathbb{R}^4_+)\hat{\otimes}\mathscr{S}(\mathbb{R})]'$.

Hence, this holomorphic function is what we desire. This completes the proof. ∎

Furthermore, we need the following lemma to guarantee the existence and temperedness of the boundary value.

Lemma 5.—Let $S(\tau + is)$ be a holomorphic function defined on the right-half plane $\tau > 0$ that can be identified with an element of $[\mathscr{S}(\mathbb{R}_+)\hat{\otimes}\mathscr{S}(\mathbb{R})]'$, the dual space of $\mathscr{S}(\mathbb{R}_+)\hat{\otimes}\mathscr{S}(\mathbb{R})$. Then, the boundary value of $S(\tau + is)$ at $\tau \to 0$ is a tempered distribution: $\lim_{\tau \downarrow 0} S(\tau + is) \in \mathscr{S}'(\mathbb{R})$. Moreover, if such a holomorphic function $S(\tau + is|h)$ is a continuous linear functional of h on another function space for each $\tau > 0, s \in \mathbb{R}$, then the smeared boundary value is also continuous in h.[24]

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[24]This proof is based on Lemma 8.7 in [37] and Theorems 2–10 in [36].

Proof.—We prove that, for any test function $f(s) \in \mathscr{S}(\mathbb{R})$, the limit

$$\lim_{\tau \downarrow 0} \int ds S(\tau + is) f(s) \qquad \text{(B21)}$$

exists and is continuous in $f(s) \in \mathscr{S}(\mathbb{R})$, i.e., $\lim_{\tau \downarrow 0} \int ds S(\tau + is) f(s) \to 0$ as $f \to 0$ in $\mathscr{S}(\mathbb{R})$.

Let us proceed with the following steps:

Step 1. Polynomial growth in s and τ^{-1}

Step 2. A bound for $S(\tau + is)$ smeared with a test function

Step 1. Polynomial growth in s and τ^{-1}

We show that the holomorphic function $S(\tau + is)$ grows at most polynomially in s and τ^{-1}.

Let $\tau_0 + is_0$ be an arbitrary point on the right-half plane. The mean-value property yields

$$S(\tau_0 + is_0) = \int_0^{2\pi} \frac{d\varphi}{2\pi} S(\tau_0 + is_0 + re^{i\varphi}), \qquad \text{(B22)}$$

for arbitrary $0 < r < \tau_0$. We may average this expression in r with some weight. Let $h(r)$ be a smooth function with $\mathrm{supp}\,h \subset [\frac{1}{4}, \frac{1}{2}]$ satisfying $\int_0^\infty dr\, rh(r) = 1$. We define $h_0(r) := \tau_0^{-2} h(\tau_0^{-1} r)$, which satisfies $\int_0^\infty dr\, rh_0(r) = 1$ and $\mathrm{supp}\,h_0 \subset [\frac{\tau_0}{4}, \frac{\tau_0}{2}] (\subset [0, \tau_0))$.

Therefore, we have

$$S(\tau_0 + is_0) = \int_0^\infty dr r h_0(r) \int_0^{2\pi} \frac{d\varphi}{2\pi} S(\tau_0 + is_0 + re^{i\varphi})$$

$$= \int d\tau ds S(\tau + is) h_0(\sqrt{(\tau - \tau_0)^2 + (s - s_0)^2}). \qquad \text{(B23)}$$

Since $h_0(\sqrt{(\tau - \tau_0)^2 + (s - s_0)^2}) \in \mathscr{S}(\mathbb{R}_+) \hat{\otimes} \mathscr{S}(\mathbb{R})$ due to the compactness of $\mathrm{supp}\,h_0$, there exist non-negative integers $M, N \in \mathbb{Z}_{\geq 0}$, and a constant $C > 0$ such that

$$|S(\tau_0 + is_0)| \leq C \|h_0(\sqrt{(\tau - \tau_0)^2 + (s - s_0)^2})\|_{M,N}, \qquad \text{(B24)}$$

where $\| \cdot \|_{M,N}$ is the norm[25] defined by

$$\|f(\tau,s)\|_{M,N} := \sum_{\substack{k_1,k_2 \in \mathbb{Z}_{\geq 0} \\ k_1 + k_2 \leq M}} \sum_{\substack{\ell_1,\ell_2 \in \mathbb{Z}_{\geq 0} \\ \ell_1 + \ell_2 \leq N}} \sup_{\tau,s} \left| \tau^{k_1} s^{k_2} \frac{\partial^{\ell_1}}{\partial \tau^{\ell_1}} \frac{\partial^{\ell_2}}{\partial s^{\ell_2}} f(\tau,s) \right|. \qquad \text{(B25)}$$

The bound for $|S(\tau_0 + is_0)|$ can be evaluated as

$$|S(\tau_0 + is_0)| \leq C \|h_0(\sqrt{(\tau - \tau_0)^2 + (s - s_0)^2})\|_{M,N}$$

$$= C\tau_0^{-2} \sum_{\substack{k_1,k_2 \in \mathbb{Z}_{\geq 0} \\ k_1 + k_2 \leq M}} \sum_{\substack{\ell_1,\ell_2 \in \mathbb{Z}_{\geq 0} \\ \ell_1 + \ell_2 \leq N}} \sup_{\tau,s} \left| \tau^{k_1} s^{k_2} \frac{\partial^{\ell_1}}{\partial \tau^{\ell_1}} \frac{\partial^{\ell_2}}{\partial s^{\ell_2}} h(\sqrt{(\tau/\tau_0 - 1)^2 + (s/\tau_0 - s_0/\tau_0)^2}) \right|$$

$$= C \sum_{\substack{k_1,k_2 \in \mathbb{Z}_{\geq 0} \\ k_1 + k_2 \leq M}} \sum_{\substack{\ell_1,\ell_2 \in \mathbb{Z}_{\geq 0} \\ \ell_1 + \ell_2 \leq N}} \tau_0^{k_1 + k_2 - \ell_1 - \ell_2 - 2} \sup_{\tau',s'} \left| (1 + \tau')^{k_1} (s_0/\tau_0 + s')^{k_2} \frac{\partial^{\ell_1}}{\partial \tau'^{\ell_1}} \frac{\partial^{\ell_2}}{\partial s'^{\ell_2}} h(\sqrt{\tau'^2 + s'^2}) \right|$$

$$= C \sum_{\substack{k_1,k_2 \in \mathbb{Z}_{\geq 0} \\ k_1 + k_2 \leq M}} \sum_{\substack{\ell_1,\ell_2 \in \mathbb{Z}_{\geq 0} \\ \ell_1 + \ell_2 \leq N}} \sum_{m=0}^{k_2} \frac{k_2!}{m!(k_2 - m)!} \tau_0^{k_1 + k_2 - \ell_1 - \ell_2 - m - 2} s_0^m \sup_{\tau',s'} \left| (1 + \tau')^{k_1} (s')^{k_2 - m} \frac{\partial^{\ell_1}}{\partial \tau'^{\ell_1}} \frac{\partial^{\ell_2}}{\partial s'^{\ell_2}} h(\sqrt{\tau'^2 + s'^2}) \right|. \qquad \text{(B26)}$$

Note that the last term of $\sup_{\tau',s'}$ does not depend on τ_0, s_0. Hence, we finally obtain that, for $0 < \tau_0 < \tau_*$ with an arbitrary fixed τ_*, there exists a polynomial $P(s_0)$ and an integer $n \in \mathbb{Z}$ such that

$$|S(\tau_0 + is_0)| \leq \tau_0^{-n} P(s_0). \qquad \text{(B27)}$$

Step 2. A bound for smeared $S(\tau + is)$

So far, we have shown that $|S(\tau + is)|$ is of at most polynomial growth in τ^{-1} as $\tau \downarrow 0$. To prove the existence and continuity of the limit (B21), we derive a stronger bound for $S(\tau + is)$ smeared by a test function.

Let us consider $S(\tau + is)$ smeared by a test function $f(s) \in \mathscr{S}(\mathbb{R})$,

$$S(\tau; f) := \int ds S(\tau + is) f(s). \qquad \text{(B28)}$$

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[25]Recall that the topology of the spaces of test functions are introduced with the family of these (semi-)norms.

By the Cauchy-Riemann equation $\frac{\partial}{\partial \tau} S(\tau + is) = -i \frac{\partial}{\partial s} S(\tau + is)$, we have

$$\frac{\partial^{n+1}}{\partial \tau^{n+1}} S(\tau; f) = \int ds S(\tau + is) \left(i \frac{\partial}{\partial s} \right)^{n+1} f(s)$$

$$= i^{n+1} S(\tau; \partial_s^{n+1} f), \tag{B29}$$

and therefore, for sufficiently small τ,

$$\left| \frac{\partial^{n+1}}{\partial \tau^{n+1}} S(\tau; f) \right| \leq \left(\int ds P(s) \left(\frac{\partial}{\partial s} \right)^{n+1} f(s) \right) \tau^{-n}$$

$$= C_{n,f} \tau^{-n}, \tag{B30}$$

where $C_{n,f} > 0$ is a positive constant, and we have used the result of the previous step (B27). Note that $C_{n,f} \to 0$ as $f \to 0$ in $\mathscr{S}(\mathbb{R})$.

Moreover, note that $S(\tau; f)$ is represented by the iterative integration

$$S(\tau; f) = (-1)^{n+1} \int_{\tau}^{\tau_*} d\tau_1 \int_{\tau_1}^{\tau_*} d\tau_2 \cdots \int_{\tau_n}^{\tau_*} d\tau_{n+1} \frac{\partial^{n+1} S}{\partial \tau^{n+1}} (\tau_{n+1}; f) + \sum_{k=0}^{n} \frac{1}{k!} (\tau - \tau_*)^k \frac{\partial^k S}{\partial \tau^k} (\tau_*; f). \tag{B31}$$

Because of the estimate (B30), the limit $\tau \to +0$ converges. Thus, the boundary value (B21) exists. For the continuity in $\mathscr{S}(\mathbb{R})$, we obtain the bound

$$|S(\tau; f)| \leq C_{n,f} \int_{\tau}^{\tau_*} d\tau_1 \int_{\tau_1}^{\tau_*} d\tau_2 \cdots \int_{\tau_n}^{\tau_*} d\tau_{n+1} \tau^{-n} + \sum_{k=0}^{n} \frac{1}{k!} |\tau - \tau_*|^k \left| \frac{\partial^k S}{\partial \tau^k} (\tau_*; f) \right|, \tag{B32}$$

which implies

$$\lim_{\tau \downarrow 0} |S(\tau; f)| \leq C_{n,f} \int_{0}^{\tau_*} d\tau_1 \int_{\tau_1}^{\tau_*} d\tau_2 \cdots \int_{\tau_n}^{\tau_*} d\tau_{n+1} \tau^{-n} + \sum_{k=0}^{n} \frac{1}{k!} |\tau_*|^k \left| \frac{\partial^k S}{\partial \tau^k} (\tau_*; f) \right|. \tag{B33}$$

Therefore, the right-hand side of (B33) tends to vanish as $f \to 0$ in $\mathscr{S}(\mathbb{R})$, which establishes the continuity of the boundary value in $\mathscr{S}(\mathbb{R})$. Hence, the boundary value of a given holomorphic function is a tempered distribution.

For the latter assertion, suppose the holomorphic function $S(\tau + is|h)$ is a continuous linear functional on another space of test functions h. Similarly to (B28), let $S(\tau|h; f)$ denote the function smeared by a test function $f(s) \in \mathscr{S}(\mathbb{R})$. From the assumed continuity, $h \to 0$ yields $S(\tau + is|h) \to 0$ for each $\tau > 0, s \in \mathbb{R}$. Thus, $C_{n,f}$ and $|\frac{\partial^k S}{\partial \tau^k} (\tau_*|h; f)|$ tend to vanish as $h \to 0$. Therefore, the bound (B33) implies that the smeared boundary value $\lim_{\tau \downarrow 0} S(\tau|h; f)$ is continuous in h.

This completes the proof. ∎

Theorem 11.—Suppose that the two-point Schwinger function $\mathcal{S}_2$ satisfies

(i) temperedness, $\mathcal{S}_2 \in {}^0\mathscr{S}'(\mathbb{R}^{4\cdot2})$,

(ii) translational invariance, $\mathcal{S}_2(x_1 + a, x_2 + a) = \mathcal{S}_2(x_1, x_2)$ for all $a \in \mathbb{R}^4$, and

(iii) the reflection positivity for the two-point sector (A20).

Then, the reconstructed Wightman function is a tempered distribution.

Proof.—It immediately follows from Lemmas 4 and 5 that the reconstructed Wightman function is a continuous bilinear functional on $(f, h) \in \mathscr{S}(\mathbb{R}) \times \mathscr{S}(\mathbb{R}^3)$. We obtain the reconstructed Wightman function as a tempered distribution $W_1(\xi^0, \vec{\xi}) = \lim_{\tau \downarrow 0} S_1(\vec{\xi}, \tau|\xi^0) \in \mathscr{S}(\mathbb{R}^4)$ by the Schwartz nuclear theorem. ∎

Note that (i) the temperedness and (ii) the translational invariance are assumed in the definition of complex singularities; only (iii) the reflection positivity can be invalid. From the nontemperedness of complex singularities, we finally obtain Theorem 6.

APPENDIX C: WHICH AXIOMS ARE VIOLATED?

In this section, we summarize which axioms are violated or not violated due to the existence of complex singularities.

(I) Osterwalder-Schrader axioms for Schwinger functions [37,38]:

[OS0] Temperedness (for the two-point function) is *assumed* in the definition Sec. III A.

[OS1] Euclidean invariance (for the two-point function) is *assumed* in the definition Sec. III A.

[OS2] Reflection positivity is *violated* (Theorem 6).

[OS3] Symmetry (for the two-point function) is *assumed* in the definition Sec. III A.

[OS4] Cluster property (for the two-point function) *depends on massless singularity* (irrelevant to complex singularities).

[OS 0'] Laplace transform condition is itself violated, since this requires temperedness of the Wightman function. (However, this condition is required only for reconstructing higher-point functions [38].)

(II) Wightman axioms for Wightman functions [[36], Theorem 2–6]:

[W0] Temperedness is *violated*.

[W1] Poincaré symmetry (for the two-point function) is *valid* [for test functions in $\mathscr{D}(\mathbb{R}^4)$].

[W2] Spectral condition is *violated*, since the spectral condition requires the temperedness as a prerequisite.

[W3] Spacelike commutativity (for the two-point function) is *valid* [for test functions in $\mathscr{D}(\mathbb{R}^4)$].

[W4] Positivity is *violated* even for test functions in $\mathscr{D}(\mathbb{R}^4)$.

[W5] Cluster property (for the two-point function) *depends on massless singularity* (irrelevant to complex singularities).

Therefore, the axioms whose violations are proved are [OS2] Reflection positivity, [OS0'] Laplace transform condition, [W0] Temperedness, [W2] Spectral condition, and [W4] Positivity.

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