

Symmetry from entanglement suppression

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Symmetry is among the most fundamental and powerful concepts in nature, whose existence is usually taken as given, without explanation. We explore whether symmetry can be derived from more fundamental principles from the perspective of quantum information. Starting with a two-qubit system, we show there are only two minimally entangling logic gates: the identity and the SWAP, which interchanges the two states of the qubits. We further demonstrate that, when viewed as an entanglement operator in the spin-space, the S -matrix in the two-body scattering of fermions in the s -wave channel is uniquely determined by unitarity and rotational invariance to be a linear combination of the identity and the SWAP. Realizing a minimally entangling S -matrix would give rise to global symmetries, as exemplified in Wigner's spin-flavor symmetry and Schrödinger's conformal invariance in low energy quantum chromodynamics. For N_q species of qubit, the identity gate is associated with an $[SU(2)]^{N_q}$ symmetry, which is enlarged to $SU(2N_q)$ when there is a species-universal coupling constant.

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I. INTRODUCTION

Symmetry plays an essential role with respect to the laws of nature. Mathematically defined as invariance under a specified group of transformations, the notion of symmetry has provided unity to vastly different physical phenomena. For example, different physical systems with the same underlying symmetries have the same critical exponents at second order phase transitions. The existence of conserved quantities, such as energy and momentum, is typically a consequence of an underlying symmetry, in this case invariance under time and spatial translations. Effective field theories, including the Standard Model, are defined by writing the most general Lagrangian consistent with known symmetries.

But what is the origin of symmetry? In a physical theory symmetries are usually given as inputs at short distances and are valid over a large range of length scales. Sometimes symmetries could be accidental and emerge only at long distances, which nonetheless cannot be predicted *a priori*. It is not known whether symmetry can be the outgrowth of more fundamental principles.

On the other hand, in attempting to answer the question of existence, J. A. Wheeler adopted the philosophy that

every physical quantity is derived from binary bits, inventing the phrase *It from bit* [1]. Indeed, remarkable connections between fundamental physics and information science have been revealed in the past decades. Central to this pursuit is quantum entanglement, a basic phenomenon in quantum mechanics. Naturally one is led to wonder: can symmetry come from qubit? [2]

In this regard there is a fascinating observation in low energy quantum chromodynamics (QCD) recently. Ref. [3] computed the entanglement power in two-nucleon scattering and found entanglement suppression in regions of parameter space with enhanced Wigner's spin-flavor symmetry and Schrödinger's non-relativistic conformal invariance. This raises the intriguing possibility of understanding symmetry from quantum entanglement.

In this work we aim to purify the observation and study the correlation between symmetry and entanglement suppression in an information-theoretic setting. As a result, our conclusions are universal, quantum mechanically speaking, and not confined to QCD.

II. ALICE AND BOB MEET CARTAN

We start by considering two qubits, Alice (A) and Bob (B), each with the basis vectors $|0\rangle$ and $|1\rangle$. For a spin-1/2 system, $|0\rangle$ and $|1\rangle$ can be identified with states having the spin-component $S_z = +1/2$ and $-1/2$, respectively. It is customary to define the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, where $|ij\rangle = |i\rangle_A \otimes |j\rangle_B$. There are several measures of entanglement of two-qubit systems

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[4] but our analysis relies only on general features of these measures, namely that they vanish if $|\Psi\rangle$ is unentangled, i.e., can be written as a direct product state, $|\Psi\rangle = |\phi\rangle \otimes |\phi'\rangle$ for some $|\phi\rangle, |\phi'\rangle$, and are maximal for maximally entangled states. Examples of maximally entangled states include the Bell basis:

$$|\Phi^\pm\rangle = \frac{(|00\rangle \pm |11\rangle)}{\sqrt{2}}, \quad |\Psi^\pm\rangle = \frac{(|01\rangle \pm |10\rangle)}{\sqrt{2}}, \quad (1)$$

as well as the magic basis [5], defined in the same way as the Bell basis except for some phases: $|\Phi_1\rangle = |\Phi^+\rangle, |\Phi_2\rangle = -i|\Phi^-\rangle, |\Phi_3\rangle = |\Psi^+\rangle, |\Phi_4\rangle = -i|\Psi^-\rangle$. An example of an entanglement measure is the concurrence $\Delta(\Psi)$ [5,6], which for a normalized state $|\Psi\rangle = \alpha|00\rangle + \beta|10\rangle + \gamma|01\rangle + \delta|11\rangle$ is defined as

$$\Delta(\Psi) = |\langle\Psi|\sigma_y \otimes \sigma_y|\Psi^*\rangle| = 2|\alpha\delta - \beta\gamma|. \quad (2)$$

The concurrence has the property that $0 \leq \Delta(\Psi) \leq 1$, where $\Delta(\Psi) = 0$ if $|\Psi\rangle$ is not entangled and $\Delta(\Psi) = 1$ if it is maximally entangled. Concurrence is reviewed in the Appendix and its relation to some other entanglement measures is discussed. The crucial property that all these measures share is that the entanglement is unchanged by single qubit unitary operators.

A quantum-mechanical operator acting on the two-qubit system is represented by a matrix U in the $SU(4)$ group, which is parametrized by 15 generators:

$$\{\sigma^a \otimes \mathbf{1}_2, \mathbf{1}_2 \otimes \sigma^b, \sigma^a \otimes \sigma^b; a, b = x, y, z\}, \quad (3)$$

where σ^a 's are the Pauli matrices and $\mathbf{1}_2$ is the 2×2 identity matrix. It turns out that not all U 's generate entanglement; operators that are local in the product space, $U = U_A \otimes U_B$, $U_{A/B} \in SU(2)$, leave the entanglement unchanged, leading to the notion of equivalent classes: two operators U and U' generate the same amount of entanglement if they are related by local operators,

$$U \sim U', \quad \text{if } U = (U_A \otimes U_B)U'(V_A \otimes V_B), \quad (4)$$

for some single qubit operators $U_{A/B}$ and $V_{A/B}$.

Classification of all nonlocal, and hence entanglement generating operators in a two-qubit system has been achieved long ago [4,7,8] and relies on the Cartan decomposition [9] of the Lie algebra of a symmetric coset space G/H , whose generators satisfy

$$[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{p}] = \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] = \mathfrak{h}, \quad (5)$$

where $\mathfrak{h} = \{T^i\}$ and $\mathfrak{p} = \{X^a\}$ are generators of H and G/H , respectively. The Lie algebra \mathfrak{g} of G can be written as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$. Let $\mathfrak{a} = \{A^a\}$ be the maximal Abelian subalgebra of \mathfrak{g} contained in \mathfrak{p} : $\mathfrak{a} \subset \mathfrak{p}$ and $[\mathfrak{a}, \mathfrak{a}] = 0$. Then an element g of G can be written as

$$g = K_1 e^{i\beta_a A^a} K_2, \quad K_1, K_2 \in H, \quad (6)$$

for some real parameters $\{\beta_a\}$. This is generally referred to as the Cartan decomposition for symmetric spaces.

The Cartan decomposition in Eq. (6) can be understood as follows. The coset structure allows for the ‘‘polar decomposition’’ for any $g \in G$,

$$g = \bar{K}_1 L, \quad \bar{K}_1 \in H \quad \text{and} \quad L \in G/H. \quad (7)$$

Next we would like to show that,

$$\mathfrak{p} = \mathfrak{a} \oplus [\mathfrak{h}, \mathfrak{a}]. \quad (8)$$

Recall that, since $\mathfrak{a} \subset \mathfrak{p}$, Eq. (5) implies $[\mathfrak{h}, \mathfrak{a}] \subset \mathfrak{p}$. Consider a generator $X^a \in \mathfrak{p}$ that is orthogonal to $[\mathfrak{h}, \mathfrak{a}]$,

$$0 = \text{Tr}(X^a [\mathfrak{h}, \mathfrak{a}]) = \text{Tr}(\mathfrak{h}[\mathfrak{a}, X^a]), \quad (9)$$

where we have used the cyclic property of the trace in the second equality. Using Eq. (5) again, $[X^a, \mathfrak{a}] \subset \mathfrak{h}$ which implies the only way Eq. (9) can hold is if $[X^a, \mathfrak{a}] = 0$. This shows X^a must be part of the maximal Cartan subalgebra $\mathfrak{a} \subset \mathfrak{p}$ and Eq. (8) follows. The last step is to apply Baker-Campbell-Hausdorff formula,

$$e^{iZ} e^{iY} e^{-iZ} = e^{i(Y+[Z,Y]+\frac{1}{2}[Z,[Z,Y]]+\frac{1}{3}[Z,[Z,[Z,Y]]+\dots)}, \quad (10)$$

with $Y = \mathfrak{a}$ and $Z = \mathfrak{h}$. Using Eqs. (5) and (8) we see $[\mathfrak{h}, [\mathfrak{h}, \mathfrak{a}]] = \mathfrak{p}$, $[\mathfrak{h}, [\mathfrak{h}, [\mathfrak{h}, \mathfrak{a}]]] = \mathfrak{p}$, etc. Therefore the right-hand side of Eq. (10) sits in G/H , leading to

$$L = (K_2)^{-1} e^{i\beta_a A^a} K_2. \quad (11)$$

Plugging Eq. (11) into Eq. (7) we arrive at Eq. (6).

III. MINIMAL ENTANGLERS

For $SU(4)/(SU(2) \otimes SU(2))$, with the $SU(2)$'s generated by $\sigma^a \otimes \mathbf{1}_2$ and $\mathbf{1}_2 \otimes \sigma^b$, $\mathfrak{a} = \{\sigma^a \otimes \sigma^a, a = x, y, z\}$. The space of inequivalent ‘‘entanglers’’ is then parametrized by only 3 parameters, $(\beta_x, \beta_y, \beta_z)$,

$$\begin{aligned} U_d \equiv e^{i\beta_a \sigma^a \otimes \sigma^a} &= (c_x c_y c_z + i s_x s_y s_z) \mathbf{1} \\ &+ [(s_z s_y c_x + i c_z c_y s_x) \sigma^x \otimes \sigma^x \\ &+ (\text{cyclic perm. of } \{x, y, z\})], \end{aligned} \quad (12)$$

where $c_a = \cos \beta_a$ and $s_a = \sin \beta_a$ and $\mathbf{1}$ is the four-dimensional identity matrix. It is worth recalling some basic properties of U_d . First, the amount of entanglement created by U_d is periodic in $\beta_i \rightarrow \beta_i + \pi/2$ [4]. To see this, consider

$$e^{i(\beta_x + \delta) \sigma^x \otimes \sigma^x} = (\mathbf{1} \cos \delta + i \sigma^x \otimes \sigma^x \sin \delta) e^{i\beta_x \sigma^x \otimes \sigma^x}. \quad (13)$$

When $\delta = \pi/2$ the above operator is related to the operator with $\delta = 0$ by an operator that is a product of local unitaries, $U_A \otimes U_B = i\sigma^x \otimes \sigma^x$, and therefore belongs to the same equivalent class. Moreover, the eigenvalues of U_d (most easily computed in the Bell basis which is the eigenbasis of U_d) are $\lambda_i = e^{i\phi_i}$, where ϕ_i are

$$\begin{aligned}\phi_1 &= \beta_x - \beta_y + \beta_z, & \phi_2 &= -\beta_x + \beta_y + \beta_z \\ \phi_3 &= \beta_x + \beta_y - \beta_z, & \phi_4 &= -\beta_x - \beta_y - \beta_z.\end{aligned}\quad (14)$$

These are the eigenvalues of $|\Phi^+\rangle$, $|\Phi^-\rangle$, $|\Psi^+\rangle$, and $|\Psi^-\rangle$, respectively. Then $(\beta_a, \beta_b) \rightarrow (-\beta_a, -\beta_b)$ or $(\beta_a, \beta_b) \rightarrow (\beta_b, \beta_a)$ generate all possible permutations of the eigenvalues. Permuting the eigenvalues of U_d does not change the entanglement properties of U_d . The geometric structure underlying all U_d is a 3-torus, $T^3 = S^1 \times S^1 \times S^1$, and we can restrict to $\pi/2 > \beta_x \geq \beta_y \geq \beta_z \geq 0$, $\pi/2 \geq \beta_x + \beta_y$. If $\beta_z = 0$ we have the additional constraint $\pi/4 \geq \beta_x$. This corresponds to one Weyl chamber of G/H [8,10], where every point corresponds to an equivalent class. We will use $[U_d]$ to denote the equivalent class containing U_d . Related discussions in the context of the S -matrix in a quantum field theory can be found in Ref. [11].

To quantify the ability of an operator to produce entanglement, it is possible to define the entanglement power of an operator by averaging over the states it acts on [12]. For our purpose, it is sufficient to consider $U_d|\psi\rangle$, where $|\psi\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$ is a general product state in the computational basis, compute the concurrence defined in Eq. (2) and average over the wave function of the qubit on the Bloch sphere. It turns out that it is easier to compute the average of the square of the concurrence [13,14], as explained in the Appendix,

$$\frac{1}{\Delta(U_d|\psi\rangle)^2} = \frac{1}{6} - \frac{1}{18} \sum_{a<b} \cos(4\beta_a) \cos(4\beta_b). \quad (15)$$

Since Eq. (15) is positive-definite, a vanishing entanglement power requires $\cos(4\beta_a) \cos(4\beta_b) = 1$ for all a and b . The only solutions are $\beta_x = \beta_y = \beta_z = 0$ or $\pi/4$. So there are only two minimally entangling U_d whose matrix representations in the computational basis are,

$$\begin{aligned}\beta_a = 0: U_d &= \mathbf{1}, \\ \beta_a = \frac{\pi}{4}: U_d &= e^{i\frac{\pi}{4}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv e^{i\frac{\pi}{4}} \times (\text{SWAP}).\end{aligned}\quad (16)$$

The SWAP interchanges the states of qubit A and qubit B , i.e., $|ij\rangle \rightarrow |ji\rangle$ in the computational basis and $\text{SWAP}^2 = \mathbf{1}$. The corresponding equivalence classes are [1] and [SWAP].

IV. TWO-BODY SCATTERING

In a scattering experiment $AB \rightarrow AB$, where A and B are fermions, the S -matrix can be considered as an entanglement operator in the spin-space. Moreover, it is well-known that the low-energy scattering of non-identical fermions is dominated by the s -wave channel, which we focus on here. See Refs. [15,16] for prior discussions of entanglement in fermion scattering.

In the s -wave channel, there is no orbital angular momentum and the S -matrix, \mathcal{S} , can be decomposed into the singlet, 1S_0 , and the triplet, 3S_1 , channels [17], $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1$. The general form of \mathcal{S} is constrained by (1) unitarity: $\mathcal{S}^\dagger \mathcal{S} = 1$, and (2) rotational invariance: $[\mathcal{S}_i, \vec{J}] = 0$, where in the s -wave $\vec{J} = \vec{s}_A + \vec{s}_B$ is the total spin of A and B . In the computational basis, $\vec{s}_A = (\vec{\sigma}/2) \otimes \mathbf{1}_2$ and $\vec{s}_B = \mathbf{1}_2 \otimes (\vec{\sigma}/2)$. Therefore \mathcal{S}_i can only be a linear combination of the identity $\mathbf{1}$ and $J^2 = (\vec{s}_A + \vec{s}_B)^2$:

$$\mathcal{S}_i = e^{i2\delta_i} (a_i \mathbf{1} + b_i \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}), \quad \boldsymbol{\sigma} \cdot \boldsymbol{\sigma} \equiv \sum_a \sigma^a \otimes \sigma^a. \quad (17)$$

where $\delta_i = \delta_i(p)$ is a phase which could depend on the center-of-mass momentum p and a_i, b_i are constants. Demanding that \mathcal{S}_0 and \mathcal{S}_1 project onto the singlet and the triplet channels, respectively, gives $a_1 = 3b_1$ and $a_0 = -b_0$. The overall normalization is fixed by the unitarity constraint, $\mathcal{S}^\dagger \mathcal{S} = 1$, and we arrive at

$$\mathcal{S} = \frac{1}{4} (3e^{i2\delta_1} + e^{i2\delta_0}) \mathbf{1} + \frac{1}{4} (e^{i2\delta_1} - e^{i2\delta_0}) \boldsymbol{\sigma} \cdot \boldsymbol{\sigma}, \quad (18)$$

$$= \frac{1}{2} (e^{2i\delta_1} + e^{2i\delta_0}) \mathbf{1} + \frac{1}{2} (e^{2i\delta_1} - e^{2i\delta_0}) \text{SWAP}, \quad (19)$$

where we have used $\text{SWAP} = (\mathbf{1} + \boldsymbol{\sigma} \cdot \boldsymbol{\sigma})/2$. Surprisingly, rotational invariance and unitarity constrains the s -wave S -matrix to be a linear combination of minimal entanglers. Realizing a minimally entangling gate in the S -matrix then requires the relations:

$$\delta_0(p) - \delta_1(p) = 0 \quad \Rightarrow \quad \mathcal{S} \sim [\mathbf{1}], \quad (20)$$

$$|\delta_0(p) - \delta_1(p)| = \pi/2 \quad \Rightarrow \quad \mathcal{S} \sim [\text{SWAP}] \quad (21)$$

Given the momentum dependence in δ_i 's, the above relations are not satisfied generically, except at certain fixed momenta. Therefore, when the relations hold independently of the momentum p , we are sitting in special regions of parameter space. Indeed, these are regions where symmetries emerge.

At sufficiently low energies, when effective range expansion is valid, $p \cot \delta_i(p)$ is analytic in p^2

$$p \cot \delta_i(p) = -\frac{1}{a_i} + \frac{1}{2} r_0^{(i)} p^2 + \dots, \quad (22)$$

where a_i is the scattering length and $r_0^{(i)}$ is the effective range. This translates into

$$\tan \delta_i(p) = -pa_i + \mathcal{O}(p^3). \quad (23)$$

The limit $a_i \rightarrow 0$ and $\delta_i \rightarrow 0$ corresponds to a free theory in that channel, while $a_i \rightarrow \pm\infty$ and $\delta_i \rightarrow \mp \pi/2$ arises from the ‘‘unitarity limit,’’ with the largest possible cross section consistent with unitarity. Then it is clear that to achieve the [SWAP] gate with $|\delta_0 - \delta_1| = \pi/2$ for more than one p one must tune $a \rightarrow \pm\infty$ in one channel and $a \rightarrow 0$ in the other channel. This corresponds to an RG fixed point of the theory, which is known to possess an emergent Schrödinger symmetry [18].

On the other hand, realizing the identity with $\delta_0(p) = \delta_1(p)$ one needs to set the scattering parameters in the two channels equal. This could be enforced through an internal symmetry, which we turn to next.

V. EFFECTIVE HAMILTONIANS

In this section we investigate what kind of Hamiltonians would give rise to the minimally entangling [1] gate in the S -matrix. In a quantum field theory the phase δ_i is computed through the scattering amplitude \mathcal{A}_i defined by $e^{2i\delta_i} = 1 + i(Mp/2\pi)\mathcal{A}_i$, where M is the mass of the fermions (which we take to be degenerate, for simplicity). We then have

$$\mathcal{A}_i = \frac{4\pi}{M} \frac{1}{p \cot \delta_i - ip}. \quad (24)$$

To realize the identity gate we need $\delta_0 = \delta_1$, which implies the amplitudes in both channels are equal, suggesting the existence of an internal symmetry.

In what follows we will consider an effective Hamiltonians for two species of qubits, labeled by the species index: $I, J = 1, 2$. Then the notation $|Ii; Jj\rangle$ represents the situation where Alice possesses a qubit $|i\rangle$ in species I and Bob has a qubit $|j\rangle$ in species J . The addition of the species quantum number opens the possibility of entanglement through the species quantum numbers.

An entangler that takes an input $|Ii; Jj\rangle$ to an output $|I'i'; J'j'\rangle$ can arise from the generic Hamiltonian,

$$\mathbf{a}_{I'i'}^\dagger \mathbf{a}_{Ii} \mathbf{a}_{J'j'}^\dagger \mathbf{a}_{Jj}, \quad (25)$$

where \mathbf{a}_{Ii}^\dagger (\mathbf{a}_{Ii}) creates (annihilates) $|Ii\rangle$ and all other quantum numbers (such as the momentum) are suppressed. The most general effective Hamiltonian is

$$H_{\text{int}} = \sum G_{i'i'jj'} K_{I'I'JJ'} \mathbf{a}_{I'i'}^\dagger \mathbf{a}_{Ii} \mathbf{a}_{J'j'}^\dagger \mathbf{a}_{Jj} + \text{H.c.}, \quad (26)$$

where $G_{i'i'jj'}$ and $K_{I'I'JJ'}$ are coupling constants in the spin- and species-space, respectively.

From Eq. (16) we see that, in order for H_{int} to generate [1], it must be diagonal in the spin-space, $G_{i'i'jj'} = \mathcal{G}_{ij} \delta^{i'i'} \delta^{j'j}$. Rotational invariance further implies \mathcal{G}_{ij} must be independent of (ij) ,

$$\mathcal{G} = g_s \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (27)$$

where g_s is a constant. It turns out to achieve entanglement suppression when there is more than one species of qubit, H_{int} must conserve the ‘‘species number’’ for each individual species [3]. To see this, consider the following species-changing one-body operators,

$$\Theta = \mathbf{a}_{I\uparrow}^\dagger \mathbf{a}_{J\uparrow} + \text{H.c.} \quad (28)$$

When Θ acts on the state $|I\uparrow; J\uparrow\rangle$,

$$\Theta |I\uparrow; J\uparrow\rangle = |I\uparrow; I\uparrow\rangle + |J\uparrow; J\uparrow\rangle, \quad (29)$$

which will result in an entangled output. Therefore, entanglement suppression also requires $K_{I'I'JJ'} = \mathcal{K}_{IJ} \delta^{I'I'} \delta^{J'J}$. If we use the notation $N_I = (\mathbf{a}_{I\uparrow}, \mathbf{a}_{I\downarrow})^T$, the Hamiltonian can now be written as

$$H_{\text{int}} = g_s \sum_{I,J} \mathcal{K}_{IJ} (N_I^\dagger N_I) (N_J^\dagger N_J) + \text{H.c.}, \quad (30)$$

which is invariant under an $SU(2)_1 \times SU(2)_2$ global symmetry,

$$N_I \rightarrow U_I N_I, \quad U_I \in U(2)_I, \quad I = 1, 2. \quad (31)$$

The rotational invariance is the diagonal subgroup of $SU(2)_1 \times SU(2)_2$, which acts on each species identically.

In some cases it is possible to further assume the existence of a ‘‘species symmetry’’ such that, much like the rotational invariance in Eq. (27), the coupling constant \mathcal{K}_{IJ} is universal and independent of (IJ) ,

$$\mathcal{K} = g_f \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \quad (32)$$

In the context of two-nucleon scattering the universality is achieved through the $SU(2)$ isospin symmetry transforming the neutron and the proton into each other. In this case the Hamiltonian becomes

$$H_{\text{int}} = g_s g_f (\mathcal{N}^\dagger \mathcal{N})^2, \quad (33)$$

where $\mathcal{N} = (N_1, N_2)^T = (\mathbf{a}_{1\uparrow}, \mathbf{a}_{1\downarrow}, \mathbf{a}_{2\uparrow}, \mathbf{a}_{2\downarrow})^T$. We see the $SU(2)_1 \times SU(2)_2$ global symmetry is enlarged to an $SU(4)$

symmetry. This is Wigner's $SU(4)$ spin-flavor symmetry in effective interactions of nucleons [19].

The above argument generalizes immediately to the case when there are N_q species of qubit: $I, J = 1, 2, \dots, N_q$. In fact, Eq. (30) remains valid in the general case and the global symmetry is N_q -copy of $SU(2)$: $SU(2)_1 \times \dots \times SU(2)_{N_q}$. Moreover, when there is a species-universal coupling constant (due to some underlying species symmetry), the Hamiltonian implementing the [1] gate must be diagonal in both the spin- and species-space, with \mathcal{N} in Eq. (33) now defined by $\mathcal{N} = (\mathbf{a}_{1\uparrow}, \mathbf{a}_{1\downarrow}, \dots, \mathbf{a}_{N_q\uparrow}, \mathbf{a}_{N_q\downarrow})^T$. The resulting global symmetry is enlarged to $SU(2N_q)$.

In low energy QCD interactions of the octet baryons indeed exhibit an enhanced $SU(16)$ spin-flavor symmetry [3]. Now we see this is a direct consequence of QCD realizing the [1] gate. The example also serves to demonstrate that, to arrive at a species-universal coupling constant among the N_q qubit, it is not necessary to impose an $SU(N_q)$ species symmetry. In fact, it suffices if the N_q species of qubit furnish an irreducible linear representation of an underlying group G_q which may be smaller than $SU(N_q)$. For the octet baryon this is achieved through the $SU(3)$ flavor symmetry in QCD.

It is interesting to entertain the possibility of realizing the [SWAP], which requires the S -matrix in one spin channel saturating the unitarity limit ($\delta = \pi/2$) and the scattering being absent in the other channel ($\delta = 0$). Both S matrices are invariant under the Schrödinger invariance [18], which is not apparent at the level of the Lagrangian, but emerges after resumming certain contact interactions to all orders and performing a fine tuning of the theory parameters. For octet baryons the interactions in question are the two double trace operators in the chiral Lagrangian [20]. The resulting theory is not $SU(16)$ symmetric and does not appear to be realized in low energy QCD.

VI. CONCLUSION

In this work we initiated an information-theoretic study on the origin of symmetry. In particular, we studied the correlation between the emergence of global symmetries and entanglement suppression in the two-body scattering of fermions in the s -wave channel. After showing there are only two minimally entangling gates in a two-qubit system: the identity and the SWAP gates, which turned out to be rotationally invariant, we argued the S -matrix is constrained by unitarity and rotational invariance to be a linear combination of the identity and the SWAP. Implementing a minimally entangling S -matrix requires certain conditions on the phases in the S -matrix in the 1S_0 and 3S_1 channels. The SWAP gate can be realized when one of the channels flows to a free fixed point in the IR and the other a unitarity fixed point with the Schrödinger symmetry. On the other hand, an S -matrix implementing the identity gate implies $[SU(2)]^{N_q}$ global symmetry in the effective

Hamiltonian, where N_q is the number of species of qubits. When there is a universal coupling among the species, the symmetry is enlarged to $SU(2N_q)$.

In nature the identity gate is realized in low energy QCD. One wonders whether the SWAP gate arises naturally in other physical systems [21]. Furthermore, while we focused on symmetries realized in the Wigner-Weyl mode [23], it would be interesting to study whether spontaneously broken symmetries, the Nambu-Goldstone mode, can also be realized in an information-theoretic setting.

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APPENDIX: ENTANGLEMENT MEASURES

This Appendix gives a brief discussion of entanglement measures, which we argue can be motivated by demanding that they be invariant under unitary transformations on single qubits. The concurrence [6] turns out to be the simplest such invariant. We then discuss its relation to other entanglement measures based on reduced density matrices, such as the von Neumann entropy and the 2-entropy. Entanglement measures for operators are defined by applying them to unentangled direct product states and averaging over all such states. This is used to prove that all minimal entanglers are equivalent to **1** or SWAP up to local unitary operations.

For a two-qubit system the most general state is

$$|\Psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle. \quad (\text{A1})$$

Any entanglement measure should vanish on direct product states and attain a maximum on maximally entangled states. It is easy to show that $\Delta(\Psi) = 2|\alpha\delta - \beta\gamma|$ defined in Eq. (2) vanishes on all direct product states:

$$|\psi\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle). \quad (\text{A2})$$

In fact we will show that $\alpha\delta - \beta\gamma$ is the only second order polynomial in α, β, γ , and δ that has this property. It is also clear that $|\alpha\delta - \beta\gamma| \leq |\alpha||\delta| + |\beta||\gamma| \leq (|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2)/2 = 1/2$, so that $0 \leq \Delta(\Psi) \leq 1$. Finally $\Delta(\Psi) = 1$ for Bell states, so $\Delta(\Psi)$ has the desired properties.

Here we want to emphasize another way of motivating $\Delta(\Psi)$, which could easily be generalized to m -level qubits. Under a unitary transformation U_A on Alice the components of the wave function transform as:

$$\begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \rightarrow U_A \begin{pmatrix} \alpha \\ \gamma \end{pmatrix}, \quad \begin{pmatrix} \beta \\ \delta \end{pmatrix} \rightarrow U_A \begin{pmatrix} \beta \\ \delta \end{pmatrix}, \quad (\text{A3})$$

while under a unitary transformation U_B on Bob the components transform as

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \rightarrow U_B \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \rightarrow U_B \begin{pmatrix} \gamma \\ \delta \end{pmatrix}. \quad (\text{A4})$$

If we assemble the components of the wave function into a matrix

$$M \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad (\text{A5})$$

these transformations can be written as

$$M \rightarrow U_A M U_B^T. \quad (\text{A6})$$

An important ingredient underlying this work is that local unitary transformations do not affect entanglement and this led us to the Cartan decomposition of a general $SU(4)$ element acting on two qubits. In the present context we see that entanglement measures must be invariants under the transformation in Eq. (A6). There are only two invariants that are second order:

$$\begin{aligned} \text{Det}[M] &= \alpha\delta - \beta\gamma \\ \text{Tr}[M^\dagger M] &= |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1. \end{aligned} \quad (\text{A7})$$

Given this, it is not surprising that other entanglement measures for two qubits can be expressed in terms of $\Delta[\Psi]$. For example, the reduced density matrices for qubits A and B are

$$\rho_A = M M^\dagger, \quad \rho_B = M^\dagger M, \quad (\text{A8})$$

whose eigenvalues are

$$\lambda_{\pm} = \frac{1}{2} (1 \pm \sqrt{1 - \Delta^2(\Psi)}). \quad (\text{A9})$$

Any entanglement measure derived from reduced density matrices can therefore be expressed in terms of the concurrence. For example the von Neumann entropy is

$$\mathcal{E}_E(|\Psi\rangle) = -\text{Tr}[\rho_A \ln \rho_A] = -\lambda_+ \ln \lambda_+ - \lambda_- \ln \lambda_-, \quad (\text{A10})$$

while the 2-entropy is

$$\mathcal{E}_R(|\Psi\rangle) = 1 - \text{Tr}[\rho_A^2] = \frac{1}{2} \Delta^2(\Psi) = 2|\alpha\delta - \beta\gamma|^2. \quad (\text{A11})$$

These share the property of $\Delta(\Psi)$ that they vanish when evaluated on an unentangled state. For maximally entangled states, $\mathcal{E}_E = \ln 2$ and $\mathcal{E}_R = 1/2$.

So far we have discussed entanglement measures for states. For operators the entanglement generated is dependent on the state they act on. For example, CNOT acting on the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ yields $\{|00\rangle, |01\rangle, |11\rangle, |10\rangle\}$ and generates no entanglement. But acting on the direct product state $(|0\rangle + |1\rangle) \otimes |0\rangle/\sqrt{2}$ the CNOT gate creates the maximally entangled state $|\Phi^+\rangle$. Forming a sensible entanglement measure for operators requires some sort of averaging over initial states. One possible approach is to average over all direct product states [12,13], which we will do below.

We parametrize the wave function of the qubits using Bloch sphere coordinates,

$$\begin{aligned} a &= \cos \frac{\theta_A}{2}, & b &= \sin \frac{\theta_A}{2} e^{i\phi_A}, \\ c &= \cos \frac{\theta_B}{2}, & d &= \sin \frac{\theta_B}{2} e^{i\phi_B}. \end{aligned} \quad (\text{A12})$$

It is simple to compute the concurrence for arbitrary U_d . Expanding the pure state $|\psi\rangle$ in Eq. (A2) in terms of the Bell states, which are eigenstates of U_d , it is straightforward to show:

$$|\alpha\delta - \beta\gamma| = \frac{1}{4} \left| \sum_{i=1}^4 v_i \lambda_i^2 \right|, \quad (\text{A13})$$

where $\lambda_i = e^{i\phi_i}$ is defined through the ϕ_i in Eq. (14) and

$$\begin{aligned} v_1 &= (ac + bd)^2, & v_2 &= -(ac - bd)^2, \\ v_3 &= -(ad + bc)^2, & v_4 &= (ad - bc)^2. \end{aligned} \quad (\text{A14})$$

For averaging over direct product initial states it is more convenient to consider the entanglement power, which is the 2-entropy in Eq. (A11) averaged over all direct product states,

$$\begin{aligned} \overline{\mathcal{E}_R(U_d|\psi\rangle)} &= \overline{2|\alpha\delta - \beta\gamma|^2} \\ &= \frac{1}{8} \overline{\left| \sum_i v_i \lambda_i^2 \right|^2} \\ &= \frac{1}{8} \sum_i \overline{|v_i|^2} + \frac{1}{8} \sum_{i \neq j} \overline{v_i v_j^*} \lambda_i^2 \lambda_j^{-2}, \end{aligned} \quad (\text{A15})$$

where the last line follows from $\lambda_i^* = \lambda_i^{-1}$. The overline denotes averaging over the solid angles $\Omega_{A,B}$. It is simple to evaluate the integral $\overline{v_i v_j^*}$,

$$\overline{v_i v_j^*} = \int \frac{d\Omega_A}{4\pi} \int \frac{d\Omega_B}{4\pi} v_i v_j^* = -\frac{1}{9} + \frac{4}{9} \delta_{ij}, \quad (\text{A16})$$

which yields [13,14]

$$\begin{aligned} & \overline{\mathcal{E}_R(U_d|\psi)} \\ &= \frac{1}{6} - \frac{1}{18} \sum_{a<b} \cos(4\beta_a) \cos(4\beta_b). \end{aligned} \quad (\text{A17})$$

The entanglement power is manifestly invariant under the transformations $\beta_a \rightarrow \beta_a + \pi/2$, $(\beta_a, \beta_b) \rightarrow (\beta_b, \beta_a)$ and $(\beta_a, \beta_b) \rightarrow (-\beta_a, -\beta_b)$ discussed in the main text. To have minimal entanglement we must have all $\cos(4\beta_a) \cos(4\beta_b)$ in Eq. (A17) equal to 1. This requires $\beta_a = 0$ or $\pi/4$ for all

a . The solution $\beta_a = 0$ yields the identity gate, $\mathbf{1}$, and $\beta_a = \pi/4$ yields the SWAP gate times $e^{i\pi/4}$. Thus we see that all minimal entanglers are either equivalent to $\mathbf{1}$ or SWAP up to local unitary operations.

For $\beta_x = \beta_y = \beta_z \equiv \beta$, the entanglement power simplifies to

$$\overline{\mathcal{E}_R(U_d|\psi)} = \frac{\sin^2 4\beta}{6}. \quad (\text{A18})$$

In this case the eigenvalues of U_d are $e^{i\beta}$ and $e^{-3i\beta}$, while the eigenvalues of S are $e^{2i\delta_1}$ and $e^{2i\delta_0}$. Identifying 4β with $2(\delta_1 - \delta_0)$ we recover the entanglement power derived in Ref. [3].

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