Analytic nonintegrability and S-matrix factorization

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We formulate an equivalence between the 2-dim σ-model spectrum expanded on a nontrivial massive vacuum and a classical particle Hamiltonian with variable mass and potential. By considering methods of analytic Galoisian nonintegrability on the appropriate geodesics of the Hamiltonian system we algebraically constrain the particle masses at fixed time, such that integrability is allowed. Through our equivalence, this explicitly constrains the masses of the excited spectrum of the dual 2-dim theory in such a way to imply the S-matrix factorization and no particle production. In particular, the integrability of the classical particle system implies the factorization of the S-matrix in the dual quantum 2-dim theory. Our proposal provides also nontrivial evidence on the connection between integrability and S-matrix factorization for large class of theories with interactions that break Lorentz invariance.

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I. INTRODUCTION

Integrable field theories have been known to play a pivotal role in understanding physical structures such as the bosonization and the properties of the factorization of scattering [1,2]. On the other hand, integrable field theories are very few and hard to find in nature; their population is usually resembled by a small region of islands in a large ocean of water. The standard study on the integrable structures is done by the construction of the Lax pair for the relevant sigma model—a complicated task with no standard methodology that can be based on the symmetries or properties of the theory. It has been practically more applicable in theories that admit a deformation of integrable parent theories where one keeps track of the Lax pair deformation, with several modern and older applications including [1,3–8].

Due to these reasons, there is extensive research of alternative methods for the study of integrability. A natural development in this direction are formalisms focusing on the necessary conditions for the existence of integrability. Such are the methods of analytic nonintegrability which are primarily based on the connection of differential Galois theory with the Hamiltonian equations and the way that the integrable systems behave under certain fluctuations [9–12]. The idea can be implemented by the Kovacic algorithm where the question of (non)integrability boils down to simpler algebraic statements [13]. More recently the method has found application in cosmological and holographic models, initiated holographically with the string dynamics in black hole environments [14], while the literature so far focuses on the application of the method in different theories with aim to classify their integrability status including [15–26].

An independent criterion of integrability is generated by the classical S-matrix on a 2-dim massive theory. The higher conserved charges imply equal sets of masses and momenta before and after the collision and the absence of particle production. The locality and causality impose factorization of the $n \to n$ amplitudes into products of $2 \to 2$ ones. Therefore, the integrability simply requires that the S-matrix factorizes and satisfies the conditions of the no particle production in Lorentz invariant theories with a massive spectrum [27–29]. However, even for theories where the Lorentz invariance is broken by the interactions and a massless spectrum exists, which still may not contribute to the amplitudes, the relation between integrability and factorization scattering is assumed [30,31]. Nevertheless, formal proofs exist only for theories that preserve the Lorentz invariance, for example [32]. Our current work provides systematic evidence for a large class of theories for the validity of this statement. It is interesting, that while the S-matrix factorization has been related to integrability since the very early studies, only recently has it been used as an application to classify nonintegrable structures [31,33–35].

In this paper we initiate a study of combining the ideas of the two methods, by applying analytic nonintegrability

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methods on configurations that contain the nontrivial vacuum of the world sheet S-matrix. In particular, in the formalism we develop we show that the σ-model massive field excitations around a nontrivial vacuum are related to the dynamics of classical particle systems of variable mass with a potential that depends on the details of the equivalent 2-dim theory and the vacuum chosen. On the resulting effective particle Hamiltonian on which the string system is reduced, the analytic Galoisian methods provide a nontrivial algebraic relation on the particle's mass parameter, imposing strong fine tuning to allow the integrability of the classical system. The formalism is fundamental, universal, and generic, and has concrete applications to a large number of theories. This is based on the general fact that the masses of the excited spectrum of the dual 2-dim theory and also because the methods of analytic nonintegrability both rely on second order expansion on certain strings.

An especially interesting question concerns the interpretation of the (non)integrability particle constraints, when mapped on the 2-dim world sheet theory. By computing the spectrum excitations on the chosen vacuum and the relevant amplitudes we find that the classical particle integrability conditions imply the no-particle production and factorization of the S-matrix of the quantum 2-dim σ-model. Our proposal therefore breaks new ground for the applications on the nonintegrability techniques on the S-matrix vacua, which goes beyond the classification of nonintegrable theories used so far as e.g., in [16–19].

II. THE SETUP

A generic background of d space-time dimensions with Minkowski signature and with n cyclic coordinates is described by

$$ds^2 = g_{ii}\mathrm{d}x^i\mathrm{d}x^i + 2g_{ij}\mathrm{d}x^i\mathrm{d}x^j,$$

where $i < j$ and $i, j = 1, \ldots, d$ and the metric fields are functions of the noncyclic coordinates, while we assign the indices $i \geq d - n$ to label the cyclic angles. The Polyakov Lagrangian $\mathcal{L}$ is given by the following integrand

$$S = \int d\sigma d\tau g_{ii}(x^{i\prime 2} - \dot{x}^{i2}) + 2g_{ij}(x^{i\prime}x^{j\prime} - \dot{x}^i\dot{x}^j),$$

where the dotted and primed derivatives are with respect to the world sheet coordinates (τ, σ).

The equations of motion for the noncyclic angles $\alpha^i \coloneqq x^i$ with $i < d - n$ are

$$\partial_{\alpha^i}\mathcal{L} + 2\partial_\tau(g_{ii}\dot{\alpha}^i + g_{ij}\dot{\alpha}^j) - 2\partial_\sigma(g_{ii}\alpha^{i\prime} + g_{ij}\alpha^{j\prime}) = 0, \quad (1)$$

where still $i < j$ and here the i index is not summed since it labels the field of the corresponding equation of motion. The cyclic angles $\phi^i \coloneqq x^i$ for $i \geq d - n$ have simpler equations of motion

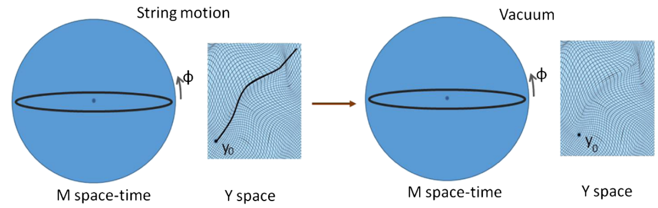

FIG. 1. An example of a string motion in the curved spacetime, to be mapped to the particle classical Hamiltonian. The nontrivial vacuum for the world sheet scattering is the localized string configuration at y_0.

$$\partial_\tau(g_{ii}\dot{\phi}_i + g_{ij}\dot{\phi}_j) - \partial_\sigma(g_{ii}\phi_i' + g_{ij}\phi_j') = 0, \quad (2)$$

where the index i is not summed. The Virasoro constraints take the compact form

$$g_{ii}\dot{x}^i x^{\prime i} + g_{ij}\dot{x}^i x^{\prime j} = 0, \quad (3)$$

$$g_{ii}(x^{i\prime 2} + \dot{x}^{i2}) + 2g_{ij}(x^{i\prime}x^{j\prime} + \dot{x}^i\dot{x}^j) = 0, \quad (4)$$

where all the indices sum.

III. THE FORMULATION

In this section we describe the method and the general mapping of the particle-string system. We consider a holographic Lorentz invariant geometry, with a metric written in the diagonal form

$$ds^2 = ds_M^2(\mathbf{x}, y) + \mathrm{d}s_Y^2(\mathbf{y}). \quad (5)$$

The space-time $\mathcal{M}$ has a boundary and Y is an internal space. The geometry is parametrized by $\mathbf{x}$ and $\mathbf{y}$ respectively, while y is one of the noncyclic angles of Y. Let us denote the nontrivial classic vacuum which we expand on for the study of the world sheet S-matrix to be parametrized by $\{\tilde{\mathbf{x}}(\tau, \sigma), y_0\}$ where y_0 denotes a particular value and could be, for example, a Gubser-Klebanov-Polyakov (GKP) type of string, or another string solution. One of the key ingredients for the equivalence we propose, is to find the appropriate extended configuration $\{\mathbf{x}(\tau, \sigma), y(\tau, \sigma)\}$ which localizes consistently to the vacuum solution at the y_0, as illustrated in Fig. 1.

Once we have chosen the vacuum and its appropriate string configuration that localizes to it, we reduce the full system of equations consistently to an effective Hamiltonian of a particle with nontrivial potential, as in Fig. 2, using the equations of Sec. II. In Appendix A we show that, in general, the Hamiltonian can be brought to the form

$$\mathcal{H} = \frac{p_x^2}{2m(y)} + p_y^2 + V(\mathbf{x}, y), \quad (6)$$

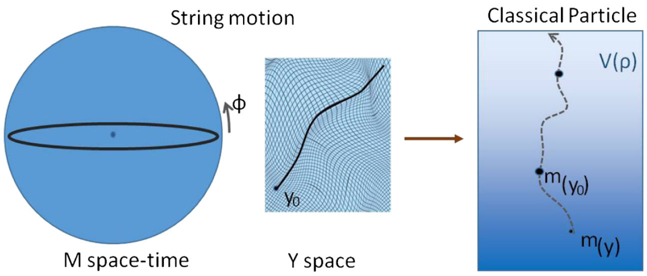

FIG. 2. A string motion in the curved spacetime and its equivalence with a particle of variable mass in an effective potential $V(\rho)$. The nontrivial vacuum for the world-sheet scattering is the localized configuration at y_0. The properties of the mass of the particle at this point, are related to the mass of the excited fields of the 2-dim σ-model.

where through the kinetic terms we can read the effective mass $m(y)$ of the particle, related to the corresponding background metric fields (5). In general, we can have a nontrivial mass in the y-kinetic term, but this does not affect the conclusions of this section. In most cases it can be rescaled to the presented form using the symmetries of the theory. The origin of the kinetic terms in general is due to motion on the noncyclic coordinates and of the potential due to the cyclic ones. At the point y_0, where we recover the vacuum, our prescription is to perform transverse fluctuations on the solution as $\delta y = y_0 + \eta(\sigma)$. The Eq. (1) of y using all the equations of motion, as we discuss in Appendix A, gives a second order homogeneous differential equation for η of the form

$$\eta''(\sigma) + h_1(\tilde{x}(\sigma))\eta'(\sigma) + \partial_y^2 m(y_0) h_2(\tilde{x}(\sigma))\eta(\sigma) = 0. \quad (7)$$

This is the normal variational equation (NVE) in the appropriate form necessary for the application of the Kovacic algorithm. The functions h_1 and h_2 depend on the effective potential V in the particle description, or equivalently on the geometry of the σ-model in the dual string picture. The parameter $m(y_0)$ is a constant at the point where the vacuum in the string picture is recovered.

The differential Galois group analysis relates the absence of integrability with the absence of Liouvillian solutions of the NVE. Therefore, we obtain the nonintegrability constraints on the second derivative of the effective mass of the particle $m(y_0)$ to restrict it to a set of values $\mathcal{A}$, such that only for these values the integrability of the classic system may exist.

In our formalism the effective particle mass is related to the mass of a field in the spectrum of the world sheet excitations on the vacuum. To confirm this statement we study the world sheet scattering where we consider the bosonic string sigma model on the 2-dim vacuum $\{\tilde{\mathbf{x}}(t,s), y_0\}$. Instead of fluctuating the extended string solutions around the vacuum, as in the method described above, here we expand the sigma model. The spectrum is read from the quadratic part of the Lagrangian

$$L_2 = (\partial x_i)^2 + m_i x_i^2 + (\partial y)^2 + \partial_y^2 m(y_0) y^2, \quad (8)$$

where m_i are constants depending on vacuum and the geometry, while the presence of the second derivative of $m(y)$ matches the second-order expansion of the Lagrangian. It is the same derivative that appears in the integrability analysis of the particle picture (7). The requirement of the no particle production and the factorization of the amplitudes requires the ones with unequal incoming and outgoing masses to vanish, constraining therefore the masses of the spectrum $(\sqrt{m_i}, \sqrt{\partial_y^2 m(y_0)})$. The quadratic, cubic, and quartic order Lagrangians provide the propagators and the vertices to analyze and compute the $2\to 2$ scattering at tree-level and the higher-loop corrections to the two-point function.

According to the generic equivalence we have described between the particle and the string picture, the requirement of (non)integrability in the classical particle system, constrains the spectrum in the dual string side such that the S-matrix factorization in the quantum 2-dim theory occurs. Below we show explicitly how our ideas can be applied to certain theories.

IV. THE STRING-PARTICLE MAPPING ON THE GKP VACUUM CASE

To demonstrate explicitly the above generic formalism we consider an example of a background included in the class of backgrounds (5) with generic warp factors as

$$ds^2 = g_\alpha(y) \mathrm{d}s^2_{\mathrm{AdS}} + g_\beta(y) \mathrm{d}s^2_Y, \quad (9)$$

where the $\mathbf{y}$ coordinates parametrize the internal space Y. We parametrize the anti–de Sitter (AdS) as $ds^2 = -\cosh^2\rho\, \mathrm{d}t^2 + d\rho^2 + \sinh^2\rho\, \mathrm{d}\phi^2$, where higher-dimension AdS spaces can be considered without loss of generality.

The appropriate string configuration is parametrized on the conformal space by $t = c\tau, \phi = c\omega t$, with a rigid rotation on the holographic direction $\rho(\sigma)$ and the internal space direction $y(\sigma)$. The equations of motion for our fields are obtained from Eqs. (1) and (2) and are consistent with the Virasoro constraints. The system is reduced to the particle Hamiltonian system where the role of the time is played by the parameter σ parametrizing the string length. The effective Hamiltonian in terms of the conjugate momenta p_y and p_ρ is given by

$$H_{\mathrm{eff}} = \frac{p_\rho^2}{4g_\alpha} + \frac{p_y^2}{4g_\beta} + c^2 g_\alpha(-\cosh^2\rho(\sigma) + \omega^2 \sinh^2\rho(\sigma)), \quad (10)$$

and is constrained to zero by the Virasoro constraint.

We localize our rotating string around an appropriate point y_0, which we set it to 0. As long as the background condition

$$\partial_y g_\alpha(0) = 0 \tag{11}$$

holds, the equations of motion are solved consistently for a folded string rigidly rotating, the GKP string [36]. The solution for $\rho(\sigma)$ takes the form of the Jacobi amplitude $\rho(\sigma) = iam(ic\sigma, 1-\omega^2)$, which is the inverse of the incomplete elliptic integral of the first kind. The constant c may be adjusted to fix the period of σ to a desirable value, for example 2π. The length of the string depends on the value of the parameter ω. For ω close to the unit the string is large, while for $\omega = 1$ it is infinite. For large ω the string is short and the solution can be approximated to the spinning string in flat space, producing the string Regge trajectory. In fact the regular string is a one-soliton sinh-Gordon solution while the long string limit corresponds to a two-soliton configuration.

The NVE is obtained at the point where the string solution is localized in the internal space and recovers the vacuum. The variation is therefore introduced in the classical particle system as $\delta y(\sigma) = 0 + \eta(\sigma)$ and after the appropriate manipulation of the system of equations and the change of variable $z \coloneqq \cosh^2\rho(\sigma) - \omega^2 \sinh^2\rho(\sigma)$, the linearized rational NVE can be identified as the linear general Heun differential equation

$$\eta''(z) + \kappa(z)\eta'(z) + \lambda(z)\eta(z) = 0,$$
$$\kappa(z) \coloneqq \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-c_0}, \qquad \lambda(z) \coloneqq \frac{\tilde{\alpha}\beta z - q}{z(z-1)(z-c_0)}, \tag{12}$$

which has four regular singular points $(0, 1, c_0, \infty)$ on the Riemann sphere when the condition $\tilde{\alpha} + \beta - \gamma - \delta - \epsilon + 1 = 0$ holds. The identification happens for $\tilde{\alpha} = 1, \gamma = \delta = \epsilon = \frac{1}{2}, c_0 = \omega^2, q = 0, \beta = -\tilde{g}_\alpha(0)/4$, where $\tilde{g}_\alpha(0) \coloneqq \partial_y^2 g_\alpha(0)$ [37]. We immediately notice that the requirement of integrability can constrain the second derivative of the warp factor $g_\alpha(y)$ at $y = 0$. Does the application of the differential Galois group analysis on the NVE provide an algebraic analytic condition, fine tuning the acceleration of change of mass $\tilde{g}_\alpha(0)$ of the classical system to values that allow integrability? The answer is positive and this is one additional key ingredient of this work.

V. THE GALOISIAN NONINTEGRABILITY OF THE PARTICLE SYSTEM

By transforming the Heun equation to a Riccati and using the differential Galois group analysis, as described in Appendix B. We apply analytically the Kovacic algorithm [38] to obtain the values of the parameters for which the differential equation has Liouvillian solutions

$$\pm(\gamma + \delta + \epsilon + \tilde{\alpha} - \beta) - 1 = 2n, \qquad n \in \mathbb{Z}. \tag{13}$$

What follows is a generic statement; the integrability of a particle system which produce an NVE of the form (12) is not be excluded only for the combination of the parameters that satisfy the above condition.

Therefore the Liouvillian criterium (13) applied to our NVE on the Riemann sphere gives the constraint

$$\tilde{g}_\alpha(0) = n, \qquad n = \{0, 2\}, \tag{14}$$

where the null value corresponds to the trivial linear fluctuation. With the given identification, the effective particle Hamiltonian can be integrable for the values of $\tilde{g}_\alpha(0)$ of the Eq. (14), which correspond to the acceleration of the change of the particle's mass [41].

VI. NONINTEGRABILITY WITH FACTORIZED SCATTERING FOR THE GKP VACUUM

Let us consider the AdS space of arbitrary dimension in Poincaré coordinates (x^μ, z) with boundary at $z = 0$ and work with the Euclidean action in the light cone gauge

$$\mathcal{L}_E = g_\alpha(y)\left(\dot{z}^2 + |\dot{x}|^2 + \frac{1}{z^4}(z'^2 + |x'|^2)\right) + z^2\dot{y}^2 + \frac{y'^2}{z^2}, \tag{15}$$

with the condition $\sqrt{-\gamma}\gamma^{\alpha\beta} = \mathrm{diag}(-z^2, z^{-2})$ on the world sheet metric. When the derivative of the warp factor of the metric satisfies the condition (11), we find that the system admits the generalized null cusp [42] solution which ends on the null cusp at the boundary of the space, as in [43,44]. Following [31,45], the fluctuations around the null cusp with $z = \sqrt{\tau/\sigma}\tilde{z}$, $\tilde{z} \equiv e^{\tilde{\phi}}$, $x = \sqrt{\tau/\sigma}\tilde{x}$, and world sheet coordinate change $\tilde{\tau} = 2\ln\tau$ and $\tilde{\sigma} = 2\ln\sigma$ that makes the induced world sheet conformally flat metric, give the quadratic action

$$\mathcal{L}_2 = (\partial_\alpha\tilde{\phi})^2 + (2\tilde{\phi})^2 + (\partial_\alpha\tilde{x})^2 + (\sqrt{2}\tilde{x})^2 + (\partial_\alpha y)^2 + (\sqrt{\tilde{g}_\alpha(0)}y)^2, \tag{16}$$

where all the derivatives ∂_α are with respect to $(\tilde{\tau}, \tilde{\sigma})$ [46]. We then identify the bosonic fluctuation spectrum from this Lagrangian where the fields are

$$(m_{\tilde{x}}^2, m_{\tilde{\phi}}^2, m_y^2) \coloneqq (2, 4, \tilde{g}_\alpha(0)) \tag{17}$$

and the bosonic propagator is diagonal. As expected by our construction the square of the mass of the fields y, is related to the variable particle mass in the dual particle Hamiltonian description (10). The factorization of scattering requires that all the $2 \to 2$ amplitudes with incoming particles of a certain mass going to different mass vanish.

Therefore according to (17) all amplitudes of an integrable theory must vanish, unless $\tilde{g}_\alpha(0) = (2, 4)$. To constrain the set of $\tilde{g}_\alpha(0)$ further it is necessary to compute the amplitudes. The cubic and quartic interaction vertices can be read off the relevant order Lagrangians

$$L_3 = -4\tilde{\phi}(\tilde{x} - \tilde{x}')^2 + 2\tilde{\phi}(\dot{\tilde{\phi}}^2 - \tilde{\phi}'^2) + 2\tilde{\phi}(\dot{y}^2 - y'^2) + \tilde{g}_\alpha(0)y^2(\dot{\tilde{\phi}} - \tilde{\phi}') + \ldots \tag{18}$$

and

$$\mathcal{L}_4 = 8\tilde{\phi}^2(\tilde{x} - \tilde{x}')^2 + 2\tilde{\phi}^2\left[(\partial_\alpha\tilde{\phi})^2 + \frac{2}{3}\tilde{\phi}^2\right] + 2\tilde{\phi}^2(\partial_\alpha y)^2 + \frac{\tilde{g}_\alpha(0)}{2}y^2\left[2\tilde{\phi}^2 + 4\tilde{\phi}\sum_\alpha \partial_\alpha\tilde{\phi} + (\partial_\alpha\tilde{\phi})^2 + (\tilde{x} + \dot{\tilde{x}})^2 + (\tilde{x} - \tilde{x}')^2\right] + \ldots. \tag{19}$$

All the tree amplitudes turn out to behave qualitatively in the same way, so let us study the tree level contributions of the amplitude $xx \to yy$ which come from a contact diagram and the s-channel to obtain

$$\mathcal{A}_c(xx \to yy) = \tilde{g}_\alpha(0)A_1(p_1, p_2),$$
$$\mathcal{A}_s(xx \to yy) = \tilde{g}_\alpha(0)A_1(p_1, p_2)A_2(p_1, p_2),$$

where $A_{1,2}$ are unequal functions of the light cone momenta of the incoming particles. By considering the total contribution to the amplitude summing both channels and since $A_2(p_1, p_2)$ is a nontrivial function of momenta, the factorization forces $\tilde{g}_\alpha(0)$ to be either 0, so that the amplitude vanishes, or 2 so that the masses are equal $m_x = m_y$ and the factorization does not impose any further condition. By summing the contact, the s and $t + u$-channel contributions of the different amplitudes, as in [31], it turns out that we have already constrained the system enough and the factorization happens for $\tilde{g}_\alpha(0) = 0$ where the amplitudes vanish, or for $\tilde{g}_\alpha(0) = 2$. Remarkably, the S-matrix factorization in string picture constrains the spectrum and the theory to the set of values (14), obtained already by the equivalent particle picture and the Galoisian integrability of the effective particle Hamiltonian (10). Moreover, the nonintegrability analysis (14) fully specifies the integrable conditions for the theory.

VII. DISCUSSION

We have formulated an equivalence between the 2-dim σ-model spectrum expanded on a nontrivial massive vacuum and an effective classical particle Hamiltonian with nontrivial masses and potential. The equivalence holds for large classes of backgrounds and vacua. We have demonstrated explicitly the formalism in a theory with warp factors of AdS and internal space and a chosen GKP vacuum, where the mass of the excitations of the 2-dim σ-model spectrum is dual to the acceleration of the particle's variable mass change. The analytic Galoisian nonintegrability on appropriate geodesics of the Hamiltonian system sets an algebraic constraint on the particle masses, such that integrability is allowed. It is remarkable that when this condition is translated to the 2-dim theory through our equivalence, it constrains the spectrum of the theory such that the factorization of the S-matrix occurs. In particular, the integrability of the classical particle system implies the factorization of the S-matrix in the dual quantum 2-dim theory. Our proposal initiates the study of analytic nonintegrability techniques on the S-matrix vacua in relation to factorization, which goes beyond the classification techniques used so far.

The particle-string equivalence we have established relies on the fact that the methods of analytic nonintegrability and the spectrum of the 2-dim σ-model, although independent, are based on the same order expansions; this is the reason our formulation applies well to the tree-level factorization. This is also the reason that this relation is entirely fundamental, universal, and generic. Moreover, our formalism provides nontrivial evidence beyond any assumptions, on the connection between integrability and the S-matrix factorization for large class of theories with interactions that may break Lorentz invariance. The (non) integrability constraints obtained in the particle picture are in agreement in the dual quantum picture with the S-matrix factorization, therefore establishing such a connection for a large classes of theories. Our proposal provides a possible ground to develop a formal proof or to clarify explicitly the requirements for the validity of this relation.

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APPENDIX A: THE HAMILTONIAN DERIVATION

Let us write the components of the metric (5) we like to focus as

$$ds^2_M(\mathbf{x}, y) \coloneqq g_{tt}(x, y)dt^2 + g_{xx}(y)dx^2 + g_{\phi\phi}(x, y)d\phi^2 + \ldots$$
$$\mathrm{d}s^2_Y(\mathbf{y}) \coloneqq g_{yy}(y)dy^2 + \ldots. \tag{A1}$$

We highlight that this is without any loss of generality, the dots represent other directions where the stings can consistently localize. We have made a rescaling on the x

coordinate since it considerably simplifies the length of the expressions below. We consider the configuration $t = \kappa\tau$, $\phi = \omega\tau$, $x = x(\sigma)$ and $y = y(\sigma)$. The equations of motion read

$$\partial_y g_{xx} x'^2 + \partial_y g_{yy} y'^2 - \partial_y g_{tt}\kappa^2 - \partial_y g_{\phi\phi}\omega^2 - 2\partial_\sigma(g_{yy}y') = 0,$$
$$-\partial_x g_{tt}\kappa^2 - \partial_x g_{xx}\omega^2 - 2\partial_\sigma(g_{xx}x') = 0,$$

while the Virasoro constraint is

$$g_{tt}\kappa^2 + g_{\phi\phi}\omega^2 + g_{xx}x'^2 + g_{yy}y'^2 = 0. \tag{A2}$$

The system of equations should be solved at the background chosen, without loss of generality let us choose the $y = 0$ where it can be seen that the system has a solution if

$$\left.\frac{\partial_y g_{xx}}{g_{xx}}\right|_{y=0} = -\left.\frac{\partial_y g_{\phi\phi}}{g_{\phi\phi}}\right|_{y=0} = -\left.\frac{\partial_y g_{tt}}{g_{tt}}\right|_{y=0}, \tag{A3}$$

where one sees that simple background solution as

$$\partial_y g_{ii}(x, 0) = 0. \tag{A4}$$

satisfy the condition. The effective particle Hamiltonian is derived by the (A2) straightforwardly as in [19,25]

$$\mathcal{H} = \frac{p_x^2}{2g_{xx}} + \frac{p_y^2}{2g_{yy}} + V(x, y), \tag{A5}$$

where the effective potential is

$$V(x, y) \coloneqq \kappa^2 g_{tt}(x, y) + \omega^2 g_{\phi\phi}(x, y). \tag{A6}$$

The Hamiltonian belongs in the form of (6) with the use of a canonical transformation.

The NVE is then obtained by applying the fluctuation $\delta y = y_0 + \eta(\sigma)$ on the equations of motion as

$$\eta''(x) + \frac{1}{2g_{yy}(x, 0)}\partial_y^2 V\eta(x) = 0, \tag{A7}$$

where all the fields are evaluated at $y = 0$. The NVE belongs to the more general family of Eq. (7).

To apply the Kovacic algorithm on the NVE (A7) sometimes a change of variables $z = f(x)$ is needed to bring it in the appropriate form. The generated first and second derivatives of $x(\sigma)$ from the change of variables can be read from the Virasoro constraints and the equations of motion, without the need of the knowledge of the analytical form of the string solution we expand on as in [19,25].

APPENDIX B: DIFFERENTIAL GALOIS GROUP ON THE HEUN EQUATION

The Heun differential equation is brought to its normal form by a transformation $\eta \to \eta \exp(-\frac{1}{2}\int \kappa(z)dz)$. The transformation between the two equations is Liouvillian, so the integrability status between the original and the transformed equations is equivalent. Nevertheless, in general the differential Galois group of the two equations are not the same. In fact the differential Galois group of the transformed normal equation is a subgroup of $SL(2, C)$, while for the initial Heun equation (12) this is true if and only if $\kappa(z) = nf/f'$ with $n \in \mathbb{Z}$ and f is a differentiable function, $f \in K$. The differential equation in its canonical form has Liouvillian solutions if and only if, its differential Galois group is a proper algebraic subgroup of $SL(2, C)$ [47,48]. The normal differential equation can be further transformed to the Riccati equation with $g = -(\log\eta)'$. The criterium for the existence of integrability now takes a more tractable form: The original Heun NVE equation has Liouvillian solutions if and only if the Riccati NVE equation has an algebraic solution with the degree of the relevant minimal polynomial that belongs to the set $\{1, 2, 4, 6, 12\}$. By applying the Kovacic algorithm on the Riccati equation we find the conditions (13). The Galois differential techniques for such special functions have been also studied in [39,40].

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