Interference in the Heisenberg picture of quantum field theory, local elements of reality, and fermions

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We describe a simple model for quantum interference of a single photon in the Mach-Zehnder interferometer using the Heisenberg picture. Our purpose is to show that the description in the Heisenberg picture is local just like in the case of the classical electromagnetic field, the only difference being that the electric and the magnetic fields are, in the quantum case, operators representing quantum observables. We then consider a simple model for a single-electron Mach-Zehnder interferometer and explain what the appropriate Heisenberg picture treatment is in this case. Interestingly, the parity superselection rule that arises in fermions due to the different spin statistics forces us to describe the electron in a radically different way to the photon in order to preserve the account in terms of local observables. A model using only local quantum observables of fermionic modes (such as the current operator) is nevertheless still viable to describe phase acquisition. We discuss how to extend this local analysis to coupled fermionic and bosonic fields within the same local formalism of quantum electrodynamics as formulated in the Heisenberg picture.

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I. INTRODUCTION

The superposition principle of quantum theory has an immediate expression in the Schrödinger picture. If two different state vectors satisfy the Schrödinger equation then their linear sum also does. Thus the Schrödinger picture lends itself naturally to the description of quantum interference. For example, if the state $|0\rangle$ of a qubit acquires a phase ϕ_0 and the state $|1\rangle$ acquires a phase ϕ_1, then according to the Schrödinger equation the superposed state of the qubit evolves into $e^{i\phi_0}|0\rangle + e^{i\phi_1}|1\rangle$. This phase difference can then be detected by performing a measurement in the $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ basis.

By contrast, in the Heisenberg picture, the initial state vector (the so-called Heisenberg state) never changes. It can always be assumed to be a fixed state ρ_0 without any loss of generality. How then is the interference phenomenon manifested? The explanation of interference lies in the dynamics of operators which evolve unitarily, consistent with the dynamical evolution of the state vectors in the Schrödinger picture. The consistency condition is given by the requirement that the Heisenberg and Schrödinger pictures must be empirically equivalent. For any given observable $\hat{O}$ of a physical system, the empirically accessible quantity at any one time is given by the expected value $\mathrm{Tr}(\hat{O}\rho(t)) = \mathrm{Tr}(\hat{O}(t)\rho_0)$, where $\hat{O}(t) = U(t)^\dagger \hat{O} U(t)$ and $\rho(t) = U(t)\rho_0 U^\dagger(t)$. Hence the empirical content of the two pictures is the same. But the mode of explanation is different, especially regarding locality [1].

Here we study a simple model of spatial interference in the Heisenberg picture of quantum field theory, showing that this picture provides a fully local account of it, both for bosons and for fermions. First, we illustrate how the Heisenberg picture provides local elements of reality in terms of *q-numbered descriptors* (quantum observables) for bosonic and fermionic fields, both free and interacting, much as in classical field theory. However, the local elements of reality are in the quantum case operators (q-numbers) and not c-numbers (the usual complex numbers, that all commute with each other).

These q-numbered descriptors satisfy the principle of no-action at a distance: given a partition of the whole universe into subsystems, operations (unitaries or *CP*-maps in general) involving the descriptors of a given subsystem cannot modify the descriptors of other non-overlapping subsystems. In [1] the proof of this fact is presented for an N-qubit system and extended to any other physical system via the universality of quantum computation. However, there are some outstanding points to clarify in the case of quantum fields. For instance, fermionic fields lack crucial qubit properties (such as local tomography [2]); hence the proof of locality by the universality of computation may not directly apply to them.

II. BOSONIC MACH-ZEHNDER INTERFEROMETRY

Consider a simple example of bosonic interferometry. A single-photon going through a Mach-Zehnder interferometer has been the foremost way of thinking about interference in quantum information, and computation [3]. It is analogous to the double-slit experiment, which in the words of Feynman contains "the only mystery" in quantum physics [4]. We shall imagine that an additional phase ϕ is introduced locally in one of the arms. How it is applied depends on the physics of the system undergoing interference and is irrelevant for our discussion about locality. So is the analysis of fluctuations in the phase, [5,6], so without loss of generality we shall consider a simple minimal model for the interference, as follows.

We consider the standard quantization procedures of the electromagnetic field, which lead to introducing the creation and annihilation bosonic operators at mode x, denoted by $a_x, a_x^\dagger$. These operators satisfy the following constraints: $[a_x, a_y] = 0$, $[a_x, a_y^\dagger] = \delta_{x,y}$, and $a_x|0\rangle = 0$, where $|0\rangle$ is the chosen vacuum state of the global Fock space and $[A, B] = AB - BA$ is the commutator of the two operators A and B. For simplicity of exposition, we shall assume that x represents a region of space where the photon can be confined to arbitrarily high accuracy; in this case, x can be either L (a region around the left arm of the interferometer) or R (a region around the right arm of the interferometer). These two regions are nonoverlapping (their separation being much larger than their respective extents). The details of the free evolution of the photons are not relevant for present purposes. Also, we do not consider the polarization of the photons, since it is irrelevant for the locality discussion.

In the Schrödinger picture, the quantum state of the photon changes after the first beam splitter (U_{BS}), then acquires the additional phase in one arm (say the left one) via $U_L^{(\phi)}$, and finally undergoes another change at the final beam splitter ($U_{BS}^\dagger$). Labeling the quantum state where the photon is on the left or right arm of the interferometer respectively as $|L\rangle \doteq a_L^\dagger|0\rangle$ and $|R\rangle \doteq a_R^\dagger|0\rangle$, the dynamical evolution of the photon is given by

$$|L\rangle \xrightarrow{U_{BS}} \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle) \xrightarrow{U_L^{(\phi)}} \frac{1}{\sqrt{2}}(|R\rangle + e^{i\phi}|L\rangle) \xrightarrow{U_{BS}} \frac{1}{\sqrt{2}}(|+\rangle + e^{i\phi}|-\rangle) = \sin\phi/2|L\rangle + \cos\phi/2|R\rangle, \tag{1}$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}(|L\rangle \pm |R\rangle)$ are equally weighted superpositions of the left and right path.

In the Heisenberg picture, we see how the phase affects the dynamical evolution of photonic quantum observables. We can consider for instance the operator representing the vector potential field of each position mode x, which is proportional to $A_x = a_x + a_x^\dagger$. (The same analysis could be done with the electric or magnetic field.) Since we have two arms of the interferometer, $x = L$ (left) and $x = R$ (right), we shall specify the field in both of these (which we think of as modes of the electromagnetic field). We will use the ordered pair notation $t\colon [\![a_L(t) + a_L(t)^\dagger; a_R(t) + a_R(t)^\dagger]\!]$ to denote the descriptors of the left and right modes at time t, where the left mode descriptor occupies the first slot of the ordered pair and right mode occupies the second slot. (Note that in general, due to unitarity, it is possible to retrieve the dynamical evolution of all relevant observables of a composite system by merely tracking the dynamical evolution of the generators of the algebra of observables of each subsystem [7]. In this case, the generators would be $a_L(t), a_R(t)$.)

At the start, let the field operators be

$$t_0\colon [\![a_L + a_L^\dagger; a_R + a_R^\dagger]\!]$$

and the Heisenberg state $|\Psi\rangle = |1_L 0_R\rangle$. This simply represents the quantum photon field operators in the left and the right modes at time t_0, with one photon existing in mode L. The unitary beam splitter U_{BS} applied at time t acts as Bogoliubov transformations on the creation and annihilation operators: $a_L(t) \xrightarrow{U_{BS}(t)} a_L(t+dt) = \frac{1}{\sqrt{2}}(a_L(t) + a_R(t))$ and $a_R(t) \xrightarrow{U_{BS}(t)} a_R(t+dt) = \frac{1}{\sqrt{2}}(a_L(t) - a_R(t))$.

So the photon field operator descriptors after the first beam splitter, expressed as functions of the initial descriptors, are

$$t_1\colon [\![\frac{1}{\sqrt{2}}(a_L + a_R + a_L^\dagger + a_R^\dagger); \frac{1}{\sqrt{2}}(a_L - a_R + a_L^\dagger - a_R^\dagger)]\!]. \tag{2}$$

The phase shift $U_L^{(\phi)}(t)$ acts only on the left arm. That is, it is a function of the operators $a_L(t)$ only. Hence, it induces a change in the left modes only: $a_L(t) \xrightarrow{U_L^{(\phi)}(t)} a_L(t+dt) = e^{i\phi}a_L(t)$ and $a_R(t) \xrightarrow{U_L^{(\phi)}(t)} a_R(t+dt) = a_R(t)$. The new field operators after the phase shift are therefore

$$t_2\colon [\![\frac{1}{\sqrt{2}}(e^{i\phi}(a_L + a_R) + e^{-i\phi}(a_L^\dagger + a_R^\dagger)); \frac{1}{\sqrt{2}}(a_L - a_R + a_L^\dagger - a_R^\dagger)]\!]. \tag{3}$$

The property of no-action at a distance is the crux of quantum field theory in the Heisenberg picture: changes induced by a phase shift acting locally on one mode do not affect operators pertaining to the other modes. In our example, only the field operators of the left mode (the first slot of the ordered pair above) contain the phase, while the field operators of the right mode do not. Hence by simple inspection of the descriptors of the two modes, we can tell where the phase shift was applied. Note also that a state tomography of the left mode would not at this stage reveal the phase (the expected value of the number operator of the left mode does not depend on the phase). Hence the phase is at this stage encoded in the left mode, but it is locally inaccessible. We need a phase reference in order to compare the phase induced in the left arm. This is precisely what the second beam splitter does. It makes the left and right arm interact in such a way that one arm is a phase reference for the other.

The final step is the second beam splitter, which induces once more a mixing of modes, and hence allows one to recover the phase. The field operators at the output of the interferometer are then

$$t_3\colon \ [\![\cos\left(\frac{\phi}{2}\right)a_L + i\sin\left(\frac{\phi}{2}\right)a_R + \text{H.c.}; \cos\left(\frac{\phi}{2}\right)a_R - i\sin\left(\frac{\phi}{2}\right)a_L + \text{H.c.}]\!]. \tag{4}$$

The usual interference is directly manifested in the expected value of the number operator $\hat{N}_x(t) = a_x^\dagger(t)a_x(t)$ at time t_3, in the Heisenberg state $|\Psi\rangle = |1_L 0_R\rangle$. This value can be calculated from $\langle A_x^2(t_3)\rangle_{|\Psi\rangle} = 1 + 2\langle \hat{N}_x(t_3)\rangle_{|\Psi\rangle}$. Thus the phase that we have tracked with $A_x(t)$ is now manifested in a direct observation. For the output left mode, we obtain

$$\langle \hat{N}_L(t_3)\rangle_\Psi = \cos^2\frac{\phi}{2}. \tag{5}$$

The expected value of the output mode R could be calculated in the same fashion (and it would yield the value of $\sin^2\phi/2$). Hence the expected values at the end of the interferometry are the same in the Heisenberg and Schrödinger pictures, as they are empirically equivalent. The significant difference in the explanation for the interference is that in the Heisenberg picture the phase introduced by the phase shift on one mode is only manifested in that mode and not others, locally. In the Schrödinger picture this would not be the case as the wave function does not allow for a separable description and the phase difference due to the beam splitter acting on mode L could equally well have been introduced by a beam splitter acting on mode R. Note that here we consider the Schrödinger picture as it is usually interpreted, with the wave function completely describing the system (see e.g., [3,8]). In [1], this fact is pointed out in regard to the quantum teleportation protocol with qubits. The key advantage of the Heisenberg picture is that it is manifestly local (i.e., it satisfies no action at a distance as well as no signaling; of course, the Schrödinger picture does not allow signaling either and, in this sense, it is also local, or microcausal in the language of field theory). If one were to expand the usual interpretation of the Schrödinger picture to also include the unitary transformations that have been applied to the system, then the physical situation where the phase shift is in the left arm could be distinguished from the opposite shift being applied to the right arm, by considering the two unitary transformations that apply such transformations, since they are manifestly different. In the Heisenberg picture this information is fully contained in the local descriptors.

Any bosonic field (say pertaining to a Bose condensate of atoms) has precisely the same description as above in the Heisenberg picture. We can interfere condensates applying this Mach-Zehnder interferometer implementation, and the operator description of this interference would be identical to the one presented above. The same is true of fermionic fields, but with one significant subtlety, due to the presence of superselection rules. To expose this subtlety, we will now proceed to describe a single electron Mach-Zehnder interferometer.

III. FERMIONIC MACH-ZEHNDER INTERFEROMETRY

For a single electron, one could naively expect that the model for interferometry is the same as in the bosonic case, replacing the descriptors with fermionic creation and annihilation operators f_x and $f_x^\dagger$ at mode x. This strategy, however, is not possible. Fermionic operators anticommute at spacelike separated points $\{f_x, f_y^\dagger\} = \delta_{x,y}$ and $\{f_x, f_y\} = 0$, where the curly brackets represent the anticommutator, $\{A, B\} \doteq AB + BA$. Hence, the operator $f_x + f_x^\dagger$ is not an observable. The term observable is used here in an operational sense, to refer to the class of operators with real eigenvalues, whose corresponding eigenvectors are physical states, i.e., they can be prepared and distinguished by a physically allowed process. If that operator were an observable, then one would be able to signal. To explain why, we present a simple argument which is originally due to Wigner [9]. It leads us to have to impose a superselection rule (the parity superselection rule), ruling out superpositions of even- and odd-numbered fermionic states.

The thrust of the argument is that, if one could superpose even and odd numbers of fermions, such as by preparing an eigenstate of $f_x + f_x^\dagger$, one would violate the no-signaling principle. For instance, one could send messages between two spacelike separated regions by the following protocol.

Consider two spacelike separated regions, A and B. Suppose that f_A is a fermionic mode at A, and f_B is one at B. First, let us consider the Schrödinger picture. Let us assume that $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)_B$ (eigenstate of $f_B + f_B^\dagger$ with eigenvalue 1) is an allowed fermionic state. This state can be interpreted physically as a superposition of a state with no fermions in mode B (state $|0\rangle_B$, even) and of a state with one fermion in mode B (state $|1\rangle_B$, odd). Then, let us prepare the product state $|0\rangle_A(|0\rangle + |1\rangle)_B$. Assuming no restrictions on the allowed quantum observables, $i(f_A^\dagger - f_A)$ is an allowed

Hamiltonian, hence the unitary $U_A = \exp(\frac{\pi}{2}(f_A^\dagger - f_A))$ is also permitted. U_A creates a fermion in the mode A when applied to the vacuum, and annihilates a fermion in the mode A if there is one. In other words, $U_A|0\rangle_A = |1\rangle_A$ and $U_A|1\rangle_A = -|0\rangle_A$. (This action defines a class of unitaries up to a phase, all of which are equivalent for the sake of this proof.) Now, let us either leave A's state as the vacuum or create one electron in A, by not applying or applying U_A. If the state remains the vacuum, B's state remains unchanged: the expected value of $f_B + f_B^\dagger$ is $+1$ at the end of the protocol. If an electron is created in the region A by applying U_A, the state becomes $|1\rangle_A(|0\rangle + |1\rangle)_B$. Now, the expected value of the observable $f_B + f_B^\dagger$ at B is -1 in this state. Thus an observer at B, if they were able to prepare and distinguish superpositions of the vacuum and one electron state, they could tell whether an electron had or had not been created at A, despite A being spacelike separated from B. Generalizing the argument, it follows that any superposition of even and odd numbers of fermions is prohibited.

An equivalent argument exists in the Heisenberg picture. Assume that the Heisenberg state is $|\psi_H\rangle = |0\rangle_A(|0\rangle + |1\rangle)_B$. We can see what happens to the observable $f_B + f_B^\dagger$ when acted upon by U_A—corresponding to the action of creating a superposition of even and odd numbers of particles. We can see by algebraic calculation and by the action of U_A to the vacuum and excited states $|0\rangle_A$, $|1\rangle_A$ that $U_A = f_A^\dagger - f_A$. One obtains $U_A^\dagger(f_B + f_B^\dagger)U_A = -(f_B + f_B^\dagger)$. This is due to the fact that the $f_A, f_A^\dagger$ anticommute with the $f_B, f_B^\dagger$. This expression is a violation of the principle of no-action at a distance because quantum observables at B's location can be modified instantaneously by the action of a unitary U_A, which is only operating on A. Taking the expected value of the field operators at B at the end of the protocol, $U_A^\dagger(f_B + f_B^\dagger)U_A$, using the Heisenberg state $|\psi_H\rangle$, we obtain -1, reaching the same conclusion as in the Schrödinger picture.

Following this line of argument, one imposes a parity superselection rule: fermionic observables of a given mode x must commute with the parity operator $\exp(-i\pi f_x^\dagger f_x)$; which implies that they have to consist of quadratic forms of fermionic operators, such as the electric charge density operator: $j_0(x) = -e f_x^\dagger f_x$. The quadratic forms of fermionic operators at different spacelike points commute (just like in the bosonic case), so there is no problem with locality.

However, there is more. Quadratic forms of fermionic operators alone cannot keep track of the phase in the Mach-Zehnder experiment in the same way as bosonic operators can. So not only would the phase not be registered in the local density operator (or any other local quadratic observable), but even in the Heisenberg picture, it would not be registered in any local quadratic fermionic descriptor. Luckily, the second-quantized Dirac field does not contain just the electron operators; it also contains the positron operators. So in the Heisenberg picture, it is still possible to track local observables of the Dirac field pertaining to each mode, in order to give an entirely local account of the interferometry.

In the bosonic case, any unitary process can be fully described by tracking observables, since the algebra generators can be found from observables by $a_x = \frac{1}{2}((a_x + a_x^\dagger) - i(i(a_x - a_x^\dagger)))$. However, for fermions, this is not entirely the case (in [10] a detailed local mathematical analysis of general fermionic systems that goes beyond the scope of the current paper shall be presented).

Going back to the fermionic Mach-Zehnder interferometer, the proper second-quantized Dirac field is described by the four-spinor operator (see [11])

$$\psi(x) = b(x) + d^\dagger(x). \tag{6}$$

This involves the electron annihilation operator $b(x)$ and the creation operator of a positron $d^\dagger(x)$ at point x.

Once more, here we deliberately omit the spinor details and the momentum representation as it is not relevant for the following argument–for details see e.g., [8]. This fermionic Dirac field descriptor is not Hermitian and thus is not an observable. Also, the superselection rules prohibit superposing odd and even numbers of fermions. Thus, no linear combination of creation and annihilation operators is allowed to represent a physical variable.

To describe the Mach-Zehnder electronic experiment we need to construct a quadratic operator out of the fundamental descriptor of the Dirac field. Consider for instance the charge density operator pertaining to each arm mode $(j_0(x_L), j_0(x_R))$, where $j_0(x) = -e : \psi(x)^\dagger\psi(x) : = -e(b_x^\dagger b_x - b_x d_x + b_x^\dagger d_x^\dagger - d_x^\dagger d_x)$ (we would in general have to use the four-vector, including the current density, but, in this case, the other three components do not add to our analysis). We denote by $:AB:$ the normal ordering of the fermionic operators A and B. Even though d_x, b_x are spinors with some orthogonality properties imposed, all the four terms are nonzero in general. We will also assume, for simplicity of exposition, that the Heisenberg state is $|\Psi\rangle_{ep} = \frac{1}{\sqrt{2}}(|0_L 1_R\rangle_e + |1_L 0_R\rangle_e)|0_L 0_R\rangle_p$, so we describe the interferometry just after having applied the first beam splitter.

Tracking now the time evolution of the density operator $[\![j_0(L), j_0(R)]\!]$ in the Heisenberg picture, we see that a phase rotation applied on the left mode now does manifest itself in the quantum observables of the Dirac field, by modifying the charge density operator. The reason is that the field transforms under the phase rotation $U_L^{(\phi)}$ at time t as $b_L(t) + d_L^\dagger(t) \longrightarrow^{U_L^{(\phi)}} b_L(t+dt) + d_L^\dagger(t+dt) = e^{i\phi} b_L(t) + e^{-i\phi} d_L^\dagger(t)$ and $b_R(t) + d_R^\dagger(t) \longrightarrow^{U_L^{(\phi)}} b_R(t+dt) + d_R^\dagger(t+dt) = b_R(t) + d_R^\dagger(t)$. Then, after applying the phase rotation $U_L^{(\phi)}$, one obtains that the charge density of the Dirac field in the left mode is

$$j_0(L) = -e(b_L^\dagger b_L + e^{-2i\phi} b_L^\dagger d_L^\dagger - e^{2i\phi} b_L d_L - d_L^\dagger d_L), \tag{7}$$

while the right mode charge density stays unchanged.

We see that the phase is present in the charge density operator of the left mode after applying the phase shift. So a perfectly valid observable, even under superselection rules, can keep track of the phase locally to each mode of the Dirac field. The positronic part of the Dirac field allows one to have a phase reference for the electron field. The different action of the phase shift on the electron and positron field operators is what allows one to keep track of the phase. It does so by providing a local phase reference to the left arm between the fermionic and positronic field. We emphasize that if, as in the case of the interferometer we are studying, the Heisenberg state consists of an electron superposed across the left and the right modes and no positrons, then the expected value of j_0 will be phase independent—the phase is locally inaccessible via empirical observation as in the bosonic case. So, in the absence of superposed positrons, or another superposed electron that acts as a reference, the phase is at this point unobservable (all we can observe is whether the electron is in the mode L or mode R). However, the local picture of quantum field theory, when considering the Dirac field in its entirety as a q-number, allows us to tell that the phase has been applied on one mode and not the other, by tracking the electron and positron field current operator of each mode.

Moreover, in the Mach-Zehnder case, we can do so by tracking a physical observable, and not requiring the tracking of the Dirac field itself. In the Mach-Zehnder interferometer, the final beam splitter mixes the left and the right modes in exactly the way to the phase to be observable in the output charge densities (or currents in general). After the final beam splitter, the density operator in the left mode is

$$\begin{aligned} j_0(L) &= -\frac{e}{2} :(e^{-i\phi} b_L^\dagger + b_R^\dagger + e^{i\phi} d_L + d_R)(e^{i\phi} b_L + b_R + e^{-i\phi} d_L^\dagger + d_R^\dagger): = \\ &= -\frac{e}{2}[(e^{-i\phi} b_L^\dagger + b_R^\dagger)(e^{i\phi} b_L + b_R + e^{-i\phi} d_L^\dagger + d_R^\dagger) - (e^{i\phi} b_L + b_R + e^{-i\phi} d_L^\dagger + d_R^\dagger)(e^{i\phi} d_L + d_R)]. \end{aligned} \tag{8}$$

Now, considering the expected value in the Heisenberg state $|\Psi\rangle_e = \frac{1}{\sqrt{2}}(|0_L 1_R\rangle_e + |1_L 0_R\rangle_e)|0_L 0_R\rangle_p$, we see that only four components of the density survive, i.e., $b_L^\dagger b_L + b_R^\dagger b_R + e^{-i\phi} b_L^\dagger b_R + e^{i\phi} b_R^\dagger b_L$. Two of these contain the phase information. Concretely, we obtain

$$\begin{aligned} \langle\Psi|_e j_0(L)|\Psi\rangle_e &= -\frac{e}{2}\langle\Psi|_e[(e^{-i\phi} b_L^\dagger + b_R^\dagger)(e^{i\phi} b_L + b_R + e^{-i\phi} d_L^\dagger + d_R^\dagger) - (e^{i\phi} b_L + b_R + e^{-i\phi} d_L^\dagger + d_R^\dagger)(e^{i\phi} d_L + d_R)]|\Psi\rangle_e \\ &= -\frac{e}{2}\frac{1}{\sqrt{2}}\langle 0_L 0_R|_p(\langle 0_L 1_R|_e + \langle 1_L 0_R|_e)[(e^{-i\phi} b_L^\dagger + b_R^\dagger)(e^{i\phi} b_L + b_R)]\left(\frac{1}{\sqrt{2}}(|0_L 1_R\rangle_e + |1_L 0_R\rangle_e)|0_L 0_R\rangle_p\right) \\ &= -\frac{e}{4}(\langle 0_L 1_R|_e + \langle 1_L 0_R|_e)[(e^{-i\phi} b_L^\dagger + b_R^\dagger)(e^{i\phi} b_L + b_R)](|0_L 1_R\rangle_e + |1_L 0_R\rangle_e) \\ &= -\frac{e}{4}\langle 0_L 0_R|_e[(e^{-i\phi} + 1)(e^{i\phi} + 1)]|0_L 0_R\rangle_e \\ &= -\frac{e}{2}(1 + \cos(\phi)) = -e\cos^2\left(\frac{\phi}{2}\right). \end{aligned} \tag{9}$$

The expected value in the state $|\Psi\rangle_e$ therefore is $-e\cos^2\phi/2$, as expected, completely analogously to the bosonic case. As a side remark, we note that the operator $b_L^\dagger b_R + b_R^\dagger b_L$ quantifies coherence between the L and R modes (or what could be called a single particle entanglement between the two modes [12]). It is therefore not surprising that it emerges as the key observable in the Heisenberg treatment of interference.

IV. DISCUSSION

What have we achieved? We are able to describe the locality of spatial interference of both bosonic and fermionic fields in terms of the relevant local elements of reality, the q-numbered valued observables of the local fields pertaining to each mode, in the Heisenberg picture. The local elements of reality are in each case, the operators that are physically allowed observables, so considering the parity superselection rules for fermions. For bosons, it is sufficient to keep track of the photon field, while for fermions we need to use the current operator (or some other quadratic operator) of the full Dirac field.

Nevertheless, one could ask if this scheme can be applied to interacting fields. The answer is yes. In QED, for example, we have to combine the quantized electromagnetic field $A_\mu(x)$ with the Dirac field $\psi(x)$

through the charge current vector $j_\mu(x) = \frac{e}{2}[\bar{\psi}, \gamma_\mu \psi(x)]$. In this case we have the quantum electrodynamics equations, that show the dependence of $A_\mu(x)$ with $A_\mu(x')$ and $j_\mu(x')$, and vice versa [13]. Therefore we could follow the same strategy, tracking how the quantum observables $A_\mu(x), j_\mu(x)$ change. Since they only change through interactions local in position, all observed phases and phenomena can be explained in a nonaction at a distance way using these observables as descriptors.

Using this scheme, it is possible to describe the Aharanov-Bohm effect in an entirely local way [14] by quantizing the electromagnetic field and the Dirac field entirely and then using $j_\mu(x)$ and $A_\mu(x)$ as descriptors that track the local changes of the system? This can be done by generalizing the Heisenberg picture treatment presented in [15], using the picture of the full quantum field theory. We leave the details of this to a future work.

We have previously questioned whether the fermionic phase due to anticommutation is also acquired locally [16]. Here, instead, we embraced the anticommutation of fermions and explored its consequences for a fully local description of quantum superpositions and interferences in quantum electrodynamics. The Heisenberg picture proved to be natural in this context. However, could even more be said? Could we argue that the Schrödinger picture does not even exist in some of the scenarios we have considered? The answer to this exciting question may be yes, since the operators describing interactions in quantum electrodynamics may not have a finite norm, i.e., they may not transform all state vectors with a finite norm into state vectors with the same property. Dirac provided an example of this when he emphasized that the Heisenberg picture helps us get rid of the deadwood in quantum electrodynamics arising due to the Schrödinger picture [17]. Our analysis adds another point in favor of the argument that the Heisenberg picture may be superior to the Schrödinger picture when it comes to showing that quantum theory and general relativity obey the same locality principle, albeit with different elements of reality: c-numbers for general relativity, q-numbers for quantum theory.

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