Chiral anomalies induced by gravitational waves

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Chiral symmetries in field theory are typically affected by an anomaly in the quantum theory. This anomaly emerges when one introduces an interaction with a Yang-Mills or gravitational background. Physical applications of this quantum effect have been traditionally connected to topological questions of the background field and the study of instantons. We show here how one can alternatively find situations of physical interest that only involve ordinary, but dynamical solutions of the background field equations. More precisely, we show that solutions to the Einstein (Maxwell) equations are able to trigger the chiral anomaly if and only if they admit a flux of gravitational (electromagnetic) radiation with net circular polarization. As a consequence, astrophysical systems that admit such radiation spontaneously generate a flux of particles with net helicity from the quantum vacuum.

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I. INTRODUCTION

It is well-known that strong gravitational fields can affect the vacuum fluctuations of quantum fields and lead to important physical phenomena. From the pioneer work of Parker on particle creation in expanding universes [1], to the subsequent discovery by Hawking of thermal emission during a gravitational collapse [2], different phenomena of quantum origin can arise if a quantum field propagates on a dynamical, gravitational background. One of these quantum effects is related to the emergence of anomalies due to spacetime curvature.

An anomaly is understood as the failure of some Noether symmetry of a classical field theory to persist after the quantization. More precisely, when the classical conservation law of a Noether current breaks down in the quantum theory, the associated symmetry is said to be anomalous [3]. Their discovery was initiated in the late 1960s with the fermion axial or chiral anomaly [4,5], motivated with the aim of understanding the observed phenomenon of the pion decay into two photons. Since then, the field has grown enormously, leading to the discovery of many more anomalous symmetries of diverse nature (conformal, gauge, etc.), and to a rich interplay with differential geometry and topology [6]. From a physical viewpoint, anomalies were found useful to address key conceptual questions in the standard model of particles [U(1) problem, strong *CP* violation in QCD [7]] and cosmology (baryogenesis). Our goal in this paper is to point out and develop an unexplored aspect of chiral anomalies that, remarkably, turns out to have a simple physical interpretation and could lead to new physical applications in gravity and electrodynamics.

For definiteness, let $\psi(x)$ be a Dirac field interacting with a classical Yang-Mills background of field strength F_{ab} in Minkowski spacetime $(\mathbb{R}^4, \eta_{ab})$, with coupling constant g. Let γ^a be the Dirac matrices, and let $\gamma^5 \coloneqq i\gamma^0\gamma^1\gamma^2\gamma^3$ be the chiral matrix. In the massless limit, the standard action of this theory possesses a (global) Noether symmetry generated by the transformation $\psi(x) \to e^{i\gamma^5\theta}\psi(x)$, $\theta \in \mathbb{R}$, that leads to a Noether current: $j_5^a(x) = \bar{\psi}(x)\gamma^a\gamma^5\psi(x)$. This is the well-known (Abelian) chiral symmetry. This current is conserved for solutions $\psi(x)$ of the Dirac equation of motion, $\nabla_a j_5^a(x) \approx 0$. In the quantum theory, however, off-shell contributions yield

$$\langle \nabla_a j_5^a \rangle = \frac{-g^2}{16\pi^2}\mathrm{Tr}F_{ab}{}^*F^{ab} \neq 0, \tag{1}$$

thus spoiling the classical conservation law. This is the indication that the classical symmetry is anomalous in the quantum theory. Denoting by $\psi_L = 1/2(\mathbb{I} + \gamma^5)\Psi$ and $\psi_R = 1/2(\mathbb{I} - \gamma^5)\Psi$ the right-handed and left-handed chiral sectors of the Dirac field, respectively, the associated Noether charge can be written as $Q_5(t) = \int_\Sigma d^3x\sqrt{h}(\psi_R^\dagger\psi_R - \psi_L^\dagger\psi_L)$, which is a measure of the net difference between right-handed (positive-helicity particles plus negative-helicity antiparticles) and left-handed fermions (negative-helicity particles plus positive-helicity antiparticles). While this difference is preserved by the equation of motion for classical fields, $\dot{Q}_5(t) \approx 0$, the emergence of the anomaly indicates that quantum fluctuations are able to induce a change in time given by

$$\langle \hat{Q}_5(t_2)\rangle - \langle \hat{Q}_5(t_1)\rangle = \int_{[t_1,t_2]\times\Sigma} d^4x\sqrt{-\eta}\langle \nabla_a j_5^a\rangle = \frac{-g^2}{16\pi^2}\int_{[t_1,t_2]\times\Sigma} d^4x\sqrt{-\eta}\mathrm{Tr}F_{ab}{}^*F^{ab}, \tag{2}$$

if and only if the right-hand side (RHS) of this equation is nonvanishing. What does the anomalous time dependence of the chiral charge imply physically? Using S-matrix theory and Bogoliubov transformations, it can be explicitly shown [8] that the Yang-Mills field creates and destroys fermions in such a way that $\langle Q_5(t_2)\rangle - \langle Q_5(t_1)\rangle$, which is a measure of asymmetric particle creation, is given precisely by the amount predicted by the RHS of the previous equation. Thus, the anomalous temporal evolution of the chiral charge is physically interpreted as a phenomenon of asymmetric particle creation by a dynamical background: a nontrivial gauge field is able to excite spontaneously a net number of right-handed fermions over left-handed ones from the quantum vacuum or vice versa.

In a similar fashion, it has recently been shown [9–12] that the analogous chiral symmetry in electrodynamics, most popularly known as electric-magnetic duality symmetry of source-free Maxwell equations, suffers from a similar anomaly when a nontrivial spacetime background $(\mathbb{R}^4, g_{ab})$ is introduced. More precisely, under a chiral rotation ${}^{\pm}F(x) \to e^{\mp i\theta}{}^{\pm}F(x)$ of the self-dual and anti–self-dual sectors of the electromagnetic field, ${}^{\pm}F_{ab} = \frac{1}{2}[F_{ab} \pm i^*F_{ab}]$, the usual action for the source-free Maxwell theory remains invariant. This leads to a Noether current that, for solutions of the field equations, reads $J_5^a \approx A_b{}^*F^{ab} - Z_bF^{ab}$. Although classically conserved, it was found in [9–12] that quantum fluctuations of the electromagnetic field produce

$$\langle \nabla_a J_5^a\rangle = \frac{-1}{96\pi^2}R_{abcd}{}^*R^{abcd}, \tag{3}$$

if spacetime curvature $R_{abc}{}^d$ is considered. The associated Noether charge can be expressed as the difference between right-handed and left-handed photons, and, while Maxwell equations guarantee that $\dot{Q}_5 \approx 0$ for the classical function $Q_5(t)$, off-shell contributions can spontaneously make this quantity change in time if and only if the RHS of

$$\langle \hat{Q}_5(t_2)\rangle - \langle \hat{Q}_5(t_1)\rangle = \frac{-1}{96\pi^2}\int_{[t_1,t_2]\times\Sigma} d^4x\sqrt{-g}R_{abcd}{}^*R^{abcd} \tag{4}$$

is different from zero. Again, the physical picture is analogous to the fermion case: a nontrivial gravitational background would be responsible to create spontaneously a difference in the number of right- and left-handed circularly polarized photons from the quantum vacuum.

In general, the physical interpretation of chiral anomalies is strongly associated with the phenomenon of "level crossing." The Hamiltonian of the quantum field, which determines the energy of field modes, depends on the background field. Then, a nontrivial temporal evolution of the latter can make a positive-chirality mode with initial negative energy transform into a positive-chirality mode with final positive energy [8,13]. In other words, the dynamical evolution of the Yang-Mills or gravitational field can reverse the helicity of field modes, producing as a result a net creation of helicity from the quantum vacuum: more particles of one helicity are created than particles of the opposite helicity.

The questions we want to answer here are as follows: what are the physical spacetime backgrounds that can induce this level crossing in the helicity of field modes? How is this dynamical evolution supposed to be? In the Yang-Mills case an important historical role has been played by instantons [14,15]. Instantons are classical solutions to the Euclidean field equations of a non-Abelian gauge theory that exhibit a nontrivial topology in the manifold of field configurations. Physically, they are interpreted as amplitudes that quantify quantum-mechanical transitions between topologically inequivalent vacua in the Hilbert space of gauge fields [14,16–18]. Their use played a fundamental role in the 70–80 in addressing several problems of the standard model of particles and QCD, most notably the $U(1)$ problem [19]. But what about the gravitational case? Analogous solutions of the Euclidean Einstein's equations are also known for a long time [20,21] and were baptized as gravitational instantons. While mathematically these solutions have a rich structure [22], their physical interpretation could be regarded as exotic, since it relies on quantum-gravity issues [23]. If one is only interested in studying applications of chiral anomalies, it can be more natural to look instead for more realistic spacetimes, i.e., for *ordinary*, Lorentzian solutions of Einstein's equations directly.

To our knowledge, this problem has not been proposed in the literature before. Since it is known that the integrand in (4) is locally a total derivative, *a priori* one expects that only nontrivial spacetime topologies can produce nonvanishing contributions, thereby the historical interest in instantons. However, in gravity this is not quite true because, at least for asymptotically flat spacetimes, the boundary of the spacetime, called null infinity [24], is nontrivial and can provide a contribution by means of the flux of gravitational waves (GWs) that enters/exits the spacetime. The goal of this paper is to fill this gap and to open a new window for applications of chiral anomalies that go beyond the realm of topology or instanton calculus.

Our main result will be to show that chiral anomalies are intimately related to the circular polarization state of ordinary gravitational radiation in the spacetime background. More precisely, we shall prove that

$$\langle \hat{Q}_5(\mathrm{scri}+)\rangle - \langle \hat{Q}_5(\mathrm{scri}-)\rangle = \int_0^\infty \frac{d\omega\omega^3}{24\pi^3}\sum_{\ell m}[|h_+^{\ell m}(\omega) - ih_\times^{\ell m}(\omega)|^2 - |h_+^{\ell m}(\omega) + ih_\times^{\ell m}(\omega)|^2], \quad (5)$$

where h_+ and $h_\times$ denote the two GW linear polarization modes that reach future null infinity, emitted by an isolated gravitational source that is stationary at both past and future timelike infinities, but otherwise arbitrary. What this formula is indicating is that the more right(left)-handed gravitational radiation is emitted by a system, the more right(left)-handed particles will be excited from the quantum vacuum through the mechanism that produces the chiral anomaly. This is a realistic gravitational setting, with a clear and unambiguous physical meaning.

That an asymptotically flat spacetime background must emit gravitational waves in order to induce the quantum anomaly is an indication that only *dynamical solutions* of Einstein's equations are relevant in this question.[1] This should come with no surprise in light of the previous physical interpretation of anomalies in terms of asymmetric particle creation, since only *dynamical gravitational fields* are able to spontaneously create particles (and hence helicity) from the quantum vacuum. On the other hand, because the study dynamical solutions of Einstein's equations is a rather involved issue that typically requires numerical techniques, this could explain why only instantonic solutions (which are known in closed form) have only been considered so far in the study of chiral anomalies.

In a given sense the result that we obtain shares some parallelisms and interpretations with instantons. Namely, our result can be understood as tunneling between degenerate vacua of the asymptotically flat spacetime (associated with the degeneracy of gravitational connections at future null infinity [25,26]), but in this case these transitions are produced *classically* simply by a flux of gravitational waves crossing null infinity (i.e., not through the usual quantum-mechanical tunneling). However, important conceptual differences exist. For instance, our result indicates that any spacetime where its manifold is homeomorphic to $\mathbb{R}^4$, but such that its metric is deformed with respect to Minkowski so as to allow the presence of circularly polarized gravitational radiation (i.e., to allow for curvature, in a specific form) will be able to induce the quantum anomaly and hence a level crossing of modes. Thus, our result has nothing to do with topological or global questions, but rather to the geometry of the spacetime.[2] Moreover, it seems unlikely that these results could simply be obtained by a wick rotation of an instanton solution, since the former are purely dynamical (gravitational waves), and hence it does not look that they could be recovered from "static" Euclidean solutions by any analytical continuation.

In light of the systematic detections of gravitational waves by the interferometers LIGO-Virgo in the last years [28], and given that gravitational backgrounds emitting these waves are intimately related to chiral anomalies, it is important to discuss astrophysical settings where these quantum effects could play a role in the underlying physics. What is the possible phenomenology that one could predict? This will be studied in detail in a separate paper. The present paper is an extended and a detailed discussion of the theoretical results presented already in [29].

We shall work with four-dimensional spacetimes and the Levi-Civita connection. We follow Wald's [30] sign conventions. Namely, the metric signature is $(-,+,+,+)$, the Riemann tensor is defined by $[\nabla_a,\nabla_b]v_c =: R^d_{abc}v_d$ for any covector field v_d, the Ricci tensor is $R_{ab} := R^c_{acb}$, and the scalar curvature is $R := g^{ab}R_{ab}$. Unless otherwise stated, all tensor fields will be considered smooth. In Sec. II we use units in which $\hbar = c = 1$, while in Sec. III we use units in which $G = c = 1$.

II. A SIMPLE CASE: THE ELECTROMAGNETIC ANALOG

Although we are ultimately interested in understanding the chiral anomaly induced by a gravitational background, the usual technical complications associated with the nonlinearities of the gravitational field and Einstein's equations makes necessary working first with a simpler model. Consequently, let us focus in this section on the original Adler-Bell-Jackiw chiral anomaly [4,5], which is the electromagnetic analog. The above complications are avoided due to the simplicity of Maxwell theory. More importantly, the final result will have such a simple physical interpretation that will guide us in the gravitational case.

A. Setup and main calculation

The Adler-Bell-Jackiw chiral anomaly is the anomalous nonconservation of the Noether current associated with the chiral symmetry of a Dirac spinor $\Psi(x)$. This anomaly arises when the spinor interacts with a background electromagnetic field $F_{ab}(x)$. The explicit expression for this anomaly can be read off from (1) if we identify the gauge group with $U(1)$ and $g^2 \to e^2$. As emphasized in the Introduction, the quantity of major physical interest is the Noether charge, whose failure to be preserved in time is determined by the integral over all spacetime M:

[1]But not all dynamical spacetimes produce a nonvanishing Chern-Pontryagin. One can easily check that Friedmann-Lemaître-Robertson-Walker metrics produce a zero result.

[2]In the context of the Atiyah-Patodi-Singer index theorem, this geometric contribution is essentially the one that must be subtracted from the Chern-Pontryagin integral in order to recover a topological quantity for manifolds with a boundary; see [27].

$$\langle \hat{Q}_5(\text{scri}+)\rangle - \langle \hat{Q}_5(\text{scri}-)\rangle = -\frac{e^2}{8\pi^2}\int_M F\wedge F = -\frac{e^2}{16\pi^2}\int_M d^4x\sqrt{-\eta}F_{ab}{}^*F^{ab}. \tag{6}$$

Our goal is to determine which class of electromagnetic backgrounds (i.e., solutions F_{ab} of Maxwell equations) produce a nonvanishing chiral anomaly. To achieve this we need to analyze this integral and to study under which conditions it is not zero. We assume that this electromagnetic background F_{ab} is produced by some electromagnetic sources J^a that are smooth and with spatial compact support, but otherwise arbitrary.

First of all, it is not difficult to prove that for stationary solutions of Maxwell equations, $dF = 0$, $d{}^*F = {}^*J$, Eq. (6) above is identically zero. To see this, let us take a Cartesian coordinate system $\{t, \vec{x}\}$. A stationary Maxwell field is a solution of Maxwell's equations that remains invariant under time translations, i.e., a solution that satisfies $\mathcal{L}_k F_{ab} = 0$, where $\mathcal{L}_k$ denotes the Lie derivative along the generator of infinitesimal time translations, $k = \partial/\partial t$. This condition is equivalent to $k^a F_{ab} = \nabla_b \Lambda$, where Λ is a function, traditionally called the electrostatic potential. The integral of interest can now be rewritten as

$$\begin{aligned}\int_M d^4x\sqrt{-\eta}F_{ab}{}^*F^{ab} &= -4\int_{-\infty}^{\infty}dt\int_{\mathbb{R}^3}d^3\vec{x}k^a F_{ab}k_c{}^*F^{cb}\\ &= -4\int_{-\infty}^{\infty}dt\int_{\mathbb{R}^3}d^3\vec{x}\nabla_b\Lambda k_c{}^*F^{cb}\\ &= -4\int_{-\infty}^{\infty}dt\int_{\mathbb{R}^3}d^3\vec{x}\nabla_b(\Lambda k_c{}^*F^{cb}),\end{aligned} \tag{7}$$

where in the first equality we used the $3+1$ decomposition of the metric, $\eta^{ab} = -k^a k^b + h^{ab}$, and ${}^*F_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd}$ to write $h^{ab}h^{cd}F_{ac}{}^*F_{bd} = -2k^a F_{ab}k_c{}^*F^{cb}$; while in the third equality we used Maxwell equation $\nabla_a{}^*F^{ab} = 0$. Assuming standard falloff conditions for the magnetic field and electrostatic potential at spatial infinity, $k^a{}^*F_{ab}\nabla^b r \sim 1/r^3$, $\Lambda \sim 1/r$, the final result is zero.

We must look then for nonstationary solutions to Maxwell's equations. To guarantee convergence for the integral in time in Eq. (6), we shall assume that both at early and late times the solution of Maxwell equations approaches a stationary configuration.[3] During the nonstationarity period, the dynamics of the electromagnetic sources will generate outgoing radiation propagating to infinity (we will assume no incoming electromagnetic radiation for simplicity). The study of outgoing radiation is most conveniently carried out within the framework of asymptotically Minkowski spacetimes [31–33]. A detailed summary of this topic can be found in Appendix B, and we will provide the key points here. Let $(\mathbb{R}^4, \hat{\eta}_{ab})$ denote our physical, Minkowski spacetime, and let (M, η_{ab}) be the unphysical spacetime constructed from the physical one by a standard conformal compactification.[4] The unphysical metric is related to the physical one by an ordinary conformal transformation: $\eta_{ab} = \Omega^2(x)\hat{\eta}_{ab}$. On the other hand, the unphysical manifold is just the physical one together with additional points attached smoothly to it: $M = \mathbb{R}^4 \cup \mathcal{J}$. The set of all these new points constitute a null hypersurface $\mathcal{J}$, locally characterized by the condition $\Omega = 0$, and with null normal $\eta^{ab}\nabla_b\Omega$. Physically, they represent the "points of (null) infinity," i.e., the points that can be asymptotically reached in the original spacetime by following outgoing, null geodesics.

The importance of this construction is that it allows one to apply ordinary techniques in differential geometry to study the behavior of fields in a neighborhood of infinity (which now is just a boundary of the spacetime manifold). To do the calculation of interest, Eq. (6), one further needs to carry the tensors of the original spacetime to the unphysical one. This is straightforward due to the invariance of the electromagnetic field under conformal transformations, $\hat{F}_{ab} = F_{ab}$, $\hat{A}_a = A_a$. Thus,

$$-\frac{1}{2}\int_{\mathbb{R}^4} d^4x\sqrt{-\hat{\eta}}\hat{F}_{ab}{}^*\hat{F}^{ab} = \int_{\mathbb{R}^4}\hat{F}\wedge\hat{F} = \int_M F\wedge F = -\frac{1}{2}\int_M d^4x\sqrt{-\eta}F_{ab}{}^*F^{ab}. \tag{8}$$

The key point now is to notice that, mathematically, $p_1(F) = -\frac{1}{8\pi^2}F\wedge F$ is an invariant polynomial [6]. The Chern-Weil theorem from the theory of characteristic classes (see Theorem 11.1 in [6], for instance) states that the difference between two invariant polynomials, $p_1(F) - p_1(F')$, associated with two connection 1-forms, A and A', is exact and determined by the transgression term $Q(A, A')$ [6,27]:

$$p_1(F) - p_1(F') = dQ(A, A'). \tag{9}$$

Because the spacetime $(\mathbb{R}^4, \hat{\eta}_{ab})$ is trivial from the topological viewpoint, it admits a global flat connection $\hat{A}'$. Due to conformal invariance and continuity, we have an electromagnetic potential that is pure gauge globally:

[3] At early times the electromagnetic field is stationary for all $\vec{x} \in \mathbb{R}^3$ and using the same arguments as above we get $\int_{\mathbb{R}^3} d^3\vec{x}\nabla_b(\Lambda k_c{}^*F^{cb}) = 0$. At late times t the field is stationary only in a spacelike open region $U(t) \subset \mathbb{R}^3$ of radius $r(t)$ that does not intersect the electromagnetic waves generated during the intermediate nonstationary period. Because the waves propagate to future infinity, $r(t) = t + \text{const}$, then $U(t\to\infty)\to\mathbb{R}^3$, and we find $\int_{U(t)} d^3\vec{x}\nabla_b(\Lambda k_c{}^*F^{cb}) = \int d\mathbb{S}^2 r(t)^2\Lambda(r(t))k_c{}^*F^{cb}(r(t))\nabla_b r \sim t^{-2}$ as $t\to\infty$, which guarantees convergence of the integral.

[4] Because we shall be working with the unphysical spacetime all the time, we use the hat symbol to denote quantities associated with the physical spacetime in order to avoid its use later.

$A' = d\alpha$. Then, the difference $\theta = A - A'$ represents the same physical electromagnetic potential.[5] The transgression term can be evaluated following its definition (see [6,27] for details) and reads

$$Q(A, A') = -\frac{1}{8\pi^2}\theta \wedge F. \tag{10}$$

It is straightforward now to check that $p_1(F') = p_1(0) = 0$. Then, by integrating (9) and applying Stokes theorem we get

$$\int_M p_1(F) = \int_{\mathcal{J}} i^* Q(A, A') = -\frac{1}{16\pi^2}\int_{\mathcal{J}} \theta_a F_{bc}\epsilon^{abc} du d\mathbb{S}^2, \tag{11}$$

where the boundary of the unphysical manifold is $\partial M = \mathcal{J}$; i^* denotes the pullback of the inclusion map $i : \mathcal{J} \hookrightarrow M$; and $du d\mathbb{S}^2$ is the canonical integration measure on $\mathcal{J} \approx \mathbb{R} \times \mathbb{S}^2$. Note that the RHS is manifestly gauge invariant, as it must be in view of the left-hand side (LHS).

This result can be further simplified if we work in a Newman-Penrose basis [34]. The electromagnetic field has 6 physical degrees of freedom per spacetime point that can be described with 3 complex scalars (these are analogous to the Weyl scalars in the gravitational case [35]). These scalars are the components of the tensor F_{ab} in a null tetrad $\{\ell^a, n^a, m^a, \bar{m}^a\}$. Without loss of generality, we can take n^a such that it equals $\eta^{ab}\nabla_b\Omega$ at $\Omega = 0$, i.e., such that it is normal to the hypersurface $\mathcal{J}$. Then ℓ^a is chosen as a null vector that satisfies $\ell^a n_a = -1$; and m^a, $\bar{m}^a$ are complex conjugate null vector fields, taken such that their real and imaginary parts are tangential to 2-spheres (hence orthogonal to n^a and ℓ^a), and normalized as $m^a\bar{m}_a = 1$. In this null tetrad, the metric takes the form $\eta_{ab} = -2n_{(a}\ell_{b)} + 2m_{(a}\bar{m}_{b)}$ and the three electromagnetic scalars are defined by

$$\Phi_2 = F_{ab}n^a\bar{m}^b, \tag{12}$$

$$\Phi_1 = \frac{1}{2}[F_{ab}n^a l^b + F_{ab}m^a\bar{m}^b], \tag{13}$$

$$\Phi_0 = F_{ab}m^a l^b. \tag{14}$$

If we restrict to smooth solutions of Maxwell equations, the peeling theorem [36] guarantees that in a neighborhood of $\mathcal{J}$ we can expand $\Phi_i(u, \Omega, \theta, \phi) = \Phi_i^0(u, 0, \theta, \phi) + \Omega\Phi_i^1(u, 0, \theta, \phi) + \cdots$, where (u, θ, ϕ) are Bondi-Sachs coordinates adapted to $\mathcal{J}$ [37,38]. Going back to the physical spacetime, it is not difficult to see that this condition requires $\Phi_2 \sim \frac{1}{r}$, so Φ_2^0 represents the 2 radiative degrees of freedom of the electromagnetic field (corresponding to real and imaginary parts of this complex number). If we further assume the same asymptotic behavior for the electromagnetic potential [39,40], $A_a \sim O(1/r)$, then the 2 radiative degrees of freedom are encoded in the component $A_2 \coloneqq A_a\bar{m}^a$. Indeed, using $F_{ab} = 2\nabla_{[a}A_{b]}$ and the above definitions for the scalars, one can see that $A_a n^a = 0$ and $\phi_2^0 = \partial_u A_2^0$ at $\mathcal{J}$.

We are now in position to evaluate (11). The tangent space of $\mathcal{J}$ is spanned by $\{n^a, m^a, \bar{m}^a\}$, so $\epsilon^{abc} = i3!n^{[a}m^b\bar{m}^{c]}$. Then,

$$\begin{aligned}\int_M p_1(F) &= \frac{-6i}{16\pi^2}\int_{\mathcal{J}} \theta_a F_{bc}(n^{[a}m^b\bar{m}^{c]})du d\mathbb{S}^2 \\ &= \frac{-2i}{16\pi^2}\int_{\mathcal{J}} \theta_a F_{bc}(n^a m^{[b}\bar{m}^{c]} + m^a\bar{m}^{[b}n^{c]} \\ &\quad + \bar{m}^a n^{[b}m^{c]})du d\mathbb{S}^2 \\ &= \frac{1}{4\pi^2}\int_{\mathcal{J}} du d\mathbb{S}^2(\theta_a n^a \mathrm{Im}\Phi_1^0 - \mathrm{Im}(A_2^0\bar{\Phi}_2^0 - \bar{\delta}\alpha\bar{\Phi}_2^0)),\end{aligned} \tag{15}$$

where we used $A'^0_2 = \bar{\delta}\alpha$. Recalling that $A_a n^a = 0$ at $\mathcal{J}$, we get

$$\int_M p_1(F) = -\frac{1}{4\pi^2}\int_{\mathcal{J}} du d\mathbb{S}^2 \mathrm{Im}(A_2^0\bar{\Phi}_2^0 - \bar{\delta}\alpha\bar{\Phi}_2^0). \tag{16}$$

Notice that the value of α is determined by the choice of gauge of A_2^0. In other words, under a gauge transformation A_2^0 transforms as $A_2 \to A_2^0 + \bar{\delta}\beta$, while α transforms as $\alpha \to \alpha + \beta$, so the role of α is to maintain gauge invariance in the full expression. In a specific gauge, α can be set to zero.

B. Physical interpretation: Circularly polarized electromagnetic waves

Since A_2^0 (or Φ_2^0) is a complex number that encodes the 2 radiative degrees of freedom of the electromagnetic field, we see that the chiral anomaly for fermions is intrinsically related to the emission of electromagnetic waves. Which properties should these waves have in order to produce a nontrivial result? To understand the physical meaning of this result, let us expand the electromagnetic field in Fourier modes as[6]

[5]The advantage of working with θ rather than with A directly is that the former is manifestly gauge invariant, while A is not. It is customary in the literature to set $A' = 0$, but this can be misleading during the calculation given that intermediate formulas would not have a manifestly gauge-invariant form.

[6]The condition $\Phi_2^0(u \to \pm\infty) \to 0$ is required from the finiteness of energy flux across $\mathcal{J}$, $\int_{-\infty}^{+\infty} du|\Phi(u, \theta, \phi)|^2 < \infty$. This condition in turn requires that Φ_2^0 belongs to $L^2(\mathbb{C})$, and its Fourier transform exists. Notice that A_2^0 does not necessarily decay at $u \to \pm\infty$, so its Fourier transform is not defined.

$$\Phi_2^0(u,\theta,\phi) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \Phi(\omega,\theta,\phi) e^{-i\omega u} = \int_0^{\infty} \frac{d\omega}{2\pi} [\Phi_R(\omega,\theta,\phi) e^{-i\omega u} + \bar{\Phi}_L(\omega,\theta,\phi) e^{i\omega u}], \tag{17}$$

where in the second equality we explicitly split the modes in terms of positive and negative definite frequencies: $\Phi_R(\omega) \coloneqq \Phi(\omega)$ for $\omega > 0$ while $\bar{\Phi}_L(-\omega) \coloneqq \Phi(\omega)$ for $\omega < 0$. The electromagnetic potential satisfies $\dot{A}_2 = \Phi_2^0$, so we can write

$$A_2(u,\theta,\phi) = \int_0^{\infty} \frac{d\omega}{2\pi} \left[\frac{\Phi_R(\omega,\theta,\phi)}{-i\omega} e^{-i\omega u} + \frac{\bar{\Phi}_L(\omega,\theta,\phi)}{i\omega} e^{i\omega u} \right] + \beta(\theta,\phi), \tag{18}$$

where the function $\beta(\theta,\phi)$ emerges as a constant of integration. Imposing $\Phi_2 = 0$, one concludes that $\beta = \bar{\eth}\alpha$. Plugging these formulas in (16), we get

$$\int d^4x \sqrt{-g} \langle \nabla_a j_5^a \rangle = \frac{e^2}{4\pi^2} \int d\mathbb{S}^2 \int_0^{\infty} \frac{d\omega}{2\pi\omega} (|\Phi_R(\omega,\theta,\phi)|^2 - |\Phi_L(\omega,\theta,\phi)|^2). \tag{19}$$

Note that this formula is reminiscent of the phenomenon of level crossing, discussed in the Introduction. What is the physical meaning of these modes, Φ_R and Φ_L? The electromagnetic field Φ_2^0 is self-dual, which means that it can be written as $\Phi_2^0 = (E + iB)$, where E, B are the electric and magnetic fields, representing the two possible, linearly independent polarization directions of the electromagnetic field. Because of this, we can also write

$$\Phi_2^0(u,\theta,\phi) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (E(\omega,\theta,\phi) + iB(\omega,\theta,\phi)) e^{-i\omega u}, \tag{20}$$

from which we identify

$$\begin{aligned} \Phi_R(\omega) &= (E(\omega) + iB(\omega)), \omega > 0, \\ \bar{\Phi}_L(-\omega) &= (E(\omega) + iB(\omega)), \omega < 0. \end{aligned} \tag{21}$$

The second equation implies $\Phi_L(\omega) = (\bar{E}(-\omega) - iB(-\omega))$, for $\omega > 0$. Because $E(u)$ and $B(u)$ are real functions, we must have then $\bar{E}(-\omega) = E(\omega)$ and $\bar{B}(-\omega) = B(\omega)$, leading to $\Phi_L(\omega) = (E(\omega) - iB(\omega))$. Taking into account all this, and expanding the fields in spin-weight spherical harmonics of modes (ℓ, m), we finally arrive at

$$\boxed{\langle \hat{Q}_5(\text{scri}+) \rangle - \langle \hat{Q}_5(\text{scri}-) \rangle = \int_0^{\infty} \frac{d\omega e^2}{8\pi^3\omega} \sum_{\ell m} [|E^{\ell m}(\omega) + iB^{\ell m}(\omega)|^2 - |E^{\ell m}(\omega) - iB^{\ell m}(\omega)|^2].}$$

The RHS represents the difference in intensity between right- and left-handed circularly polarized electromagnetic waves reaching future null infinity, i.e., the Stokes V parameter. The LHS represents the amount of net helicity spontaneously created on the fermion field. We conclude that the emission of circularly polarized electromagnetic radiation implies the spontaneous creation of massless charged fermions (highly energetic electrons, for instance) with net helicity. The more right- or left-handed electromagnetic radiation the spacetime contains, the more left- or right-handed massless fermions will be excited from the quantum vacuum.

C. A concrete example: Electric-magnetic oscillating dipole

Consider an electric dipole of moment p_0 pointing in the z direction, oscillating with frequency ω. The electromagnetic vector potential is [41]

$$\vec{A}_a^E = -\frac{p_0\omega}{4\pi e^2 r} \cos(\omega u) \nabla_a z. \tag{22}$$

On top of this, consider a magnetic dipole of moment m_0 located in the $x - y$ plane, oscillating with the same frequency ω but in opposite phase. In the radiation-zone approximation, the electromagnetic vector potential yields [41]

$$\vec{A}_a^M = -\frac{m_0\omega}{4\pi e^2} \sin(\omega u) \sin^2\theta \nabla_a \phi. \tag{23}$$

One expects this configuration to provide a nontrivial Chern-Pontryagin because the magnetic fields of both the electric and magnetic oscillating dipoles considered here are entangled, leading to a nonzero magnetic helicity [42].

To calculate (16), let us work in spherical coordinates (u, r, θ, ϕ). A (conformal) Newman-Penrose basis can be constructed such that $m^a = \frac{q^{ab}}{\sqrt{2}}(\nabla_b\theta + i\sin\theta\nabla_a\phi)$, where $q_{ab} = \mathrm{diag}\{1, \sin^2\theta\}$ is the standard metric on the unit, homogeneous 2-sphere. The radiative component of the total electromagnetic potential is

$$A_2^0 = \bar{m}^a\vec{A}_a = \frac{\omega\sin\theta}{4\pi e^2\sqrt{2}}(im_0\sin(\omega u) + p_0\cos(\omega u)). \quad (24)$$

Then,

$$\int_{\mathcal{J}^+} du d\mathbb{S}^2 \mathrm{Im}(A_2^0\partial_u\bar{A}_2^0) = -\frac{\omega^3 m_0 p_0}{16\pi e^4}(u_2 - u_1), \quad (25)$$

where $u_2 - u_1$ is the period of (finite) time during which the system operates (we let $\omega \to 0$ for $u < u_1$ and $u > u_2$). This result is gauge invariant, so $\alpha = 0$ in (16).

III. THE GRAVITATIONAL CASE

The fact that net fermion helicity can be spontaneously created from the quantum vacuum in a background of circularly polarized electromagnetic waves is a physically interesting result. In particular, it suggests that a similar phenomenon may occur for photon helicity in a background of gravitational waves, by simply noticing the parallelism with the chiral electromagnetic anomaly of (3). In this section we prove this in detail, following a similar strategy as in the electromagnetic case.

A. Setup and main calculation

Let $(\hat{M}, \hat{g}_{ab})$ denote our physical, curved spacetime. The quantity of interest is

$$\langle\hat{Q}_5(\mathrm{scri}+)\rangle - \langle\hat{Q}_5(\mathrm{scri}-)\rangle = \frac{\hbar}{48\pi^2}\int_{\hat{M}}\mathrm{Tr}(\hat{R}\wedge\hat{R}) = \frac{-\hbar}{96\pi^2}\int_{\hat{M}} d^4x\sqrt{-g}\hat{R}_{abcd}{}^*\hat{R}^{abcd}, \quad (26)$$

where $\hat{R} = \frac{1}{2}\hat{R}_{ab}dx^a\wedge dx^b$ is the curvature 2-form. Our goal is to compute this integral in an astrophysically relevant setting in order to know under which circumstances a given gravitational system may generate a flux of photons with net helicity. In particular, we restrict to asymptotically flat spacetimes. As in the previous section, the above integral is identically zero for stationary spacetimes. The proof is similar to the electromagnetic case but technically more tedious, so it is relegated to Appendix A.

We must then focus on dynamical solutions of Einstein's equations and, to guarantee convergence, we consider spacetimes that asymptotically reach stationary regimes at both future and past timelike infinities. An example of this is a binary merger of two black holes which, ideally, are initially separated an infinite distance away, and end up merging to form a final stationary Kerr black hole.

As in the electromagnetic case, it is convenient to work instead with a conformally compactified spacetime, (M, g_{ab}), constructed from the physical one by the standard procedure: $M = \hat{M}\cup\mathcal{J}$, $g_{ab} = \Omega^2\hat{g}_{ab}$. Our physical spacetime will be globally hyperbolic, so that $\hat{M}\simeq\mathbb{R}\times\Sigma$ [30], and in particular we shall restrict to $\Sigma\simeq\mathbb{R}^3$ (physically one does not expect more sophisticated spaces). The next step is to carry the relevant tensors of the physical spacetime to the unphysical one. It is useful to note that, due to the totally antisymmetric tensor $\hat{\epsilon}^{abmn}$, the Ricci part in the physical Riemann tensor, $\hat{R}^d_{abc}$, does not contribute in this problem:

$$\sqrt{\hat{g}}\hat{R}^d_{abc}\hat{\epsilon}^{abmn}\hat{R}^c_{mnd} = \sqrt{\hat{g}}\hat{C}^d_{abc}\hat{\epsilon}^{abmn}\hat{C}^c_{mnd}, \quad (27)$$

where $\hat{C}_{abcd}$ is the physical Weyl tensor. Using now the conformal invariance of the Weyl tensor, $\hat{C}^d_{abc} = C^d_{abc}$, that $\sqrt{-\hat{g}} = \Omega^{-4}\sqrt{-g}$, and that $\hat{\epsilon}^{abcd} = \Omega^4\epsilon^{abcd}$, then the quantity of interest turns out to be conformal invariant:

$$\sqrt{\hat{g}}\hat{R}_{abc}{}^d\hat{\epsilon}^{abmn}\hat{R}_{mnd}{}^c = \sqrt{g}R_{abc}{}^d\epsilon^{abmn}R_{mnd}{}^c, \quad (28)$$

and therefore

$$\langle\hat{Q}_5(\mathrm{scri}+)\rangle - \langle\hat{Q}_5(\mathrm{scri}-)\rangle = \frac{\hbar}{48\pi^2}\int_{\hat{M}}\mathrm{Tr}(\hat{R}\wedge\hat{R}) = \frac{\hbar}{48\pi^2}\int_M\mathrm{Tr}(R\wedge R). \quad (29)$$

Mathematically, $p_1(R) = -\frac{1}{8\pi^2}\mathrm{Tr}(R\wedge R)$ is another invariant polynomial. To calculate this quantity we shall recall again the Chern-Weil theorem. This theorem tells us that the difference between two invariant polynomials, $p_1(R) - p_1(R')$, associated with any two given connection 1-forms, ω and ω' in M, is exact and is determined by the transgression term $Q(\omega,\omega')$ [6]:

$$p_1(R) - p_1(R') = dQ(\omega,\omega'). \quad (30)$$

If we introduce the difference $\theta = \omega - \omega'$, then the RHS can be evaluated in the standard way and yields [27]

$$Q(\omega,\omega') = -\frac{1}{8\pi^2}\mathrm{Tr}\left(2\theta\wedge R + \frac{2}{3}\theta\wedge\theta\wedge\theta - 2\theta\wedge\omega\wedge\theta - \theta\wedge d\theta\right). \quad (31)$$

Because the physical spacetime manifold is $\hat{M}\simeq\mathbb{R}^4$, it admits a flat Minkowskian metric $\hat{\eta}_{ab}$. Its associated

conformal compactification, $(\mathbb{R}^4 \cup \mathcal{J}^+, \eta_{ab})$, with $\eta_{ab} = \Omega^2 \hat{\eta}_{ab}$, will represent our auxiliary spacetime in this calculation. We denote all associated quantities with prime indices. Given that $\hat{C}^d_{abc} = 0$ in $(\mathbb{R}^4, \hat{\eta}_{ab})$ one has $C'^d_{abc} = 0$ at any point of $\mathbb{R}^4 \cup \mathcal{J}^+$ due to conformal invariance and continuity, and from (27) we deduce $p_1(R') = p_1(C') = p_1(0) = 0$. Thus, the value of $\int_M p_1(R)$ is simply determined by the flux $Q(\omega, \omega')$ at future null infinity (i.e., at $\Omega = 0$):

$$\int_M p_1(R) = \int_{\mathcal{J}^+} i^* Q(\omega, \omega') = -\frac{1}{8\pi^2} \int_{\mathcal{J}^+} i^* \mathrm{Tr}\left(2\theta \wedge R + \frac{2}{3}\theta \wedge \theta \wedge \theta - 2\theta \wedge \omega \wedge \theta - \theta \wedge d\theta\right). \tag{32}$$

The previous formula can be simplified in a more convenient manner. First, note that $\theta \wedge d\theta = \theta \wedge d\omega - \theta \wedge d\omega' = \theta \wedge R - \theta \wedge \omega \wedge \omega + \theta \wedge \omega' \wedge \omega'$ (recall $R' = d\omega' + \omega' \wedge \omega' = 0$ for Minkowski). Then

$$\begin{aligned} \mathrm{Tr}(-2\theta \wedge \omega \wedge \theta - \theta \wedge d\theta) &= \mathrm{Tr}\theta \wedge (-2\omega \wedge \theta + \omega \wedge \omega - \omega' \wedge \omega') - \mathrm{Tr}\theta \wedge R \\ &= -\mathrm{Tr}\theta \wedge (\omega \wedge \omega - 2\omega \wedge \omega' + \omega' \wedge \omega') - \mathrm{Tr}\theta \wedge R \\ &= -\mathrm{Tr}\theta \wedge (\omega - \omega') \wedge (\omega - \omega') - \mathrm{Tr}\theta \wedge R, \end{aligned} \tag{33}$$

where in the last step we noticed that $\mathrm{Tr}\theta \wedge \omega \wedge \omega' = \mathrm{Tr}\theta \wedge \omega' \wedge \omega$. Equation (32) can now be written as

$$\int_M p_1(R) = \int_{\mathcal{J}^+} i^* Q(\omega, \omega') = -\frac{1}{8\pi^2} \int_{\mathcal{J}^+} i^* \mathrm{Tr}\left(\theta \wedge R - \frac{1}{3}\theta \wedge \theta \wedge \theta\right). \tag{34}$$

It is convenient to introduce a $3+1$ splitting of (M, g_{ab}) by $\{\Omega = \mathrm{const}\}$ hypersurfaces in order to simplify the integrand. Let $\hat{n}_a = \frac{1}{\sqrt{g^{ab} n_a n_b}} n_a$, with $n_a := \nabla_a \Omega$, be the normalized transversal vector to the $\Omega = \mathrm{const}$ hypersurfaces. The induced metric on these hypersurfaces is $h_{ab} = -\hat{n}_a \hat{n}_b + g_{ab}$, and its associated Levi-Civita derivative operator will be denoted by D_a. For any two vectors u^a and v^a that are tangent to $\{\Omega = \mathrm{const}\}$ we can write the decomposition:

$$u^a D_a v^b = u^a h^b_c \nabla_a v^c = u^a (g^b_c - \hat{n}^b \hat{n}_c) \nabla_a v^c = u^a \nabla_a v^b + u^a (\nabla_a \hat{n}_c) \hat{n}^b v^c, \tag{35}$$

where in the last equality $v^a \hat{n}_a = 0$ was used. This leads to $D_a v_b = h^c_a \nabla_c v_b + \hat{n}_b v^c K_{ac}$, where $K_{ac} = D_a \hat{n}_c$ is the extrinsic curvature of $\{\Omega = \mathrm{const}\}$ as a hypersurface embedded in M (as usual, it satisfies $K_{ac} = K_{ca}$ and $K_{ab}\hat{n}^b = 0$, as can easily be checked). Consider now an orthonormal frame $\{e^a_I\}_{I=0,\ldots,3}$ in (M, g_{ab}); i.e., a set of four vectors labeled by I that at each point x of M satisfy $g_{ab}(x) e^a_I(x) e^b_J(x) = \eta_{IJ}$. The dual frame is defined via $e_{a,I} = g_{ab} e^b_I$, and latin indices $I, J, \ldots$ can be lowered and raised with η_{IJ}. It is convenient to choose this frame as a Newman-Penrose tetrad ($\eta_{01} = \eta_{10} = -\eta_{23} = -\eta_{32} = -1$, zero otherwise) such that for $\Omega = 0$ the tangent space at future null infinity is spanned by $\{n^a, m^a, \bar{m}^a\}$. Given this tetrad, a (torsion-free) connection 1-form ω_a is defined by the equation $\nabla_a e^I_b + \omega^I_{aJ} e^J_b = 0$, and the metric-compatibility condition $\nabla_a g_{bc} = 0$ gives the antisymmetry property $\omega^{IJ}_a = -\omega^{JI}_a$. Taking $v_b = e^I_b$ in (35) we find

$$h^d_a (\omega_d)_{IJ} = -e^b_J D_a e_{bI} + K_{aI} \hat{n}_b e^b_J, \tag{36}$$

where K_{aJ} is shorthand for $K_{ac} e^c_J$. Using the antisymmetry of ω^{IJ}_a between I and J, one can deduce $h^d_a (\omega_d)^I_J = -\delta^I_K (e^b_J D_a e^K_b) - K_{aJ}\hat{n}^I + K^I_a \hat{n}_J$, where δ^I_J and $\hat{n}^I$ are shorthand for $h_{ab} e^{aI} e^b_J$ and $\hat{n}^a e^I_a$. Let us introduce the additional notation $({}^3\omega_a)^I_J \equiv -\delta^I_K (e^b_J D_a e^K_b)$. Thus

$$(\omega_a)^I_J = ({}^3\omega_a)^I_J - K_{aJ}\hat{n}^I + K^I_a \hat{n}_J + \hat{n}_a \hat{n}^b (\omega_b)^I_J. \tag{37}$$

Repeating this procedure exactly in the auxiliary Minkowskian spacetime $(\mathbb{R}^4 \cup \mathcal{J}^+, \eta_{ab})$ we get $(\omega'_a)^I_J = ({}^3\omega'_a)^I_J - K'_{aJ}\hat{n}'^I + K'^I{}_a \hat{n}'_J + \hat{n}'_a \hat{n}'^b (\omega'_b)^I_J$, where prime indices denote quantities defined with respect to the metric η_{ab}. Taking the difference between the two (note that $h^a_b \hat{n}_a = h^a_b \hat{n}'_a = 0$),

$$h^d_a (\theta_d)^{IJ} = ({}^3\theta_a)^{IJ} - 2(K^{[J}_a \hat{n}^{I]} - K'^{[J}_a \hat{n}'^{I]}). \tag{38}$$

As discussed in [31,32] and summarized in Appendix B, $\mathcal{J} = \{\Omega = 0\}$ is a three-dimensional null hypersurface that is endowed with a universal geometric structure, which

consists in a collection of pairs $(\underline{g}_{ab}, \underline{n}^a)$ satisfying a set of properties. Each pair consists of a degenerate metric $\underline{g}_{ab}$ on $\{\Omega = 0\}$ and the corresponding null normal $\underline{n}^a$. This geometric structure is common and available for any asymptotically flat spacetime. Consequently, we can fix the same conformal frame $(\underline{g}_{ab}, \underline{n}^a)$ for both $(\mathbb{R}^4 \cup \mathcal{J}^+, g_{ab})$ and $(\mathbb{R}^4 \cup \mathcal{J}^+, \eta_{ab})$ spacetimes. Furthermore, without loss of generality we can choose this conformal frame such that $i^*(\Omega^{-2} g_{ab} n^a n^b) = 1$ (i^* being the pullback of $i : \mathcal{J} \hookrightarrow M$), which will allow some simplifications in the next calculation.

On the other hand, because the specification of the degenerate metric $\underline{g}_{ab}$ is equivalent to the specification of two complex-conjugate vectors m^a and $\bar{m}^a$ whose real and imaginary parts are tangential to the sphere, fixing this conformal frame $(\underline{g}_{ab}, \underline{n}^a)$ is equivalent to fixing a common basis $\{n^a, m^a, \bar{m}^a\}$ for both spacetimes. Consequently, the two tetrads introduced above, e^a_I and e'^a_I, agree for $\Omega = 0$. Taking this into account and the orthogonality properties, $({}^3\omega_a)^I_J \hat{n}^b e^J_b = ({}^3\omega'_a)^I_J \hat{n'}^b e'^J_b = K_{aJ} \hat{n}^b e^J_b = K'_{aJ} \hat{n'}^b e'^J_b = 0$, one finds

$$i^* \mathrm{Tr}\theta \wedge \theta \wedge \theta = i^*[({}^3\theta_a)^I_J ({}^3\theta_b)^J_K ({}^3\theta_c)^K_I - 3({}^3\theta_a)^{IJ}(K_{bI} - K'_{bI})(K_{cJ} - K'_{cJ})]\epsilon^{abc}\sqrt{h} d^3x \tag{39}$$

and

$$i^* \mathrm{Tr}\theta \wedge R = \frac{1}{2} i^*[({}^3\theta_a)^I_J R^J_{bcJ} - 2(K_{aI}\hat{n}^J - K'_{aI}\hat{n'}^J) R^I_{bcJ}]\epsilon^{abc}\sqrt{h} d^3x. \tag{40}$$

The 1-form $({}^3\theta_a)^I_J$ can be determined at $\Omega = 0$ from the intrinsic geometry of future null infinity. As discussed in Appendix B, for any covector e^I_b at null infinity, the difference $D'_a - D_a$ between two (equivalence classes of) connections is completely characterized by a traceless, symmetric tensor σ_{ab}: $(D_a - D'_a)e^I_b = -\sigma_{ab} n^c e^I_c$. Since, as discussed above, at $\Omega = 0$ the tetrad e^I_a is equal to the tetrad e'^I_a, and D_a is tangential to $\Omega = 0$, we have $({}^3\theta_a)^I_J = -\delta^I_K e^b_J (D_a - D'_a) e^K_b$ at $\Omega = 0$, and hence $({}^3\theta_a)^I_J e^d_I e^J_e = -e^d_I \delta^I_K (D_a - D'_a) e^K_e = -e^d_I \delta^I_K \sigma_{ae} n^g e^K_g$ at $\Omega = 0$. Now $e^d_I \delta^I_K n^g e^K_g = e^d_I (\eta^I_K - \hat{n}^I \hat{n}_K) n^g e^K_g = n^d - \hat{n}_g n^g \hat{n}^d = n^d - \frac{n^g \nabla_g \Omega g^{db} \nabla_b \Omega}{g^{ef} \nabla_e \Omega \nabla_f \Omega}$, and since at $\Omega = 0$ we have $n^a = g^{ab}\nabla_b \Omega$, we conclude

$$i^*[({}^3\theta_a)^I_J e^d_I e^J_e] = 0, \tag{41}$$

and we are led to

$$\int_M p_1(R) = \frac{1}{8\pi^2} \int_{\mathcal{J}^+} i^*[(K_{ad} - K'_{ad}) \hat{n}^e R_{bc}{}^d{}_e \epsilon^{abc} \sqrt{h} d^3x]. \tag{42}$$

We obtain now a compact expression for the extrinsic curvature of $\mathcal{J}^+$ as a hypersurface of M. First note that

$$i^*(K_{ab} - K'_{ab}) = i^*(D_a \hat{n}_b - D'_a \hat{n'}_b) = i^*[(D_a \alpha) n_b + \alpha D_a n_b - (D'_a \alpha') n_b - \alpha' D'_a n_b] = i^*(\Omega^{-1}(D_a n_b - D'_a n_b)), \tag{43}$$

where we denoted $\alpha \equiv \frac{1}{\sqrt{g^{ab}\nabla_a \Omega \nabla_b \Omega}}$ for brevity and in the last step we took into account that $i^*(\Omega\alpha) = i^*(\Omega\alpha') = 1$. The term inside parentheses vanishes as $O(\Omega)$ at null infinity but the prefactor diverges as Ω^{-1}, so the product is a well-defined, smooth quantity at null infinity. To calculate its value let us use Eq. (B4) from Appendix B:

$$\Omega S_{ab} + 2\nabla_a n_b - \Omega^{-1} n^c n_c g_{ab} = O(\Omega^3). \tag{44}$$

The pullback of $i : \mathcal{J}^+ \to M$ on this expression provides us with the value of $\Omega^{-1}\nabla_a n_b$ at future null infinity,

$$i^*(\Omega^{-1}\nabla_a n_b) = -\frac{1}{2}\underline{S}_{ab} + \frac{1}{2}\underline{g}_{ab} i^*(\Omega^{-2} n^a n_a) = -\frac{1}{2}\underline{S}_{ab} + \frac{1}{2}\underline{g}_{ab}. \tag{45}$$

Repeating the same with the auxiliary Minkowski space, one gets [recall that we fixed the same conformal frame $(\underline{g}_{ab}, \underline{n}^c)$ for both spacetimes]:

$$i^*(\Omega^{-1}\nabla_a n_b) - i^*(\Omega^{-1}\nabla'_a n_b) = i^*(\Omega^{-1}(D_a n_b - D'_a n_b)) = -\frac{1}{2}(\underline{S}_{ab} - S'_{ab}). \tag{46}$$

But for Minkowski, $S'_{ab} = \rho_{ab}$, where ρ_{ab} is the "gauge" part of S_{ab}. The combination $S_{ab} - \rho_{ab} =: N_{ab}$ is manifestly invariant under conformal gauge transformations of the form $\Omega \to \omega\Omega$, and defines what is known as the Bondi News tensor, N_{ab} (see Appendix B). This is the quantity that determines whether an asymptotically flat spacetime contains nontrivial gravitational radiation. We find

$$i^*(K_{ab} - K'_{ab}) = -\frac{1}{2}N_{ab}. \tag{47}$$

On the other hand,

$$i^*(\hat{n}^e R_{bc}{}^d{}_e \epsilon^{abc}\sqrt{h}) = i^*(n^e\Omega^{-1}(C_{bce}{}^d + g_{e[b}S^d_{c]} - \delta^d_{[b}S_{c]e})\epsilon^{abc}\sqrt{h}). \tag{48}$$

Because $\hat{n}_a\epsilon^{abc} = 0$, the second term does not contribute. On the other hand, the third term can be written as $i^*(\hat{n}^e S^n_e g_{cn}\epsilon^{adc}) = i^*(g_{cn}\epsilon^{adc}\Omega^{-1})i^*(n^e S^n_e) \propto i^*(g_{cb}\epsilon^{adc}\Omega^{-1})i^*(n^b) = i^*(\epsilon^{adc})i^*(\Omega^{-1}n^b g_{bc}) = 0$, where we used $S^a_b n^b \propto n^a$ and $\underline{n}^a i^*(\Omega^{-2}n_a) = 1$ in the last step. Thus,

$$\begin{aligned} i^*(\hat{n}^e R_{bc}{}^d{}_e \epsilon^{abc}\sqrt{h}) &= i^*(n^e\Omega^{-1}C_{bce}{}^d\epsilon^{abc}\sqrt{h}) \\ &= -i^*(n_e\Omega^{-1}C^{**\,ed}_{bc}\epsilon^{abc}\sqrt{h}) \\ &= \frac{1}{2}i^*(\epsilon^{dpqe}n_e\Omega^{-1}C^*_{bcpq}\epsilon^{abc}\sqrt{h}), \end{aligned}$$

where $*$ denotes the Hodge dual. Note that $\epsilon^{abc} = \epsilon^{abcd}\hat{n}_d = \Omega^{-1}\epsilon^{abcd}n_d$ but $\sqrt{h} = \Omega\sqrt{q}$, where q_{ab} is the metric of the two-dimensional spheres, so that $\epsilon^{abc}\sqrt{h} = \epsilon^{abcd}n_d\sqrt{q}$. Now, the quantity $(\epsilon^{dpqe}n_e)(\epsilon^{abcm}n_m)(\Omega^{-1}C^*_{bcpq})$ is smooth in all M, and thus it exists in $\mathcal{J}^+$ (see Appendix B). Its pullback to null infinity is denoted as ${}^*K^{ad}$. Taking into account all this and using ${}^*K^{ab} = 2\epsilon^{pqa}D_p N_q{}^b$ [31],

$$i^*(K_{ad} - K'_{ad})i^*(\hat{n}^e R_{bc}{}^d{}_e\epsilon^{abc}\sqrt{h}) = -\frac{1}{4}N_{mn}{}^*K^{mn}\sqrt{q} = -\frac{1}{2}N_{mn}\epsilon^{pqm}D_p N_q{}^n\sqrt{q}. \tag{49}$$

This expression can be simplified further. The basis $\{n^a, m^a, \bar{m}^a\}$ satisfies $0 = \mathcal{L}_n m^a = n^d D_d m^a$. Using $\epsilon^{abc} = i3!n^{[a}m^b\bar{m}^{c]}$ and $N_{ab}n^b = 0$, $D_a n^b = 0$, we can get

$$i^*(K_{ad} - K'_{ad})i^*(\hat{n}^e R_{bc}{}^d{}_e\epsilon^{abc}\sqrt{h}) = -iN_{ab}m^{[n}\bar{m}^{a]}n^d D_d N^b_n = -\mathrm{Im}(N_{44}\partial_u N_{33}), \tag{50}$$

where $N_{33} \coloneqq N_{ab}m^a m^b$ and $N_{44} = \bar{N}_{33}$, following the usual Newman-Penrose notation.

Taking into account these results, we can rewrite (29) in the final form

$$\int d^4x\sqrt{-g}\langle\nabla_a j^a_5\rangle = -\frac{\hbar}{6}\int_M p_1(R) = \frac{\hbar}{48\pi^2}\int_{\mathcal{J}^+} du d\mathbb{S}^2 \mathrm{Im}(N_{44}\partial_u N_{33}). \tag{51}$$

Note the strong analogy with the electromagnetic case, Eq. (16), and also that this result is manifestly gauge invariant (N_{ab} is invariant under conformal gauge transformations of the form $\Omega \to \omega\Omega$). On the other hand, notice that this result is purely geometrical. In other words, the topological information encoded in the Chern-Pontryagin is here trivial (zero) because we are just working with $\mathbb{R}^4$ with the usual differentiable structure. It is the contribution of the boundary (physically, null infinity) that contributes nontrivially to the final result, but this contribution is not topological. For manifolds with boundary, the Chern-Pontryagin is not purely topological, and its utility as a topological invariant is recovered only when a surface correction is added [27]. This correction is precisely equal to the result that we obtain with a sign reversed.

B. Physical interpretation: Circularly polarized gravitational waves

The result (51) tells us that the electromagnetic duality anomaly (4) is fully determined by the radiative content of the spacetime. To write the result in terms of the Ψ^0_4 Weyl scalar, widely used in the gravitational-wave literature, we notice that $N_{33} = 2\dot{\sigma}$ and $\Psi^0_4 = -\ddot{\bar{\sigma}}$, where σ is the shear of the gravitational radiation (see Appendix C). Then,

$$\int d^4x\sqrt{-g}\langle\nabla_a j_5^a\rangle = \frac{\hbar}{12\pi^2}\int_{\mathcal{J}^+} du d\mathbb{S}^2 \int^u du' \mathrm{Im}(\Psi_4^0(u',\theta,\phi)\bar{\Psi}_4^0(u,\theta,\phi)). \tag{52}$$

Despite its apparent form, the physical interpretation of this result is remarkably simple. Expand in Fourier modes as

$$\Psi_4^0(u,\theta,\phi) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}h(\omega,\theta,\phi)e^{-i\omega u} = \int_0^{\infty}\frac{d\omega}{2\pi}[h_R(\omega,\theta,\phi)e^{-i\omega u} + \bar{h}_L(\omega,\theta,\phi)e^{i\omega u}], \tag{53}$$

where in the second equality we split the modes explicitly in terms of positive and negative definite modes: $h_R(\omega) \coloneqq h(\omega)$ for $\omega > 0$ while $\bar{h}_L(-\omega) \coloneqq h(\omega)$ for $\omega < 0$. The News scalar satisfies $\dot{N}_{33} = -2\bar{\Psi}_4^0$, so we can write

$$N_{33}(u,\theta,\phi) = 2\int_0^{\infty}\frac{d\omega}{2\pi}\left[\frac{h_L(\omega,\theta,\phi)}{i\omega}e^{-i\omega u} + \frac{\bar{h}_R(\omega,\theta,\phi)}{-i\omega}e^{i\omega u}\right] + \beta(\theta,\phi), \tag{54}$$

where the function $\beta(\theta,\phi)$ emerges as a constant of integration. This function, however, does not contribute to (51) because the physical requirement of finite GW energy crossing null infinity implies $N_{33}(u\to\pm\infty)\to 0$ (this is inferred from the Bondi mass formula, [31]), and leads to $\beta(\theta,\phi) = 0$. From (51),

$$\int d^4x\sqrt{-g}\langle\nabla_a j_5^a\rangle = -\frac{\hbar}{12\pi^2}\int d\mathbb{S}^2\int_0^{\infty}\frac{d\omega}{2\pi\omega}(|h_R(\omega,\theta,\phi)|^2 - |h_L(\omega,\theta,\phi)|^2). \tag{55}$$

What is the physical meaning of these modes, h_R and h_L? Because $\Psi_4^0 = -\ddot{\bar{\sigma}} = -(\ddot{h}_+ - i\ddot{h}_\times)$, where h_+, $h_\times$ are the two standard linear polarization modes of the GWs, we also have

$$\Psi_4^0(u,\theta,\phi) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\omega^2(h_+(\omega,\theta,\phi) - ih_\times(\omega,\theta,\phi))e^{-i\omega u}, \tag{56}$$

from which we identify

$$\begin{aligned} h_R(\omega) &= \omega^2(h_+(\omega) - ih_\times(\omega)), \omega > 0, \\ \bar{h}_L(-\omega) &= \omega^2(h_+(\omega) - ih_\times(\omega)), \omega < 0. \end{aligned} \tag{57}$$

The second equation implies $h_L(\omega) = \omega^2(\bar{h}_+(-\omega) + i\bar{h}_\times(-\omega))$, for $\omega > 0$. Because $h_+(u)$ and $h_\times(u)$ are real functions, we must have $\bar{h}_+(-\omega) = h_+(\omega)$ and $\bar{h}_\times(-\omega) = h_\times(\omega)$, leading to $h_L(\omega) = \omega^2(h_+(\omega) + ih_\times(\omega))$, and

$$\boxed{\langle Q_5(\mathrm{scri}+)\rangle - \langle Q_5(\mathrm{scri}-)\rangle = \hbar\int_0^{\infty}\frac{d\omega\omega^3}{24\pi^3}\sum_{\ell m}[|h_+^{\ell m}(\omega) + ih_\times^{\ell m}(\omega)|^2 - |h_+^{\ell m}(\omega) - ih_\times^{\ell m}(\omega)|^2],} \tag{58}$$

where we expanded the field variables in spin-weight spherical harmonics of modes (ℓ, m). The physical interpretation of this result is again clear: while the LHS represents the net amount of photon circular polarization created, the RHS is the difference in intensity between right- and left-handed circularly polarized GWs reaching future null infinity, i.e., the Stokes V parameter of GWs. Thus, we conclude that the emission of chiral gravitational radiation by astrophysical systems implies the spontaneous creation of photons with net helicity. The more right- or left-handed GWs the spacetime contains, the more left- or right-handed photons will be excited from the quantum vacuum.

C. An example: Precessing binary black hole systems

Let us consider a binary black hole merger. The system emits GWs, which are analyzed in modes of frequency ω and angular numbers (ℓ, m). During the inspiral phase the frequency spectrum is determined by the angular velocity Ω as $\omega_m \sim m\Omega$. The shear of the gravitational waves can be decomposed as $\sigma(u,\theta,\phi) = \sum_{\ell m}(A^+_{\ell m-2}Y_{\ell m}(\theta,\phi)e^{-i\omega_m u} + \bar{A}^-_{\ell m 2}\bar{Y}_{\ell m}(\theta,\phi)e^{i\omega_m u})$. Self-consistency requires $\bar{A}^-_{\ell m} = (-1)^m A^+_{\ell -m}$, which is deduced using ${}_s\bar{Y}_{lm} = (-1)^{s+m}{}_{-s}Y_{l(-m)}$. Equation (58) gives

$$\langle Q_5(\mathrm{scri}+)\rangle - \langle Q_5(\mathrm{scri}-)\rangle \propto \sum_{\ell m}\omega_m^3(|A^+_{\ell m}|^2 - |A^+_{\ell(-m)}|^2). \tag{59}$$

The parameters $A^{\pm}_{\ell m}$ can be understood as "excitation" factors for the generation of each GW polarization of mode (ℓ, m), and they depend on the details of the physical system under consideration (initial data). If the binary system is invariant under mirror symmetry with respect to some plane, choosing angular coordinates such that $\theta = \pi/2$ represents that plane, this invariance is equivalent to say that $C_{abcd}(u,\theta,\phi) = C_{abcd}(u,\pi-\theta,\phi)$, which implies $\sigma(u,\theta,\phi) = \bar{\sigma}(u,\pi-\theta,\phi)$. Using ${}_sY_{lm}(\pi-\theta,\phi+\pi) = (-1)^l {}_{-s}Y_{lm}(\theta,\phi)$, the previous condition leads to $A^{+}_{\ell m} = A^{+}_{\ell(-m)}(-1)^{\ell}$, which makes (59) vanish. In other words, the chiral anomaly emerges in binary mergers that do not have any mirror symmetry.[7] Examples of this are precessing binary systems, in which the individual spins of the black holes are not aligned with the total angular momentum [43] and break any potential symmetry under mirror transformations (see [29] for more details and implications in astrophysics).

IV. CONCLUSIONS

Chiral anomalies are a long-standing prediction of quantum field theory that have provided rich physical consequences along the past decades in several branches of physics. Despite this, their use has been considerably restricted to nontrivial topological issues, with instantons playing a dominant role. While this has been fruitful in many aspects, as for instance in unraveling the vacuum structure in Yang-Mills theories and solving problems of major importance in particle physics, it is not the whole story, at least in gravity (and electrodynamics). In this paper we characterized which class of solutions to Einstein's (and Maxwell) equations are able to induce the chiral anomaly on fermion and electromagnetic fields. On the one hand, we found that stationary solutions cannot trigger this anomaly. On the other hand, we found that, among all dynamical solutions, only those which involve radiation with net circular polarization are able to induce the quantum anomaly, and we provided specific examples of physical interest where this occurs. The physical interpretation of this quantum effect is associated with spontaneous creation of particles, but in sharp contrast to the familiar Hawking radiation of black holes, a net amount of helicity can be originated from the quantum vacuum. This new aspect of chiral anomalies could be useful in the search for phenomenology, but this is out of the scope of this paper and will be left for future studies.

[7]Note that we are neglecting the backreaction of GWs on the evolution of the inspiral. When this is taken into account, the radius of the orbit shrinks for any binary merger, and to some extent this breaks the symmetry under spatial reversals. However, this process can be considered adiabatic, and its contribution insignificant.

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APPENDIX A: STATIONARY SPACETIMES

In this Appendix we prove that in stationary, asymptotically flat spacetimes (M, g_{ab}) with $M \simeq (t_1, t_2) \times \mathbb{R}^3$ one has

$$\int_M d^4x\sqrt{-g} R_{abcd} {}^*R^{abcd} = 0. \tag{A1}$$

The argument follows in close analogy to the electromagnetic case (see Sec. II A).

Given a local orthonormal frame ("vierbein") $\{e^a_I\}$, we can define the curvature 2-form from the Riemann tensor as $R_{ab}{}^I{}_J = R_{abc}{}^d e^I_d e^c_J$. For notational simplicity we will frequently omit the internal indices I, J of the curvature 2-form and/or work directly with $R = \frac{1}{2} R_{ab} dx^a \wedge dx^b$. If the spacetime is stationary there exists a timelike killing vector k^a that leaves the metric invariant along its integral curves, $\mathcal{L}_k g_{ab} = 0$. We construct our tetrad basis $\{e^a_I\}$ such that $\mathcal{L}_k e^a_I = 0$ as well. The stationarity condition leads to $\mathcal{L}_k R_{abcd} = 0$. Together with the previous equation it gives $\mathcal{L}_k R_{ab} = 0$, or equivalently $d i_k R + i_k dR = 0$. For a general matrix-valued, p-form V we can introduce the covariant derivative $DV = dV + \omega \wedge V - (-1)^p V \wedge \omega$ [27], under which the familiar Bianchi identity $\nabla_a {}^*R^{abcd} = 0$ is equivalent to $DR = dR + \omega \wedge R - R \wedge \omega = 0$. Using these equations we can write $i_k dR = i_k(-\omega \wedge R + R \wedge \omega) = -i_k\omega \wedge R + \omega \wedge i_k R + i_k R \wedge \omega + R \wedge i_k \omega$ and $d(i_k R) = D(i_k R) - \omega \wedge i_k R - i_k R \wedge \omega$. Joining both results,

$$D(i_k R) = i_k \omega R - R i_k \omega. \tag{A2}$$

For any matrix Λ one has $D(D\Lambda) = -\Lambda R + R\Lambda$, so one can deduce from the above that $i_k R = -D(i_k \omega)$, or $k^a R_{ab} = -\nabla_b i_k \omega$.

On the other hand, let us use the normalized vector $\hat{k}^a = \frac{1}{\alpha} k^a$, with $\alpha = \sqrt{-k^a k_a}$, to make a $3+1$ decomposition of the metric, $g_{ab} = -\hat{k}_a \hat{k}_b + h_{ab}$. This decomposition allows the simplification

$$\begin{aligned} &\int_M d^4x\sqrt{-g} R_{abcd} {}^*R^{abcd} \\ &\quad = \mathrm{Tr} \int_M d^4x\sqrt{-g} R_{ab} {}^*R^{ab} \\ &\quad = -4\mathrm{Tr} \int_M d^4x\sqrt{-g} \hat{k}^a R_{ab} {}^*R^b_c \hat{k}^c, \end{aligned} \tag{A3}$$

where in the last equality we used ${}^*R_{ab} = \frac{1}{2}\epsilon_{abcd}R^{cd}$ to write $\mathrm{Tr}h^{ab}h^{cd}R_{ac}R_{bd} = -2\mathrm{Tr}\hat{k}^a R_{ab}\hat{k}_c {}^*R^{cb}$. Doing some manipulations one gets

$$\mathrm{Tr}((D_b i_k\omega)\alpha^{-1}\hat{k}_c {}^*R^{cb}) = \nabla_b \mathrm{Tr}(i_k\omega\alpha^{-1}\hat{k}_c {}^*R^{cb}) - \nabla_b(\alpha^{-1}\hat{k}_c)\mathrm{Tr}({}^*R^{cb}i_k\omega). \tag{A4}$$

It turns out that the second term is identically zero. To see this expand as

$$\nabla_b(\alpha^{-1}\hat{k}_c) = \alpha^{-1}\nabla_b\hat{k}_c - \frac{1}{\alpha^2}\hat{k}_c\nabla_b\alpha. \tag{A5}$$

Since $\nabla_b\alpha = -\frac{1}{\alpha}k^d\nabla_b k_d$ and $\nabla_a k_b = -\nabla_b k_a$ (k^a is a Killing vector field),

$$\nabla_b(\alpha^{-1}\hat{k}_c) = \alpha^{-1}\left(\nabla_b\hat{k}_c - \frac{1}{\alpha}\hat{k}_c\hat{k}^d\nabla_d k_b\right) = \frac{1}{\alpha}(2\nabla_{(b}\hat{k}_{c)} - D_c\hat{k}_b), \tag{A6}$$

where in the last equality we used $k^a\nabla_a\alpha = 0$ (which can be deduced after expanding $k^a\mathcal{L}_k k_a = k^a g_{ab}\mathcal{L}_k k^b = 0$), $\hat{k}_a\hat{k}_b = -g_{ab} + h_{ab}$, and introduced the spatial covariant derivative $D_a\hat{k}_b = h_a^c h_b^d\nabla_c\hat{k}_d = h_a^c\nabla_c\hat{k}_b$. The RHS is a symmetric tensor, so when contracting with ${}^*R^{cb}$ in (A4) the result will be zero.

We end up with the integral of a total derivative, which can be solved using Stokes theorem. Let us work in coordinates $\{t, r, \theta, \phi\}$, where t is the time measured by static observers at spatial infinity: $k \to \frac{\partial}{\partial t}$ as $r \to \infty$. Then

$$\int_M d^4x\sqrt{-g}R_{abcd}{}^*R^{abcd} = \lim_{r\to\infty}\int_{t_1}^{t_2} dt \int d\mathbb{S}^2 r^2 \mathrm{Tr}(i_k\omega\hat{k}_c {}^*R^{cd}\nabla_d r) \tag{A7}$$

(notice that for asymptotically flat spacetimes $\alpha \to 1$ as $r \to \infty$). Using $\mathcal{L}_k e_I^a = [k, e_I]^a = 0$ one can further deduce that $(i_k\omega)_{IJ} = e_I^a e_J^b\nabla_a k_b$ so

$$\int_M d^4x\sqrt{-g}R_{abcd}{}^*R^{abcd} = \lim_{r\to\infty}\int_{t_1}^{t_2} dt \int d\mathbb{S}^2 r^2\nabla_a k_b {}^*R^{abcd}k_c\nabla_d r. \tag{A8}$$

At spatial infinity we have $\nabla_a k_b = \nabla_a\nabla_b t = 0$ so $\nabla_a k_b \sim O(r^{-1})$. Assuming standard falloff conditions at spatial infinity for the Weyl tensor [44], $C_{abcd} \sim O(r^{-3})$, we finally see that

$$\int_M d^4x\sqrt{-g}R_{abcd}{}^*R^{abcd} = 0. \tag{A9}$$

APPENDIX B: ASYMPTOTIC MINKOWSKIAN SPACETIMES

We summarize here the basic points of [31–33] that are needed to follow the calculation in the main text.

A spacetime $(\hat{M}, \hat{g}_{ab})$ is called asymptotically flat at null infinity if there exists a manifold M with boundary I endowed with a metric tensor g_{ab}, and a diffeomorphism from $\hat{M}$ onto $M - I$ (with which we identify $\hat{M}$ and $M - I$) that satisfies the following:

(a) (a) there exists a smooth function Ω on M with $g_{ab} = \Omega^2\hat{g}_{ab}$ on $\hat{M}$; with $\Omega = 0$ on I; and with $n_a := \nabla_a\Omega$ as nonvanishing at I.
(b) I is homeomorphic to $\mathbb{S}^2 \times \mathbb{R}$.
(c) $\tilde{g}_{ab}$ satisfies Einstein's equations $\hat{R}_{ab} - \frac{1}{2}\hat{R}\hat{g}_{ab} = 8\pi G\hat{T}_{ab}$, and $\Omega^{-2}\hat{T}_{ab}$ has a smooth limit to I.

One refers to $(\hat{M}, \hat{g}_{ab})$ as the physical spacetime, and to (M, g_{ab}) as the unphysical one, or the conformal completion of $(\hat{M}, \hat{g}_{ab})$. Using the known conformal transformation rules for the Ricci tensor and scalar curvature, it is easy to find that these conditions imply $n^a n_a = 0$ on I. Thus, I is a three-dimensional null hypersurface in M.

Notice that within this definition there is freedom to perform conformal rescalings: if Ω is an allowed conformal factor for a physical spacetime $(\hat{M}, \hat{g}_{ab})$, so is $\Omega' = \omega\Omega$, where ω is a smooth function on M and nonvanishing at I. Under this *conformal gauge transformation*, it is easy to check that $g_{ab} \to \omega^2 g_{ab}$, $n^a \to \omega^{-1}n^a + \omega^{-2}\Omega\nabla^a\omega$. Using this freedom, it is always possible to consider a conformal completion so that $\nabla_a n^a = 0$ on I. This gauge fixing will be preserved under conformal gauge transformations as long as we restrict consideration to functions ω that satisfy $n^a\nabla_a\omega = \mathcal{L}_n\omega = 0$ on I. This gauge-fixing condition, together with property (c) above and the formula for the Ricci tensor under conformal transformations, implies $\nabla_a n_b = 0$ on I, or equivalently $\nabla_a n_b = \nabla_{(a}n_{b)} = \frac{1}{2}\mathcal{L}_n g_{ab} = 0$ on I. Furthermore, suppose we have any two divergence-free conformal frames associated with Ω and Ω'. Because the relative conformal factor ω obeys $\mathcal{L}_n\omega = 0$ on

I, the vector field n^a is complete if and only if n'^a is complete. An asymptotically flat spacetime is called asymptotically Minkowski if I is complete in any divergence-free conformal frame.

Denote by $\mathcal{I}$ a diffeomorphic copy of I, and let $\xi:\mathcal{I}\to M$ the corresponding smooth map. The pullback, denoted by ξ^*, is defined on all covariant tensor fields in M in a natural way. It can also be extended to those contravariant tensor fields such that their contraction of each of their indices with n_a gives zero at I. Set $\underline{\mathrm{n}}^a := \xi^*(n^a)$, $\underline{\mathrm{g}}_{ab} := \xi^*(g_{ab})$, and $\underline{\omega} := \xi^*(\omega)$. It follows from the discussion above that $\mathcal{I}$ is endowed with the following *universal structure*. It is homeomorphic to $\mathbb{S}^2\times\mathbb{R}$, and equipped with pairs of fields $(\underline{\mathrm{g}}_{ab},\underline{\mathrm{n}}^c)$ such that:

(i) $\underline{\mathrm{g}}_{ab}$ is a degenerate metric of signature $0,+,+$ with $\underline{\mathrm{g}}_{ab}\underline{\mathrm{n}}^b=0$ and $\mathcal{L}_{\underline{\mathrm{n}}}\underline{\mathrm{g}}_{ab}=0$;
(ii) $\underline{\mathrm{n}}^a$ is complete; and
(iii) any two pairs $(\underline{\mathrm{g}}_{ab},\underline{\mathrm{n}}^c)$ and $(\underline{\mathrm{g}}'_{ab},\underline{\mathrm{n}}'^c)$ in the collection are related by a conformal rescaling $\Omega\to\underline{\omega}\Omega$ as $\underline{\mathrm{g}}'_{ab}=\underline{\omega}^2\underline{\mathrm{g}}_{ab}$, $\underline{\mathrm{n}}'^a=\underline{\omega}^{-1}\underline{\mathrm{n}}^a$, with $\mathcal{L}_n\underline{\omega}=0$.[8]

This collection exists in any asymptotically Minkowskian spacetime, and thereby receives the name of *universal structure*. A choice of one element $(\underline{\mathrm{g}}_{ab},\underline{\mathrm{n}}^c)$ of the collection $\{(\underline{\mathrm{g}}_{ab},\underline{\mathrm{n}}^c)_i\}_{i\in I}$ will be called a choice of conformal frame. Note that, since 2-spheres carry a unique conformal structure, every $\underline{\mathrm{g}}_{ab}$ in this collection is conformal to a unit 2-sphere metric. Because of this, it is sometimes convenient to restrict the remaining conformal freedom at I (i.e., to fully fix the gauge function ω) by demanding that the metric q_{ab} on these 2-spheres be the metric of the unit radius 2-sphere. This is always possible, and this conformal frame is known as the Bondi frame.

The metric in (M,g_{ab}) allows the raising and lowering of indices, introduces an alternating tensor field ϵ^{abcd} unique up to a sign, and leads to a preferred derivative operator ∇_a and its associated curvature tensor R^d_{abc}. Suppose we are given a fixed conformal frame. We study now what the corresponding apparatus is for $(\mathcal{I},\underline{\mathrm{g}}_{ab},\underline{\mathrm{n}}^a)$. This is not a trivial question since $\underline{\mathrm{g}}_{ab}$ is a degenerate metric. In the following we will define what fields, operations, etc., one can construct from this conformal frame, and then study their behavior under a conformal gauge transformation.

First of all, we can lower indices with $\underline{\mathrm{g}}_{ab}$, but we cannot raise indices *a priori* since $\underline{\mathrm{g}}_{ab}$ is degenerate, and hence it does not have an inverse. Define a tensor $\underline{\mathrm{g}}^{ab}$ by the property $\underline{\mathrm{g}}_{am}\underline{\mathrm{g}}^{mn}\underline{\mathrm{g}}_{nb}=\underline{\mathrm{g}}_{mn}$. This is unique up to the addition of a tensor of the form $v^{(a}\underline{\mathrm{n}}^{b)}$, for any vector field v^a. We will use this $\underline{\mathrm{g}}^{ab}$ to raise indices whenever the lack of uniqueness does not lead to an ambiguous result. Next, we introduce an alternating tensor field ϵ^{abc}, up to a sign, by the equation $\epsilon^{amn}\epsilon^{bpq}\underline{\mathrm{g}}_{mp}\underline{\mathrm{g}}_{nq}=2\underline{\mathrm{n}}^a\underline{\mathrm{n}}^b$ and demanding antisymmetry. Having fixed the sign, we can define uniquely the tensor ϵ_{abc} by $\epsilon^{abc}\epsilon_{abc}=6$ and the condition of antisymmetry. The above definition implies that $\epsilon^{abc}=\xi^*(\epsilon^{abcd}n_d)$, but note that $\epsilon_{abc}\neq\epsilon^{a'b'c'}\underline{\mathrm{g}}_{aa'}\underline{\mathrm{g}}_{bb'}\underline{\mathrm{g}}_{cc'}=\xi^*(\epsilon_{abcd}n^d)=0$. The usual identities for ϵ_{abc} and ϵ^{abc} hold.

As commented above, the universal structure of I is common to *every* asymptotic Minkowski spacetime. The $\mathbb{S}^2\times\mathbb{R}$ differentiable structure together with the collection of pairs $(\underline{\mathrm{g}}_{ab},\underline{\mathrm{n}}^c)$ is called the *zeroth order* structure of I and is available in any asymptotic Minkowski spacetime. We shall describe now higher order geometrical structures that are not universal and that contain specific physical information of the given spacetime. The connection D defined intrinsically on I in any given conformal completion, induced by the torsion-free connection ∇ compatible with g_{ab}, will be regarded as the first order structure. As we shall see, it contains the "radiative information" of the physical spacetime $(\tilde{M},\tilde{g}_{ab})$, and consequently it changes from one spacetime to another.

We define the derivative operator in $\mathcal{I}$ by $D_a\mu_b := \xi^*(\nabla_a\nu_b)$, where μ is any 1-form in $\mathcal{I}$ and ν_a is a 1-form in M such that $\mu_a=\xi^*(\nu_a)$. This derivative operator is defined intrinsically in $\mathcal{I}$. Notice that given any μ_a in $\mathcal{I}$, there exists many ν_a in M that satisfies $\mu_a=\xi^*(\nu_a)$. However, it can be shown that the derivative operator is a well-defined operation: given two ν_a and ν'_a that leads to μ_a in $\mathcal{I}$, one actually has $\xi^*(\nabla_a\nu_b)=\xi^*(\nabla_a\nu'_b)$. Having seen this, we can now extend the derivative operator to all tensor fields in the usual manner. In particular, given that $\nabla_a g_{bc}=0$, and $\nabla_a n^b=0$ on I, we find $D_a\underline{\mathrm{g}}_{bc}=0$ and $D_a\underline{\mathrm{n}}^b=0$ (it is also not difficult to prove also that $D_a\epsilon_{bcd}=0, D_a\epsilon^{bcd}=0$). In other words, this derivative operator is compatible with the metric $\underline{\mathrm{g}}_{ab}$. However, it should be remarked that this connection is not uniquely defined because $\underline{\mathrm{g}}_{ab}$ is degenerate. We shall now characterize in physical terms the different allowed derivative operators.

First of all, we need to know how any derivative operator changes under a conformal gauge transformation $\Omega\to\omega\Omega$. For any covector k_a, the transformation rule, at points of I, is

$$D'_ak_b=D_ak_b-2\underline{\omega}^{-1}k_{(a}D_{b)}\underline{\omega}+\underline{\omega}^{-1}(\nabla^m\underline{\omega})k_m\underline{\mathrm{g}}_{ab}. \tag{B1}$$

Notice that, even when $\underline{\omega}=1$ so that $\underline{\mathrm{g}}_{ab}$ and $\underline{\mathrm{n}}^a$ are invariant, D_a changes nontrivially as $D'_ak_b=D_ak_b+f(\underline{\mathrm{n}}^mk_m)\underline{\mathrm{g}}_{ab}$, where we introduced $\nabla^a\underline{\omega}=:f\underline{\mathrm{n}}^a$. This shows that the derivative operator at I is not invariant under conformal gauge transformations (in analogy to the magnetic potential in electrodynamics). Because this residual transformation of the derivative operator is just pure gauge, one is motivated to define an equivalence class of

[8]The result $\xi^*(n_a)=D_a\xi^*(\Omega)=0$ implies $\underline{\mathrm{g}}_{ab}\underline{\mathrm{n}}^b=0$. On the other hand, because the pullback commutes with the Lie derivative, we automatically inherit $\mathcal{L}_{\underline{\mathrm{n}}}\underline{\mathrm{g}}_{ab}=0$ and $\mathcal{L}_{\underline{\mathrm{n}}}\underline{\omega}=0$.

connections $\{D_a\}$, in a given conformal frame $(\underline{g}_{ab}, \underline{n}^c)$, by the equivalence relation

$$D_a \sim D'_a \quad \text{iff } (D'_a - D_a)k_b = (f\underline{n}^c k_c)\underline{g}_{ab}, \tag{B2}$$

where f is an arbitrary function on $\mathcal{I}$. Now, given two connections D_a and D'_a belonging to different equivalence classes, their difference when acting on any covector is linear, and thus it must be determined by another tensor C^c_{ab}: $(D_a - D'_a)k_b = C^c_{ab}k_c$. The torsion-free derivative ∇_a implies that $C_{ab}{}^c = C_{(ab)}{}^c$ (just take $k_b = D_b g$ in the previous equation for some function g to find $C_{ab}{}^c = C_{ba}{}^c$). On the other hand, the condition $D_a\underline{n}^b = D'_a\underline{n}^b = 0$ implies $C^c_{ab}\underline{n}^b = 0$, and the metric compatibility $D_a\underline{g}_{bc} = D'_a\underline{g}_{bc} = 0$ implies $C_{a(b}{}^d\underline{g}_{c)d} = 0$. Since the only vector that annihilates the metric is $\underline{n}^d$, then

$$(D_a - D'_a)k_b = \Sigma_{ab}\underline{n}^c k_c, \tag{B3}$$

for some tensor Σ_{ab} with $\Sigma_{ab} = \Sigma_{(ab)}$ and $\Sigma_{ab}\underline{n}^b = 0$. Consequently, due to (B2) the difference $\{D'_a\} - \{D_a\}$ between the equivalence classes of connections is fully characterized by the trace-free tensor: $\sigma_{ab} := \Sigma_{ab} - \frac{1}{2}\Sigma_{cd}\underline{g}^{cd}\underline{g}_{ab}$. The space of equivalence classes $\{D_a\}$ is an affine space, we can select any $\{D^0_a\}$ as an origin, and then any other $\{D_a\}$ is labeled uniquely by a transverse $(\sigma_{ab}\underline{n}^b = 0)$ trace-free symmetric tensor σ_{ab} on $\mathcal{J}^+$. These properties allow one to write $\sigma_{ab} = \sigma m_a m_b + \text{c.c.}$, for some complex function σ. In physical terms, the two independent components of σ_{ab} represent the two radiative degrees of freedom of the gravitational field in full general relativity.

We turn now to study the second order structure of an asymptotically Minkowski spacetime. Let $R_{abc}{}^d$ be the Riemann tensor of the unphysical spacetime, defined by $\nabla_{[a}\nabla_{b]}k_c = \frac{1}{2}R_{abc}{}^d k_d$ for any covector k_c. The Riemann tensor can be split into a totally traceless part (the Weyl tensor $C_{abc}{}^d$) plus a traceful part (the Ricci R_{ab} or, alternatively, Schouten tensor S_{ab}) as $R_{abcd} = C_{abcd} + g_{a[c}S_{d]b} - g_{b[c}S_{d]a}$. It is a fundamental result [31] that the Weyl tensor vanishes at I, and consequently all the information about the curvature of I will be determined by S^a_b. On the other hand, let us introduce the combination $L^b_a := \hat{R}^b_a - \frac{1}{6}\hat{R}\delta^a_b$ and $L_{ab} = g_{ac}L^c_b$, where $\hat{R}^b_a$ and $\hat{R}$ denote the Ricci tensor and scalar curvature of the unphysical spacetime. From the standard formula for the behavior of the Ricci tensor under conformal transformations, and using property (c) above, one can find[9]

$$\Omega S_{ab} + 2\nabla_a n_b - \Omega^{-1}n^c n_c g_{ab} = \Omega^{-1}L_{ab} = O(\Omega^3). \tag{B4}$$

[9]The definition of asymptotically Minkowski spacetimes requires that $\hat{T}_{ab} = O(\Omega^2)$, and so $\hat{R}_{ab} = O(\Omega^2)$. Then $L_{ab} = g_{bc}L^c_a = O(\Omega^4)$.

From this equation one deduces that, at points of I, $2\nabla_a n_b = 2\nabla_{(a}n_{b)} = \mathcal{L}_n g_{ab} = \Omega^{-1}n^c n_c g_{ab}$. But remember that $\mathcal{L}_n g_{ab} = 0$ at points of I, so $f := \Omega^{-1}n^c n_c = 0$ at points of I, too. Now, contracting the above equation with n^b and rearranging terms, one arrives at

$$S_{ab}n^b + \nabla_a f = O(\Omega^2). \tag{B5}$$

Since f vanishes at I, it serves to define this hypersurface, and so its gradient must be transverse to it. Since the only transverse covector to I is n_a, we necessarily have $\nabla_a f \propto n_a$. Then, $S^b_a n_b \propto n_a$ and vanishes at I. This means that the pullback is well-defined on the tensor S^b_a, so we define $\underline{S}^b_a := \xi^*(S^b_a)$ and also $\underline{S}_{ab} := \underline{g}_{ac}\underline{S}^c_b$. Notice the properties $\underline{S}_{ab}\underline{n}^b = 0$ and $\underline{S}_{ab} = \underline{S}_{(ab)}$. There is one further property of S_{ab} that is important to keep in mind. By taking the pullback of the Riemann tensor, and recalling the vanishing of C_{abcd} at I, one gets

$$\underline{R}_{abc}{}^d = \underline{g}_{c[a}\underline{S}^d_{b]} + \underline{S}_{c[a}\underline{\delta}^d_{b]}. \tag{B6}$$

If we define $\underline{R}_{abcd} := \underline{R}_{abc}{}^e\underline{g}_{de}$, then the contraction of any of its indices with $\underline{n}^a$ is zero. The corresponding Ricci tensor $R_{ab} := \underline{g}^{cd}\underline{R}_{abcd}$ and scalar curvature $\underline{R} := \underline{g}^{cd}R_{cd}$ are thus unambiguous. Since $\underline{R}_{abcd}$ lives in the two dimensions orthogonal to $\underline{n}^a$, it can be reconstructed from its scalar curvature alone, $\underline{R}_{abcd} = \underline{R}\underline{g}_{a[c}\underline{g}_{d]b}$. Combining this with (B6), one finally gets $\underline{g}^{ab}\underline{S}_{ab} = \underline{R}$.

The tensor $\underline{S}_{ab}$ carries information of major importance about gravitational radiation in the given spacetime, but there is still a small complication. If we change the conformal frame, this tensor transforms in a complicated way:

$$\begin{aligned}\underline{S}'_{ab} = \underline{S}_{ab} &- 2\underline{\omega}^{-1}D_aD_b\underline{\omega} + 4\underline{\omega}^{-2}D_a\underline{\omega}D_b\underline{\omega} \\ &- \underline{\omega}^{-2}(\underline{g}^{mn}D_m\underline{\omega}D_n\underline{\omega})\underline{g}_{ab}.\end{aligned} \tag{B7}$$

Consequently, a portion of this curvature is "gauge" in the sense that it contains information that is not truly physical. The goal is to extract information from this curvature tensor that remains invariant under conformal gauge transformations. This was successfully done in [31]: given any conformal frame $(\underline{g}_{ab}, \underline{n}^c)$, it can be proven that there exists a unique tensor field ρ_{ab} on I that fulfills

$$\rho_{[ab]} = 0, \quad \rho_{ab}\underline{n}^b = 0, \quad \rho_{ab}\underline{g}^{ab} = \underline{R}, \quad D_{[a}\rho_{b]c} = 0, \tag{B8}$$

and, most importantly, transforms exactly as $\underline{S}_{ab}$ does under a conformal gauge reescaling. Therefore, the combination

$$N_{ab} := \underline{S}_{ab} - \rho_{ab}, \tag{B9}$$

is *conformally gauge invariant*. Consequently, the role of ρ_{ab} is to subtract from S_{ab} the pure gauge-dependent contribution. In a Bondi conformal frame, in particular,

one has $\rho_{ab} = \frac{1}{2}\underline{g}_{ab}\underline{R}$. N_{ab} is referred to as the Bondi news tensor and is regarded as the second order structure at $\mathcal{J}$. It satisfies

$$N_{[ab]} = 0, \qquad N_{ab}\underline{n}^b = 0, \qquad N_{ab}\underline{g}^{ab} = 0. \tag{B10}$$

This is all the physical information we can extract from S_{ab}. Nevertheless, the full information of the curvature of $\{D_a\}$ is actually contained in $\underline{S}^a_b$, and not in $\underline{S}_{ab}$ (notice that since $\underline{g}_{ab}$ is not invertible it is not possible to reconstruct $\underline{S}^a_b$ from $\underline{S}_{ab}$). This information is encoded in what we shall call the third (and last) geometric asymptotic structure, which can be worked out from the Weyl tensor. Since the Weyl tensor vanishes at I, the tensor $\Omega^{-1}C_{abcd}$ is smooth up to and including I. If we define

$$K^{ab} := \epsilon^{amn}\epsilon^{bpq}\xi^*(\Omega^{-1}C_{mnpq}), \tag{B11}$$

$$^*K^{ab} := \epsilon^{amn}\epsilon^{bpq}\xi^*(\Omega^{-1}{}^*C_{mnpq}), \tag{B12}$$

then we immediately see that they are symmetric and that $K^{ab}\underline{g}_{ab} = {}^*K^{ab}\underline{g}_{ab} = 0$. Taking the curl of (B4), using the definition of Riemann tensor, expressing it in terms of the Weyl tensor, and doing some manipulations, it is possible to show that $D_{[a}\underline{S}^c_{b]} = \frac{1}{4}\epsilon_{amn}{}^*K^{mc}$, which automatically leads to

$$D_{[a}N_{b]c} = \frac{1}{4}\epsilon_{amn}{}^*K^{mn}\underline{g}_{nc}, \tag{B13}$$

or, equivalently, ${}^*K^{ab} = 2\epsilon^{pqa}D_pN^b_q$. Furthermore, a straightforward calculation shows that ${}^*K^{ab}$ remains invariant under conformal gauge transformations with $\omega = 1$, so it is a physically meaningful quantity. Because ${}^*K^{ab}$ involves derivatives of S^b_a, it is called the third order structure at I.

If ${}^*K^{ab} = 0$, then $N_{ab} = 0$, and the associated equivalence class $\{D_a\}$ of connections is said to be trivial. In this case, the physical spacetime $(\hat{M}, \hat{g}_{ab})$ does not contain gravitational radiation. In particular, every stationary, asymptotically flat spacetime produces a trivial connection on I. Conversely, if $N_{ab} = 0$ (i.e., no gravitational waves), it can be shown that the spacetime is stationary [45].

APPENDIX C: SPIN-COEFFICIENT FORMALISM AND ASYMPTOTIC BEHAVIOR

Let (M, g_{ab}) be a spacetime and $\{\ell^a, n^a, m^a, \bar{m}^a\}$ a Newman-Penrose basis, i.e., a null tetrad satisfying $n^a\ell_a = 1$, $m^a\bar{m}_a = -1$, and zero otherwise.[10] If we introduce the notation $e^a_1 = \ell^a$, $e^a_2 = n^a$, $e^a_3 = m^a$, $e^a_4 = \bar{m}^a$, then this basis of null vectors satisfies $g_{ab} = \eta_{ij}e^i_ae^j_b$ with $\eta_{12} = \eta_{21} = 1$, $\eta_{34} = \eta_{43} = -1$. Internal indices $(i, j, \ldots)$ are raised and lowered with η_{ij}, while spacetime indices $(a, b, \ldots)$ are raised and lowered with g_{ab}.

Given this tetrad we can introduce the connection 1-form by $\gamma_{c\ a}^{\ b} := -e^i_c\nabla_ae^b_i$, which satisfies $\gamma_{abc} = -\gamma_{bac}$. In this basis there are 12 independent (complex) components of the connection 1-form, which are called spin coefficients. They are designated by

$$\kappa = \gamma_{311} = -m^a\ell^b\nabla_b\ell_a, \qquad \rho = \gamma_{314} = -m^a\bar{m}^b\nabla_b\ell_a, \qquad \epsilon = \frac{1}{2}(\gamma_{211} + \gamma_{341}) = -\frac{1}{2}(n^a\ell^b\nabla_b\ell_a - \bar{m}^a\ell^b\nabla_bm_a),$$

$$\sigma = \gamma_{313} = -m^am^b\nabla_b\ell_a, \qquad \mu = \gamma_{243} = \bar{m}^am^b\nabla_bn_a, \qquad \gamma = \frac{1}{2}(\gamma_{212} + \gamma_{342}) = -\frac{1}{2}(n^an^b\nabla_b\ell_a - \bar{m}^an^b\nabla_bm_a),$$

$$\lambda = \gamma_{244} = \bar{m}^a\bar{m}^b\nabla_bn_a, \qquad \tau = \gamma_{312} = -m^an^b\nabla_b\ell_a, \qquad \alpha = \frac{1}{2}(\gamma_{214} + \gamma_{344}) = -\frac{1}{2}(n^a\bar{m}^b\nabla_b\ell_a - \bar{m}^a\bar{m}^b\nabla_bm_a),$$

$$\nu = \gamma_{242} = \bar{m}^an^b\nabla_bn_a, \qquad \pi = \gamma_{241} = \bar{m}^a\ell^b\nabla_bn_a, \qquad \beta = \frac{1}{2}(\gamma_{213} + \gamma_{343}) = -\frac{1}{2}(n^am^b\nabla_b\ell_a - \bar{m}^am^b\nabla_bm_a).$$

Note that $\gamma_{311} = \bar{\gamma}_{411}$, $\gamma_{314} = \bar{\gamma}_{413}$, etc. On the other hand, the Weyl tensor has ten independent components that are represented in this framework by five complex scalars:

$$\Psi_0 := -C_{1313} = -C_{abcd}\ell^am^b\ell^cm^d, \tag{C1}$$

$$\Psi_1 := -C_{1213} = -C_{abcd}\ell^an^b\ell^cm^d, \tag{C2}$$

$$\Psi_2 := -C_{1342} = -C_{abcd}\ell^am^b\bar{m}^cn^d, \tag{C3}$$

$$\Psi_3 := -C_{1242} = -C_{abcd}\ell^an^b\bar{m}^cn^d, \tag{C4}$$

$$\Psi_4 := -C_{2424} = -C_{abcd}n^a\bar{m}^bn^c\bar{m}^d. \tag{C5}$$

The remaining components are determined using the symmetry properties of the Weyl tensor. In particular, it is not difficult to show that

[10] In this Appendix and in the next one we follow the Newman-Penrose [34] notation. In particular, the metric signature will be $(+,-,-,-)$ in order to use the asymptotic expressions for the spin-coefficients calculated in [46].

$$\Psi_1 = C_{1334} = C_{1231}, \tag{C6}$$

$$\Psi_3 = C_{2443}, \tag{C7}$$

$$\mathrm{Re}\Psi_2 = \frac{-1}{2}C_{1212} = \frac{-1}{2}C_{3434}, \tag{C8}$$

$$\mathrm{Im}\Psi_2 = \frac{1}{2i}C_{1234}, \tag{C9}$$

$$C_{1314} = C_{2324} = C_{1332} = C_{1442} = 0. \tag{C10}$$

In asymptotically flat spacetimes, a preferred coordinate system and an associated null tetrad can always be considered. Following Bondi and Sachs, we can always introduce a foliation of the asymptotic region of M by outgoing null hypersurfaces $\{u = \text{const}\}$. Denoting the corresponding geodesic null normal by ℓ^a, we can introduce an affine parameter r of ℓ^a (i.e., $\ell = \frac{\partial}{\partial r}$ so that $\ell^a \nabla_a r = 1$) such that each null surface $u = \text{const}$ is foliated by a family of (spacelike) 2-spheres $\{r = \text{const}\}$. The set $\{u, r, \theta, \phi\}$ is called Bondi-Sachs coordinates. Let us denote the intrinsic $(-,-)$ metric of these 2-spheres by q_{ab} and the other null normal to each of these 2-spheres by n^a, normalized so that $g_{ab}\ell^a n^b = 1$. If ℓ^a is normal to the $\{u = \text{const}\}$ hypersurfaces, necessarily $\ell^a = g^{ab}\nabla_b u$, so that $\ell_a := g_{ab}\ell^b = \nabla_a u$ and we can write $n = V\frac{\partial}{\partial u} + U\frac{\partial}{\partial r} + X^A\frac{\partial}{\partial X^A}$ with $V = 1$ (n_a is not simply given by $\nabla_a r$ since n^a is not normal to $\{r = \text{const}\}$ hypersurfaces in general). Finally, introduce a null complex vector field m^a and its complex conjugate $\bar{m}^a$ such that their real and imaginary parts are tangential to these 2-spheres, and they are normalized such that $g_{ab}m^a\bar{m}^b = -1$. Thus, at each point in the asymptotic region we have a null tetrad $\{\ell^a, n^a, m^a, \bar{m}^a\}$ for which the only nonzero contractions are $\ell \cdot n = 1$ and $m \cdot \bar{m} = -1$. In terms of the null tetrad, the metric takes the form $g_{ab} = 2n_{(a}\ell_{b)} - 2m_{(a}\bar{m}_{b)}$.

The spin-coefficient formalism is particularly useful for asymptotically flat spacetimes. If the Weyl scalars are smooth functions on the spacetime manifold, their asymptotic behavior as $r \to \infty$, keeping u, θ, ϕ constant, is determined by the Peeling theorem [36]:

$$\Psi_i(u, r, \theta, \phi) \sim \Psi_i^0(u, \theta, \phi)/r^{5-i}, \quad i = 0, 1, 2, 3, 4. \tag{C11}$$

Furthermore, the asymptotic behavior of the spin coefficients can be systematically obtained by integrating asymptotically a set of equations in the Newman-Penrose framework that are equivalent to Einstein's field equations [46]. The results read

$$\lambda = \lambda^0/r + O(r^{-2}), \qquad \lambda^0 = \dot{\bar{\sigma}}^0, \tag{C12}$$

$$\mu = \mu^0/r + O(r^{-2}), \qquad \mu^0 = -1, \tag{C13}$$

$$\sigma = \sigma^0/r^2 + O(r^{-4}), \qquad \sigma^0 = \text{free data}, \tag{C14}$$

$$\rho = \rho^0/r + \rho^1/r^3 + O(r^{-5}), \qquad \rho^0 = -1,$$
$$\rho^1 = -|\sigma^0|^2, \tag{C15}$$

$$\kappa = 0, \tag{C16}$$

$$\pi = 0, \tag{C17}$$

$$\nu = \nu^0 + O(r^{-1}), \qquad \nu^0 = 0, \tag{C18}$$

$$\tau = \bar{\alpha} + \beta = (\bar{\alpha}^0 + \beta^0)/r + O(r^{-2}), \qquad (\bar{\alpha}^0 + \beta^0) = 0, \tag{C19}$$

and $\Psi_4^0 = -\ddot{\bar{\sigma}}^0$.

The relation with the Bondi News $N_{33} = N_{ab}m^a m^b$ introduced in Appendix A can be obtained using (B12) and the result ${}^*K^{ab} = 2\epsilon^{pqa}D_p N_q^b$. Using the first equation we get ${}^*K^{ab}m_a m_b = 4in^a m^b n^c m^d \xi^*(\Omega^{-1}C_{abcd}) = -4i\bar{\Psi}_4^0$; on the other hand, the second equation yields ${}^*K^{ab}m_a m_b = 2in^p D_p(N_{ab}m^a m^b) = 2i\partial_u N_{33}$. Combining both we get $\dot{N}_{33} = -2\bar{\Psi}_4^0 = 2\ddot{\sigma}^0$. Furthermore, from the Bondi mass formula $\int_{\mathcal{J}} du d\mathbb{S}^2 |N_{33}|^2 < \infty$ one infers $N_{33} \to 0$ at $u \to \pm\infty$ so $N_{33} = 2\dot{\sigma}^0$.

APPENDIX D: ALTERNATIVE DERIVATION USING THE SPIN-COEFFICIENT FORMALISM

In this Appendix we derive the result (51) using the spin-coefficient formalism and the corresponding asymptotic behavior summarized in the previous Appendix. Our starting point is Eq. (42), which in the physical spacetime $(\hat{M}, \hat{g}_{ab})$ can be rewritten in a similar way (in this Appendix we only work with the physical spacetime so we will omit the hat symbol in all associated geometric quantities for convenience),

$$\int_{\hat{M}} p_1(\hat{R})$$
$$= -\lim_{r_0 \to \infty} \frac{1}{8\pi^2} \int_{r=r_0} (K_{ad} - K'_{ad})\hat{n}^e R_{bc}{}^d{}_e \epsilon^{abcf} \hat{n}_f \sqrt{h} d^3x, \tag{D1}$$

where here $\hat{n}_b = \frac{1}{\sqrt{g^{ab}\nabla_a r \nabla_b r}}\nabla_b r$ is the normal vector to $\{r = r_0\}$ hypersurfaces and $K_{ab} = D_a \hat{n}_b$ the extrinsic curvature. In terms of the Newman-Penrose basis constructed in Appendix C we must have $\nabla_a r = a_1 n_a + a_2 \ell_a$. Given that $\ell^a \nabla_a r = 1$, then $\nabla_a r = n_a + a_2 \ell_a$; and squaring $n_a = \nabla_a r - a_2 \ell_a$ we get $a_2 = \frac{1}{2}g^{rr}$ so that $\nabla_a r = n_a + \frac{g^{rr}}{2}\ell_a$. The asymptotic behavior of g^{rr} in Bondi-Sachs coordinates can be found in [37,38] and is

given by $-g^{rr} = 1 - \frac{2M}{r} + O(r^{-2})$. Note that $D_a\hat{n}_b = h_a^{a'}h_b^{b'}\nabla_b n_{b'} = h_a^{a'}\nabla_a\hat{n}_b$ and $\epsilon^{abch}\hat{n}_h D_a\hat{n}_e = \epsilon^{abch}\hat{n}_h\nabla_a\hat{n}_e$. We can also take the prefactor of $\hat{n}_b$ out of the derivative operator thanks to the antisymmetry of the Riemann tensor, $(\nabla_a\hat{n}_e)\hat{n}_d R_{bc}{}^{de} = \nabla_a[(g^{rr})^{-1/2}\nabla_e r]\hat{n}_d R_{bc}{}^{de} = (g^{rr})^{-1/2}\nabla_a[\nabla_e r]\hat{n}_d R_{bc}{}^{de} + 0$. Finally, if we take the $\{r = r_0\}$ surface outside the gravitational sources (we assume they have spatial compact support), then $R_{abcd} = C_{abcd}$. Taking into account all this

$$\int_{\hat{M}} p_1(\hat{R}) = \lim_{r_0\to\infty}\frac{1}{8\pi^2}\int_{r=r_0}\frac{\sqrt{h}}{(g^{rr})^{3/2}}d^3x C^{de}_{bc}\epsilon^{abch}\left(n_h + \frac{g^{rr}}{2}l_h\right)\left(n_d + \frac{g^{rr}}{2}l_d\right)\nabla_a\left(n_e + \frac{g^{rr}}{2}l_e\right). \tag{D2}$$

We have $\nabla_a(n_e + \frac{g^{rr}}{2}l_e) = -(\gamma_{2ea} + \frac{g^{rr}}{2}\gamma_{1ea}) + \frac{1}{2}l_e\nabla_a g^{rr}$. Then, using $\epsilon^{abcd} = 4!il^{[a}n^b m^c\bar{m}^{d]}$, we find

$$\begin{aligned}\frac{-1}{8\pi^2}\int\frac{6\sqrt{h}d^3x}{(g^{rr})^{3/2}}\Bigg[&\left(C_{2ebc} + \frac{g^{rr}}{2}C_{1ebc}\right)i\left(n^{[a}m^b\bar{m}^{c]} - \frac{g^{rr}}{2}l^{[a}m^b\bar{m}^{c]}\right)\left(\gamma^e_{2a} + \frac{g^{rr}}{2}\gamma^e_{1a}\right)\\ &- C_{21bc}i\left(n^{[a}m^b\bar{m}^{c]} - \frac{g^{rr}}{2}l^{[a}m^b\bar{m}^{c]}\right)\frac{\nabla_a g^{rr}}{2}\Bigg].\end{aligned} \tag{D3}$$

We do the calculation term by term:

(A)

$$\begin{aligned}C_{2ebc}n^{[a}m^b\bar{m}^{c]} &= \frac{1}{3}C_{2ebc}[n^a m^{[b}\bar{m}^{c]} + m^a\bar{m}^{[b}n^{c]} + \bar{m}^a n^{[b}m^{c]}]\\ &= \frac{1}{3}[n^a C_{2e34} + m^a C_{2e42} + \bar{m}^a C_{2e23}] = \frac{1}{3}[n^a C_{2e34} + 2i\mathrm{Im}(m^a C_{2e42})];\end{aligned} \tag{D4}$$

(B) same as (A) but changing $n^a \to l^a$

$$\begin{aligned}C_{2ebc}l^{[a}m^b\bar{m}^{c]} &= \frac{1}{3}C_{2ebc}[l^a m^{[b}\bar{m}^{c]} + m^a\bar{m}^{[b}l^{c]} + \bar{m}^a l^{[b}m^{c]}]\\ &= \frac{1}{3}[l^a C_{2e34} + m^a C_{2e41} + \bar{m}^a C_{2e13}] = \frac{1}{3}[l^a C_{2e34} + 2i\mathrm{Im}(m^a C_{2e41})];\end{aligned} \tag{D5}$$

(C) same as (A) but changing $C_{2\ldots} \to C_{1\ldots}$.

$$\begin{aligned}C_{1ebc}n^{[a}m^b\bar{m}^{c]} &= \frac{1}{3}C_{1ebc}[n^a m^{[b}\bar{m}^{c]} + m^a\bar{m}^{[b}n^{c]} + \bar{m}^a n^{[b}m^{c]}]\\ &= \frac{1}{3}[n^a C_{1e34} + m^a C_{1e42} + \bar{m}^a C_{1e23}] = \frac{1}{3}[n^a C_{1e34} + 2i\mathrm{Im}(m^a C_{1e42})];\end{aligned} \tag{D6}$$

(D) same as (B) but changing $C_{2\ldots} \to C_{1\ldots}$.

$$\begin{aligned}C_{1ebc}l^{[a}m^b\bar{m}^{c]} &= \frac{1}{3}C_{1ebc}[l^a m^{[b}\bar{m}^{c]} + m^a\bar{m}^{[b}l^{c]} + \bar{m}^a l^{[b}m^{c]}]\\ &= \frac{1}{3}[l^a C_{1e34} + m^a C_{1e41} + \bar{m}^a C_{1e13}] = \frac{1}{3}[l^a C_{1e34} + 2i\mathrm{Im}(m^a C_{1e41})];\end{aligned} \tag{D7}$$

(E) same as (A) but changing $C_{2e\ldots} \to C_{21\ldots}$.

$$C_{21bc}n^{[a}m^b\bar{m}^{c]} = \frac{1}{3}[n^a C_{2134} + 2i\mathrm{Im}(m^a C_{2142})] = \frac{1}{3}[-2in^a\mathrm{Im}\Psi_2 + 2i\mathrm{Im}(m^a\Psi_3)]; \tag{D8}$$

(F) same as (B) but changing $C_{2e\ldots} \to C_{21..}$.

$$C_{21bc}l^{[a}m^b\bar{m}^{c]} = \frac{1}{3}[l^aC_{2134} + 2i\mathrm{Im}(m^aC_{2141})] = \frac{1}{3}[-2il^a\mathrm{Im}\Psi_2 - 2i\mathrm{Im}(m^a\bar{\Psi}_1)]. \tag{D9}$$

We now elaborate in detail each of these terms:
(A)

$$\begin{aligned}
&\frac{1}{3}[n^aC_{2e34} + 2i\mathrm{Im}(m^aC_{2e42})]\left(\gamma^e_{2a} + \frac{g^{rr}}{2}\gamma^e_{1a}\right)\\
&= \frac{1}{3}\left[\gamma^e_{22}C_{2e34} + 2i\mathrm{Im}(\gamma^e_{23}C_{2e42}) + \frac{g^{rr}}{2}\gamma^e_{12}C_{2e34} + g^{rr}i\mathrm{Im}(\gamma^e_{13}C_{2e42})\right]\\
&= \frac{1}{3}\left[2i\mathrm{Im}(\gamma^3_{22}C_{2334}) + 2i\mathrm{Im}(\gamma^3_{23}C_{2342} + \gamma^4_{23}C_{2442})\right.\\
&\quad\left. + \frac{g^{rr}}{2}(\gamma^1_{12}C_{2134} + 2i\mathrm{Im}(\gamma^3_{12}C_{2334})) + g^{rr}i\mathrm{Im}(\gamma^1_{13}C_{2142} + \gamma^3_{13}C_{2342} + \gamma^4_{13}C_{2442})\right]\\
&= \frac{1}{3}[-2i\mathrm{Im}(\nu\bar{\Psi}_3) - 2i\mathrm{Im}(\bar{\lambda}\Psi_4) + 2i\mathrm{Re}\gamma\mathrm{Im}\Psi_2g^{rr} + ig^{rr}\mathrm{Im}(\bar{\tau}\bar{\Psi}_3) - 2ig^{rr}\mathrm{Re}\beta\mathrm{Im}(\Psi_3) + g^{rr}i\mathrm{Im}(\sigma\Psi_4)];
\end{aligned} \tag{D10}$$

(C) similar to (A) changing $C_{2\ldots} \to C_{1..}$.

$$\begin{aligned}
&\frac{1}{3}[n^aC_{1e34} + 2i\mathrm{Im}(m^aC_{1e42})]\left(\gamma^e_{2a} + \frac{g^{rr}}{2}\gamma^e_{1a}\right)\\
&= \frac{1}{3}\left[\gamma^e_{22}C_{1e34} + 2i\mathrm{Im}(\gamma^e_{23}C_{1e42}) + \frac{g^{rr}}{2}\gamma^e_{12}C_{1e34} + g^{rr}i\mathrm{Im}(\gamma^e_{13}C_{1e42})\right]\\
&= \frac{1}{3}[\gamma^2_{22}C_{1234} + 2i\mathrm{Im}(\gamma^3_{22}C_{1334}) + 2i\mathrm{Im}(\gamma^2_{23}C_{1242} + \gamma^3_{23}C_{1342} + \gamma^4_{23}C_{1442})\\
&\quad + \frac{g^{rr}}{2}2i\mathrm{Im}(\gamma^3_{12}C_{1334}) + g^{rr}i\mathrm{Im}(\gamma^3_{13}C_{1342} + \gamma^4_{13}C_{1442})]\\
&= \frac{1}{3}[4i\mathrm{Re}\gamma\mathrm{Im}\Psi_2 - 2i\mathrm{Im}(\nu\Psi_1) - 4i\mathrm{Re}\beta\mathrm{Im}\Psi_3 + 2i\mathrm{Im}(\mu\Psi_2) + ig^{rr}\mathrm{Im}(\bar{\tau}\Psi_1) - ig^{rr}\mathrm{Im}(\bar{\rho}\Psi_2)];
\end{aligned} \tag{D11}$$

(B)

$$\begin{aligned}
&\frac{1}{3}[l^aC_{2e34} + 2i\mathrm{Im}(m^aC_{2e41})]\left(\gamma^e_{2a} + \frac{g^{rr}}{2}\gamma^e_{1a}\right)\\
&= \frac{1}{3}\left[\gamma^e_{21}C_{2e34} + 2i\mathrm{Im}(\gamma^e_{23}C_{2e41}) + \frac{g^{rr}}{2}\gamma^e_{11}C_{2e34} + g^{rr}i\mathrm{Im}(\gamma^e_{13}C_{2e41})\right]\\
&= \frac{1}{3}[2i\mathrm{Im}(\gamma^3_{21}C_{2334}) + 2i\mathrm{Im}(\gamma^3_{23}C_{2341} + \gamma^4_{23}C_{2441})\\
&\quad + \frac{g^{rr}}{2}(\gamma^1_{11}C_{2134} + 2i\mathrm{Im}(\gamma^3_{11}C_{2334})) + g^{rr}i\mathrm{Im}(\gamma^1_{13}C_{2141} + \gamma^3_{13}C_{2341} + \gamma^4_{13}C_{2441})]\\
&= \frac{1}{3}[-2i\mathrm{Im}(\pi\bar{\Psi}_3) + 2i\mathrm{Im}(\mu\bar{\Psi}_2) + 2ig^{rr}\mathrm{Re}\epsilon\mathrm{Im}\Psi_2 + ig^{rr}\mathrm{im}(\bar{\kappa}\bar{\Psi}_3) + 2ig^{rr}\mathrm{Re}\beta\mathrm{Im}(\bar{\Psi}_1) - ig^{rr}\mathrm{Im}(\bar{\rho}\bar{\Psi}_2)];
\end{aligned} \tag{D12}$$

(D)

$$\begin{aligned}
&\frac{1}{3}[l^a C_{1e34} + 2i\mathrm{Im}(m^a C_{1e41})]\left(\gamma^e_{2a} + \frac{g^{rr}}{2}\gamma^e_{1a}\right)\\
&= \frac{1}{3}\left[\gamma^e_{21}C_{1e34} + 2i\mathrm{Im}(\gamma^e_{23}C_{1e41}) + \frac{g^{rr}}{2}\gamma^e_{11}C_{1e34} + g^{rr}i\mathrm{Im}(\gamma^e_{13}C_{1e41})\right]\\
&= \frac{1}{3}[\gamma^2_{21}C_{1234} + 2i\mathrm{Im}(\gamma^3_{21}C_{1334}) + 2i\mathrm{Im}(\gamma^2_{23}C_{1241} + \gamma^3_{23}C_{1341} + \gamma^4_{23}C_{1441})\\
&\quad + \frac{g^{rr}}{2}2i\mathrm{Im}(\gamma^3_{11}C_{1334}) + g^{rr}i\mathrm{Im}(\gamma^3_{13}C_{1341} + \gamma^4_{13}C_{1441})]\\
&= \frac{1}{3}[4i\mathrm{Re}\epsilon\mathrm{Im}\Psi_2 - 2i\mathrm{Im}(\pi\Psi_1) + 4i\mathrm{Re}\beta\mathrm{Im}\bar{\Psi}_1 - 2i\mathrm{Im}(\bar{\lambda}\bar{\Psi}_0) + ig^{rr}\mathrm{Im}(\bar{\kappa}\Psi_1) + ig^{rr}\mathrm{Im}(\sigma\bar{\Psi}_0)];
\end{aligned} \tag{D13}$$

(E)

$$\frac{1}{3}[-2in^a\mathrm{Im}\Psi_2 + 2i\mathrm{Im}(m^a\Psi_3)]\nabla_a g^{rr} = \frac{-2i}{3}\left[\frac{-2M}{r^2} + O(r^{-3})\right]n^a\nabla_a r\mathrm{Im}\Psi_2 = \frac{2Mi}{3r}[1 + O(r^{-1})]g^{rr}\mathrm{Im}\Psi_2; \tag{D14}$$

(F)

$$\frac{1}{3}[-2in^a\mathrm{Im}\Psi_2 - 2i\mathrm{Im}(m^a\bar{\Psi}_1)]\nabla_a g^{rr} = \frac{-2i}{3}\left[\frac{-2M}{r^2} + O(r^{-3})\right]n^a\nabla_a r\mathrm{Im}\Psi_2 = \frac{2Mi}{3r}[1 + O(r^{-1})]g^{rr}\mathrm{Im}\Psi_2. \tag{D15}$$

We use now the asymptotic properties of the spin coefficients (see Appendix C) in the limit to future null infinity to simplify all these quantities. Notice that $\sqrt{h} \sim r^2$, so all terms that decay faster than $1/r^2$ vanish at $\mathcal{J}^+$. On the other hand, all the spin coefficients decay at least as $1/r$, and because all terms above are of the form spin coefficient $\times$ Weyl scalar, the only nonvanishing contributions are those that involve $\Psi_4 = O(\frac{1}{r})$. Among all of them, we have to take the one whose spin coefficient only decays as $O(\frac{1}{r})$, which is λ. Doing this we get

$$\int d^4x\sqrt{-g}\langle\nabla_a j^a_5\rangle = -\frac{\hbar}{6}\int_{\hat{M}} p_1(\hat{R}) = \frac{\hbar}{12\pi^2}\int_{\mathcal{J}^+} dud\mathbb{S}^2\mathrm{Im}(\ddot{\sigma}^0\dot{\bar{\sigma}}^0). \tag{D16}$$

Recalling that $N_{33} = 2\dot{\sigma}^0$ (see Appendix C), we recover Eq. (51).

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