Odd-parity stability of black holes in Einstein-aether gravity

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In Einstein-aether theory, we study the stability of black holes against odd-parity perturbations on a spherically symmetric and static background. For odd-parity modes, there are two dynamical degrees of freedom arising from the tensor gravitational sector and aether vector field. We derive general conditions under which neither ghosts nor Laplacian instabilities are present for these dynamical fields. We apply these results to concrete black hole solutions known in the literature and show that some of those solutions can be excluded by the violation of stability conditions. The exact Schwarzschild solution present for $c_{13} = c_{14} = 0$, where c_i's are the four coupling constants of the theory with $c_{ij} = c_i + c_j$, is prone to Laplacian instabilities along the angular direction throughout the horizon exterior. However, we find that the odd-parity instability of high radial and angular momentum modes is absent for black hole solutions with $c_{13} = c_4 = 0$ and $c_1 \geq 0$.

I. INTRODUCTION

General Relativity (GR) is a fundamental theory of gravity well tested by solar-system experiments. With the dawn of gravitational-wave astronomy, it is now possible to probe the validity of GR around black holes (BHs) and neutron stars [1,2]. Recently, there has been growing interest in searching for extra degrees of freedom beyond GR and standard model of particle physics in such a strong gravity regime [3,4]. The existence of new degrees of freedom is also motivated by the firm observational evidence of dark matter and dark energy [5–7].

The construction of GR is based on Lorentz invariance (LI), which is a continuous symmetry invariant under the 4-dimensional diffeomorphism. In discrete spacetime that can arise from the quantization of gravity, the Lorentz symmetry can be broken at very high energy. The violation of LI in standard model fields is tightly limited from various experiments [8,9], but the Lorentz violation in the gravity sector is much less constrained [10,11]. Hořava gravity [12,13] is an example of allowing for gravitational Lorentz violation at high energy, in which a Lifshitz-type anisotropic scaling is introduced to realize a power-counting renormalizable theory of gravity (for a recent review of Hořava gravity, see, for example, [14] and references therein).

There is yet the other type of a gravitational Lorentz-violating scenario dubbed Einstein-aether theory [15,16]. In this scenario there is a unit timelike vector (aether) field u^α at every point in spacetime characterized by the metric tensor $g_{\alpha\beta}$, so it breaks local Lorentz symmetry under a rotation. This is a subclass of vector-tensor theories possessing two derivative terms of the aether field. The existence of a unit aether field is ensured by the constraint $g_{\alpha\beta}u^\alpha u^\beta = -1$ [with the metric signature $(-, +, +, +)$], which appears as the Lagrange multiplier $\lambda(g_{\alpha\beta}u^\alpha u^\beta + 1)$ in the action. We note that generalized Proca theories with a broken $U(1)$ gauge symmetry [17–21] do not have such a constraint, so the vector-field dynamics is generally different from that in Einstein-aether theory.

In Einstein-aether theory there are scalar, transverse vector, and tensor perturbations, whose propagation speeds c_S, c_V, c_T on the Minkowski background are generally different from that of light [22]. To ensure the stability of Minkowski spacetime, we require that all of c_S^2, c_V^2, and c_T^2 are positive. Moreover, the observations of gravitational Cerenkov radiation [23], solar system tests [24], big bang nucleosynthesis [25], binary pulsars [26,27], and gravitational waves [28,29] put constraints on the dimensionless coupling constants $c_{1,2,3,4}$ of aether derivative interactions. In particular, the gravitational-wave event GW170817 [2] together with the gamma-ray burst 170817A [30] placed the upper limit $|c_T - 1| \lesssim 10^{-15}$, which translates to $|c_{13}| \lesssim 10^{-15}$ [28,29], where $c_{ij} := c_i + c_j$. However, there are still

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theoretically viable parameter spaces in which all the observational constraints are satisfied.

In Einstein-aether theory, the existence and properties of spherically symmetric vacuum solutions have been extensively studied in the literature [31–44]. Some of them were already excluded by the combination of observational bounds mentioned above. However, the recent papers [45,46] have shown the presence of spherically symmetric and static BH solutions compatible with current observational constraints. Since the speeds of scalar and transverse vector perturbations can be arbitrarily large, there exists a universal horizon corresponding to a causal boundary of any large speeds of propagation [14,35,36]. The universal horizon can exist inside the event horizon, so that particles can cross the event horizon to escape toward infinity. It is expected that this unique feature of Einstein-aether BHs may leave some distinguished signatures in the gravitational-wave measurements of binary BHs.

In this paper, we study the stability of spherically symmetric and static BHs against odd-parity perturbations in Einstein-aether theory. We first identify two dynamical gauge-invariant perturbations corresponding to the tensor and vector propagations. Then, we obtain the second-order action of odd-parity perturbations and explicitly derive stability conditions for the absence of ghosts and Laplacian instabilities. The tensor and vector propagation speeds along the radial and angular directions are different from those in Minkowski spacetime. Thus, our analysis of BH perturbations in the odd-parity sector provides new stability conditions for Einstein-aether BHs. We also note that our general formulation of odd-parity perturbations will be useful to study the propagation of gravitational waves during the inspiral and ringdown phases of binary BHs.

We apply our conditions to the Einstein-aether BH solutions known in the literature. We show that an exact Schwarzschild BH present for the couplings $c_{13} = 0$ and $c_{14} = 0$ is excluded by the Laplacian instability along the angular direction. The BH solutions with $c_{13} = 0$, $c_{14} \neq 0$, and $c_4 \neq 0$ are prone to the ghost instability by imposing a superluminal propagation of the transverse vector mode ($c_4 < 0$) to avoid the gravitational Cerenkov radiation. However, provided that $c_1 \geq 0$, the BH solutions with $c_{13} = 0$ and $c_4 = 0$ are stable against odd-parity perturbations with high radial and angular momentum modes. Thus, our general stability conditions are sufficiently powerful to distinguish between unstable and stable BHs in Einstein-aether theory.

II. BACKGROUND EQUATIONS OF MOTION

We begin with the Einstein-aether theory given by the action [22]

$$S = \frac{1}{16\pi G_æ} \int \sqrt{-g} \mathrm{d}^4 x [R + \mathcal{L}_æ + \lambda(g_{\alpha\beta}u^\alpha u^\beta + 1)],$$
(2.1)

where $G_æ$ is a constant, R is the Ricci scalar, g is the determinant of metric tensor $g_{\alpha\beta}$, λ is a Lagrange multipler, u^α is the aether vector field, and

$$\mathcal{L}_æ = -M^{\alpha\beta}{}_{\mu\nu}\nabla_\alpha u^\mu \nabla_\beta u^\nu,$$
(2.2)

with

$$M^{\alpha\beta}{}_{\mu\nu} := c_1 g^{\alpha\beta}g_{\mu\nu} + c_2\delta^\alpha_\mu\delta^\beta_\nu + c_3\delta^\alpha_\nu\delta^\beta_\mu - c_4 u^\alpha u^\beta g_{\mu\nu}.$$
(2.3)

Here, the Greek indices represent from 0 to 3, ∇_α is a covariant derivative operator with respect to the metric tensor $g_{\mu\nu}$, and c_i's are four dimensionless coupling constants.

Variation of the action (2.1) with respect to λ leads to

$$u^\alpha u_\alpha + 1 = 0.$$
(2.4)

This constraint ensures the existence of a timelike unit vector field, so that there is a preferred frame responsible for the breaking of LI. Varying Eq. (2.1) with respect to u^μ, it follows that

$$\nabla_\mu J^\mu{}_\alpha + \lambda u_\alpha + c_4 \dot{u}^\mu \nabla_\alpha u_\mu = 0,$$
(2.5)

where

$$J^\mu{}_\alpha := M^{\mu\nu}{}_{\alpha\beta}\nabla_\nu u^\beta,$$
(2.6)

$$\dot{u}^\mu := u^\beta \nabla_\beta u^\mu.$$
(2.7)

Multiplying Eq. (2.5) by u^α and using Eq. (2.4), the Lagrange multiplier can be expressed as

$$\lambda = u^\alpha \nabla_\mu J^\mu{}_\alpha + c_4 \dot{u}^\mu \dot{u}_\mu.$$
(2.8)

For the general line element $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$, the gravitational field equations derived by the variation of (2.1) with respect to $g_{\mu\nu}$ are

$$G_{\alpha\beta} = \nabla_\mu[u_{(\alpha}J^\mu{}_{\beta)} + u^\mu J_{(\alpha\beta)} - u_{(\alpha}J_{\beta)}{}^\mu]$$
$$+ c_1(\nabla_\alpha u^\nu \nabla_\beta u_\nu - \nabla^\nu u_\alpha \nabla_\nu u_\beta)$$
$$+ c_4 \dot{u}_\alpha \dot{u}_\beta + \frac{1}{2}g_{\alpha\beta}\mathcal{L}_æ + \lambda u_\alpha u_\beta,$$
(2.9)

where $G_{\alpha\beta}$ is the Einstein tensor.

In general, the theory admits three different species of gravitons, the spin-0, spin-1, and spin-2 ones. According to the perturbative analysis on the Minkowski background, their squared speeds are given by [22]

$$c_S^2 = \frac{c_{123}(2 - c_{14})}{c_{14}(1 - c_{13})(2 + c_{13} + 3c_2)},$$
(2.10)

$$c_V^2 = \frac{2c_1 - c_{13}(2c_1 - c_{13})}{2c_{14}(1 - c_{13})},$$
(2.11)

$$c_T^2 = \frac{1}{1 - c_{13}}, \qquad (2.12)$$

where $c_{ijk} := c_i + c_j + c_k$, and $c_{S,V,T}$ represent the speeds of the spin-0, spin-1, and spin-2 gravitons, respectively. If we require that the theory: (i) be self-consistent, such as free of ghosts and instability; and (ii) be compatible with all the observational constraints obtained so far, it was found that the parameters c_i's must satisfy the conditions [29]

$$|c_{13}| \lesssim 10^{-15}, \qquad (2.13)$$

$$0 < c_{14} \le 2.5 \times 10^{-5}, \qquad (2.14)$$

$$c_{14} \le c_2 \le 0.095, \qquad (2.15)$$

$$c_4 \le 0. \qquad (2.16)$$

It should be noted that the above conditions assure $c_{S,V,T} \ge 1$, that is, all the propagation speeds are not subluminal, in order to avoid the gravitational Cerenkov radiation [23]. Later, we shall come to this point again when we study the odd-parity stability of BHs in Sec. V.

With the above in mind, let us consider a spherically symmetric and static background given by

$$ds^2 = -f(r)dt^2 + h^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (2.17)$$

where f and h depend on the distance r from the center of symmetry. The aether-field profile compatible with the background (2.17) is of the form

$$u^\mu = (a(r), b(r), 0, 0), \qquad (2.18)$$

where a and b are functions of r. The constraint (2.4) gives the following relation

$$b = \epsilon\sqrt{(a^2 f - 1)h}, \qquad (2.19)$$

where $\epsilon = \pm 1$. The existence of the aether-field profile (2.19) requires that $(a^2 f - 1)h \ge 0$.

Under the constraint (2.19), there are three independent background equations of motion following from (2.5) and (2.9), with the Lagrange multiplier λ determined by Eq. (2.8). Then, the $\alpha = 0$ component of Eq. (2.5) and $(\alpha, \beta) = (1, 1), (2, 2)$ components of Eq. (2.9) lead to

$$2h\alpha_1 a'' + \alpha_1 h'a' + 2h\alpha_2 f'' + \alpha_2 f'h' + \alpha_3 a'^2 + \alpha_4 f'^2$$
$$+ \alpha_5 a'f' + \alpha_6 f' + \alpha_7 h' + \alpha_8 a' + \alpha_9 = 0, \qquad (2.20)$$

$$2h\beta_1 a'' + \beta_1 h'a' + 2h\beta_2 f'' + \beta_2 f'h' + \beta_3 a'^2 + \beta_4 f'^2$$
$$+ \beta_5 a'f' + \beta_6 f' + \beta_7 h' + \beta_8 a' + \beta_9 = 0, \qquad (2.21)$$

$$2h\mu_1 a'' + \mu_1 h'a' + 2h\mu_2 f'' + \mu_2 f'h' + \mu_3 a'^2 + \mu_4 f'^2$$
$$+ \mu_5 a'f' + \mu_6 f' + \mu_7 h' + \mu_8 a' = 0, \qquad (2.22)$$

where a prime represents the derivative with respect to r. The explicit form of λ as well as the coefficients $\alpha_1, \cdots \alpha_9$, $\beta_1, \ldots, \beta_9$, and $\mu_1, \cdots \mu_8$ are given in Appendix A. We note that Eqs. (2.20)–(2.22) hold irrespective of the sign of ϵ in Eq. (2.19). For given coupling constants c_i's, the variables f, h, and a are known by integrating Eqs. (2.20)–(2.22) with appropriate boundary conditions.

III. SECOND-ORDER ACTION OF ODD-PARITY PERTURBATIONS AND GENERAL STABILITY CONDITIONS

In this section, we derive the second-order action of dynamical perturbations in the odd-parity sector to study the stability of spherically symmetric and static BH solutions in Einstein-aether theory. Analogous to the analysis performed in Ref. [47] in the context of generalized Proca theories, we consider metric perturbations $h_{\mu\nu}$ on the background (2.17) as well as the perturbation of the aether field. We express the perturbations in terms of the sum of spherical harmonics $Y_{lm}(\theta, \varphi)$.

For $l \ge 2$, we choose the Regge-Wheeler gauge in which the components h_{ij}, where i and j correspond to either θ or φ, vanish [48,49]. For the dipole ($l = 1$), the metric components h_{ij} vanish identically, so we need to handle this case separately. In the following, we first study the case $l \ge 2$ and then proceed to the discussion for $l = 1$.

A. $l \ge 2$

In the Regge-Wheeler gauge, the nonvanishing components of metric perturbations are given by

$$h_{ti} = \sum_{l,m} Q_{lm}(t, r) E_{ij} \partial^j Y_{lm}(\theta, \varphi), \qquad (3.1)$$

$$h_{ri} = \sum_{l,m} W_{lm}(t, r) E_{ij} \partial^j Y_{lm}(\theta, \varphi), \qquad (3.2)$$

where the subscripts i, j represent either θ or φ with the notation $\partial^j Y_{lm} = \partial Y_{lm}/\partial x_j$, and Q_{lm} and W_{lm} are functions of t and r. The tensor E_{ij} is defined by $E_{ij} = \sqrt{\gamma}\varepsilon_{ij}$, where $\gamma = \sin^2\theta$ is the determinant of two dimensional metric γ_{ij} on the sphere and ε_{ij} is the antisymmetric symbol with $\varepsilon_{\theta\varphi} = 1$.

In the presence of odd-parity perturbations, the covariant aether field is expressed as

$$u_\mu = \left(-a(r)f(r), \frac{b(r)}{h(r)}, u_\theta, u_\varphi\right), \qquad (3.3)$$

where the $i = \theta, \varphi$ components are

$$u_i = \sum_{l,m} \delta u_{lm}(t,r) E_{ij} \partial^j Y_{lm}(\theta, \varphi). \qquad (3.4)$$

The perturbation δu_{lm} is a function of t and r. We expand the action (2.1) up to second order in odd-parity perturbations. In doing so, we can set $m = 0$ without loss of generality and multiply the action 2π for the integral with respect to φ [47]. In the following, we also omit the subscripts "lm" from the variables Q_{lm}, W_{lm}, and δu_{lm} for the simplification of notation. On using the background

Eqs. (2.20)–(2.22), the resulting second-order action of odd-parity perturbations is expressed in the form

$$\mathcal{S}_{\text{odd}} = \sum_{l,m} L \int dt dr \mathcal{L}_{\text{odd}}, \qquad (3.5)$$

where

$$L := l(l+1), \qquad (3.6)$$

and

$$\mathcal{L}_{\text{odd}} = \frac{r^2}{16\pi G_{\text{æ}}} \sqrt{\frac{f}{h}} \left[C_1 \left(\dot{W} - Q' + \frac{2}{r}Q \right)^2 + 2(C_2 \dot{\delta u} + C_3 \delta u' + C_4 \delta u) \left(\dot{W} - Q' + \frac{2}{r}Q \right) + C_5 \dot{\delta u}^2 + C_6 \dot{\delta u} \delta u' \right.$$
$$\left. + C_7 \delta u'^2 + (L-2)(C_8 W^2 + C_9 W \delta u - a C_9 W Q + C_{10} Q^2 + C_{11} Q \delta u) + (L C_{12} + C_{13}) \delta u^2 \right], \qquad (3.7)$$

with a dot being the derivative with respect to t. The coefficients C_i's in Eq. (3.7) are given in Appendix B. Even with the unit-vector constraint $u^\alpha u_\alpha + 1 = 0$ in Einstein-aether theory, the Lagrangian (3.7) is of the same form as that derived for generalized Proca theories [47], with the correspondence of the temporal vector component $A_0 \to -af$. The difference appears only for the coefficients C_i's, so we can resort to the prescription exploited in Ref. [47] for the derivation of stability conditions of dynamical perturbations.

Let us consider the gauge transformation $x_\mu \to x_\mu + \xi_\mu$, where

$$\xi_t = \xi_r = 0, \qquad \xi_i = \sum_{lm} \Lambda(t,r) E_{ij} \partial^j Y_{lm}(\theta, \varphi). \qquad (3.8)$$

Then, the perturbations Q, W, and δu transform as [47,50,51]

$$Q \to Q + \dot{\Lambda}, \qquad (3.9)$$

$$W \to W + \Lambda' - \frac{2\Lambda}{r}, \qquad (3.10)$$

$$\delta u \to \delta u. \qquad (3.11)$$

Besides the aether perturbation δu, we consider the following gauge-invariant combination

$$\chi = \dot{W} - Q' + \frac{2}{r}Q + \frac{C_2 \dot{\delta u} + C_3 \delta u' + C_4 \delta u}{C_1}, \qquad (3.12)$$

which is associated with the tensor perturbation in the odd-parity gravity sector. The gauge-invariant perturbation (3.12) is introduced to combine the first and second contributions to the square brackets of Eq. (3.7). We express the Lagrangian (3.7) in the form

$$\mathcal{L}_{\text{odd}} = \frac{r^2}{16\pi G_{\text{æ}}} \sqrt{\frac{f}{h}} \left[C_1 \left\{ 2\chi \left(\dot{W} - Q' + \frac{2}{r}Q + \frac{C_2 \dot{\delta u} + C_3 \delta u' + C_4 \delta u}{C_1} \right) - \chi^2 \right\} - \frac{(C_2 \dot{\delta u} + C_3 \delta u' + C_4 \delta u)^2}{C_1} + C_5 \dot{\delta u}^2 \right.$$
$$\left. + C_6 \dot{\delta u} \delta u' + C_7 \delta u'^2 + (L-2)(C_8 W^2 + C_9 W \delta u - a C_9 W Q + C_{10} Q^2 + C_{11} Q \delta u) + (L C_{12} + C_{13}) \delta u^2 \right], \qquad (3.13)$$

where χ is regarded as a Lagrange multiplier independent of the fields W and Q in Eq. (3.13). The similar treatment was also performed in the context of scalar-tensor theories [50–52] and generalized Proca theories [47].

Varying Eq. (3.13) with respect to W and Q, it follows that

$$2C_1 \dot{\chi} - (L-2)[2C_8 W + C_9(\delta u - aQ)] = 0, \qquad (3.14)$$

$$2C_1 \chi' + \frac{2rfhC_1' + (8fh + rf'h - rfh')C_1}{rfh} \chi - (L-2)(aC_9 W - 2C_{10}Q - C_{11}\delta u) = 0. \qquad (3.15)$$

These equations can be solved for W and Q to express them in terms of χ, $\dot{\chi}$, χ', and δu. Substituting them into Eq. (3.13) and integrating it by parts, we obtain the reduced Lagrangian

$$\mathcal{L}_{\text{odd}} = \frac{r^2}{16\pi G_{\text{æ}}(L-2)} \sqrt{\frac{f}{h}} (\dot{\vec{\mathcal{X}}}^t \boldsymbol{K} \dot{\vec{\mathcal{X}}} + \dot{\vec{\mathcal{X}}}^t \boldsymbol{R} \vec{\mathcal{X}}'$$
$$+ \vec{\mathcal{X}}'^t \boldsymbol{G} \vec{\mathcal{X}}' + \vec{\mathcal{X}}^t \boldsymbol{M} \vec{\mathcal{X}}), \tag{3.16}$$

where

$$\vec{\mathcal{X}}^t = (\chi, \delta u), \tag{3.17}$$

and $\boldsymbol{K}, \boldsymbol{R}, \boldsymbol{G}, \boldsymbol{M}$ are 2×2 symmetric matrices. We note that the contributions to Eq. (3.16) of the forms $\dot{\vec{\mathcal{X}}}^t \boldsymbol{T} \vec{\mathcal{X}}$ and $\vec{\mathcal{X}}'^t \boldsymbol{S} \vec{\mathcal{X}}$, which appear in generalized Proca theories [47], vanish in Einstein-aether theory. The Lagrangian (3.16) can now be used to study the stability of dynamical fields χ and δu.

The nonvanishing components of $\boldsymbol{K}$ are given by

$$K_{11} = q_1, \qquad K_{22} = (L-2)q_2, \tag{3.18}$$

where

$$q_1 := \frac{4C_1^2 C_{10}}{a^2 C_9^2 - 4C_8 C_{10}}, \qquad q_2 := \frac{C_1 C_5 - C_2^2}{C_1}. \tag{3.19}$$

To avoid the appearance of ghosts, we require that

$$q_1 > 0, \tag{3.20}$$

$$q_2 > 0, \tag{3.21}$$

where the former and latter correspond to the no-ghost conditions of gravity and vector-field sectors, respectively.

The matrices $\boldsymbol{R}$ and $\boldsymbol{G}$ have the following nonvanishing components

$$R_{11} = \mathcal{R}_{11} q_1, \qquad R_{22} = (L-2)\mathcal{R}_{22}, \tag{3.22}$$

$$G_{11} = \mathcal{G}_{11} q_1, \qquad G_{22} = (L-2)\mathcal{G}_{22}, \tag{3.23}$$

where

$$\mathcal{R}_{11} := -\frac{aC_9}{C_{10}}, \qquad \mathcal{R}_{22} := \frac{C_1 C_6 - 2C_2 C_3}{C_1}, \tag{3.24}$$

$$\mathcal{G}_{11} := \frac{C_8}{C_{10}}, \qquad \mathcal{G}_{22} := \frac{C_1 C_7 - C_3^2}{C_1}. \tag{3.25}$$

To derive the dispersion relation along the radial direction, we assume the solutions of Eq. (3.17) in the form

$\vec{\mathcal{X}}^t = \vec{\mathcal{X}}_0^t e^{i(\omega t - kr)}$, where $\vec{\mathcal{X}}_0^t$ is a constant vector, and ω and k are the constant frequency and wave number respectively. In the limits $k \to \infty$ and $\omega \to \infty$, the existence of nonvanishing solutions of $\vec{\mathcal{X}}^t$ requires that $\det(\omega^2 \boldsymbol{K} - \omega k \boldsymbol{R} + k^2 \boldsymbol{G}) = 0$. Since there are no off-diagonal components in $\boldsymbol{K}$, $\boldsymbol{R}$, and $\boldsymbol{G}$, it follows that

$$\omega^2 - \omega k \mathcal{R}_{11} + k^2 \mathcal{G}_{11} = 0, \tag{3.26}$$

$$\omega^2 q_2 - \omega k \mathcal{R}_{22} + k^2 \mathcal{G}_{22} = 0. \tag{3.27}$$

In terms of the proper time $\tau = \int \sqrt{f} dt$ and the rescaled radial coordinate $r_* = \int dr/\sqrt{h}$, the propagation speed of perturbations along the radial direction is given by $c_r = dr_*/d\tau = \hat{c}_r/\sqrt{fh}$, where $\hat{c}_r = dr/dt = \omega/k$ is the propagation speed in the coordinates t and r. Substituting $\omega = k\sqrt{fh}c_r$ into Eqs. (3.26) and (3.27), the solutions to c_r are given, respectively, by

$$c_{r1} = \frac{\mathcal{R}_{11} \pm \sqrt{\mathcal{F}_1}}{2\sqrt{fh}}, \tag{3.28}$$

$$c_{r2} = \frac{\mathcal{R}_{22} \pm \sqrt{\mathcal{F}_2}}{2q_2\sqrt{fh}}, \tag{3.29}$$

where

$$\mathcal{F}_1 := \mathcal{R}_{11}^2 - 4\mathcal{G}_{11}, \tag{3.30}$$

$$\mathcal{F}_2 := \mathcal{R}_{22}^2 - 4q_2\mathcal{G}_{22}. \tag{3.31}$$

The speeds c_{r1} and c_{r2} correspond to the radial sound speeds associated with the propagation of gravity and vector-field sectors, respectively. Depending on the direction of radial propagation, the signs of c_{r1} and c_{r2} can be either positive or negative. As long as c_{r1} and c_{r2} are real, we have $c_{r1}^2 \geq 0$ and $c_{r2}^2 \geq 0$. Hence the absence of Laplacian instabilities along the radial direction requires that

$$\mathcal{F}_1 \geq 0, \tag{3.32}$$

$$\mathcal{F}_2 \geq 0. \tag{3.33}$$

The propagation speed c_Ω along the angular direction can be derived by taking the limits $L \to \infty$ and $\omega \to \infty$ in Eq. (3.16). In these limits, the dominant contributions to the matrix components of $\boldsymbol{M}$ are given by

$$M_{11} = -LC_1, \qquad M_{22} = L^2 D_1, \tag{3.34}$$

where

$$D_1 := C_{12} + \frac{C_8 C_{11}^2 + C_9^2 (C_{10} + aC_{11})}{4C_1^2 C_{10}} q_1. \qquad (3.35)$$

There are also the matrix components $M_{12}(=M_{21})$ proportional to L, but they do not affect the angular sound speeds derived below. Substituting the solution of the form $\vec{\mathcal{X}}^t = \vec{\mathcal{X}}_0^t e^{i(\omega t - l\theta)}$ into the perturbation equations following from (3.16), we obtain the dispersion relation $\det(\omega^2 \boldsymbol{K} + \boldsymbol{M}) = 0$. There are no off-diagonal components of $\boldsymbol{K}$ and $\boldsymbol{M}$, so that

$$\omega^2 q_1 - LC_1 = 0, \qquad (3.36)$$

$$\omega^2 q_2 + LD_1 = 0. \qquad (3.37)$$

The angular propagation speed in proper time is given by $c_\Omega = r d\theta/d\tau = \hat{c}_\Omega/\sqrt{f}$, where $\hat{c}_\Omega = r d\theta/dt$ satisfies $\omega^2 = \hat{c}_\Omega^2 l^2/r^2$. Taking the limit $L \to \infty$ and substituting the relation $\omega^2 = c_\Omega^2 f L/r^2$ into Eqs. (3.36) and (3.37), the solutions to c_Ω^2 are given, respectively, by

$$c_{\Omega 1}^2 = \frac{C_1 r^2}{f q_1}, \qquad (3.38)$$

$$c_{\Omega 2}^2 = -\frac{D_1 r^2}{f q_2}. \qquad (3.39)$$

To avoid the Laplacian instabilities along the angular direction, we require that

$$c_{\Omega 1}^2 \geq 0, \qquad (3.40)$$

$$c_{\Omega 2}^2 \geq 0, \qquad (3.41)$$

which translate to $C_1/q_1 \geq 0$ and $D_1/q_2 \leq 0$, respectively, outside the horizon ($f > 0$).

So far, we have considered the stabilities of perturbations χ and δu along the radial and angular directions by separately taking the limits $k \to \infty$ or $L \to \infty$. We will also study the propagation of inclined modes where the limits $k \to \infty$ and $L \to \infty$ are taken, with the ratio

$$\xi := \frac{k}{\sqrt{L}}, \qquad (3.42)$$

being constant. In this case, we substitute the solution $\vec{\mathcal{X}}^t = \vec{\mathcal{X}}_0^t e^{i(\omega t - kr - l\theta)}$ into the perturbation equations following from Eq. (3.16). Then, the dispersion relation yields $\det(\omega^2 \boldsymbol{K} - \omega k \boldsymbol{R} + k^2 \boldsymbol{G} + \boldsymbol{M}) = 0$, so that

$$q_1(\omega^2 - \omega k \mathcal{R}_{11} + k^2 \mathcal{G}_{11}) - LC_1 = 0, \qquad (3.43)$$

$$\omega^2 q_2 - \omega k \mathcal{R}_{22} + k^2 \mathcal{G}_{22} + LD_1 = 0. \qquad (3.44)$$

Solving Eqs. (3.43) and (3.44) for ω respectively, we obtain the dispersion relations for the perturbations χ and δu, as

$$\omega = \frac{\sqrt{L}}{2} \left[\mathcal{R}_{11} \xi \pm \sqrt{\mathcal{F}_1 \xi^2 + \frac{4C_1}{q_1}} \right], \qquad (3.45)$$

$$\omega = \frac{\sqrt{L}}{2q_2} \left[\mathcal{R}_{22} \xi \pm \sqrt{\mathcal{F}_2 \xi^2 - 4q_2 D_1} \right], \qquad (3.46)$$

where $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined by Eqs. (3.30) and (3.31). The absence of Laplacian instabilities for the perturbations χ and δu can be ensured under the conditions

$$\mathcal{F}_1 \xi^2 + \frac{4C_1}{q_1} \geq 0, \qquad (3.47)$$

$$\mathcal{F}_2 \xi^2 - 4q_2 \mathcal{D}_1 \geq 0, \qquad (3.48)$$

respectively. Let us first consider the stability of the perturbation χ. In the limit $\xi \to \infty$, there is no Laplacian instability for $\mathcal{F}_1 \geq 0$, see Eq. (3.32). In the other limit $\xi \to 0$, the angular propagation speed squared is given by Eq. (3.38), so the stability is ensured for $C_1/q_1 \geq 0$. Under these two conditions, the inequality (3.47) holds for any arbitrary values of ξ. On using Eq. (3.48), we also find that the same property holds for the perturbation δu, i.e., the conditions (3.33) and (3.41) are sufficient to ensure the Laplacian stability of the inclined mode. Hence the inclined mode does not give rise to additional conditions to those derived for purely radial and angular modes.

In summary, for $l \geq 2$, the stabilities of perturbations for high radial and angular momentum modes are ensured under the conditions (3.20), (3.21), (3.32), (3.33), (3.40) and (3.41). We caution that these conditions are derived in the large k or (and) l limits, so they are not sufficient to guarantee all the stabilities for finite values of k and l. Due to the complexity of matrix components of $\boldsymbol{M}$, we do not consider the stability of perturbations for such an intermediate range of k and l.

In addition, instabilities might arise when we consider the spectrum of $\vec{\mathcal{X}}^t$, by solving the corresponding differential equations for χ and δu with boundary conditions. However, such studies are out of the scope of the current paper, and we wish to return to this important issue in another occasion.

B. $l = 1$

Since the metric components h_{ij} vanish identically for the dipole perturbation ($l = 1$), there is a gauge degree of freedom to be fixed. In this case, we choose the gauge $W = 0$. From Eq. (3.10), the gauge-transformation scalar $\Lambda(t, r)$ is constrained to be

$$\Lambda = -r^2 \int d\tilde{r} \frac{W(t,\tilde{r})}{\tilde{r}^2} + r^2 \mathcal{C}(t), \qquad (3.49)$$

where $\mathcal{C}(t)$ is a function of t. The Lagrangian (3.7) has been derived for $l \geq 2$ with a nonvanishing W, but it is also valid for $l = 1$ by setting $W = 0$ for the above gauge choice. An alternative procedure to be taken for $l = 1$ is that we literally exploit the Lagrangian (3.7), vary it with respect to W and Q, and set $W = 0$ at the end. This process leads to

$$\dot{\mathcal{E}} = 0, \qquad (r^2 \mathcal{E})' = 0, \qquad (3.50)$$

where

$$\mathcal{E} := r^2 \sqrt{\frac{f}{h}} \left[C_1 \left(Q' - \frac{2}{r} Q \right) - (C_2 \delta u + C_3 \delta u' + C_4 \delta u) \right]. \qquad (3.51)$$

From Eq. (3.50), we obtain the integrated solution

$$\mathcal{E} = \frac{\mathcal{E}_0}{r^2}, \qquad (3.52)$$

where $\mathcal{E}_0$ is a constant. On using Eq. (3.51), the perturbation Q can be expressed as

$$Q = r^2 \int d\tilde{r} \frac{1}{C_1 \tilde{r}^2} \left(\frac{\mathcal{E}_0}{\tilde{r}^4} \sqrt{\frac{h}{f}} + C_2 \delta u + C_3 \delta u' + C_4 \delta u \right) + r^2 \mathcal{C}_2(t), \qquad (3.53)$$

where $\mathcal{C}_2$ is a function of t. From Eqs. (3.9) and (3.53), the residual gauge mode $\mathcal{C}(t)$ in Eq. (3.49) can be removed by setting

$$\mathcal{C}(t) = \int d\tilde{t} \mathcal{C}_2(\tilde{t}). \qquad (3.54)$$

On using Eq. (3.52) with Eq. (3.51), we can eliminate the terms containing $-Q' + 2Q/r$ in Eq. (3.7). This process leads to the reduced Lagrangian

$$\mathcal{L}_{odd} = \frac{r^2}{16\pi G_{\ae}} \sqrt{\frac{f}{h}} \left[\frac{C_1 C_5 - C_2^2}{C_1} \dot{\delta u}^2 + \left(C_6 - \frac{2C_2 C_3}{C_1} \right) \dot{\delta u} \delta u' + \left(C_7 - \frac{C_3^2}{C_1} \right) \delta u'^2 \right. \\ \left. - \frac{2C_3 C_4}{C_1} \delta u' \delta u + \left(2C_{12} + C_{13} - \frac{C_4^2}{C_1} \right) \delta u^2 + \frac{h \mathcal{E}_0^2}{C_1 r^8 f} \right]. \qquad (3.55)$$

This shows that only the aether perturbation δu propagates. From the coefficient of $\dot{\delta u}^2$ in Eq. (3.55), we find that the ghost is absent for $(C_1 C_5 - C_2^2)/C_1 > 0$, which is equivalent to the condition $q_2 > 0$ derived for $l \geq 2$. From the first three terms in Eq. (3.55), we can also show that the radial propagation speed is equivalent to c_{r2} given by Eq. (3.29). This is analogous to the result found for generalized Proca theories [47]. In summary, for $l = 1$, there are neither ghost nor Laplacian instabilities under the conditions $q_2 > 0$ and $\mathcal{F}_2 \geq 0$.

IV. STABILITY CONDITIONS IN EINSTEIN-AETHER THEORIES

To study the odd-parity stabilities of BHs, we use the explicit forms of C_i's given in Appendix B together with the value of b constrained by Eq. (2.19). Then, the no-ghost conditions (3.20) and (3.21) translate to

$$q_1 = \frac{h(1 - c_{13})(1 - c_{13} a^2 f)}{2f^2} > 0, \qquad (4.1)$$

$$q_2 = \frac{c_1 + c_4 a^2 f}{r^2 f} - \frac{c_{13}^2 (a^2 f - 1)}{2r^2 f (1 - c_{13})} > 0. \qquad (4.2)$$

The conditions (3.32) and (3.33), which ensure the absence of Laplacian instabilities along the radial direction, are given by

$$\mathcal{F}_1 = \frac{4fh(1 - c_{13})}{(1 - c_{13} a^2 f)^2} \geq 0, \qquad (4.3)$$

$$\mathcal{F}_2 = \frac{2hc_{14}[2c_1 - c_{13}(2c_1 - c_{13})]}{r^4 f (1 - c_{13})} \geq 0. \qquad (4.4)$$

The conditions (3.40) and (3.41) for the absence of Laplacian instabilities along the angular direction translate to

$$c_{\Omega 1}^2 = \frac{1}{1 - c_{13} a^2 f} \geq 0, \qquad (4.5)$$

$$c_{\Omega 2}^2 = \frac{2c_1 - c_{13}(2c_1 - c_{13})}{2(1 - c_{13})(c_1 + c_4 a^2 f) - c_{13}^2 (a^2 f - 1)} \geq 0. \qquad (4.6)$$

For the dipole ($l = 1$), only the two conditions (4.2) and (4.4) need to be satisfied for the aether perturbation δu.

On the Minkowski background characterized by the metric components $f = h = 1$, the aether field is given by $u^\mu = (+1, 0, 0, 0)$ and hence $a = 1$ and $b = 0$. Then, in Minkowski spacetime, the stability conditions (4.1)–(4.6) reduce, respectively, to

$$(q_1)_{\text{Min}} = \frac{(1 - c_{13})^2}{2} > 0, \qquad (4.7)$$

$$(q_2)_{\text{Min}} = \frac{c_{14}}{r^2} > 0, \qquad (4.8)$$

$$(\mathcal{F}_1)_{\text{Min}} = \frac{4}{1 - c_{13}} \geq 0, \qquad (4.9)$$

$$(\mathcal{F}_2)_{\text{Min}} = \frac{2c_{14}[2c_1 - c_{13}(2c_1 - c_{13})]}{r^4(1 - c_{13})} \geq 0, \qquad (4.10)$$

$$(c_{\Omega 1}^2)_{\text{Min}} = \frac{1}{1 - c_{13}} \geq 0, \qquad (4.11)$$

$$(c_{\Omega 2}^2)_{\text{Min}} = \frac{2c_1 - c_{13}(2c_1 - c_{13})}{2c_{14}(1 - c_{13})} \geq 0, \qquad (4.12)$$

which are satisfied for

$$c_{13} < 1, \qquad (4.13)$$

$$c_{14} > 0, \qquad (4.14)$$

$$2c_1 - c_{13}(2c_1 - c_{13}) \geq 0. \qquad (4.15)$$

These conditions coincide with those derived in Refs. [22,29] by expanding the action (2.1) up to quadratic order in tensor and vector perturbations on the Minkowski background. On using the fact that the coefficients C_9, C_2, and C_6 vanish in Eqs. (3.28) and (3.29), the radial sound speed squares in Minkowski spacetime are given by

$$(c_{r1}^2)_{\text{Min}} = \frac{1}{1 - c_{13}}, \qquad (4.16)$$

$$(c_{r2}^2)_{\text{Min}} = \frac{2c_1 - c_{13}(2c_1 - c_{13})}{2c_{14}(1 - c_{13})}. \qquad (4.17)$$

Then we have that $(c_{r1}^2)_{\text{Min}} = (c_{\Omega 1}^2)_{\text{Min}}$ and $(c_{r2}^2)_{\text{Min}} = (c_{\Omega 2}^2)_{\text{Min}}$, while this equality does not generally hold on the curved background (2.17).

As shown in Refs. [22,29], $(c_{r1}^2)_{\text{Min}}$ and $(c_{r2}^2)_{\text{Min}}$ correspond to the propagation speed squares of tensor and vector perturbations on the Minkowski background, respectively, see Eqs. (2.12) and (2.11). From the gravitational-wave

event GW170817 [2] together with its electromagnetic counterpart [30], the speed of the tensor perturbations is in the range $-3 \times 10^{-15} < (c_{r1})_{\text{Min}} - 1 < 7 \times 10^{-16}$, so the coupling c_{13} is constrained to be

$$|c_{13}| \lesssim 10^{-15}. \qquad (4.18)$$

We note that there are also stability conditions in the Minkowski spacetime arising from scalar perturbations [22,29]. They can be derived by considering even-parity perturbations on the curved background (2.17) and taking the Minkowski limit $f \to 1$, $h \to 1$, and $a \to 1$.

In this paper, we will not carry out the analysis of even-parity perturbations, but we will show in Sec. V that the stability analysis based on odd-parity perturbations alone is sufficiently powerful to exclude some BH solutions in Einstein-aether theory.

V. STABILITY OF EINSTEIN-AETHER BLACK HOLES

Let us consider the stability of spherically symmetric and static BH solutions known in the literature. Performing the transformation

$$dv = dt + \frac{dr}{\sqrt{fh}}, \qquad (5.1)$$

the line element (2.17) is transformed to the Eddington-Finkelstein coordinate of the form

$$ds^2 = -f(r)dv^2 + 2B(r)dvdr + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (5.2)$$

where

$$B(r) = \sqrt{\frac{f}{h}}. \qquad (5.3)$$

From Eq. (3.3), the nonvanishing components of the background aether field u_μ are given by $u_t = -af$ and $u_r = b/h$, where b is constrained as Eq. (2.19). On using Eq. (5.1), we have

$$u_t dt + u_r dr = u_v dv + \tilde{u}_r dr, \qquad (5.4)$$

where

$$u_v = -af, \qquad \tilde{u}_r = a\sqrt{\frac{f}{h}} + \frac{b}{h}. \qquad (5.5)$$

Since $g^{vv} = 0$, $g^{vr} = g^{rv} = \sqrt{h/f}$, and $g^{rr} = h$, the nonvanishing components of u^μ in the (v, r) coordinate are given by

$$u^v = a + \frac{b}{\sqrt{fh}}, \qquad \tilde{u}^r = b. \qquad (5.6)$$

In the notation of Ref. [45], the variable A is used for u^v, in which case we have

$$u^v = A = a + \frac{b}{\sqrt{fh}}, \qquad \tilde{u}^r = b = \frac{fA^2 - 1}{2AB}. \qquad (5.7)$$

Since we would like to consider the case in which the gravitational-wave bound (4.18) is satisfied, we will focus on the BH solutions satisfying the conditions

$$c_{13} = 0, \qquad (5.8)$$

in the following analysis.

A. Stealth Schwarzschild solution

We first consider the coupling constants satisfying

$$c_{14} = 0. \qquad (5.9)$$

For this choice we have $(q_2)_{\text{Min}} = 0$ from Eq. (4.8), so there is a strong coupling problem on the Minkowski background. In curved spacetime the stability conditions are different from those on the Minkowski background, so we will study whether BH solutions satisfying the condition (5.9) are stable or not.

The background Eqs. (2.20)–(2.22) admit the existence of an exact stealth BH solution characterized by

$$f = h = 1 - \frac{r_s}{r}, \qquad (5.10)$$

$$a = \frac{\sqrt{4r^3(r - r_s) + w_2^2}}{2r(r - r_s)}, \qquad b = \epsilon \frac{w_2}{2r^2}, \qquad (5.11)$$

where r_s is the Schrawarzschild radius, $\epsilon = \pm 1$, and w_2 is a positive constant. From Eqs. (5.3) and (5.7), we have

$$A = \frac{\epsilon w_2 + \sqrt{4r^3(r - r_s) + w_2^2}}{2r(r - r_s)}, \qquad B = 1, \qquad (5.12)$$

so that $A \to 1$ as $r \to \infty$. For $\epsilon = +1$, the temporal vector component diverges as $u^v = A \propto (r - r_s)^{-1}$ around $r = r_s$. In this case, the quantity $J := A^2 f$ is in the range $J > 1$ outside the horizon and it exhibits the divergence $J \to (w_2^2/r_s^3)(r - r_s)^{-1}$ for $r \to r_s$. On the other hand, for $\epsilon = -1$, the expansion of A around $r = r_s$ gives

$$A = \frac{r_s^2}{w_2} - \frac{r_s(r_s^4 - 2w_2^2)}{w_2^3}(r - r_s) + \mathcal{O}((r - r_s)^2), \qquad (5.13)$$

and hence A is finite at $r = r_s$. In this case, $J < 1$ outside the horizon and $J = 0$ at $r = r_s$. For this latter branch ($b < 0$), the above exact BH solution with $w_2 = 3\sqrt{3}r_s^2/8$ gives rise to a universal horizon at $r = 3r_s/4$ [45].

From Eq. (5.11), the temporal vector component u^t in the (t, r) coordinate has the divergent behavior $u^t = a \propto (r - r_s)^{-1}$ as $r \to r_s$, irrespective of the signs of b. Defining the quantity

$$j := a^2 f, \qquad (5.14)$$

we have

$$j = \frac{w_2^2}{4r^3(r - r_s)} + 1, \qquad (5.15)$$

and hence $j > 1$ outside the horizon. There is the divergence $j \to \infty$ as $r \to +r_s$, with the asymptotic behavior $j \to 1$ at spatial infinity.

From Eqs. (4.1)–(4.6), the quantities associated with the stability conditions are given by

$$q_1 = \frac{1}{2f}, \qquad q_2 = -\frac{c_1(j - 1)}{r^2 f}, \qquad (5.16)$$

$$\mathcal{F}_1 = 4f^2, \qquad \mathcal{F}_2 = 0, \qquad (5.17)$$

$$c_{\Omega 1}^2 = 1, \qquad c_{\Omega 2}^2 = -\frac{1}{j - 1}. \qquad (5.18)$$

The conditions $q_1 > 0$, $\mathcal{F}_1 \geq 0$, $\mathcal{F}_2 \geq 0$, and $c_{\Omega 1}^2 \geq 0$ are satisfied for $r > r_s$. Since $j > 1$ outside the horizon, we have $c_{\Omega 2}^2 < 0$ and hence there is a Laplacian instability along the angular direction. In particular, as r increases from r_s to spatial infinity, $c_{\Omega 2}^2$ changes from -0 to $-\infty$. For $c_1 > 0$, we have $q_2 < 0$ outside the horizon, so the ghost instability is also present. From Eqs. (3.28) and (3.29), the radial propagation speed squares are given by

$$c_{r1}^2 = 1, \qquad c_{r2}^2 = \frac{j}{j - 1}. \qquad (5.19)$$

Since both c_{r1}^2 and c_{r2}^2 are positive outside the horizon, the Laplacian instabilities are absent along the radial direction.

In summary, the stealth Schwarzschild solution with the vector-field profile (5.11) is unstable due to the Laplacian instability associated with the negative propagation speed squared $c_{\Omega 2}^2$ outside the horizon. In addition, for $c_1 > 0$, the ghost instability for the aether perturbation also exists. It is interesting to note that stealth Schwarzschild solutions present in the context of generalized Proca theories [53–55] are also unstable against odd-parity perturbations [47].

B. BH solutions with $c_{14} \neq 0$

We proceed to study the stability of BH solutions for the couplings

$$c_{14} \neq 0. \tag{5.20}$$

Then, the quantities in Eqs. (4.1)–(4.6) reduce to

$$q_1 = \frac{h}{2f^2}, \qquad q_2 = \frac{c_1 + c_4 j}{r^2 f}, \tag{5.21}$$

$$\mathcal{F}_1 = 4fh, \qquad \mathcal{F}_2 = \frac{4hc_1 c_{14}}{r^4 f}, \tag{5.22}$$

$$c_{\Omega 1}^2 = 1, \qquad c_{\Omega 2}^2 = \frac{c_1}{c_1 + c_4 j}, \tag{5.23}$$

where j is defined by Eq. (5.14). Since $f > 0$ and $h > 0$ outside the horizon, the conditions $q_1 > 0$, $\mathcal{F}_1 \geq 0$, and $c_{\Omega 1}^2 \geq 0$ are satisfied. The condition $\mathcal{F}_2 \geq 0$ translates to

$$c_1 c_{14} \geq 0. \tag{5.24}$$

Taking the asymptotically flat (Minkowski) limit $a \to 1$, $f \to 1$, and $h \to 1$ in Eqs. (5.21) and (5.23), it follows that $q_2 \to c_{14}/r^2$ and $c_{\Omega 2}^2 \to c_1/c_{14}$. Then, the stability conditions $q_2 > 0$ and $c_{\Omega 2}^2 \geq 0$ translate to

$$c_{14} > 0, \qquad c_1 \geq 0, \tag{5.25}$$

which are compatible with Eq. (5.24). In the Minkowski limit the propagation speed squares of the transverse vector mode along both radial and angular directions are $(c_{r2}^2)_{\text{Min}} = (c_{\Omega 2}^2)_{\text{Min}} = c_1/c_{14}$, so the propagation is subluminal (or superluminal) for $c_4 > 0$ (or for $c_4 < 0$). In Einstein-aether theory, the gravitational Cerenkov radiation can occur for the subluminal propagation of transverse vector mode. For an interaction between a fermion and a graviton studied in Ref. [23], the emission rate Γ from a fermion for the transverse vector mode is proportional to $c_{13}^2[1 - (c_{r2}^2)_{\text{Min}}]$ [23], so that $\Gamma = 0$ for $c_{13} = 0$. When $(c_{r2}^2)_{\text{Min}} < 1$, however, there may be a possibility that other higher-order interactions give rise to the gravitational Cerenkov radiation even for $c_{13} = 0$. In the superluminal range realized by the coupling $c_4 < 0$, there is no constraint arising from the gravitational Cerenkov radiation.

To discuss the stability of BH solutions around the horizon, we search for background solutions where the temporal vector component $u^v = A$ in the (v, r) coordinate is regular at $r = r_s$ like Eq. (5.13). In doing so, we expand A, f, h around $r = r_s$ in the forms

$$A = A_0 + A_1(r - r_s) + A_2(r - r_s)^2 + \cdots, \tag{5.26}$$

$$f = f_1(r - r_s) + f_2(r - r_s)^2 + f_3(r - r_s)^3 \cdots, \tag{5.27}$$

$$h = h_1(r - r_s) + h_2(r - r_s)^2 + h_3(r - r_s)^3 \cdots, \tag{5.28}$$

where A_i, f_i, h_i are constants. On using Eqs. (5.3) and (5.7), there is the following relation

$$a = \frac{A^2 f + 1}{2Af}. \tag{5.29}$$

Then, we can express Eqs. (2.20)–(2.22) as the differential equations for A, f, h, instead of those for a, f, h. Substituting Eqs. (5.26)–(5.28) into such differential equations, we find that there are solutions where the coefficients $A_{1,2,3,\ldots}$ and metric components are related to the constant A_0. For the special case with $c_{14} = 0$, we confirmed that the iterative solutions derived by this prescription coincide with those obtained by expanding Eqs. (5.10) and (5.12) around $r = r_s$.

From Eqs. (5.26)–(5.28), the temporal metric component a in the (t, r) coordinate and the quantity $j = a^2 f$ have the following dependence around the horizon:

$$a = \frac{1}{2A_0 f_1}(r - r_s)^{-1} + \mathcal{O}((r - r_s)^0), \tag{5.30}$$

$$j = \frac{1}{4A_0^2 f_1}(r - r_s)^{-1} + \mathcal{O}((r - r_s)^0). \tag{5.31}$$

Since j diverges at $r = r_s$, the quantities q_2 and $c_{\Omega 2}^2$ around the horizon can be estimated as

$$q_2 = \frac{c_4}{4A_0^2 r_s^2 f_1^2}(r - r_s)^{-2} + \mathcal{O}((r - r_s)^{-1}), \tag{5.32}$$

$$c_{\Omega 2}^2 = \frac{4c_1 A_0^2 f_1}{c_4}(r - r_s) + \mathcal{O}((r - r_s)^2). \tag{5.33}$$

From Eq. (5.32) the ghost is absent for

$$c_4 > 0. \tag{5.34}$$

Provided that $c_1 \geq 0$, we also have $c_{\Omega 2}^2 \geq 0$ around $r = r_s$. Indeed, for $c_1 \geq 0$ and $c_4 > 0$, the two conditions $q_2 > 0$ and $c_{\Omega 2}^2 \geq 0$ hold throughout the horizon exterior, since j is positive. We note that the odd-parity stability about the Minkowski spacetime, which is satisfied under the conditions (5.25), does not necessarily require that $c_4 > 0$ (unless the superluminality of $c_{\Omega 2}^2$ is imposed). For BHs the term $c_4 j$ in Eq. (5.21) dominates over c_1 around the horizon, so the positivity of q_2 demands that $c_4 > 0$. In other words, the inequality (5.34) is a new stability condition derived by the analysis on the curved background.

The BH solution with $c_4 < 0$ and $c_1 > 0$ is plagued by the ghost instability as well as the Laplacian instability around the horizon. If we restrict the superluminal propagation of transverse vector mode, we have $c_4 < 0$ and hence the BH solution in this case is unstable.

So far, we have performed the expansion of Taylor series of A as Eq. (5.26) with a finite value of A at $r = r_s$. Suppose that there is a solution of A diverging at $r = r_s$ in the form

$$A = \frac{A_0}{(r - r_s)^p}, \qquad (5.35)$$

where A_0 and p are constants. Here we are considering positive values of p, but we also include the case $p = 0$ in the analysis below. Analogous to the discussion in scalar-tensor theories [56], we consider the scalar product $J^{\mu\nu} J_{\mu\nu}$ for the current tensor $J^\mu{}_\alpha$ defined in Eq. (2.6) and impose the regularity of $J^{\mu\nu} J_{\mu\nu}$ at $r = r_s$. On using Eq. (5.35) and regular expansions of f and h as those in Eqs. (5.27) and (5.28), the scalar product $J^{\mu\nu} J_{\mu\nu}$ diverges at $r = r_s$ apart from the special cases $p = 0$ and $p = 1$. For p close to be 0 or 1, there is the power-law dependence $J^{\mu\nu} J_{\mu\nu} \propto 1/(r - r_s)^q$ with q close to 2. The powers $p = 0$ and $p = 1$ are the special cases in which $J^{\mu\nu} J_{\mu\nu}$ is regular at $r = r_s$. For $p > 1$, the scalar product diverges as $J^{\mu\nu} J_{\mu\nu} \propto 1/(r - r_s)^{2p}$ at $r = r_s$.

The expansion of A performed in Eq. (5.26) corresponds to the power $p = 0$, in which case the regularity of $J^{\mu\nu} J_{\mu\nu}$ is ensured at the horizon. For $p = 1$, expanding the quantities q_2 and $c_{\Omega 2}^2$ around $r = r_s$ gives

$$q_2 = \frac{c_4 A_0^2}{4r_s^2}(r - r_s)^{-2} + \mathcal{O}((r - r_s)^{-1}), \qquad (5.36)$$

$$c_{\Omega 2}^2 = \frac{4c_1}{c_4 A_0^2 f_1}(r - r_s) + \mathcal{O}((r - r_s)^2). \qquad (5.37)$$

Under the superluminal condition $c_4 < 0$, there is the ghost instability ($q_2 < 0$) as well as the Laplacian instability ($c_{\Omega 2}^2 < 0$) for $c_1 > 0$. As in the case of $p = 0$, we require the conditions (5.25) and (5.34) to ensure the odd-parity stability of BHs, but in this case the propagation of transverse vector mode is subluminal.

C. BH solutions with $c_4 = 0$

Let us finally discuss the stability of BH solutions for the coupling

$$c_4 = 0. \qquad (5.38)$$

In this case, the quantities q_1, $\mathcal{F}_1$, and $c_{\Omega 1}^2$ are the same as those given in Eqs. (5.21), (5.22), and (5.23), which are all positive outside the horizon. For $c_1 \neq 0$, the other quantities are given by

$$q_2 = \frac{c_1}{r^2 f}, \qquad \mathcal{F}_2 = \frac{4hc_1^2}{r^4 f}, \qquad c_{\Omega 2}^2 = 1. \qquad (5.39)$$

Provided that $c_1 > 0$, the ghost is absent.

When $c_1 = 0$, both the denominator and numerator of $c_{\Omega 2}^2$ in Eq. (5.23) vanish. This reflects the fact that, for $c_1 = 0$, the vector perturbation does not propagate as in the case of GR. The coupling constant c_2 does not appear in any of the stability conditions obtained in Sec. IV, so the case $c_1 = 0$ can be regarded as the GR limit for the couplings under consideration now (i.e., $c_1 = 0$, $c_3 = 0$, and $c_4 = 0$). In this case, we only need to consider the stability conditions $q_1 > 0$, $\mathcal{F}_1 \geq 0$, and $c_{\Omega 1}^2 \geq 0$ in the odd-parity sector, all of which are trivially satisfied outside the horizon.

In summary, for $c_4 = 0$, the stability of BHs against odd-parity perturbations with large values of k and l is ensured for

$$c_1 \geq 0. \qquad (5.40)$$

There are numerically obtained BH solutions consistent with this range of couplings [31,45].

VI. CONCLUSIONS

In this paper, we studied the stability of spherically symmetric and static BHs against odd-parity perturbations in Einstein-aether theory. On the background (2.17), the presence of a unit vector constraint (2.4) gives the relation (2.19) between the temporal and radial components of the aether field. At the background level, there are three independent Eqs. (2.20)–(2.22) to be solved for a and the metric components f and h.

In Sec. III, we derived the second-order action of odd-parity perturbations by using the expansion in terms of the spherical harmonics $Y_{lm}(\theta, \varphi)$. Choosing the Regge-Wheeler gauge for $l \geq 2$, we obtained the second-order Lagrangian of the form (3.7) and identified χ and δu as the two dynamical perturbations associated with the gravity sector and the aether field, respectively. After the integration by parts, the Lagrangian of these dynamical fields is given by Eq. (3.16). We showed that there are neither ghost nor Laplacian instabilities under the conditions (3.20), (3.21), (3.32), (3.33), (3.40) and (3.41) for large values of k and l. For the dipole ($l = 1$) the propagating degree of freedom is the aether perturbation δu alone, which does not give additional constraints to those derived for $l \geq 2$.

Using the explicit forms of coefficients C_i's given in Appendix B, the stability conditions in Einstein-aether theory reduce to Eqs. (4.1)–(4.6). In the limit of the Minkowski spacetime, we also showed in Sec. IV that the propagation speeds along both radial and angular directions coincide with those of tensor and vector perturbations already derived in the literature. The combination of coupling

constants $c_{13} = c_1 + c_3$ is tightly constrained to be $|c_{13}| \lesssim 10^{-15}$ from the GW170817 event together with 170817A.

In Sec. V, the odd-parity stabilities of BHs in Einstein-aether theory were studied for the couplings satisfying

$$c_{13} = 0. \tag{6.1}$$

This choice is consistent with the constraint (4.18) on the speed of tensor perturbations in the range $-3 \times 10^{-15} < (c_{r1})_{\text{Min}} - 1 < 7 \times 10^{-16}$ obtained from the gravitational-wave event GW170817 [2] and its electromagnetic counterpart [30]. In doing so, we used the relations of metric and vector-field components between the two different coordinates (2.17) and (5.2). In Sec. V A, we considered the exact Schwarzschild solution present for $c_{14} = 0$ and found that there is a Laplacian instability along the angular direction throughout the horizon exterior. Moreover, for $c_1 > 0$, the ghost instability also exists for the aether perturbation. In Sec. V B, we discussed the BH solutions for $c_{14} \neq 0$ and showed that their stabilities require the conditions (5.25) and $c_4 > 0$. In this case, the propagation of the vector perturbation is subluminal in the asymptotically flat regime, so there is a possibility for the gravitational Cerenkov radiation to occur. In other words, the superluminal propagation of transverse vector mode occurring for $c_4 < 0$, under which the gravitational Cerenkov radiation is avoided, is incompatible with the BH stability conditions. In Sec. V C, we showed that the BH solutions with

$$c_4 = 0, \qquad c_1 \geq 0, \tag{6.2}$$

are stable against odd-parity perturbations for high radial and angular momentum modes. Clearly, if we demand the odd-parity stability of BHs, the viable region of the parameter space of Eqs. (2.13)–(2.16), obtained recently in Ref. [29], is reduced further.

It is interesting to note that the instability of perturbations mainly happens in the aether field, represented by δu, while the metric part, represented by χ, behaves well for the coupling with $c_{13} = 0$.

The Lagrangian (3.7) of odd-parity perturbations can be applied to the computation of quasinormal modes of BHs. Moreover, the analysis of even-parity perturbations will provide us additional stability conditions of BHs to those derived in this paper. We leave these issues, together with the analyses of their corresponding quasinormal mode spectra, for future separate publications.

APPENDIX A: QUANTITIES IN BACKGROUND EQUATIONS

On the spherically symmetric and static background (2.17), the Lagrange multiplier (2.8) is given by

$$
\begin{aligned}
\lambda = \{ & fr^2[a^2(6c_1 + 3c_2 + 2c_{13} - 8c_{14})h(f')^2 + c_2(2hf'' + f'h')] \\
& + 2a^2 f^4[-2a^2(2(c_2 + c_{13})h - c_2 rh') + 4(c_{14} - c_1)hr^2(a')^2 + a(-c_1 + c_2 + c_{13})r(2hra'' + a'(rh' + 4h))] \\
& - f^2[rh'(a^2(-2c_1 + 3c_2 + 2c_{13})rf' - 4c_2) \\
& + 2h(a^4(3c_1 + c_2 + c_{13} - 4c_{14})r^2(f')^2 + a^2 r((-2c_1 + 3c_2 + 2c_{13})rf'' + 2(-2c_1 + c_2 + 2c_{13})f') \\
& - a(11c_1 - 5c_2 - 5c_{13} - 8c_{14})r^2 a'f' + 4(c_2 + c_{13})) - (2c_2 + c_{13})hr^2(f')^2 \\
& + 2f^3[a^4 r((-c_1 + c_2 + c_{13})rf'h' + 2h((-c_1 + c_2 + c_{13})rf'' + (-2c_1 + c_2 + 2c_{13})f')) \\
& + a^2(8(c_2 + c_{13})h - 4c_2 rh') - 2(-2c_1 + c_2 + c_{13} + 2c_{14})hr^2(a')^2 \\
& + a^3(-11c_1 + 3c_2 + 3c_{13} + 8c_{14})hr^2 a'f' + a(c_1 - c_2 - c_{13})r(2hra'' + a'(rh' + 4h))]\}/[4f^2 r^2(a^2 f - 1)], \tag{A1}
\end{aligned}
$$

where $c_{ij} = c_i + c_j$.

Introducing the following quantity

$$\beta := (c_2 + c_3 - c_4)fa^2 + c_{14}, \tag{A2}$$

We find that the coefficients in Eqs. (2.20)–(2.22) can be expressed as

$\alpha_1 = -2\beta(a^2f - 1)r^2f^2,$

$\alpha_2 = (c_2 + c_{13} - 2\beta)(a^2f - 1)r^2af$

$\alpha_3 = 4(c_2 + c_{13})r^2f^3ah,$

$\alpha_4 = -[(c_2 + c_{13} - 2\beta)a^2f - 2(c_2 + c_{13} - \beta)]r^2ah,$

$\alpha_5 = 2[(2c_2 + 2c_{13} - 3\beta)a^2f + 3\beta]r^2fh,$

$\alpha_6 = 4(c_2a^2f + c_{13} - 2\beta)(a^2f - 1)rafh,$

$\alpha_7 = -4c_2(a^2f - 1)^2raf^2,$

$\alpha_8 = -8\beta(a^2f - 1)rf^2h,$

$\alpha_9 = 8(c_2 + c_{13})(a^2f - 1)^2af^2h, \tag{A3}$

$\beta_1 = 4\beta(a^2f - 1)r^2af^3,$

$\beta_2 = -2(c_2 + c_{13} - 2\beta)(a^2f - 1)r^2a^2f^2,$

$\beta_3 = -4[2(c_2 + c_{13})a^2f - \beta]r^2f^3h,$

$\beta_4 = [2(c_2 + c_{13} - 2\beta)a^4f^2 - 8(c_2 + c_{13} - \beta)a^2f$
$\qquad + c_2 + c_{13}]r^2h,$

$\beta_5 = -4[(2c_2 + 2c_{13} - 3\beta)a^2f + c_2 + c_{13} + \beta]r^2af^2h,$

$\beta_6 = -4(a^2f - 1)[2 + 2c_2 - 2(2c_2 - c_{13} + 2\beta)a^2f$
$\qquad + 2c_2a^4f^2]rfh,$

$\beta_7 = 8c_2(a^2f - 1)^2rf^3a^2,$

$\beta_8 = 16(c_2 + \beta)(a^2f - 1)raf^3h,$

$\beta_9 = 4[2 - \{2 + 4c_2 + 2c_{13} - 2(4c_2 + 3c_{13})a^2f$
$\qquad + 4(c_2 + c_{13})a^4f^2\}h]f^2(a^2f - 1), \tag{A4}$

$\mu_1 = -4c_2(a^2f - 1)raf^3,$

$\mu_2 = 2(1 + c_2 - 2c_2a^2f)(a^2f - 1)rf,$

$\mu_3 = 4(2c_2 + \beta - 2c_2a^2f)rf^3h,$

$\mu_4 = [2 + 3c_2 + c_{13} - 2(1 + c_2 + 2c_{13} - 2\beta)a^2f$
$\qquad - 4c_2a^4f^2]rh,$

$\mu_5 = 4(6c_2 - 7c_2a^2f - c_{13} + 2\beta)raf^2h,$

$\mu_6 = 4(a^2f - 1)[1 + 2c_2 + c_{13} - 3(2c_2 + c_{13})a^2f]fh,$

$\mu_7 = 4(a^2f - 1)[1 + 2c_2 + c_{13} - (2c_2 + c_{13})a^2f]f^2,$

$\mu_8 = -16(2c_2 + c_{13})(a^2f - 1)af^3h. \tag{A5}$

APPENDIX B: COEFFICIENTS IN PERTURBATION EQUATIONS

The coefficients in Eq. (3.7), which should be evaluated on the background (2.17), are

$$C_1 = \frac{(1 - c_{13})h}{2r^2f},$$

$$C_2 = -\frac{c_{13}b}{2r^2f},$$

$$C_3 = -\frac{c_{13}ah}{2r^2},$$

$$C_4 = \frac{[(2c_{14} - c_{13})(fa' + af')r + 2c_{13}af]h}{2r^3f},$$

$$C_5 = \frac{c_1 + c_4a^2f}{r^2f},$$

$$C_6 = \frac{2c_4ab}{r^2},$$

$$C_7 = \frac{[c_4(a^2f - 1) - c_1]h}{r^2},$$

$$C_8 = -\frac{[c_{13}(a^2f - 1) + 1]h}{2r^4},$$

$$C_9 = \frac{c_{13}b}{r^4},$$

$$C_{10} = \frac{1 - c_{13}a^2f}{2r^4f},$$

$$C_{11} = \frac{c_{13}a}{r^4},$$

$$C_{12} = -\frac{c_1}{r^4},$$

$$C_{13} = \frac{\lambda}{r^2} - \frac{c_{13}[(rh' + 2h - 2)f + rhf']}{2r^4f}$$
$$\qquad - \frac{2c_4(fa' + f'a)ah}{r^3}, \tag{B1}$$

where b is given by Eq. (2.19).

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