Dark gravitational sectors on a generalized scalar-tensor vector bundle model and cosmological applications

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In this work we present the foundations of generalized scalar-tensor theories arising from vector bundle constructions, and we study the kinematic, dynamical, and cosmological consequences. In particular, over a pseudo-Riemannian space-time base manifold, we define a fiber structure with two scalar fields. The resulting space is a 6-dimensional vector bundle endowed with a nonlinear connection. We provide the form of the geodesics and the Raychaudhuri and general field equations, both in the Palatini and metrical methods. When applied at a cosmological framework, this novel geometrical structure induces extra terms in the modified Friedmann equations, leading to the appearance of an effective dark energy sector, as well as of an interaction of the dark mater sector with the metric. We show that we can obtain the standard thermal history of the universe, with the sequence of matter and dark-energy epochs, and furthermore the effective dark-energy equation-of-state parameter can lie in the quintessence or phantom regimes, or exhibit the phantom-divide crossing.

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I. INTRODUCTION

Modified gravity has attracted a large amount of research for two reasons and thus motivations. First, at the purely theoretical level, it improves the renormalizability of general relativity and hence it may be the first step toward gravitational quantization [1]. Second, at the phenomenological, cosmological, level, it is one of the two main ways that can offer an explanation for the early- and late-time accelerated phases of the expansion of the universe [2,3]. Hence, it has an advantage comparing to the alternative way, which is to introduce by hand the inflaton or/and dark energy sectors while maintaining general relativity as the underlying gravitational theory [4,5].

Modified gravity theories can be obtained as extensions of the Einstein-Hilbert Lagrangian through the addition of extra terms, such as in $f(R)$ gravity [6,7], in $f(G)$ gravity [8], in Weyl gravity [9], in Lovelock gravity [10], etc. Additionally, they can be obtained through the insertion of extra scalar fields, coupled with curvature invariants, such as in the general class of scalar-tensor theories [11–14]. However, one interesting class of modified gravity arises from the consideration of alternative geometries, beyond the Riemannian framework of general relativity. Thus, one can start from the equivalent, torsional formulation of gravity and extend it obtaining $f(T)$ gravity [15], $f(T, T_G)$ gravity [16], etc. Similarly, one can allow for nonmetricity, obtaining symmetric teleparallel gravity [17], $f(Q)$ gravity [18], etc.

Inspired by the above, one may proceed to the construction of gravitational modifications through a more radical modification of the underlying geometrical structure, namely considering Finsler or Finsler-like geometries [19–41]. In the framework of these generalized metric structures in a vector bundle, scalar-tensor theories can naturally appear, and in particular the scalar fields play the role of fibers or internal variables [42–45].

On the other hand, theoretical and observational cosmological evidence have indicated the existence of dark matter sector [46–56]. Based on observational results, dark

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matter plays a significant role in the evolution of the universe, especially concerning the growth of structures [57]. Additionally, since its microphysics is unknown one could have the interesting case in which dark matter interacts with dark energy [58], a case that has significant advantages since it can lead to the alleviation of the coincidence problem [59] as well as of the H_0 tension [60]. Hence, the investigation of dark sectors in modified theories of gravity and cosmology is a fundamental subject for cosmological phenomena.

In the present work we are interested in constructing Finsler-like geometrical structures, which will induce scalar-tensor theory with two scalars-fibers models. In particular, we consider a pseudo-Riemannian 4-dimensional space-time with two fibers and we investigate the properties of F^6 space-time, with non-holonomic structures, extracting the Raychaudhuri and field equations. Finally, we apply these geometrical generalized scalar-tensor theories on vector bundle constructions on a cosmological framework, in order to examine their cosmological implications on the effective dark energy and dark matter sectors.

The paper is organized as follows. In Sec. II we present the basic geometrical concepts of the theory, analyzing the metric decomposition and the appearance of the geometric dark sectors, investigating also the geodesic structure. In Sec. III we consider the action on the fiber bundle, we derive the field equations with both Palatini and metrical methods, in holonomic and nonholonomic forms, and finally construct the involved energy-momentum tensor, incorporating the contributions of the dark matter sector. In Sec. IV we examine the Raychaudhuri equations in the context of the F^6 bundle geometry. In Sec. V we proceed to the application on a cosmological framework, showing the appearance of an effective dark sector that has a purely geometrical origin and which can lead to a universe behavior in agreement with observations. Finally, in Sec. VI we discuss the concluding remarks.

II. SCALAR-TENSOR THEORIES INDUCED FROM THE VECTOR BUNDLE

In this section we present the basics of the geometrical framework under consideration [42–45]. Firstly we will review the basic structure of the Lorentz fiber bundle, then we will describe the metric splitting and the appearance of the geometric dark sectors, and finally we will proceed to the geodesic investigation.

A. Basic structure of the Lorentz scalar tensor fiber bundle

We consider a 4-dimensional manifold M equipped with coordinates x^μ, $\mu = 0, \ldots, 3$ and a Lorentzian metric $g_{\mu\nu}(x)$ with signature $(-,+,+,+)$ on it. Over any open subset of M we define a fiber structure with two scalar degrees of freedom $\phi^{(1)}$ and $\phi^{(2)}$. The resulting space is a 6 dimensional space-time fiber bundle, F^6, over the pseudo-Riemannian base manifold M, with local coordinates $\{U^M\} = \{x^\mu, \phi^a\}$, which trivializes locally to the product, $M \times \{\phi^{(1)}\} \times \{\phi^{(2)}\}$. Capital indices $K, L, M, N, Z\ldots$ span all the range of values of indices on a fiber bundle's tangent space. Additionally, a coordinate transformation on the fiber bundle maps the old coordinates to the new as:

$$x^\mu \mapsto x'^\mu(x^\nu) \tag{1}$$

$$\phi^a(x) \mapsto \phi'^a(x') = \delta^a_b \phi^b(x) \tag{2}$$

where δ^a_b is the Kronecker symbol for the corresponding latin indices a, b which take values in the range $\{(1), (2)\}$ and the Jacobian matrix $\frac{\partial x'^\mu}{\partial x^\nu}$ is nondegenerate.

In the space at hand, the adapted basis is defined as

$$\{X_M\} = \{\delta_\mu, \partial_{(1)}, \partial_{(2)}\} \tag{3}$$

where

$$\delta_\mu = \partial_\mu - N^{(1)}_\mu(x^\nu, \phi^a)\partial_{(1)} - N^{(2)}_\mu(x^\nu, \phi^a)\partial_{(2)} \tag{4}$$

with $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and $\partial_a \equiv \frac{\partial}{\partial \phi^a}$. The fields $N^a_\mu(x^\nu, \phi^b)$ comprise a special type of nonlinear connection and it is a fundamental structure of the framework under consideration, since it connects the base manifold's tangent space with the one of the fiber. Furthermore, the dual basis is $\{X^M\} = \{\mathrm{d}x^\mu, \delta\phi^{(1)}, \delta\phi^{(2)}\}$ where $\delta\phi^a = \mathrm{d}\phi^a + N^a_\mu(x^\nu, \phi^b)\mathrm{d}x^\mu$ and $a = 1, 2$. Finally, the basis vectors transform as:

$$\delta'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\delta_\nu, \qquad \partial'_a = \delta^b_a \partial_b \tag{5}$$

where summations are implied over the ranges of values of μ and a.

From its defining relations (3), (4), the noncommutative nature of the adapted basis can be easily revealed. Specifically we obtain

$$[X_M, X_N] = \mathcal{W}^L{}_M N X_L \tag{6}$$

where $\mathcal{W}^L{}_M N$ are the structure functions of the adapted base algebra which obey the Jacobi identity,[1]

$$\circlearrowleft_{M,N,L}\{X_M \mathcal{W}^R{}_{NL} + \mathcal{W}^R{}_{MS}\mathcal{W}^S{}_{NL}\} = 0 \tag{7}$$

As can be directly observed, the nonzero components of the structure functions are,

[1] $\circlearrowleft_{M,N,L}$ indicates summation with respect to the cyclic permutation of the indices M, N, L.

$$\mathcal{W}^L{}_{MN} = \{\tilde{W}^a{}_{\mu\nu}, W^a{}_{\mu b}\} \tag{8}$$

where

$$\begin{aligned}\tilde{W}^a{}_{\mu\nu} &= \delta_\nu N^a_\mu - \delta_\mu N^a_\nu \\ W^a{}_{\mu b} &= \partial_b N^a_\mu \end{aligned} \tag{9}$$

The metric structure of the fiber bundle is defined as

$$\boldsymbol{G} = g_{\mu\nu}(x)\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + v_{ab}(x)\delta\phi^a \otimes \delta\phi^b \tag{10}$$

Furthermore, the form of the fiber metric is assumed to be

$$v_{ab}(x) = \delta_{ab}\phi(x) \tag{11}$$

and transforms as $v'_{ab}(x') = \delta^c_a\delta^d_b v_{cd}(x)$. This particular choice (11) encodes the mutual independence of the fiber scalar fields and their equivalent contribution in the internal space geometry.

The covariant derivative of a base vector X_M over E, with respect to a base vector X_N, is in general

$$D_{X_N} X_M = \boldsymbol{\Gamma}^L{}_{MN} X_L. \tag{12}$$

A special connection structure is chosen [45], such that the nonvanishing components of the vector bundle connection are[2]

$$\boldsymbol{\Gamma}^L{}_{MN} = \{L^\lambda{}_{\mu\nu}, L^c{}_{a\nu}, C^c{}_{\mu b}, C^\lambda{}_{ab}\}. \tag{13}$$

The above local connections determine the action of the covariant derivatives upon the adapted basis of the bundle. Further details about the geometrical structure of our consideration is given in the Appendix A.

Alongside with the general symmetry property $\boldsymbol{\Gamma}^L{}_{[MN]} = 0$ and under a trivial permutation of the indices, the general metricity condition

$$D_{X_M}\boldsymbol{G} = 0 \tag{14}$$

leads to the result

$$\boldsymbol{\Gamma}^L{}_{MN} = \frac{1}{2}\mathcal{G}^{RL}(X_M\mathcal{G}_{NR} + X_N\mathcal{G}_{RM} - X_R\mathcal{G}_{MN}). \tag{15}$$

As will soon be illustrated, relation (15) does not imply a Levi-Civita tensor. The nonholonomic nature of the adapted basis (6) gives rise to torsion contributions [see relation (21)]. Taking into account the presumed special connection structure (13) we arrive at the following explicit expressions for the nonvanishing components of the vector bundle special, linear connection:

$$L^\mu{}_{\nu\lambda}(x) = \Gamma^\mu{}_{\nu\lambda}(x) \tag{16}$$

$$C^a{}_{\mu b} = L^a{}_{b\mu} = \delta^a_b \frac{1}{2\phi}\partial_\mu\phi \tag{17}$$

$$C^\mu{}_{ab} = -\frac{1}{2}\delta_{ab}g^{\mu\nu}\partial_\nu\phi \tag{18}$$

where $\Gamma^\mu{}_{\nu\lambda}$ is the Levi-Civita connection of the second kind[3]

The curvature tensor of a linear connection is defined as

$$\begin{aligned}\mathcal{R}^K{}_{LMN} = {} & X_M\boldsymbol{\Gamma}^K{}_{LN} - X_N\boldsymbol{\Gamma}^K{}_{LM} + \boldsymbol{\Gamma}^R{}_{LN}\boldsymbol{\Gamma}^K{}_{RM} - \boldsymbol{\Gamma}^R{}_{LM}\boldsymbol{\Gamma}^K{}_{RN} \\ & - \mathcal{W}^R{}_{MN}\boldsymbol{\Gamma}^K{}_{LR}.\end{aligned} \tag{19}$$

In the holonomic base limit, $\mathcal{W}^L{}_{MN} = 0$, the generalized curvature tensor (19) reduces to the standard Riemann tensor.

The torsion tensor of the vector bundle is defined as

$$\mathcal{T}^L{}_{MN} = 2\boldsymbol{\Gamma}^L{}_{[MN]} + \mathcal{W}^L{}_{MN} \tag{20}$$

Since in our case $\boldsymbol{\Gamma}^L{}_{[MN]} = 0$, we have

$$\mathcal{T}^L{}_{MN} = \mathcal{W}^L{}_{MN} \tag{21}$$

Analogously, we define the generalized Ricci tensor

$$\begin{aligned}\mathcal{R}_{MN} &\equiv \mathcal{G}^L{}_K\mathcal{R}^K{}_{MLN} = \mathcal{R}^L{}_{MLN} \\ &= X_L\boldsymbol{\Gamma}^L{}_{MN} - X_N\boldsymbol{\Gamma}^L{}_{ML} + \boldsymbol{\Gamma}^L{}_{MN}\boldsymbol{\Gamma}^R{}_{LR} - \boldsymbol{\Gamma}^L{}_{MR}\boldsymbol{\Gamma}^R{}_{LN} \\ &\quad + \boldsymbol{\Gamma}^L{}_{MR}\mathcal{W}^R{}_{NL}\end{aligned} \tag{22}$$

The last term casts the tensor nonsymmetric as can be directly seen in (C9).

For the linear connection (16)–(18) we obtain the nonzero components of the generalized Ricci tensor[4]

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + R^{(\phi)}{}_{\mu\nu} \tag{23}$$

$$\mathcal{R}_{ab} = -\frac{1}{2}\Box\phi\delta_{ab} + \frac{1}{2}\delta_{ac}(\partial^\lambda\phi)W^c{}_{\lambda b} \tag{24}$$

$$\mathcal{R}_{a\mu} = C^\nu{}_{ab}\tilde{W}^b{}_{\mu\nu} \tag{25}$$

[2]Note that the selected connection structure is not the usual d-connection which preserves by parallelism the horizontal and vertical distributions [19].

[3]It is obvious from (8) that the structure functions $\mathcal{W}^L{}_{MN}$ nullify if all indices are space-time. Therefore, they do not add torsion if restricted in the base manifold.

[4]Note that as is obvious from (C9) the generalized Ricci tensor is nonsymmetric. Despite the fact that $\mathcal{R}_{\mu a} = 0$, we see that $\mathcal{R}_{a\mu} \neq 0$

where $\Box \equiv D^\mu D_\mu$, $R_{\mu\nu}$ is the Ricci tensor of Levi-Civita connection and

$$R^{(\phi)}{}_{\mu\nu} = \frac{1}{2\phi^2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{\phi}D_\mu D_\nu\phi + \frac{1}{2\phi}\partial_\mu\phi W^a{}_{\nu a} \tag{26}$$

the contribution of the pure scalar field.

Multiplying (18) with v^{ab} we can express the quantity $\partial^\mu\phi$ in terms of $C^\mu{}_{ab}$. Indeed, it is easy to see that

$$\partial^\mu\phi = -\phi v^{ab}C^\mu{}_{ab} = -\delta^{ab}C^\mu{}_{ab} \tag{27}$$

The corresponding scalar curvature is

$$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu} + v^{ab}\mathcal{R}_{ab} = R + R^{(\phi)} \tag{28}$$

where R is the Levi-Civita curvature and with the aid of (27),

$$R^{(\phi)} = -\frac{2}{\phi}\Box\phi - v^{ab}\left(\frac{1}{2\phi}\partial_\mu\phi + W^c{}_{\mu c}\right)C^\mu{}_{ab} \tag{29}$$

Lastly, the generalized Einstein's tensor is

$$\mathcal{E}_{MN} = \mathcal{R}_{MN} - \frac{1}{2}\mathcal{R}\mathcal{G}_{MN} \tag{30}$$

The tensor $\mathcal{E}_{MN}$ contains extra terms that come from the introduction of internal variables $\phi^{(1)}, \phi^{(2)}$ and their derivatives, giving a possible locally anisotropic contribution.

B. The geometrical effects of dark gravitational field

In the previous subsection we presented the underlying geometrical structure of the scalar-tensor theories that are induced from the vector bundle. Hence, we can now proceed to the investigation of their effects on the physical quantities such as the metric, and in particular of the appearance of dark sectors.

In order to account for the effects of the geometry of space-time on dark sectors, we follow the general study elaborated in [61]. In particular, the metric $g_{\mu\nu}(x)$ of the base manifold M is assumed to decompose into an "ordinary" (O) and a "dark" matter sector (D)

$$g_{\mu\nu}(x) = g^{(O)}{}_{\mu\nu}(x) + g^{(D)}{}_{\mu\nu}(x) \tag{31}$$

since from a physical point of view a unified description of gravity may include the gravitational interaction of both [62]. In the following, we postulate that the fiber space remains unaffected by the dark sector.

In analogy with (31), the Levi-Civita connection admits contributions from the ordinary (O) and dark matter (D) energy densities alongside a term that arises from their mutual interaction:

$$\Gamma^\mu{}_{\nu\lambda}(x) = \Gamma^{(O)\mu}{}_{\nu\lambda}(x) + \Gamma^{(D)\mu}{}_{\nu\lambda}(x) + \gamma^\mu{}_{\nu\lambda}(x) \tag{32}$$

Substituting (32) in the definition relation $\Gamma^\mu{}_{\nu\lambda} = g^{\mu\rho}\Gamma_{\rho\nu\lambda}$, the interaction part $\gamma^\mu{}_{\nu\lambda}$ can be easily expressed in terms of the inverse of the total, ordinary and dark matter metrics, as well as the respective connection parts of the second kind, namely

$$\gamma^\mu{}_{\nu\lambda} = (g^{\mu\rho} - g^{(O)\mu\rho})\Gamma^{(O)}{}_{\rho\nu\lambda} + (g^{\mu\rho} - g^{(D)\mu\rho})\Gamma^{(D)}{}_{\rho\nu\lambda} \tag{33}$$

As it is evident from (27) the connection $C^\mu{}_{ab}$ depends linearly on the inverse of the space-time metric. Therefore, it should split in a manner similar to (32), i.e.,

$$C^\mu{}_{ab}(x) = C^{(O)\mu}{}_{ab}(x) + C^{(D)\mu}{}_{ab}(x) + c^\mu{}_{ab}(x) \tag{34}$$

where

$$\begin{aligned} C^{(O)\mu}{}_{ab} &= -\frac{1}{2}\delta_{ab}g^{(O)\mu\nu}\partial_\nu\phi \\ C^{(D)\mu}{}_{ab} &= -\frac{1}{2}\delta_{ab}g^{(D)\mu\nu}\partial_\nu\phi \end{aligned} \tag{35}$$

and

$$c^\mu{}_{ab} = -\frac{1}{2}\delta_{ab}(g^{\mu\nu} - g^{(O)\mu\nu} - g^{(D)\mu\nu})\partial_\nu\phi \tag{36}$$

All other connections are not conditioned in such splittings, since, as can be seen from (17), they are not directly related to the metric of the base manifold.

In order to make manifest the contributions of the ordinary, dark matter, scalar and interaction sectors in the Ricci tensor, let us reformulate the above expressions in a spirit analogous to [61]. We have,

$$R_{\mu\nu} = R^{(O)}{}_{\mu\nu} + R^{(D)}{}_{\mu\nu} + r_{\mu\nu} \tag{37}$$

where $r_{\mu\nu}$ expresses the interactions between ordinary and dark matter.

In similar lines, we assume an analogous splitting for the $\Box$ operator, namely

$$\Box\phi = (\Box^{(O)} + \Box^{(D)} + \Box)\phi \tag{38}$$

where

$$\begin{aligned} \Box\phi &= g^{\mu\nu}D_\mu D_\nu\phi = g^{\mu\nu}(\partial_\mu\partial_\nu\phi - \Gamma^\lambda{}_{\mu\nu}\partial_\lambda\phi) \\ \Box^{(O)}\phi &= g^{(O)\mu\nu}(\partial_\mu\partial_\nu\phi - \Gamma^{(O)\lambda}{}_{\mu\nu}\partial_\lambda\phi) \\ \Box^{(D)}\phi &= g^{(D)\mu\nu}(\partial_\mu\partial_\nu\phi - \Gamma^{(D)\lambda}{}_{\mu\nu}\partial_\lambda\phi) \end{aligned} \tag{39}$$

Accordingly to (33), the interaction part $\Box$ can be expressed in terms of Christofell symbols and of the inverse of the total, ordinary and dark matter metrics.

Considering all the above, we can now write

$$\mathcal{R}_{ab} = \mathcal{R}^{(O)}{}_{ab} + \mathcal{R}^{(D)}{}_{ab} + \mathfrak{r}_{ab} \tag{40}$$

where

$$\mathcal{R}^{(O)}{}_{ab} = -\frac{1}{2}\delta_{ab}\Box^{(O)}\phi - \frac{1}{2}\delta_{ac}\delta^{de}W^{c}{}_{\mu b}C^{(O)\mu}{}_{de} \tag{41}$$

$$\mathcal{R}^{(D)}{}_{ab} = -\frac{1}{2}\delta_{ab}\Box^{(D)}\phi - \frac{1}{2}\delta_{ac}\delta^{de}W^{c}{}_{\mu b}C^{(D)\mu}{}_{de}$$
$$\mathfrak{r}_{ab} = -\frac{1}{2}\delta_{ab}\Box\phi - \frac{1}{2}\delta_{ac}\delta^{de}W^{c}{}_{\mu b}c^{\mu}{}_{de} \tag{42}$$

Analogously with the above, the extra fiber contribution to the Ricci scalar and the Einstein tensor can be straightforwardly decomposed into ordinary, dark and interaction sectors.

C. Geodesics

We close this section with an investigation of the geodesic structure of the theory. In particular, we will derive the geodesic equations imposing the autoparallel condition on the vector tangent to the geodesic curve. Let

$$Y = Y^{\mu}\delta_{\mu} + Y^{a}\partial_{a} \tag{43}$$

be the tangent vector. Then from the autoparallel condition $D_Y Y = 0$ we obtain the pair of geodesic equations

$$\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\nu\lambda}\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau} + C^{\mu}{}_{ab}\frac{\delta\phi^a}{d\tau}\frac{\delta\phi^b}{d\tau} = 0 \tag{44}$$

$$\frac{d}{d\tau}\left(\frac{\delta\phi^a}{d\tau}\right) + L^{a}{}_{b\mu}\frac{dx^{\mu}}{d\tau}\frac{\delta\phi^b}{d\tau} = 0 \tag{45}$$

where

$$\frac{d}{d\tau} \equiv \frac{dx^{\mu}}{d\tau}\delta_{\mu} + \frac{\delta\phi^a}{d\tau}\partial_a = Y \tag{46}$$

Multiplying (44) with the mass of a test particle and inserting (32), we can reveal the kinematic influence of each of the sectors of our geometrical structure. Indeed we acquire

$$m\left(\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{(O)\mu}{}_{\nu\lambda}\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau}\right)$$
$$= -m(\Gamma^{(D)\mu}{}_{\nu\lambda} + \gamma^{\mu}{}_{\nu\lambda})\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau} - mC^{\mu}{}_{ab}\frac{\delta\phi^a}{d\tau}\frac{\delta\phi^b}{d\tau} \tag{47}$$

The three terms that appear on the right hand side of the above equation account for the deviation from Riemannian geometry. This deviation reflects the presence of dark matter and its interaction with the ordinary sector and reveal the influence of the hidden scalar fields on the motion of particles. From the point of view of an observer who does not take into account the existence of these hidden entities, the three terms on the right are interpreted as inertial forces.

Substituting (17) and (18), into (44) and (45) we obtain

$$\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\nu\lambda}\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau} - \frac{1}{2}\delta_{ab}\partial^{\mu}\phi\frac{\delta\phi^a}{d\tau}\frac{\delta\phi^b}{d\tau} = 0 \tag{48}$$

$$\frac{d}{d\tau}\left(\frac{\delta\phi^a}{d\tau}\right) + \frac{1}{\phi}\partial_{\mu}\phi\frac{dx^{\mu}}{d\tau}\frac{\delta\phi^a}{d\tau} = 0 \tag{49}$$

It can be easily verified that (49) has the exact solution

$$\frac{\delta\phi^a}{d\tau} = \frac{C^a}{\phi} \tag{50}$$

where C^a are constants of integration. Inserting the above solution into (48) leads to

$$\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\nu\lambda}\frac{dx^{\nu}}{d\tau}\frac{dx^{\lambda}}{d\tau} - \frac{1}{2}\partial^{\mu}\phi\frac{C^2}{\phi^2} = 0 \tag{51}$$

where $C^2 = (C^{(1)})^2 + (C^{(2)})^2$ and $\Gamma^{\mu}{}_{\nu\lambda}$ is given in (32).

Additionally, it is instructive to examine separately the special case where the geodesics for the Riemannian part is given by

$$\frac{d^2x^{\mu}}{d\tau^2} + \Gamma^{\mu}{}_{\kappa\lambda}\frac{dx^{\kappa}}{d\tau}\frac{dx^{\lambda}}{d\tau} = 0 \tag{52}$$

and for the internal structure by

$$\frac{\delta\phi^a}{d\tau} = 0 \tag{53}$$

or equivalently,

$$\frac{d\phi^a}{d\tau} = -N^a_{\mu}\frac{dx^{\mu}}{d\tau} \tag{54}$$

In our model the form of the geodesics is given by both the relations (52), (54). The nonlinear connection N^a_{μ} interconnects the differential of the internal quantity ϕ^a with the velocity of the observer. Such an interconnection can be interpreted as a manifestation of a condition of parallelism (53).

Lastly, note that considering a specific form for the nonlinear connection, for instance

$$N^a_{\mu} = \frac{A(\phi)}{2\phi}\partial_{\mu}\phi\phi^a \tag{55}$$

Eq. (54) has the solution

$$\phi^a(x) = \phi^a_0(x) e^{-\frac{1}{2}\int_0^\tau A(\phi) d(\ln \phi)} \tag{56}$$

where $\phi^a_0(x) = \phi^a(x)|_{\tau=0}$.

III. FIELD EQUATIONS

In the previous section we presented the geometric structure and the kinematic variables of the examined construction. In the present section we proceed to physics. In particular, we consider the action on the fiber bundle, we derive the field equations with both Palatini and metrical methods, we examine the Raychaudhuri equations, and we finally construct the involved energy-momentum tensor.

The total action of the theory is

$$\begin{aligned} S &= S_G + 2\kappa S_M \\ &= \int_Q d^6U \sqrt{|\mathcal{G}|}\mathcal{G}^{AB}\mathcal{R}_{AB} + 2\kappa \int_Q d^6U \sqrt{|\mathcal{G}|}\mathcal{L}_M \end{aligned} \tag{57}$$

where $\mathcal{L}_M(\mathcal{G}^{MN}, \Psi^i)$ is the matter Lagrangian, Ψ^i the various matter fields described collectively, and Q a closed subspace of F^6. For additional details we refer to the Appendix B.

A. Palatini method

Firstly, we follow the Palatini method in which the variation is performed independently for the fields $\mathcal{G}_{AB}$ and $\mathbf{\Gamma}^L_{MN}$ (see Appendix B). If we assume a metrical compatible connection we acquire

$$\mathcal{R}_{(MN)} - \frac{1}{2}\mathcal{G}_{MN}\mathcal{R} = \kappa\mathcal{T}_{MN} \tag{58}$$

and

$$\mathcal{G}^{MN}\mathcal{T}^A{}_{KA} + \mathcal{G}^{ML}(\mathscr{T}^N{}_{LK} - \mathscr{T}^A{}_{LA}\delta^N_K) = 0 \tag{59}$$

where $\mathscr{T}^K{}_{MN}$ is the torsion of the connection, given in (20), and specifically in our case (21). The coupling constant κ in (58) will be determined in the general relativity (GR) limit of the theory. As is evident from (8), the only independent nonzero components of the torsion tensor are the following:

$$\begin{aligned} \mathscr{T}^a{}_{\lambda b} &= W^a{}_{\lambda b} = \partial_b N^a_\lambda \\ \mathscr{T}^a{}_{\lambda\nu} &= \tilde{W}^a{}_{\lambda\nu} = \delta_\nu N^a_\lambda - \delta_\lambda N^a_\nu. \end{aligned} \tag{60}$$

From these expressions, as well as (59) we acquire

$$W^a{}_{\lambda b} = 0 \tag{61}$$

$$\tilde{W}^a{}_{\lambda\nu} = 0, \tag{62}$$

i.e., we find that all the torsion components vanish. We see that the Palatini field equations force the connection to coincide with the Levi-Civita connection, in total agreement with the Levi-Civita theorem. Therefore, if one wishes to study a nonholonomic structure of the adapted basis, one has to abandon the Palatini method of variation. Nevertheless, let us continue the study and analyze the field equations (58). On the spacetime manifold we have

$$\begin{aligned} E_{\mu\nu} &+ \frac{1}{\phi}g_{\mu\nu}\left[\Box\phi - \frac{1}{4\phi}\partial^\lambda\phi\partial_\lambda\phi\right] - \frac{1}{\phi}D_\mu D_\nu\phi \\ &+ \frac{1}{2\phi^2}\partial_\mu\phi\partial_\nu\phi = 8\pi G\mathcal{T}_{\mu\nu} \end{aligned} \tag{63}$$

while on the fiber

$$\left(-R + \frac{1}{\phi}\Box\phi - \frac{1}{2\phi^2}\partial_\mu\phi\partial^\mu\phi\right)v_{ab} = 16\pi G\mathcal{T}_{ab} \tag{64}$$

where $E_{\mu\nu}$ is the standard Einstein's tensor of GR while the extra terms in (63) come from the spacetime components of the generalized tensor (30). One can recover the standard Einstein field equations of GR from (63) in the limit $\partial_\mu\phi \to 0$, in which the coupling constant is determined as $\kappa = 8\pi G$, with G the Newtonian gravitational constant. We mention that the energy-momentum tensor corresponding to the Lagrangian of the matter fields $\mathcal{L}_M(\mathcal{G}^{MN}, \Psi^i)$ is defined in the standard way. Specifically, we have

$$\mathcal{T}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_M)}{\delta g^{\mu\nu}} \tag{65}$$

for its space-time components, and

$$\mathcal{T}_{ab} = -\frac{2}{\sqrt{v}}\frac{\delta(\sqrt{v}\mathcal{L}_M)}{\delta v^{ab}} \tag{66}$$

for its fiber components.

In summary, from relations (63), (64) we deduce that the field equations include additional terms because of fiber fields and dark matter considerations.

B. Metrical method

In this subsection we proceed to the extraction of the field equations following the metrical method. In particular, we will derive the bundle field equations by varying the action (57) with only respect to the metric $\mathcal{G}^{MN}$. Doing so, we obtain (see Appendix B):

$$\begin{aligned} &\mathcal{E}_{(MN)} + (\delta^L{}_{(M}\delta^R{}_{N)} - \mathcal{G}^{LR}\mathcal{G}_{MN})(D_L\mathcal{W}^A{}_{RA} - \mathcal{W}^B{}_{LB}\mathcal{W}^C{}_{RC}) \\ &\quad = \kappa\mathcal{T}_{MN} \end{aligned} \tag{67}$$

The fields of curvature and torsion must obey the Bianchi identities (C1), (C2). Specifically, the first identity takes the form (see Appendix C):

$$D^A\mathcal{E}_{AN} + \mathcal{R}^A{}_R\mathcal{W}^R{}_{NA} + \frac{1}{2}\mathcal{R}^{KA}{}_{NR}\mathcal{W}^R{}_{AK} = 0 \quad (68)$$

In order to derive a generalization of the continuity equation we isolate the symmetric part of the tensor $\mathcal{E}_{AN}$. Employing (C9) we write,

$$D^A\mathcal{E}_{(AN)} + \mathcal{R}^A{}_R\mathcal{W}^R{}_{NA} + \frac{1}{2}\mathcal{R}^{KA}{}_{NR}\mathcal{W}^R{}_{AK} + \frac{1}{2}D^A(\mathcal{W}^L{}_{RA}\mathbf{\Gamma}^R{}_{LN} - \mathcal{W}^L{}_{RN}\mathbf{\Gamma}^R{}_{LA}) = 0. \quad (69)$$

Now, from (67) we see that,

$$D^A\mathcal{E}_{(AN)} + D^A H_{AN} = Q_N \quad (70)$$

where

$$H_{AN} = \Delta^{LR}{}_{AN}(D_L\mathcal{W}^K{}_{RK} - \mathcal{W}^K{}_{LK}\mathcal{W}^S{}_{RS})$$
$$\Delta^{LR}{}_{AN} = \frac{1}{2}(\mathcal{G}^L{}_A\mathcal{G}^R{}_N + \mathcal{G}^L{}_N\mathcal{G}^R{}_A) - \mathcal{G}^{LR}\mathcal{G}_{AN}$$
$$Q_N = \kappa D^A\mathcal{T}_{AN}. \quad (71)$$

Thus, inserting (69) into (70) we arrive at a final expression for the dissipation vector, namely

$$Q_N = \mathcal{R}^A{}_R\mathcal{W}^R{}_{AN} + \frac{1}{2}\mathcal{R}^{AK}{}_{NR}\mathcal{W}^R{}_{AK} + \frac{1}{2}D^A[\mathcal{W}^L{}_{RN}\mathbf{\Gamma}^R{}_{LA} - \mathcal{W}^L{}_{RA}\mathbf{\Gamma}^R{}_{LN} + 2\Delta^{LR}{}_{AN}(D_L\mathcal{W}^K{}_{RK} - \mathcal{W}^K{}_{LK}\mathcal{W}^S{}_{RS})]. \quad (72)$$

As it is evident from the above expression, the conservation of energy is restored, namely $Q_N = 0$, when the torsions $\mathcal{W}^L{}_{MN}$ are set to zero.

In terms of the space-time and fiber components the generalized field equation (67) respectively reads,

$$\mathcal{E}_{(\mu\nu)} + (\delta^\lambda_{(\mu}\delta^\rho_{\nu)} - g^{\lambda\rho}g_{\mu\nu})(D_\lambda W^c{}_{\rho c} - W^d{}_{\lambda d}W^c{}_{\rho c}) + v^{ab}g_{\mu\nu}C^\lambda{}_{ab}W^c{}_{\lambda c} = \kappa\mathcal{T}_{\mu\nu} \quad (73)$$

$$\mathcal{E}_{(ab)} - g^{\lambda\rho}v_{ab}(D_\lambda W^c{}_{\rho c} - W^d{}_{\lambda d}W^c{}_{\rho c}) + C^\lambda{}_{ab}W^c{}_{\lambda c} = \kappa\mathcal{T}_{ab} \quad (74)$$

where $\mathcal{E}_{\mu\nu}$ and $\mathcal{E}_{ab}$ are the spacetime and fiber components respectively of the generalized Einstein's tensor (30). These equations must reproduce general relativity in the appropriate limit. We find that for $\dot{\phi} \to 0$ and $\mathcal{W}^K_{MN} \to 0$ Eqs. (73) reduce to the Einstein field equations of GR for the metric $g_{\mu\nu}$, provided that the coupling constant takes the value $\kappa = 8\pi G$, where G is the Newtonian gravitational constant. In general, the value of κ depends on the structure of the geometry. Moreover, in this limit, Eq. (74) gives the condition:

$$\overset{H}{\mathcal{T}} = \overset{V}{\mathcal{T}} \quad (75)$$

with

$$\overset{H}{\mathcal{T}} = g^{\mu\nu}\mathcal{T}_{\mu\nu} = \mathcal{T}^\mu{}_\mu$$

and

$$\overset{V}{\mathcal{T}} = v^{ab}\mathcal{T}_{ab} = \mathcal{T}^a{}_a$$

i.e., the traces of the spacetime energy momentum $\overset{H}{\mathcal{T}}$ and of the fiber one $\overset{V}{\mathcal{T}}$ are equal in the GR limit.

From (B8) and assuming that the matter Lagrangian $\mathcal{L}_M$ depends on the metric $\mathcal{G}_{MN}$ but not on its derivatives we acquire:

$$\mathcal{T}_{\mu\nu} = -2\frac{\partial\mathcal{L}_M}{\partial g^{\mu\nu}} + \mathcal{L}_M g_{\mu\nu} \quad (76)$$

$$\mathcal{T}_{ab} = -2\frac{\partial\mathcal{L}_M}{\partial v^{ab}} + \mathcal{L}_M v_{ab} \quad (77)$$

From (75), (76) and (77) we obtain the GR limit condition for the matter Lagrangian:

$$v^{ab}\frac{\partial\mathcal{L}_M}{\partial v^{ab}} = -\mathcal{L}_M + g^{\mu\nu}\frac{\partial\mathcal{L}_M}{\partial g^{\mu\nu}} \quad (78)$$

In this limit and for a matter fluid with a barotropic equation of state $P_m(\rho^{(0)})$ and a conserved current $D_\mu(\rho^{(0)}Y^\mu) = 0$, with $\rho^{(0)}$ the rest mass energy density, the energy-momentum tensor reads [63]:

$$\mathcal{T}^{\mu\nu} = -\rho^{(0)}\frac{\partial\mathcal{L}_M}{\partial\rho^{(0)}}Y^\mu Y^\nu + \left(\mathcal{L}_M\frac{\partial\mathcal{L}_M}{\partial\rho^{(0)}} - \rho^{(0)}\right)g^{\mu\nu} \quad (79)$$

where the following relation has been used:

$$\frac{d\rho^{(0)}}{dg^{\mu\nu}} = \frac{1}{2}\rho^{(0)}(g_{\mu\nu} + Y_\mu Y_\nu). \quad (80)$$

Comparison of (79) with (117) gives

$$\frac{\partial\mathcal{L}_M}{\partial\rho^{(0)}} = -\frac{\rho_m + P_m}{\rho^{(0)}}, \qquad \mathcal{L}_M = -\rho_m. \quad (81)$$

Finally from (81) and (78) we obtain

$$v^{ab}\frac{\partial\mathcal{L}_M}{\partial v^{ab}} = \rho_m + g^{\mu\nu}\frac{\partial\mathcal{L}_M}{\partial\rho^{(0)}}\frac{d\rho^{(0)}}{dg^{\mu\nu}} \Leftrightarrow$$
$$\frac{2}{\phi}\frac{\partial\rho_m}{\partial\phi} = \frac{\rho_m}{2} + \frac{3P_m}{2}. \tag{82}$$

This equation determines the dependence of the barotropic fluid's energy density ρ_m on the scalar field ϕ at the GR limit.

C. Incorporation of dark matter in energy-momentum tensor

We close this section by discussing the energy-momentum tensor. The theory at hand allows for two sources of dark matter. A purely geometrical one, in which dark matter is attributed to the effective properties of the bundle structure, and a fluid/particle one in which dark matter contributes directly to the energy momentum tensor.

Following the geometrical method, we rearrange the terms of (73), so that only the standard GR Einstein's tensor appears on the lhs:

$$E_{\mu\nu} = \kappa\tilde{\mathcal{T}}_{\mu\nu} \tag{83}$$

where

$$\tilde{\mathcal{T}}_{\mu\nu} = \mathcal{T}_{\mu\nu} + \mathcal{T}^{(\phi)}{}_{\mu\nu} \tag{84}$$

and

$$\mathcal{T}^{(\phi)}{}_{\mu\nu} = -\frac{1}{\kappa}[E^{\phi}{}_{(\mu\nu)} + (\delta^{\kappa}_{(\mu}\delta^{\lambda}_{\nu)} - g^{\kappa\lambda}g_{\mu\nu})(D_{\kappa}W^{a}{}_{\lambda a} - W^{b}{}_{\kappa b}W^{c}{}_{\lambda c}) + v^{ab}g_{\mu\nu}C^{\lambda}{}_{ab}W^{c}{}_{\lambda c}]. \tag{85}$$

Hence, the geometrical properties of our model can be viewed as additional terms to the energy momentum tensor and therefore, in the GR framework, interpreted as effective dark matter.

In addition to this, one can directly include dark matter contributions to the energy momentum tensor [61],

$$\mathcal{T}_{\mu\nu} = \mathcal{T}^{(O)}{}_{\mu\nu} + \mathcal{T}^{(D)}{}_{\mu\nu} + \tau_{\mu\nu} \tag{86}$$

so that (84) becomes

$$\tilde{\mathcal{T}}_{\mu\nu} = \mathcal{T}^{(O)}{}_{\mu\nu} + \mathcal{T}^{(D)}{}_{\mu\nu} + \mathcal{T}^{(\phi)}{}_{\mu\nu} + \tau_{\mu\nu}. \tag{87}$$

The above sectorial decomposition of the energy-momentum tensor induces the corresponding decomposition of the generalized Einstein's tensor. From the above relation we notice that the total form of the energy momentum tensor $\tilde{\mathcal{T}}_{\mu\nu}$ includes the fiber contributions as well as the dark matter sector and its interactions with ordinary matter. It is possible that a conformal relation between ordinary and dark matter exists [53].

IV. RAYCHAUDHURI EQUATIONS

It is known that the Raychaudhuri's equations describe the evolution of the acceleration of the universe through the gravitating fluid. Their form depends on the metrical structure of space, i.e., in spaces with generalized metric structure and torsion as in a Finsler space-time [44]. The Raychaudhuri's equations are produced by the deviation of nearby geodesics or fluid lines and monitor their evolution. In our case, they are twofold extended. On the one hand, with the introduction of the scalars $\phi^{(1)}$, $\phi^{(2)}$ and on the other, with the inclusion of the dark sector.

In order to examine the local behavior of a single, timelike geodesic among the congruence, let us assume the tangent vector,

$$Y^M \equiv \left(\frac{dx^\mu}{d\tau}, \frac{\delta\phi^a}{d\tau}\right) \tag{88}$$

which satisfies the autoparallel condition along the track of the geodesic

$$D_Y Y = 0. \tag{89}$$

Furthermore, we assume that τ is properly chosen in order for Y^M to have a unit norm,[5] namely

$$\mathcal{G}_{MN}Y^M Y^N = -1. \tag{90}$$

The 2nd rank tensor

$$\mathcal{B}^M{}_N = D_N Y^M = X_N Y^M + \Gamma^M{}_{LN}Y^L \tag{91}$$

measures the failure of the separation vector between adjacent geodesics to be parallelly transported along the congruence [64,65]. From the autoparallel condition, we obtain

$$Y^N\mathcal{B}_{MN} = 0 \tag{92}$$

and from (90) we get

$$D_M(Y_N Y^N) = 0 \Rightarrow Y^M\mathcal{B}_{MN} = 0. \tag{93}$$

Additionally, we can separate the space part of the metric, making use of the projective tensor $\mathcal{H}_{MN}$,

[5]This assumption is consistent with the definition of the geodesic parameter, (46). It does not alter the signature of the Riemannian metric, and forces the extra fiber variables to behave as space-like components. The fact that the extra degrees of freedom do not transform covariantly is not incompatible with the existence of a comoving observer in the bundle F^6.

$$\mathcal{G}_{MN} = \mathcal{H}_{MN} - Y_M Y_N. \tag{94}$$

Indeed, it is easy to see that,

$$\mathcal{H}_{MN} Y^N = 0. \tag{95}$$

As a 2nd rank tensor, $\mathcal{B}_{MN}$ can be decomposed into its irreducible components, namely its trace, traceless symmetric and antisymmetric part. In particular, the trace of the tensor $\mathcal{B}_{MN}$ is called *expansion*, i.e.,

$$\Theta = \mathcal{G}^{ML}\mathcal{G}^{NR}\mathcal{B}_{MN}\mathcal{H}_{LR} = \mathcal{B}^{MN}\mathcal{H}_{MN} = \mathcal{B}^M{}_M = D_M Y^M \tag{96}$$

and is a measure of the volume change of a sphere of test particles centered on the geodesic. The symmetric, traceless part of the same tensor is called *shear*:

$$\mathcal{S}_{MN} = \mathcal{B}_{(MN)} - \frac{1}{5}\Theta \mathcal{H}_{MN} \tag{97}$$

and describes the shape distortion of the test particles from the initial sphere to an ellipsoid. Lastly, the antisymmetric part of the tensor is called *rotation*

$$\Omega_{MN} = \mathcal{B}_{[MN]} \tag{98}$$

and describes the rotation of the initial sphere of test particles.

Now, the 2nd rank tensor $\mathcal{B}_{MN}$ can be written in terms of its irreducible components as

$$\mathcal{B}_{MN} = \frac{1}{5}\Theta\mathcal{H}_{MN} + \mathcal{S}_{MN} + \Omega_{MN} \tag{99}$$

Taking into account (89), the definition of the Riemann tensor (19) and the fact that

$$[D_L, D_N]Y_M = \mathcal{W}^R{}_{LN} D_R Y_M - \mathcal{R}^R{}_{MLN} Y_R \tag{100}$$

we obtain that the covariant derivative of $\mathcal{B}_{MN}$ along the geodesic is

$$Y^L D_L \mathcal{B}_{MN} = \mathcal{W}^R{}_{LN} Y^L \mathcal{B}_{MR} - \mathcal{R}^R{}_{MLN} Y^L Y_R - \mathcal{B}^L{}_N \mathcal{B}_{ML}. \tag{101}$$

Taking the trace of the above equation we result to

$$\frac{d\Theta}{d\tau} = \mathcal{W}^L{}_{MN} Y^M \mathcal{B}^N{}_L - \mathcal{R}_{MN} Y^M Y^N - \mathcal{B}^{MN}\mathcal{B}_{NM}. \tag{102}$$

Written in terms of the irreducible components of $\mathcal{B}_{MN}$, the above equation provides the extension of the *Raychaudhuri equation* on a general space-time vector bundle, namely

$$\frac{d\Theta}{d\tau} = \mathcal{W}^L{}_{MN} Y^M \mathcal{B}^N{}_L - \mathcal{R}_{MN} Y^M Y^N - \frac{1}{5}\Theta^2 - S^{MN}\mathcal{S}_{MN} + \Omega^{MN}\Omega_{MN}. \tag{103}$$

For the specific choice of special connection structure (13) we acquire

$$\Theta = \mathrm{div} Y + \frac{d}{d\tau}[\ln(\sqrt{-\mathcal{G}})] = \theta + \theta^{(\phi)} \tag{104}$$

where $\mathrm{div} Y = X_M Y^M$, $\mathcal{G} = \phi^2 g$ is the determinant of the bundle metric, and

$$\theta = \nabla_\mu Y^\mu = \partial_\mu Y^\mu + \frac{d}{d\tau}[\ln\sqrt{-g}] \tag{105}$$

$$\theta^{(\phi)} = \partial_a Y^a - N^a_\mu \partial_a Y^\mu + \frac{d}{d\tau}(\ln\phi). \tag{106}$$

To the standard expansion θ, a contribution of purely geometric origin $\theta^{(\phi)}$ is added. It is produced by the scalars $\phi^{(a)}$, the nonlinear connection N^a_μ and the fiber components of the tangent vector Y^a. The form and the overall sign of this contribution (106) is directly related to the kinematics of the universal evolution and under certain circumstances it provides a triggering inflation mechanism. Especially, in the case of an inflaton scalar field, the contribution of the volume $\theta^{(\phi)}$ will be positive and an increase of volume can appear.

In the same manner we can calculate each of the terms of (103). For the nonholonomic term we obtain

$$\mathcal{W}^L{}_{MN} Y^M \mathcal{B}^N{}_L = \tilde{W}^a{}_{\mu\nu} Y^\mu D_a Y^\nu + W^a{}_{\mu b}(Y^\mu D_a Y^b - Y^b D_a Y^\mu). \tag{107}$$

The generalized tidal term decomposes into its Riemannian part plus the additional contributions that rise from the additional geometric structure

$$\mathcal{R}_{MN} Y^M Y^N = R_{\mu\nu} Y^\mu Y^\nu + R^{(\phi)}{}_{\mu\nu} Y^\mu Y^\nu + \mathcal{R}_{ab} Y^a Y^b + C^\nu{}_{ab} \tilde{W}^b{}_{\mu\nu} Y^a Y^\mu \tag{108}$$

where

$$\mathcal{S}_{\mu\nu} = \sigma_{\mu\nu} + S^{(\phi)}{}_{\mu\nu}$$
$$\mathcal{S}_{ab} = \frac{1}{2}(\partial_b Y_a + \partial_a Y_b - 2C^\mu{}_{ab} Y_\mu) - \frac{1}{5}\Theta\mathcal{H}_{ab}$$
$$\mathcal{S}_{\mu a} = (\partial_a Y_\mu + \delta_\mu Y_a - 2C^b{}_{\mu a} Y_b) - \frac{1}{5}\Theta\mathcal{H}_{\mu a} \tag{109}$$

and

$$\sigma_{\mu\nu} = \nabla_{(\nu} Y_{\mu)} - \frac{1}{3}\theta \mathcal{H}_{\mu\nu}$$
$$S^{(\phi)}{}_{\mu\nu} = \frac{1}{15}(2\theta - 3\theta^{(\phi)})\mathcal{H}_{\mu\nu} - \frac{1}{2}(N^a_\mu \partial_a Y_\nu + N^a_\nu \partial_a Y_\mu). \tag{110}$$

Finally, we can acquire a similar decompositions for the generalized rotation too, namely

$$\Omega_{\mu\nu} = \omega_{\mu\nu} + \Omega^{(\phi)}{}_{\mu\nu}$$
$$\Omega_{ab} = \frac{1}{2}(\partial_b Y_a - \partial_a Y_b)$$
$$\Omega_{\mu a} = \frac{1}{2}(\partial_a Y_\mu - \delta_\mu Y_a) \tag{111}$$

where

$$\omega_{\mu\nu} = \nabla_{[\nu} Y_{\mu]}$$
$$\Omega^{(\phi)}{}_{\mu\nu} = \frac{1}{2}(N^a_\mu \partial_a Y_\nu - N^a_\nu \partial_a Y_\mu). \tag{112}$$

Assembling all the pieces together we obtain

$$\frac{d}{d\tau}(\theta + \theta^{(\phi)}) = -R_{\mu\nu} Y^\mu Y^\nu - \frac{1}{3}\theta^2 - \sigma^{\mu\nu}\sigma_{\mu\nu} + \omega^{\mu\nu}\omega_{\mu\nu} + \mathscr{Q} \tag{113}$$

with

$$\begin{aligned}\mathscr{Q} = {} & \tilde{W}^a{}_{\mu\nu} Y^\mu \partial_a Y^\nu + W^a{}_{\mu b}(Y^\mu D_a Y_b - Y_b D_a Y^\mu) \\ & - R^{(\phi)}{}_{\mu\nu} Y^\mu Y^\nu - \mathcal{R}_{ab} Y^a Y^b + \frac{2}{15}\theta^2 - \frac{1}{5}[2\theta\theta^{(\phi)} + (\theta^{(\phi)})^2] \\ & - S^{(\phi)\mu\nu} S^{(\phi)}{}_{\mu\nu} - 2\sigma^{\mu\nu} S^{(\phi)}_{\mu\nu} - S^{ab} S_{ab} - 2S^{\mu a} S_{\mu a} \\ & + \Omega^{(\phi)\mu\nu}\Omega^{(\phi)}{}_{\mu\nu} + 2\omega^{\mu\nu}\Omega^{(\phi)}{}_{\mu\nu} \\ & + \Omega^{ab}\Omega_{ab} + 2\Omega^{\mu a}\Omega_{\mu a}. \end{aligned} \tag{114}$$

As we can see from (113) $\mathscr{Q}$ disturbs the rate of the volume change for a number of reasons. First, because of the interaction between the volumes θ and $\theta^{(\phi)}$, second due to the contribution of the scalar fields $\phi^{(a)}$ and lastly because of the presence of the nonlinear connection N^a_μ and the torsion functions $\tilde{W}^a{}_{\mu\nu}$, $W^a{}_{\mu b}$.

The generalized tidal field (108) includes the standard Riemann contribution (37) and additional terms which can affect the evolution of the gravitational fluid for possible singularities/conjugate points in the universe. It is obvious, because of extra internal geometrical concepts of fiber-fields $\phi(x)$, of the nonlinear connection N^a_μ in the metrical structure of our model F^6 and of the introduction of the dark gravitational field. In the framework of our space, F^6 and for a given congruence of timelike geodesics, the expansion Θ, shear $S_{\mu\nu}$ and rotation Ω are described, in a generalized form, in Eqs (96), (97), (98) which provide us the generalized type of Raychaudhuri equation (113). The extra terms affect the variation of the volume during the evolution of fluid lines (focusing/defocusing) in the accelerating expansion of the universe. This is possible due to the perturbation of the deviation equation of nearby geodesics or curves.

In a comoving frame, the term in Eq. (114) involving the structure functions $\tilde{W}^a{}_{\mu\nu}$ vanishes. As we will see later, this is in agreement with the generalized Friedmann equations for this model. Specifically, in those equations, in which the matter fluid is at rest, no such term appears. This is an important test for the consistency of this model because the generalized Raychaudhuri equation (113) should not give additional information on the kinematics of the FRW comoving frame. It is worthwhile to mention that for a constant nonlinear connection the above equations can be drastically simplified.

With the aid of the bundle field equations (67) we acquire

$$\begin{aligned} & \mathcal{R}_{MN} Y^M Y^N \\ & \quad + (D_M \mathcal{W}^A{}_{NA} - \mathcal{W}^B{}_{MB}\mathcal{W}^C{}_{NC})\left(Y^M Y^N - \frac{1}{4}\mathcal{G}^{MN}\right) \\ & = \kappa\left(\mathcal{T}_{MN} Y^M Y^N + \frac{1}{4}\mathcal{T}\right) \end{aligned} \tag{115}$$

where $\mathcal{T} = \mathcal{T}_{MN}\mathcal{G}^{MN}$. The sign of the generalized tidal field determines the evolution of the volume of the fluid lines. It is evident that it does not only depend on the energy and current density of matter, but also on the structure of the algebra of the adapted basis.

V. COSMOLOGY WITH NONLINEAR CONNECTION

In the previous sections we presented the geometric formalism in which generalized scalar-tensor theories and dark gravitational sectors are induced from the vector bundle. In this section we proceed to the explicit cosmological application of such constructions.

In order to construct a cosmological framework, we need to extend the standard spatially homogeneous and isotropic Friedmann-Robertson-Walker (FRW) metric of ordinary Riemannian geometry and GR on the fiber bundle E. In particular, we consider the flat case of the former ($k = 0$), as the simplest one in GR, and extend it to account for the additional degrees of freedom of E in the following way:

$$\begin{aligned}\mathbf{G} = {} & -\mathrm{d}t \otimes \mathrm{d}t + a^2(t)(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}z \otimes \mathrm{d}z) \\ & + \phi(t)(\delta\phi^{(1)} \otimes \delta\phi^{(1)} + \delta\phi^{(2)} \otimes \delta\phi^{(2)}) \end{aligned} \tag{116}$$

We mention here that the observational constraints on the (almost zero) spatial curvature have been extracted under the consideration of the usual FRW metric in Riemannian geometry, and thus in principle one cannot deduce that the same feature would necessarily hold in the case of the present extended geometric structure. Nevertheless, since in our work we are interested in performing a first cosmological application, we impose zero spatial curvature. As one can see, the first line of (116) is the standard 4-dimensional spatially flat FRW metric, while the second line arises from the additional structure of the Lorentz fiber bundle. The additional degrees of freedom of the metric as well as the anholonomicity of the adapted basis are expected to enrich the dynamics of space-time, compared to the standard spatially isotropic and flat FRW cosmology [66][6]

Moreover, we consider the matter sector to correspond to a perfect fluid, with energy-momentum tensor of the form

$$\mathcal{T}_{\mu\nu} = (\rho_m + P_m) Y_\mu Y_\nu + P_m g_{\mu\nu} \tag{117}$$

with ρ_m is the energy density, P_m the pressure and Y^μ the bulk 4-velocity of the fluid.

We will first study the equations derived from the metrical method, since the Palatini equations occur as a special case of the former. For the spacetime (116), and with the perfect fluid (117), the nondiagonal components of the field equations (73), (74) give

$$\begin{aligned} \tilde{W}^a{}_{0i}\dot{\phi} &= 0, \\ [W^{(1)}{}_{0(2)} + W^{(2)}{}_{0(1)}]\dot{\phi} &= 0 \\ W^a{}_{ia} W^b{}_{jb} &= 0 \end{aligned} \tag{118}$$

for $i \neq j$, and

$$W^a{}_{ia}\left(\frac{\dot{\phi}}{4\phi} - H - W^b{}_{0b}\right) = 0 \tag{119}$$

where 0 stands for the coordinate time component, $i, j = 1, 2, 3$ for the spatial components, $a, b = (1), (2)$ for the fiber components, and a dot denotes differentiation with respect to time: $\dot{\phi} = \frac{d\phi}{dt}$. Furthermore from (118) and the spatial isotropy of (116) and assuming that $\dot{\phi} \neq 0$, we acquire

$$W^a{}_{ia} = 0 = \tilde{W}^a_{0i}$$

and

$$W^{(1)}{}_{0(2)} = -W^{(2)}{}_{0(1)}.$$

Applying the general field equations (73) and (74) for a nontrivial nonlinear connection in the case of the metric (116), and taking into account the above relations, we finally obtain:

$$3H^2 + 3H\left(\frac{\dot{\phi}}{\phi} - W_+\right) - W_+\frac{\dot{\phi}}{\phi} + \frac{\dot{\phi}^2}{4\phi^2} = 8\pi G\rho_m \tag{120}$$

$$\begin{aligned} &2\dot{H} + (W_+)^2 - \dot{W}_+ - \frac{\dot{\phi}^2}{2\phi^2} + H\left(W_+ - \frac{\dot{\phi}}{\phi}\right) \\ &\quad - \frac{1}{2\phi}(W_+\dot{\phi} - 2\ddot{\phi}) = -8\pi G(\rho_m + P_m) \end{aligned} \tag{121}$$

and

$$\begin{aligned} &\frac{1}{\phi}(\ddot{\phi} + 3H\dot{\phi}) - \frac{\dot{\phi}}{2\phi}\left(3W_+ + \frac{\dot{\phi}}{\phi}\right) + 6(\dot{H} + 2H^2) \\ &\quad - 6HW_+ + 2(W_+)^2 - 2\dot{W}_+ = -8\pi G\overset{V}{\mathcal{T}} \end{aligned} \tag{122}$$

where we have defined

$$W_+ = W^a{}_{0a}. \tag{123}$$

These are the two modified Friedmann equations and the scalar-field (Klein-Gordon) equation, for the scenario at hand. Indeed, as we can see we do obtain generalized scalar-tensor theories from the specific vector bundle model that we have constructed. Note, that according to (75) and (117), in the general relativity limit we have,

$$\overset{V}{\mathcal{T}} = -\rho_m + 3P_m$$

Therefore in the general case we can consider the trace as

$$\overset{V}{\mathcal{T}} = -\rho_m + 3P_m + \tilde{\mathcal{T}}$$

where we explicitly see that $\tilde{\mathcal{T}}$ is a correction over the GR limit.

A. Dark energy

Let us now proceed to the investigation of the modified Friedmann equations (120), (121). Observing their form, we deduce that we can write them in the standard way, namely

$$3H^2 = 8\pi G(\rho_m + \rho_{\text{eff}}) \tag{124}$$

$$2\dot{H} = -8\pi G(\rho_m + \rho_{\text{eff}} + P_m + P_{\text{eff}}) \tag{125}$$

[6]To examine whether the symmetries of the standard FRW solution persist, a careful and meaningful definition of these symmetries should be given in the current framework of extended space-time. The most consistent way to do this is by means of Lie derivatives and extended Killing vectors on the bundle E or by direct implementation of the method of complete lifts [35]. Using these tools, we could construct spatially homogeneous and isotropic cosmological solutions that may even extrapolate the classification into spatially flat, closed or open. This would be an interesting topic for a future project.

having defined an effective dark energy sector with energy density and pressure respectively as

$$\rho_{\rm eff} = \frac{1}{8\pi G}\left[\frac{\dot{\phi}}{\phi}W_+ - \frac{\dot{\phi}^2}{4\phi^2} - 3H\left(\frac{\dot{\phi}}{\phi} - W_+\right)\right] \qquad (126)$$

$$P_{\rm eff} = \frac{1}{8\pi G}\left[(W_+)^2 - \dot{W}_+ - 2HW_+ - \frac{\dot{\phi}^2}{4\phi^2} + \frac{1}{2\phi}(4H\dot{\phi} - 3W_+\dot{\phi} + 2\ddot{\phi})\right]. \qquad (127)$$

Hence, the effective dark energy sector incorporates all the extra geometrical information that arises from the vector bundle construction.

We can define the equation-of-state parameter for the effective dark-energy sector as

$$w_{\rm eff} \equiv \frac{P_{\rm eff}}{\rho_{\rm eff}}. \qquad (128)$$

According to the definitions (126), (127), we can see that $w_{\rm eff}$ can lie in the quintessence ($w_{\rm eff} > -1$) or in the phantom ($w_{\rm eff} < -1$) regime, or experience the phantom-divide crossing during the evolution. The fact that we can effectively obtain a phantom behavior without imposing by hand phantom fields, is an advantage of the scenario and reveals the capabilities of the bundle constructions. Note that $w_{\rm eff}$ can be even exactly equal to -1 if one imposes the specific condition

$$\dot{\psi} + \frac{1}{2}\psi^2 - H\psi - \frac{1}{2}W_+\psi = \dot{W}_+ - W_+^2 - HW_+ \qquad (129)$$

where $\psi \equiv \frac{\dot{\phi}}{\phi}$, in which case we obtain a cosmological constant of effective origin, although our initial action does not contain an effective cosmological constant.

Finally, using the above definitions we can examine the validity of the energy conditions:

(i) Weak: $\rho_{\rm eff} \geq 0$, $\rho_{\rm eff} + P_{\rm eff} \geq 0$
(ii) Strong: $\rho_{\rm eff} + P_{\rm eff} \geq 0$, $\rho_{\rm eff} + 3P_{\rm eff} \geq 0$
(iii) Null: $\rho_{\rm eff} + P_{\rm eff} \geq 0$
(iv) Dominant: $\rho_{\rm eff} \geq |P_{\rm eff}|$

We proceed to the specific investigation the cosmological behavior that is induced from the scenario at hand. In particular, we elaborate the Friedmann equations (124), (125) numerically, and we use the usual expression for the redshift $1+z = 1/a$ (the present scale factor is set to $a_0 = 1$) as the independent variable. This expression for the redshift is justified by two points: First, we consider trajectories of the form (52), (53) which effectively describe classic GR geodesics, and second the spacetime part of the metric (116) is identical to the classic spatially flat FRW metric of GR. Moreover, we introduce the standard density parameters, namely $\Omega_m \equiv 8\pi G\rho_m/(3H^2)$ and $\Omega_{\rm eff} \equiv \Omega_{\rm DE} = 8\pi G\rho_{\rm eff}/(3H^2)$, for the matter and effective dark energy sector respectively. Concerning the initial conditions we choose them in order to obtain $\Omega_{\rm eff}(z=0) \equiv \Omega_{\rm eff0} \approx 0.69$ and $\Omega_m(z=0) \equiv \Omega_{m0} \approx 0.31$ in agreement with observations [67], while for the matter sector we impose dust equation of state, namely $w_m \equiv P_m/\rho_m = 0$.

In the upper graph Fig. 1 we present $\Omega_{\rm DE}(z)$ and $\Omega_m(z)$ where we observe that we obtain the standard thermal history of the universe, namely the matter and dark energy eras. Additionally, in the lower graph Fig. 1 we depict the effective dark-energy equation-of-state parameter $w_{\rm eff} \equiv w_{\rm DE}$, where we can see that in this specific example the effective dark energy sector experiences the phantom-divide crossing during the cosmological evolution. In order to examine in more detail the behavior of $w_{\rm DE}$, in Fig. 2 we present its evolution for various small corrections $\tilde{\mathcal{T}}$. As we can see, we can obtain a rich behavior, and an effective dark energy sector that can be quintessencelike, phantomlike, or experience the phantom-divide crossing. These properties cannot be easily acquired in the usual scalar-tensor theories, and this reveals the capabilities of the construction at hand.

Let us examine in more detail the conservation equations in the scenario at hand. As expected, the energy densities and pressure appearing in the Friedmann equations (124), (125) satisfy the continuity equation

$$\dot{\rho}_m + \dot{\rho}_{\rm eff} + 3H(\rho_m + \rho_{\rm eff} + P_m + P_{\rm eff}) = 0. \qquad (130)$$

Using (122) we can rewrite it as

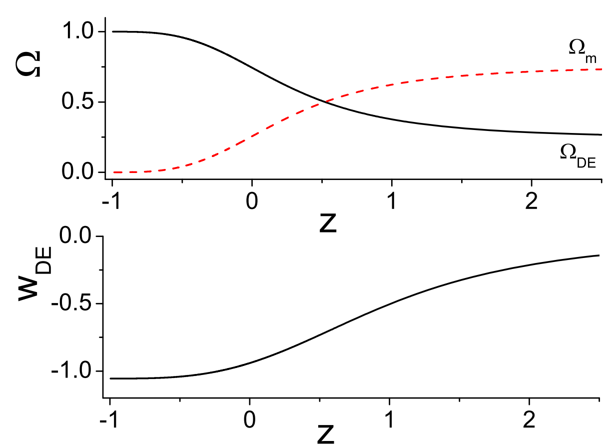

FIG. 1. Upper graph: the evolution of the effective dark energy density parameter $\Omega_{\rm DE}$ (black-solid), as well as of the matter density parameter Ω_m (red-dashed), as a function of the redshift z. Lower graph: The evolution of the corresponding dark-energy equation-of-state parameter $w_{\rm DE}$. We have imposed the initial conditions $\Omega_{\rm DE}(z=0) \equiv \Omega_{\rm DE0} \approx 0.69$ [67].

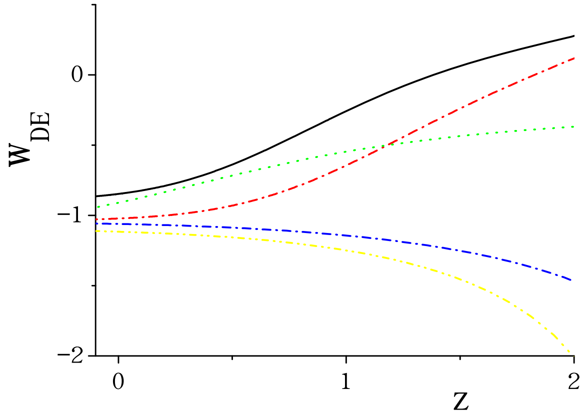

FIG. 2. The evolution of the dark-energy equation-of-state parameter $w_{\rm DE}$ as a function of the redshift z, for various small corrections $\tilde{\mathcal{T}}$. We have imposed the initial conditions $\Omega_{\rm DE}(z=0) \equiv \Omega_{\rm DE0} \approx 0.69$ [67].

$$\dot{\rho}_m + 3H(\rho_m + P_m) + \frac{\dot{\phi}}{2\phi}(\rho_m + 3P_m + \tilde{\mathcal{T}}) = -\frac{1}{8\pi G}Q_0 \tag{131}$$

where the time component of the dissipation vector (72) is calculated as

$$Q_0 = \frac{1}{4\phi^2}[12H^2\phi^2 + 12\phi^2\dot{H} + 6H\phi(2W_+\phi - \dot{\phi}) - 3\dot{\phi}^2 + 4\phi(W_+\dot{\phi} + \ddot{\phi})]W_+ \tag{132}$$

Note that this equation can also be obtained from the time component of (71) (all other components of (71) give trivial equations). The dissipation vector encodes the energy-momentum tensor potential nonconservation with respect to the special connection of F^6. An interesting observation is that in the absence of matter, Eqs. (120), (121), and (122) are independent, contrary to the standard scalar-tensor models where only two out of the three equations are. Therefore, in the absence of matter, Eq. (131) implies that Q_0 should vanish, a condition that makes (122) dependent on (120) and (121), which is then a self-consistency verification of the scenario.

In the general case the combination of (120) and (121) does not reproduce (122), exactly due to Q_0. However, observing the form of (131), we deduce that if we define

$$\tilde{Q} \equiv -\frac{\dot{\phi}}{2\phi}(\rho_m + 3P_m + \tilde{\mathcal{T}}) - \frac{1}{8\pi G}Q_0 \tag{133}$$

then (130) and (131) can be rewritten as

$$\dot{\rho}_m + 3H(\rho_m + P_m) = \tilde{Q} \tag{134}$$

$$\dot{\rho}_{\rm eff} + 3H(\rho_{\rm eff} + P_{\rm eff}) = -\tilde{Q} \tag{135}$$

As we can see, $\tilde{Q}$ represents the interaction rate between matter and effective dark energy sector, which lies at the basis of the matter nonconservation [68,69]. Therefore, in the general case the scenario at hand exhibits an interaction between the matter component and the dark energy sector that quantifies the novel geometric structure of the vector bundle. This reveals the capabilities of the model, since interacting cosmology is known to lead to very rich phenomenology [59,70–73] and among others it can alleviate the coincidence problem [59,74] as well as the H_0 tension [60,75]. However, we stress that in the scenario at hand the interaction between the dark sectors is not imposed by hand, but it naturally arises from the intrinsic geometrical structure of the bundle construction. Finally, in the particular case where $\tilde{Q} = 0$, we obtain conservation of matter and effective dark energy sectors, i.e., we obtain the standard, noninteracting, cosmology.

We close this subsection by examining the special case where the condition $\partial_{(1)}N_0^{(1)} = -\partial_{(2)}N_0^{(2)}$ is imposed on the nonlinear connection, which leads to $W_+ = 0$. This is also true when N_0^a is constant, which is a solution of the Palatini field equations. In such a case, the modified Friedmann equations (120), (121) become

$$3H^2 + 3H\frac{\dot{\phi}}{\phi} + \frac{\dot{\phi}^2}{4\phi^2} = 8\pi G\rho_m \tag{136}$$

$$2\dot{H} - \frac{\dot{\phi}^2}{2\phi^2} - H\frac{\dot{\phi}}{\phi} + \frac{\ddot{\phi}}{\phi} = -8\pi G(\rho_m + P_m) \tag{137}$$

while the Klein-Gordon equation (122) is simplified to

$$\frac{1}{\phi}(\ddot{\phi} + 3H\dot{\phi}) - \frac{\dot{\phi}^2}{2\phi^2} + 6(\dot{H} + 2H^2) = 8\pi G(\rho_m - 3P_m - \tilde{\mathcal{T}}) \tag{138}$$

Note that the interaction between matter and effective dark energy sector is maintained.

B. Cold dark matter

One of the features of the construction at hand is that the metric of the base manifold can be decomposed into an ordinary and a dark matter piece according to (31). As a result, the perfect fluid (117) can be decomposed into ordinary and cold dark matter (CDM) sectors [61], which using Eq. (86) leads to

$$\mathcal{T}_{\mu\nu} = (\rho_m^{(O)} + P_m^{(O)})Y_\mu Y_\nu + P_m^{(O)}(g^{(O)}{}_{\mu\nu}(x) + g^{(D)}{}_{\mu\nu}(x)) + \rho_m^{(D)}Y_\mu Y_\nu. \tag{139}$$

Note that we have assumed that, due to spatial isotropy, the ordinary matter and CDM fluids are at rest with respect to the comoving grid, and thus they have the same 4-velocity

Y^μ, and moreover that the CDM fluid is pressureless as usual ($P_m^{(D)} = 0$). Expression (139) can be decomposed into ordinary, dark and interaction terms, respectively as

$$\mathcal{T}^{(O)}{}_{\mu\nu} = (\rho_m^{(O)} + P_m^{(O)}) Y_\mu Y_\nu + P_m^{(O)} g^{(O)}{}_{\mu\nu}(x) \tag{140}$$

$$\mathcal{T}^{(D)}{}_{\mu\nu} = \rho_m^{(D)} Y_\mu Y_\nu \tag{141}$$

$$\tau_{\mu\nu} = P_m^{(O)} g^{(D)}{}_{\mu\nu}(x). \tag{142}$$

In this case, the modified Friedmann equations (124), (125) take the form

$$3H^2 = 8\pi G[\rho_m^{(O)} + \rho_m^{(D)} + \rho_{\rm eff}] \tag{143}$$

$$2\dot{H} = -8\pi G[\rho_m^{(O)} + \rho_m^{(D)} + \rho_{\rm eff} + P_m^{(O)} + P_{\rm eff}]. \tag{144}$$

Additionally, the continuity equation (131) becomes

$$\begin{aligned}&\dot{\rho}_m^{(O)} + \dot{\rho}_m^{(D)} + 3H(\rho_m^{(O)} + \rho_m^{(D)} + P_m^{(O)}) \\ &\quad = -\frac{\dot{\phi}}{2\phi}(\rho_m^{(O)} + \rho_m^{(D)} + 3P_m^{(O)} + \tilde{\mathcal{T}}) - \frac{1}{8\pi G} Q_0.\end{aligned} \tag{145}$$

We observe that this relation provides an effective source term with respect to General Relativity. This term can be traced to the fiber components of our special connection, which provide the first term on the right hand side, and to the dissipation term Q_0, hence to the nonconservation of the energy-momentum tensor with respect to the connection of F^6. Focusing on the CDM sector, assuming that the dark matter content is close to its GR limit ($\tilde{\mathcal{T}} \approx 0$), and considering the special case where W_+ vanishes, which according to (132) leads to $Q_0 = 0$, Eq. (145) becomes:

$$\dot{\rho}_m^{(D)} + 3H\rho_m^{(D)} = -\frac{\dot{\phi}}{2\phi}\rho_m^{(D)} \tag{146}$$

Observing Eq. (146), we find a parallelism with models of CDM creation in GR [53]. In particular, the continuity equation of CDM in these models reads [69,76–78]:

$$\dot{\rho}_m^{(D)} + 3H\rho_m^{(D)} = \Gamma\rho_m^{(D)} \tag{147}$$

where Γ is the CDM creation rate. Comparing (146) with (147), we find that our model provides a dynamics for CDM creation similar to the aforementioned models, namely

$$\Gamma = -\frac{\dot{\phi}}{2\phi} \tag{148}$$

From the point of view of an observer who interprets the creation mechanism in the framework of general relativity and standard FRW cosmology, it would appear that (146) violates the conservation of energy-momentum due the appearance of the source term in the rhs. However, from the point of view of our construction, the same mechanism can be seen as a result of energy-momentum conservation with respect to the special connection of the total space TF^6. Once again we mention that this behavior has not be imposed by hand, but it arises naturally from the geometrical structure of the bundle construction.

VI. CONCLUDING REMARKS

In this article we studied the gravitational and cosmological consequences of a, Finsler-like, scalar tensor theory on a vector bundle F^6, which consists of a pseudo-Riemannian space-time manifold with two scalars in the role of fibers or internal variables. In this approach, we used a nonlinear connection form of a nonholonomic bundle structure. Under this framework, the properties of a sectorized gravitational field are analyzed for both the ordinary and dark sectors.

The extra geometrical structure is imprinted in the field equations (67), Raychaudhuri (103) and FRW equations (120), (121), (122). Due to the introduction of the scalar fields $\phi^{(1)}$, $\phi^{(2)}$ we obtain extra degrees of freedom which affect the volume of congruence geodesics, the form of the accelerating universe and potentially lead to Lorentz violating and locally anisotropic effects [79–83]. An interesting topic for the upcoming projects would be to examine whether the symmetries of usual spatial homogeneity and isotropy persist on the vector bundle E. A careful and meaningful definition of these symmetries should be given in the current framework and the most consistent way to achieve this is through the proper extension of the concepts of Lie derivatives and Killing vector fields. We remark that the kind of isotropy we are discussing here differs from the concept of internal space-time anisotropy encountered in Finsler gravity.

Applying this construction at a cosmological framework, we showed that the induced generalized scalar-tensor theory from the bundle structure and the nonlinear connection leads to the appearance of an effective dark energy sector in the modified Friedmann equations. Hence, we were able to reproduce the thermal history of the universe, with the sequence of the matter and dark energy eras, and we showed that the resulting dark-energy equation-of-state parameter can lie in the quintessence or phantom regime, or even exhibit the phantom-divide crossing. Furthermore, we showed that this novel intrinsic geometrical structure leads to an effective interaction between the dark matter and the metric and for the particular case of cold dark matter the relation (148) was found between the scalar fields and the CDM creation rate.

There are many things that one should do in order to further investigate generalized scalar-tensor theories arising from vector bundle constructions. The first is to study the specifically symmetric and black hole solutions, and examine the differences comparing to general relativity. The second is to consider specific examples of nonlinear connections and examine whether they can lead to distinguishable behavior. Finally, one should investigate in more detail the cosmological applications, incorporating data from type Ia supernovae (SNIa), baryon acoustic oscillations (BAO), cosmic microwave background (CMB) observations. These interesting and necessary studies are left for future projects.

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APPENDIX A: CONNECTION AND CURVATURE

One can define a special type of linear connection in this space, where the following rules hold:

$$D_{\delta_\nu}\delta_\mu = L^\kappa_{\mu\nu}\delta_\kappa \qquad D_{\delta_\nu}\partial_a = L^c_{a\nu}\partial_c \tag{A1}$$

$$D_{\partial_b}\delta_\mu = C^c_{\mu b}\partial_c \qquad D_{\partial_b}\partial_a = C^\kappa_{ab}\delta_\kappa \tag{A2}$$

Differentiation of the inner product $D_{X_K}\langle X^M, X_N\rangle = 0$ and use of (A1), (A2) leads to the rules:

$$D_{\delta_\nu}\mathrm{d}x^\kappa = -L^\kappa_{\mu\nu}\mathrm{d}x^\mu \qquad D_{\delta_\nu}\delta\phi^c = -L^c_{a\nu}\delta\phi^a \tag{A3}$$

$$D_{\partial_b}\mathrm{d}x^\kappa = -C^\kappa_{ab}\delta\phi^a \qquad D_{\partial_b}\delta\phi^c = -C^c_{\mu b}\mathrm{d}x^\mu \tag{A4}$$

It is apparent from the above relations that D_{δ_ν} preserves the horizontal and vertical distributions, while D_{∂_b} maps one to the other.

Following the above rules, covariant differentiation of a vector $V = V^\mu\delta_\mu + V^a\partial_a$ along a horizontal direction gives:

$$\begin{aligned} D_{\delta_\nu}V &= (\delta_\nu V^\mu + V^\kappa L^\mu_{\kappa\nu})\delta_\mu + (\delta_\nu V^a + V^c L^a_{c\nu})\partial_a \\ &= D_\nu V^\mu\delta_\mu + D_\nu V^a\partial_a \end{aligned} \tag{A5}$$

where we have defined

$$D_\nu V^\mu = \delta_\nu V^\mu + V^\kappa L^\mu_{\kappa\nu} \tag{A6}$$

$$D_\nu V^a = \delta_\nu V^a + V^c L^a_{c\nu} \tag{A7}$$

Similarly, for the covariant differentiation of V along a vertical direction we obtain

$$\begin{aligned} D_{\partial_b}V &= [\partial_b V^\mu + V^a C^\mu_{ab}]\delta_\mu + [\partial_b V^a + V^\mu C^a_{\mu b}]\partial_a \\ &= D_b V^\mu\delta_\mu + D_b V^a\partial_a \end{aligned} \tag{A8}$$

where we have defined

$$D_b V^\mu = \partial_b V^\mu + V^a C^\mu_{ab} \tag{A9}$$

$$D_b V^a = \partial_b V^a + V^\mu C^a_{\mu b}. \tag{A10}$$

The covariant derivative over the full range of indices in F^6 reads:

$$D_{X_M}V = [X_M V^N + \mathbf{\Gamma}^N{}_{LM}V^L]X_N = (D_M V^N)X_N \tag{A11}$$

where

$$D_M V^N = X_M V^N + \mathbf{\Gamma}^N{}_{LM}V^L. \tag{A12}$$

Finally, the covariant derivative for a tensor of general rank is obtained in a similar way.

APPENDIX B: FIELD EQUATIONS

In this Appendix we present the steps which lead to the field equations, (58), (59), (73) and (74). A Hilbert-like action with a matter sector on the bundle F^6 is

$$S = \int_Q d^6U\sqrt{|\mathcal{G}|}\mathcal{R} + 2\kappa\int_Q d^6U\sqrt{|\mathcal{G}|}\mathcal{L}_M(\mathcal{G}^{MN}, \Psi^i) \tag{B1}$$

where $\mathcal{L}_M(\mathcal{G}^{MN}, \Psi^i)$ is the Lagrangian of the matter fields Ψ^i, and Q is a closed subspace of F^6. Variation of the action gives

$$\begin{aligned} \delta S = &\int_Q d^6U\sqrt{|\mathcal{G}|}\left(\mathcal{R}_{MN} - \frac{1}{2}\mathcal{G}_{MN}\mathcal{R}\right)\delta\mathcal{G}^{MN} \\ &+ \int_Q d^6U\sqrt{|\mathcal{G}|}\mathcal{G}^{MN}\delta\mathcal{R}_{MN} \\ &+ 2\kappa\int_Q d^6U\delta(\sqrt{|\mathcal{G}|}\mathcal{L}_M) = 0 \end{aligned} \tag{B2}$$

After a straightforward calculation, we acquire

$$\begin{aligned}\mathcal{G}^{MN}\delta\mathcal{R}_{MN} =& D_K[\mathcal{G}^{MN}\delta\mathbf{\Gamma}^K_{MN} - \mathcal{G}^{\mu\nu}\mathbf{\Gamma}^K_{\mu b}\delta N^b_\nu] \\ &- D_N[\mathcal{G}^{MN}\delta\mathbf{\Gamma}^K_{MK} - \mathcal{G}^{MN}\mathbf{\Gamma}^\kappa_{Mb}\delta N^b_\kappa] \\ &+ \mathcal{G}^{MN}\mathscr{T}^Z{}_{NK}\delta\mathbf{\Gamma}^K_{MZ} \\ &+ \mathcal{G}^{\mu\nu}[\delta N^b_\nu\mathcal{R}_{\mu b} + \delta N^b_\kappa(\mathscr{T}^\kappa{}_{A\nu}\mathbf{\Gamma}^A_{\mu b} - \mathcal{R}^\kappa_{\mu\nu b})]\end{aligned} \tag{B3}$$

Applying Stoke's theorem to the above result and (B2), and assuming that the boundary terms vanish, leads to the following relation:

$$\begin{aligned}\mathcal{G}^{MN}\delta\mathcal{R}_{MN} =& \mathscr{T}^A{}_{KA}[\mathcal{G}^{MN}\delta\mathbf{\Gamma}^K_{MN} - \mathcal{G}^{\mu\nu}\mathbf{\Gamma}^K_{\mu b}\delta N^b_\nu] \\ &- \mathscr{T}^A{}_{NA}[\mathcal{G}^{MN}\delta\mathbf{\Gamma}^K_{MK} - \mathcal{G}^{MN}\mathbf{\Gamma}^\kappa_{Mb}\delta N^b_\kappa] \\ &+ \mathcal{G}^{MN}\mathscr{T}^Z{}_{NK}\delta\mathbf{\Gamma}^K_{MZ} \\ &+ \mathcal{G}^{\mu\nu}[\delta N^b_\nu\mathcal{R}_{\mu b} + \delta N^b_\kappa(\mathscr{T}^\kappa{}_{A\nu}\mathbf{\Gamma}^A_{\mu b} - \mathcal{R}^\kappa_{\mu\nu b})].\end{aligned} \tag{B4}$$

In the Palatini method, the fields $\mathcal{G}^{MN}$, $\mathbf{\Gamma}^L_{MN}$ and N^a_μ are varied independently from each other, therefore (B2) and (B4) provide the equations

$$\mathcal{R}_{(MN)} - \frac{1}{2}\mathcal{G}_{MN}\mathcal{R} = \kappa\mathcal{T}_{MN} \tag{B5}$$

$$\mathcal{G}^{MN}\mathscr{T}^A{}_{KA} + \mathcal{G}^{ML}(\mathscr{T}^N{}_{LK} - \mathscr{T}^A{}_{LA}\delta^N_K) = 0 \tag{B6}$$

and

$$\begin{aligned}&\mathscr{T}^A{}_{NA}\mathcal{G}^{MN}\mathbf{\Gamma}^\kappa_{Mb} - \mathcal{T}^A{}_{KA}\mathcal{G}^{\mu\kappa}\mathbf{\Gamma}^K_{\mu b} \\ &\quad + \mathcal{G}^{\mu\nu}(\delta^\kappa_\nu\mathcal{R}_{\mu b} + \mathcal{T}^\kappa{}_{A\nu}\mathbf{\Gamma}^A_{\mu b} - \mathcal{R}^\kappa_{\mu\nu b}) = 0\end{aligned} \tag{B7}$$

where

$$\mathcal{T}_{MN} = -\frac{2}{\sqrt{|\mathcal{G}|}}\frac{\delta(\sqrt{|\mathcal{G}|}\mathcal{L}_M)}{\delta\mathcal{G}^{MN}} \tag{B8}$$

We remark that the lhs of (B7) vanishes identically for the choice of connection (15).

Alternatively, we can variate the action by considering all the fields dependent on the metric $\mathcal{G}^{MN}$. For the specific connection components given in (15), the nonvanishing part of Eq. (B4) reads:

$$\begin{aligned}&\mathcal{G}^{MN}\delta\mathcal{R}_{MN} \\ &\quad = (\delta^M_A\delta^K_B - \mathcal{G}^{MK}\mathcal{G}_{AB})(D_M\mathcal{T}^Z{}_{KZ} - \mathscr{T}^L{}_{ML}\mathscr{T}^Z{}_{KZ})\delta\mathcal{G}^{AB}\end{aligned} \tag{B9}$$

where we have used Stoke's theorem twice and eliminated all the boundary terms. Combining (B2) and (B9) gives Eqs. (73) and (74).

APPENDIX C: GENERALIZED BIANCHI IDENTITIES

The Bianchi identities constrain the curvature and torsion tensors via the relations [84]:

$$\circlearrowleft_{A,M,N}\{D_A\mathcal{R}^K{}_{LMN} + \mathcal{R}^K{}_{LAR}\mathcal{W}^R{}_{MN}\} = 0 \tag{C1}$$

$$\circlearrowleft_{A,M,N}\{D_A\mathcal{W}^L{}_{MN} + \mathcal{W}^K{}_{AM}\mathcal{W}^L{}_{NK} + \mathcal{R}^L{}_{AMN}\} = 0 \tag{C2}$$

In our calculations we will use the symmetry

$$\mathcal{R}^K{}_{LMN} = -\mathcal{R}^K{}_{LNM} \tag{C3}$$

which is obvious from the defining relation of the Riemann tensor (19). Manipulating

$$\mathcal{R}_{KLMN} = \mathcal{G}_{KR}\mathcal{R}^R{}_{LMN} \tag{C4}$$

with the aid of (15) and (19), it can be shown that

$$\begin{aligned}\mathcal{R}_{KLMN} =& \frac{1}{2}(X_MX_L\mathcal{G}_{NK} - X_MX_K\mathcal{G}_{LN} - X_NX_L\mathcal{G}_{MK} \\ &+ X_NX_K\mathcal{G}_{LM}) + \mathbf{\Gamma}^R{}_{LM}\mathbf{\Gamma}_{RNK} - \mathbf{\Gamma}^R{}_{LN}\mathbf{\Gamma}_{RMK} \\ &+ \mathcal{W}^R{}_{MN}\mathbf{\Gamma}_{LRK}\end{aligned} \tag{C5}$$

From the above, it can be seen that generally

$$\mathcal{R}_{(KL)MN} = \frac{1}{2}\mathcal{W}^R{}_MNX_R\mathcal{G}_{KL} \tag{C6}$$

However, it is obvious from (8) that only the latin upper index elements of $\mathcal{W}$ are nonzero. Since this is index is contracted with the derivative of the metric with respect to the fiber variables, $\mathcal{R}_{(KL)MN}$ is always zero. Therefore, we deduce the antisymmetry of $\mathcal{R}_{KLMN}$ with respect to its first two indices. i.e.,

$$\mathcal{R}_{KLMN} = -\mathcal{R}_{LKMN} \tag{C7}$$

Again from (C5), and taking advantage of (C6) and (C7), we prove

$$\circlearrowleft_{L,M,N}\{\mathcal{R}_{KLMN} + \mathcal{W}^R{}_{MN}\mathbf{\Gamma}_{KRL}\} = 0 \tag{C8}$$

which is equivalent to (C2), and

$$\mathcal{R}_{KLMN} = \mathcal{R}_{MNKL} + \mathcal{W}^R{}_{MN}\mathbf{\Gamma}_{LRK} - \mathcal{W}^R{}_{KL}\mathbf{\Gamma}_{NRM}$$

Lastly, from the above equation we can derive the symmetry properties of the generalized Ricci tensor:

$$\mathcal{R}_{MN} = \mathcal{R}_{NM} + \mathcal{W}^L{}_{RM}\mathbf{\Gamma}^R{}_{LN} - \mathcal{W}^L{}_{RN}\mathbf{\Gamma}^R{}_{LM} \tag{C9}$$

From the first identity (C1) we have:

$$\mathcal{G}^{AL}\mathcal{G}^{M}{}_{K}(D_A\mathcal{R}^{K}{}_{LMN} + D_M\mathcal{R}^{K}{}_{LNA} + D_N\mathcal{R}^{K}{}_{LAM} + \mathcal{R}^{K}{}_{LAR}\mathcal{W}^{R}{}_{MN} + \mathcal{R}^{K}{}_{LMR}\mathcal{W}^{R}{}_{NA} + \mathcal{R}^{K}{}_{LNR}\mathcal{W}^{R}{}_{AM}) = 0$$

and after some algebra we finally obtain

$$D^A\mathcal{E}_{AN} + \mathcal{R}^{A}{}_{R}\mathcal{W}^{R}{}_{NA} + \frac{1}{2}\mathcal{R}^{KA}{}_{NR}\mathcal{W}^{R}{}_{AK} = 0. \quad \text{(C10)}$$

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