

Conformal gravity does not predict flat galaxy rotation curves

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We reconsider the widely held view that the Mannheim–Kazanas (MK) vacuum solution for a static, spherically symmetric system in conformal gravity (CG) predicts flat rotation curves, such as those observed in galaxies, without the need for dark matter. This prediction assumes that test particles have fixed rest mass and follow timelike geodesics in the MK metric in the vacuum region exterior to a spherically symmetric representation of the galactic mass distribution. Such geodesics are not conformally invariant, however, which leads to an apparent discrepancy with the analogous calculation performed in the conformally equivalent Schwarzschild–de-Sitter (SdS) metric, where the latter does not predict flat rotation curves. This difference arises since the mass of particles in CG must instead be generated dynamically through interaction with a scalar field. The energy-momentum of this required scalar field means that, in a general conformal frame from the equivalence class of CG solutions outside a static, spherically symmetric matter distribution, the spacetime is not given by the MK vacuum solution. A unique frame does exist, however, for which the metric retains the MK form, since the scalar field energy-momentum vanishes despite the field being nonzero and radially dependent. Nonetheless, we show that in both this MK frame and the Einstein frame, in which the scalar field is constant, massive particles follow timelike geodesics of the SdS metric, thereby resolving the apparent frame dependence of physical predictions and unambiguously yielding rotation curves with no flat region. Moreover, we show that attempts to model rising rotation curves by fitting the coefficient of the quadratic term in the SdS metric individually for each galaxy are also precluded, since the scalar field equation of motion introduces an additional constraint relative to the vacuum case, such that the coefficient of the quadratic term in the SdS metric is most naturally interpreted as proportional to a global cosmological constant. We also comment briefly on how our analysis resolves the long-standing uncertainty regarding gravitational lensing in the MK metric.

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I. INTRODUCTION

Conformal gravity (CG) (also known as Weyl gravity or Weyl-squared gravity) was first proposed in 1921 by Bach [1], who took Weyl’s idea of a conformally invariant gravity theory [2], but eliminated Weyl’s additional vector (gauge) field, to which Einstein had raised some theoretical objections (see [3], however). Over the past 30 years or so, CG has attracted considerable interest as an alternative to general relativity (GR), since it is claimed, most notably by Mannheim and collaborators, to address several shortcomings of GR [4–6].

From a theoretical perspective, CG differs from GR both in incorporating the local conformal symmetry that holds for the strong, weak and electromagnetic interactions and in being renormalizable [7]. Conversely, whereas GR has field equations that contain second-order derivatives of the metric and is thus unitary, CG has fourth-order field

equations and possesses a classical ghost [8]; it is claimed that one can nonetheless construct a unitary quantum theory by redefining its Fock space [9,10], although this suggestion is controversial [11].

It is from a phenomenological viewpoint, however, that CG has generated the most interest, since it is claimed to explain various astrophysical and cosmological observations without the need for dark matter or dark energy. These analyses rely primarily on several exact solutions that have been found for systems with sufficient symmetry [4,12–18]. A number of studies have, however, called into question many of the claimed advantages of CG over GR, prompting a reconsideration of their theoretical basis, most notably in the areas of cosmology, gravitational lensing and galactic dynamics.

In a cosmological context, for homogeneous and isotropic spacetimes the CG field equations arise solely from the energy-momentum tensor of matter, which consists of a perfect fluid approximation both to radiation and to a Dirac field representing “ordinary” matter, together with a conformally coupled scalar field [5,13,19]. The resulting

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background cosmological evolution equations are identical to those of the Λ CDM model derived from GR, except that the Friedmann equation has a negative effective gravitational constant $G_{\text{eff}} = -3/(4\pi\varphi_0^2)$, where φ_0 is the vacuum expectation value of the scalar field, so that isotropic radiation and matter are repulsive, and the cosmological constant is derived from the scalar field vacuum energy, which is proportional to φ_0^4 . This leads to a somewhat different cosmological model to Λ CDM: the universe is open, radiation dominates at early times to prevent a big-bang singularity, matter is always subdominant, and the scalar field dominates at late times, driving an accelerated expansion with an effective dark energy density in the range $0 < \Omega_{\Lambda,0} < 1$ at the current epoch, which is compatible with observations. Indeed, it is claimed that the CG cosmology provides a better fit to cosmological data, such as luminosity distances from Type IA supernovae and gamma-ray bursts, and with less fine tuning than the standard Λ CDM model, all without resort to the dark sector [19–24]. The study of the growth of cosmological perturbations in CG is still in its infancy, however, and so no predictions yet exist for the cosmic microwave background radiation [6,25,26]. The CG background cosmology does, however, already have several shortcomings. From early on it was found to be incompatible with nucleosynthesis constraints [27–31], and more recently it has been shown to deviate significantly from high-redshift distance moduli data derived from gamma-ray bursts and quasars, yielding a far poorer fit than Λ CDM, and also having its own fine-tuning and cosmic coincidence problems of a similar magnitude to those of Λ CDM [32,33].

The majority of CG phenomenology is, however, based on the so-called Mannheim–Kazanas (MK) vacuum solution [4] of the CG field equations for a static, spherically symmetric spacetime, a solution that was, in fact, first found by Riegert [12] (note that any vacuum solution of GR, even including a cosmological constant, is also a vacuum solution of CG, but the converse is not true). The MK metric is compatible with solar-system tests of gravity [34–37], but is claimed to lead to observable differences from GR on larger scales for the trajectories of both massless and massive particles.¹

Massless particles follow null geodesics, which are invariant under conformal transformations, and have been extensively studied for the MK metric, particularly in the context of gravitational lensing [35–37,40–47]. Nonetheless, the literature remains inconclusive, with different studies leading to strongly contradictory conclusions, even regarding basic issues such as the required sign of the linear term in the MK metric, which is key to much of its phenomenology.

¹It has been suggested that CG is repulsive in the Newtonian limit and so fails solar-system tests [38], but it was later claimed that this conclusion is mistaken and results from a subtlety in taking the limit in isotropic coordinates [39].

These disagreements arise both from the association of the mass of the lens with different combinations of the parameters in the MK metric and from the choice of the geometric definition of the deflection angle [48]; the latter is related to the fact that the MK metric is not asymptotically flat, which is a complication that also (in part) underlies the longstanding confusion regarding the contribution of the cosmological constant to gravitational lensing in the Schwarzschild–de-Sitter (SdS) metric [49,50], a debate that has only recently been satisfactorily resolved [51].

CG is most celebrated phenomenologically, however, for its fitting of flat galaxy rotation curves without the need for dark matter [52–57]. These fits are based on the trajectories of massive particles in the MK metric, with parameter values that are consistent with solar-system tests, although the requirement for matching the MK metric onto a static, spherically symmetric matter source suggests that it may not be possible to set the parameters in a consistent manner [58–60]. More troubling, however, is that the galaxy rotation curve analyses assume simply that massive (test) particles follow timelike geodesics, which *are* affected by conformal transformations, as is well known. Since CG is (by construction) conformally invariant, such transformations should not change the observable consequences of the theory, unless the conformal symmetry is broken in some way.

A related issue is that, in any conformally invariant theory, particle rest masses cannot be fundamental, but must instead arise dynamically, typically through interaction with a scalar field, the mass of which also cannot be fundamental, but arises through dynamical symmetry breaking [61]. Thus, taking inspiration from the Standard Model of particle physics, the matter action in CG usually includes a Dirac field to represent ordinary matter, which has a Yukawa coupling to a (conformal) scalar field with a quartic self-interaction term [5,62,63]. In this way, one may incorporate ordinary matter with mass, such as that making up astrophysical objects such as stars and galaxies, as well as observers and test particles, in a way that maintains conformal invariance. Thus, CG is not only compatible with dynamic mass generation, but *requires* it and hence also the presence of the scalar field, without which massive (test) particles would not be possible. Indeed, as mentioned above, the scalar field is also central to cosmological applications of CG.

The requirement of dynamical mass generation through the presence of a scalar field has two immediate and profound consequences for the fitting of galaxy rotation curves with the MK metric [64–66]. First, one cannot ignore the energy-momentum of the scalar field and so, in general, the spacetime outside a static, spherically symmetric matter distribution in CG is *not* given by the MK vacuum solution. Second, the mass of a test particle depends on the value of the scalar field and hence varies with spacetime position, in general, so the particle does *not* follow a timelike geodesic unless the scalar field has the

same constant value everywhere [62]. It is therefore surprising that these effects are omitted in much of the CG literature devoted to fitting galaxy rotation curves.

Some of these issues have, however, been discussed recently in this context. In particular, [66] considered a special analytical solution of the CG field equations found in [65] for both the metric and the scalar field in a static, spherically symmetric system, for which the metric still has a form equivalent to the MK solution. Since the corresponding scalar field has a radial dependence, however, massive (test) particles do not follow timelike geodesics, as discussed above. Indeed, it was shown that on making a conformal transformation to the Einstein gauge, in which the scalar field takes a constant value everywhere and so massive particles do follow timelike geodesics, the resulting metric is equivalent merely to the standard SdS form, which lacks the linear term in the MK metric that is key to the successful fitting of flat galaxy rotation curves. This analysis was criticized in [67], however, who pointed out that the MK metric is conformally equivalent to the SdS metric, without the need to introduce a scalar field [1,14,68,69], and that the scalar field in the special solution investigated in [66] has vanishing energy-momentum, which was therefore considered to be trivial because it has no effect on the geometry. It is unclear from [67], however, whether these criticisms are considered to validate previous analyses [52–57] using the MK metric to fit flat galaxy rotation curves.

In this paper, we revisit and extend the analysis in [66] and address the criticisms made in [67]. In particular, we discuss how the conformal equivalence of the MK and SdS metrics, even in the absence of a scalar field, raises concerns about the use of timelike geodesics of the MK metric in fitting galaxy rotation curves. Indeed, the fact that timelike geodesics of the conformally equivalent SdS metric are well known not to produce flat rotation curves leads to the suspicion that the prediction in the “MK frame” may be merely a gauge artefact. Accepting the need to include a scalar field to facilitate dynamical mass generation, we confirm that the scalar field in the analytical solution considered in [66] has vanishing energy-momentum, as it must in order for the MK “vacuum” solution for the metric to remain valid. Such so-called “ghost solutions” for fields are, however, found to exist in other physical contexts [70–72], and it does not follow that the scalar field in [66] is dynamically unimportant to massive particle trajectories. Indeed, we verify that the conformal transformation required to reach the Einstein gauge, in which the scalar field is constant everywhere and so its energy-momentum vanishes trivially, transforms the metric into the SdS form, after performing a radial coordinate transformation to recover the usual angular part of the metric. We also note that this joint conformal and coordinate transformation is equivalent to that previously identified in [14,69] as connecting the MK and SdS

metrics, but merely performed in the opposite order. Moreover, *independently* of the considerations of [65], we show that the general form of the conformal transformation used in [66] is picked out *uniquely* as that which preserves the structure of *any* diagonal static, spherically symmetric metric with a radial coefficient that is (minus) the reciprocal of its temporal one (of which both the MK and SdS metrics are examples). We then “close the loop” in the above considerations, by investigating the motion of a Dirac particle with a dynamically generated mass in the presence of a radially dependent scalar field in a static, spherically symmetric spacetime. Applying this formalism to the analytical solution considered in [66], we show directly that the equations of motion in the “MK frame” are identical to those of timelike geodesics in the SdS metric, as they must be for conformal invariance to hold. Hence, this both removes the dependence on conformal frame of the predicted rotation curves that occurs in the absence of a scalar field, and unambiguously identifies the timelike geodesics of the SdS metric as those that are physically realized, which do *not* predict flat galaxy rotation curves. Moreover, we show that the scalar field equation of motion introduces an additional constraint relative to the vacuum case, such that the coefficient of the quadratic term in the SdS metric is most naturally interpreted as proportional to a global cosmological constant, which thus also undermines attempts to model rising rotation curves by fitting this coefficient separately for each galaxy, as has been considered previously in the context of Weyl–Dirac gravity [73]. More generally, for any of type of galactic rotation curve usually considered (see, e.g., [74] for a discussion of the overall family of rotation curves), our analysis implies that all attempts to use the MK vacuum solution to avoid the need for dark matter will fail, since the real physical motion is just that of the conventional SdS metric.² We also briefly discuss the consequences of our analysis for null geodesics in the MK metric, and how it may be used to resolve the long-standing disagreements in the literature regarding gravitational lensing in CG.

Thus, in summary, the outline of our argument is as follows. We assume the physics to be described (everywhere) by the sum of the CG gravitational action (1) and the matter action (5), which contains a Dirac field ψ to represent ordinary (fermionic) matter and a scalar (compensator) field φ that enables the mass of ψ to be generated dynamically, as required by conformal invariance. For a region with $\psi = 0$ (apart from test particles and/or observers) outside a static, spherically symmetric system, the spacetime geometry is not described by the MK vacuum metric in a general conformal frame from the equivalence

²This does *not* mean that conformal gravity itself cannot produce interesting dynamical effects, but we are concerned here only with the solution for a true vacuum, in which the scalar field has a vanishing energy-momentum tensor, and for which the MK solution can therefore be used.

class of solutions in GC, since the energy-momentum of the scalar field does not vanish [65]. A unique frame does exist, however, where the metric retains the MK form (12)–(13), since the scalar field is given by (18), for which the energy-momentum tensor vanishes; the scalar field equation of motion requires that the additional relation (19) must also hold in this case. In both this MK frame and the Einstein frame, for which the scalar field is constant and the metric has the SdS form (15), massive and massless particles follow the timelike and null geodesics, respectively, of the SdS metric. This resolves the apparent frame dependence of physical predictions, unambiguously yields rotation curves for massive particles with no flat region, and resolves the long-standing debate regarding gravitational lensing in the MK metric.

The remainder of this paper is arranged as follows. In Sec. II, we give a brief outline of conformal gravity, including a description of its gravitational and matter actions and the associated equations of motion. We then consider the MK vacuum solution [4,12] for a static, spherically symmetric system in Sec. III, and discuss its conformal equivalence to the SdS metric without reference to any scalar field [14,67,69]. In Sec. IV, we summarize the nature of the galaxy rotation curves predicted in the MK and SdS vacuum solutions, respectively. We then discuss the necessity to introduce a scalar field to facilitate the dynamical generation of massive (test) particles in Sec. V, and describe the special analytical solution of the CG field equations considered in [65,66] for both the metric and the scalar field in a static, spherically symmetric system. In Sec. VI, we describe the conformal transformation of this solution to the Einstein frame and the resulting rotation curves, before discussing rotation curves in the MK frame directly in the presence of a radially varying scalar field in Sec. VII. We briefly comment on the implications of our analysis for gravitational lensing in the MK metric in Sec. VIII, before concluding in Sec. IX.

II. CONFORMAL GRAVITY

Conformal gravity is interpreted geometrically in terms of a Riemannian spacetime with metric $g_{\mu\nu}$ and has the free gravitational action [1,4,5]

$$S_G = \alpha \int d^4x \sqrt{-g} C_{\rho\sigma\mu\nu} C^{\rho\sigma\mu\nu}, \quad (1)$$

where α is a dimensionless parameter and $C_{\rho\sigma\mu\nu}$ is the Weyl tensor, which may be written in terms of the Riemann (or curvature) tensor $R_{\rho\sigma\mu\nu}$ and its contractions³ as

³We adopt the following sign conventions: (+, −, −, −) metric signature, $R^\rho{}_{\sigma\mu\nu} = 2(\partial_{[\mu}\Gamma^\rho{}_{|\sigma|\nu]} + \Gamma^\rho{}_{\lambda[\mu}\Gamma^\lambda{}_{|\sigma|\nu]})$, where the metric (Christoffel) connection $\Gamma^\rho{}_{\lambda\mu} = \frac{1}{2}g^{\rho\sigma}(\partial_\lambda g_{\mu\sigma} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma g_{\lambda\mu})$, and $R^\rho{}_\mu = R^{\rho\sigma}{}_{\mu\sigma}$. We also employ natural units $c = \hbar = 1$ throughout, unless otherwise stated.

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{1}{2}(g_{\rho\mu}R_{\sigma\nu} - g_{\rho\nu}R_{\sigma\mu} - g_{\sigma\mu}R_{\rho\nu} + g_{\sigma\nu}R_{\rho\mu}) + \frac{1}{6}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu})R. \quad (2)$$

It is straightforward to show that under a conformal (scale) transformation $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = e^{2\rho}g_{\mu\nu}$, where $\rho = \rho(x)$ is an arbitrary scalar function, the Weyl tensor transforms covariantly as $\tilde{C}_{\rho\sigma\mu\nu} = e^{-2\rho}C_{\rho\sigma\mu\nu}$, so that the gravitational action S_G in (1) is invariant. Indeed, S_G is the unique conformally invariant action in Riemannian spacetime. Substituting (2) into (1), one finds that S_G may be written as

$$S_G = \alpha \int d^4x \sqrt{-g} \left(R_{\rho\sigma\mu\nu} R^{\rho\sigma\mu\nu} - 2R_{\rho\sigma} R^{\rho\sigma} + \frac{1}{3}R^2 \right) \quad (3)$$

$$= 2\alpha \int d^4x \sqrt{-g} \left(R_{\rho\sigma} R^{\rho\sigma} - \frac{1}{3}R^2 \right) + \text{surface term}, \quad (4)$$

where in the second line we have made use of the fact that the Gauss–Bonnet term $R^2 - 4R_{\rho\sigma}R^{\rho\sigma} + R_{\rho\sigma\mu\nu}R^{\rho\sigma\mu\nu}$ contributes a total derivative in $D \leq 4$ dimensions.

The matter action in conformal gravity is usually taken to be [5]

$$S_M = \int d^4x \sqrt{-g} \left[\frac{1}{2} i\bar{\psi} \gamma^\rho \overleftrightarrow{D}_\rho \psi - \mu \varphi \bar{\psi} \psi + \frac{1}{2} \nu (\nabla_\rho \varphi)(\nabla^\rho \varphi) - \lambda \varphi^4 + \frac{1}{12} \nu \varphi^2 R \right], \quad (5)$$

in which the parameters μ , ν and λ are dimensionless and the numerical factors ensure that S_M varies only by a surface term under a conformal transformation. In this action, ψ is a Dirac field, which has Weyl weight $w = -3/2$, and the covariant derivative in its kinetic term has the form $D_\mu \psi = (\partial_\mu + \Gamma_\mu) \psi$, where the fermion spin connection $\Gamma_\mu = \frac{1}{8}([\gamma^\lambda, \partial_\mu \gamma_\lambda] - \Gamma^\lambda{}_{\nu\mu}[\gamma^\nu, \gamma_\lambda])$ and the position dependent quantities $\gamma_\mu = e^a{}_\mu \gamma_a$ are related to the standard Dirac matrices γ_a using the tetrad components $e^a{}_\mu$.

With a slight abuse of notation, we define $\bar{\psi} \gamma^\rho \overleftrightarrow{D}_\rho \psi \equiv \bar{\psi} \gamma^\rho D_\rho \psi - (D_\rho \bar{\psi}) \gamma^\rho \psi$, where the bidirectional derivative acts only on the spinor field ψ and its conjugate $\bar{\psi}$, and not on the position-dependent gamma matrices γ^ρ . The (compensator) scalar field φ , with Weyl weight $w = -1$, has both a kinetic term and quartic self-interaction term. The covariant derivative ∇_ρ in the former reduces to the ordinary partial derivative, so the only direct interaction of φ with the gravitational field is through its nonminimal (conformal) coupling to the Ricci scalar. Finally, the Yukawa coupling term between ψ and φ is worthy of comment, since it allows the Dirac field to acquire a mass

$\mu\varphi$ dynamically. In particular, if one adopts the Einstein gauge $\varphi = \varphi_0$ (a constant), the Dirac field has a mass $m = \mu\varphi_0$ that is independent of spacetime position.

It is worth noting that the field φ in the action (5) is a fundamental scalar field. In some of the more recent CG literature, however, the scalar field is instead taken to be a long-range order parameter that arises when a fermion bilinear associated with the Dirac field ψ takes a nonzero expectation value $\varphi = \langle 0|\bar{\psi}\psi|0\rangle$ in a spontaneously broken vacuum $|0\rangle$ filled with negative energy fermion states [56]. In this case, φ does not appear in the fundamental matter action (5), but an action of an analogous form for φ instead holds only within each fermion. It is then claimed that each fermion has its own scalar order parameter, which is constant outside of the fermion, where both the kinetic and Ricci scalar terms are absent from the action. Here, irrespective of the nature of the scalar field, we will confine our attention to the case where the total matter action has the form (5) everywhere, as proposed in [5].

The equations of motion for the fields $g^{\mu\nu}$, ψ and φ are obtained by varying the total action $S_T = S_G + S_M = \int d^4x(\mathcal{L}_G + \mathcal{L}_M)$ with respect to them. On varying with respect to $g^{\mu\nu}$ one finds that $(T_G)_{\mu\nu} + (T_M)_{\mu\nu} = 0$, in which

$$(T_G)_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_G}{\delta g^{\mu\nu}} = 4\alpha(2\nabla^\rho \nabla^\sigma + R^{\rho\sigma})C_{\mu\rho\nu\sigma}, \quad (6)$$

where the operation of the quantity in parentheses on the Weyl tensor yields the standard expression for the Bach tensor, which is manifestly symmetric and traceless, as expected, and scales as $e^{-4\rho}$ under a conformal transformation (i.e., it has Weyl weight $w = -4$). It also clearly contains up to fourth-order derivatives of the metric. Using (2) and the contracted Bianchi identity $\nabla_\rho R^{\rho\sigma}{}_{\mu\nu} - 2\nabla_{[\mu} R^{\sigma}{}_{\nu]} = 0$, it is straightforward to show that any Einstein spacetime, for which $R_{\mu\nu} = \beta g_{\mu\nu}$ with β a constant (and is thus a solution of the GR vacuum field equations including a cosmological constant), is a solution of the CG vacuum field equations. The converse is not true, however, since there exist solutions of $(T_G)_{\mu\nu} = 0$ that are not Einstein spacetimes, nor conformally equivalent to them [75–77]. Nonetheless, it is worth noting that imposing a simple Neumann boundary condition on the metric selects the vacuum solutions of GR from the wider set of vacuum solutions of CG, thereby removing ghosts and rendering CG and GR with a cosmological constant equivalent in a vacuum [78].

On including matter, its energy-momentum tensor is given by

$$(T_M)_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} = (T_\psi)_{\mu\nu} + \nu(T_\varphi)_{\mu\nu}, \quad (7)$$

where the contributions from the Dirac and scalar fields are, respectively,

$$(T_\psi)_{\mu\nu} = \frac{1}{2} i\bar{\psi}\gamma_{(\mu}\overleftrightarrow{D}_{\nu)}\psi - g_{\mu\nu} \left(\frac{1}{2} i\bar{\psi}\gamma^\rho \overleftrightarrow{D}_\rho \psi - \mu\varphi\bar{\psi}\psi \right), \quad (8)$$

$$(T_\varphi)_{\mu\nu} = \frac{1}{6} \varphi^2 G_{\mu\nu} + \frac{2}{3} (\nabla_\mu \varphi)(\nabla_\nu \varphi) - \frac{1}{3} \varphi \nabla_\mu \nabla_\nu \varphi + g_{\mu\nu} \left[\frac{1}{3} \varphi \square^2 \varphi - \frac{1}{6} (\nabla_\rho \varphi)(\nabla^\rho \varphi) + \frac{\lambda}{\nu} \varphi^4 \right], \quad (9)$$

in which $\square^2 \equiv \nabla_\rho \nabla^\rho$ and $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor. One should note that to determine the variational derivative with respect to $g^{\mu\nu}$ of the kinetic term for the Dirac spinor field ψ in (5), one must first vary with respect to the tetrad components to obtain $t^a{}_\mu \equiv \delta \mathcal{L}_M / \delta e^a{}_\mu$, from which one then has $2\delta \mathcal{L}_M / \delta g^{\mu\nu} = \eta_{ab} e^a{}_{(\mu} t^b{}_{\nu)}$. It is straightforward to show that $(T_M)_{\mu\nu}$ scales as $e^{-4\rho}$ under a conformal transformation, like $(T_G)_{\mu\nu}$, and that its trace vanishes by virtue of the matter equations of motion.

The latter are obtained by varying the action with respect to the fields ψ and φ , respectively, which yields

$$i\gamma^\mu D_\mu \psi - \mu\varphi\psi = 0, \quad (10)$$

$$\square^2 \varphi - \frac{1}{6} \varphi R + \frac{4\lambda}{\nu} \varphi^3 + \frac{\mu}{\nu} \bar{\psi}\psi = 0. \quad (11)$$

One sees that (10) is the Dirac equation for a fermion field with mass $m = \mu\varphi$ induced by Yukawa coupling to φ , and (11) is a Klein–Gordon equation for a massless scalar field φ with a Dirac source $\mu\bar{\psi}\psi/\nu$ and a position-dependent ‘Mexican hat’ potential $V(\varphi) = -\frac{1}{12} R\varphi^2 + \frac{\lambda}{\nu} \varphi^4$.

As noted in [66], by rescaling φ one can set the dimensionless parameter $\nu = \pm 1$ throughout, if desired, but only the positive value yields the ‘correct’ sign for the kinetic energy of the compensator scalar field. Moreover, as mentioned above, one may use the conformal invariance of the theory to set the scalar field to a constant $\varphi = \varphi_0$, which is usually termed the Einstein gauge.

III. STATIC SPHERICALLY SYMMETRIC VACUUM SOLUTION

For any static spherically symmetric spacetime, a particular conformal transformation brings the line-element into the special form [4]

$$ds^2 = B(r)dt^2 - \frac{dr^2}{B(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (12)$$

As first shown by Riegert [12] and later by Mannheim and Kazanas [4], on substituting the corresponding metric $g_{\mu\nu}$ into the vacuum CG field equations $(T_G)_{\mu\nu} = 0$, one finds that the function $B(r)$ may be written as

$$B(r) = 1 - 3\beta\gamma - \frac{\beta(2 - 3\beta\gamma)}{r} + \gamma r - kr^2, \quad (13)$$

where β , γ and k are integration constants.⁴ In particular, to describe the spacetime outside of a central mass M one identifies the coefficient of the $1/r$ term in (13) with $-2GM/c^2$ (reinstating c for the moment), in which case $\beta \neq 0$, and the MK metric is then in agreement with the classic solar system tests of GR provided $|\beta\gamma| \ll 1$ [34–37]. As is clear from (13), β and γ have dimensions of length and inverse length, respectively, so that the product $\beta\gamma$ is dimensionless.⁵ The constant k has units of $(\text{length})^{-2}$ and the corresponding quadratic term $-kr^2$ in (13) embeds the solution in a curved background at large coordinate radius. We note that Birkhoff's theorem holds in CG, so that (13) is the most general spherically symmetric vacuum solution [12], which thus holds in any region where $(T_M)_{\mu\nu} = 0$, including exterior or interior to an arbitrary spherically symmetric matter distribution.

As expected, (13) includes the Schwarzschild solution as a special case ($\gamma = k = 0$). Also anticipated, but more interesting, is that it further includes the Schwarzschild–deSitter (SdS) solution ($\gamma = 0$), despite the absence of a cosmological constant term in (1). As Bach originally showed [1], however, every static, spherically symmetric spacetime that is conformally related to the SdS metric is a solution of the vacuum field equations of CG for such a system, with the converse being proved some years later by Buchdahl [68]. Thus, as later verified explicitly [14,67,69], the MK metric (13) is *conformally equivalent* to the SdS solution. In particular, if one redefines the radial coordinate as $r = r'/\Omega'(r')$, where

$$\Omega'(r') = 1 - \frac{\gamma}{2 - 3\beta\gamma} r', \quad (14)$$

and then makes the conformal transformation $\tilde{g}'_{\mu\nu}(x') = \Omega'^2(r')g'_{\mu\nu}(x')$, with $g'_{\mu\nu}(x') = X^\rho{}_\mu X^\sigma{}_\nu g_{\rho\sigma}(x(x'))$ and $X^\rho{}_\mu = \partial x^\rho / \partial x'^\mu$, one again obtains a line-element in the special form (12), but expressed in terms of r' and $\tilde{B}'(r')$, where

$$\tilde{B}'(r') = 1 - \frac{\beta(2 - 3\beta\gamma)}{r'} - k' r'^2, \quad (15)$$

with $k' \equiv k + \gamma^2(1 - \beta\gamma)/(2 - 3\beta\gamma)^2$, which has the usual SdS form that lacks the linear term present in the MK metric (13). One should note, however, that in addition to identifying the coefficient of $1/r'$ as $-2GM/c^2$, where M is

⁴Some of the CG literature, including [66], uses the slightly different parametrization $B(r) = \sqrt{1 - 6\beta'\gamma - 2\beta'/r + \gamma r - kr^2}$, where $\beta' = \beta(1 - \frac{3}{2}\beta\gamma) \approx \beta$ if the condition $|\beta\gamma| \ll 1$ is satisfied.

⁵On reinstating c , the more physically relevant quantities β/c^2 and γc^2 have dimensions of inverse acceleration and acceleration, respectively.

the mass of the central object, the constant k' in the r'^2 term may *also* be system dependent in CG, and need not be identified as $\frac{1}{3}\Lambda$, for some global cosmological constant Λ , which is necessary in GR.

IV. GALAXY ROTATION CURVES IN A VACUUM

The modelling of galaxy rotation curves in CG is typically performed without invoking dark matter by using the MK line-element (12)–(13) in the (assumed) vacuum region exterior to a spherically symmetric representation of the galactic matter distribution, and possibly also interior to a spherically symmetric approximation to the matter distribution on much larger scales, representing the cluster or supercluster in which the galaxy resides and potentially extending to cosmological scales and including the Hubble flow [52–57]. In the CG literature, a somewhat complicated approach is taken to the description of the matter distribution interior and exterior to the vacuum region. The galactic matter distribution is often considered as a collection of $\sim 10^{11}$ stars, which contribute terms to the metric coefficient $B(r)$ in the vacuum region that are proportional to a constant, $1/r$ and r , respectively, whereas the matter distribution on larger scales contributes a term proportional to r^2 that arises from inhomogeneities, and a further term proportional to r that is due to the Hubble flow. Irrespective of the origins of these various terms, however, in the (assumed) vacuum region where the rotation curve is modelled, the line element must simply be of the general MK form (12)–(13), owing to Birkhoff's theorem in CG [12].

The conformal equivalence discussed above between the MK and SdS metrics then immediately raises some concerns, however, regarding such analyses. In particular, a key assumption is that a matter test particle (or star) follows a timelike geodesic in the spacetime geometry defined by the MK line-element. This leads to the conclusion that, for a circular orbit of coordinate radius r (in the equatorial plane $\theta = \pi/2$), the velocity v of the test particle (as measured by a stationary observer at that radius) satisfies

$$\begin{aligned} v^2 &= \frac{r^2}{B} \left(\frac{d\phi}{dt} \right)^2 = \frac{r}{2B} \frac{dB}{dr} \\ &= \frac{1}{2} \frac{\beta(2 - 3\beta\gamma)r^{-1} + \gamma r - 2kr^2}{\beta(3\beta\gamma - 2)r^{-1} + (1 - 3\beta\gamma) + \gamma r - kr^2}. \end{aligned} \quad (16)$$

When considering a galaxy rotation curve, the weak-field limit $B \approx 1$ holds, so that the three terms in the numerator of (16) determine its shape [66]. Recalling that $|\beta\gamma| \ll 1$, the first term in the numerator recovers the standard Keplerian rotation curve $v^2 = \beta/r$, whereas the second term contributes a rising component $v^2 = \frac{1}{2}\gamma r$. Indeed, it is the transition between these two régimes, which occurs around $r^2 \sim 2\beta/\gamma$, that produces the approximately flat rotation curve that resembles those observed in

the outskirts of large spiral galaxies [79]. In this context, $M \sim 10^{11} M_\odot$ and so $\beta \sim 10^{14} \text{ m}$ (hence $\beta/c^2 \sim 10^{-3} \text{ m}^{-1} \text{ s}^2$), and γ has typically been associated with the inverse Hubble length, such that $\gamma \sim 10^{-26} \text{ m}^{-1}$ (hence $\gamma c^2 \sim 10^{-9} \text{ m s}^{-2}$). Thus, $|\beta\gamma| \sim 10^{-12}$, which amply satisfies the requirement $|\beta\gamma| \ll 1$. With these values of β and γ , the two contributions to the overall rotation curve are of a similar magnitude for $r \sim 10^{20} \text{ m}$ or $\sim 5 \text{ kpc}$, which corresponds roughly to the size of a galaxy. Some refinement of the model is necessary to accommodate the rising rotation curves observed in smaller dwarf galaxies, so more recent analyses assume $\gamma(M) = \gamma_0(1 + M/M_0)$, where $\gamma_0 \sim 10^{-28} \text{ m}^{-1}$ and $M_0 \sim 10^{10} M_\odot$ [52,53], such that $\gamma \sim 10^{-27} \text{ m}^{-1}$ for $M \sim 10^{11} M_\odot$ and the transition between régimes occurs at $r \sim 15 \text{ kpc}$. Moreover, to model the flat rotation curves in the outskirts of particularly large galaxies [6], one requires $k > 0$ so that the falling quadratic term $-kr^2$ counters the rising term γr in the numerator of (13). A reasonable fit is obtained if $k \sim 10^{-49} \text{ m}^{-2} \sim (100 \text{ Mpc})^{-2}$, where 100 Mpc coincides with the typical size of structures in the cosmic web; this also serves to eliminate bound circular orbits beyond the “watershed” radius $r = |\gamma/2k| \sim 150 \text{ kpc}$.

In any case, it is clear that the linear term γr in (13) is crucial for producing flat rotation curves that resemble those observed in galaxies. It is therefore concerning that the MK metric is conformally equivalent to the SdS form (15), for which the linear term is absent. If one again assumes simply that a matter test particle follows a timelike geodesic in the spacetime geometry, but now that defined by the SdS line-element (15), the corresponding circular velocity \tilde{v}' of the test particle satisfies

$$\tilde{v}'^2 = \frac{r'}{2\tilde{B}'} \frac{d\tilde{B}'}{dr'} = \frac{1}{2} \frac{\beta(2 - 3\beta\gamma)r'^{-1} - 2k'r'^2}{\beta(3\beta\gamma - 2)r'^{-1} + 1 - k'r'^2}. \quad (17)$$

By analogy with the argument given above, $\tilde{B}' \approx 1$ in this astrophysical application, so that the rotation curve is determined by the two terms in the numerator of (17). Again assuming $|\beta\gamma| \ll 1$, the first term similarly recovers the standard Keplerian result $\tilde{v}'^2 = \beta/r'$, and the second term contributes $\tilde{v}'^2 = -k'r'^2$, where $k' \approx k + \frac{1}{4}\gamma^2$. Moreover, assuming similar values for β , γ and k as used above, then $k' \approx k$ and the rotation curve falls for all values of r' until bound circular orbits are eliminated beyond the new watershed radius $r' = |\beta/k|^{1/3} \sim 20 \text{ kpc}$. Thus, in the “SdS frame,” there is no region with a flat rotation curve, as expected.⁶

⁶The SdS metric has nonetheless been used to model a small number of spiral galaxies with rotation curves that are rising for all observed values of r' , by fitting a *negative* value of k' separately for each galaxy, albeit in the context of Weyl–Dirac gravity [73].

Since the transformations linking the two metrics (13) and (15) leave the CG gravitational action (1) invariant, however, they should not change the observable physical consequences of the theory, unless conformal invariance is broken in some way. The ambiguity in the predicted rotation curves arises from the assumption that a test particle of fixed rest mass m follows a timelike geodesic, which is based on the standard postulate in GR that the worldline extremizes the particle action $S_p = -m \int ds$, where ds is the spacetime interval. This action is unsuitable in CG, however, since it is not invariant under conformal transformations [34]. This leads to the suspicion that the flat rotation curves predicted in the “MK frame” may be merely a gauge artefact.

V. DYNAMIC GENERATION OF TEST PARTICLE MASSES

The question then naturally arises as to which of the rotation curves (16) or (17), if either, is physically realized. The key to resolving this ambiguity is to recognize that the (massive) test particle in either scenario represents some form of “ordinary” matter, typically described by a Dirac field. Thus, even when using the MK and SdS metrics, which both satisfy the vacuum field equations of CG, one must still consider how to introduce matter in the form of a Dirac field into the theory in a consistent manner, in order to model correctly the trajectories of massive test particles in the vacuum region.

As discussed in Sec. II, the appropriate form for the matter action in CG has the form (5). In particular, to satisfy conformal invariance, the Dirac field must acquire a mass dynamically through the Yukawa coupling term $\mu\phi\bar{\psi}\psi$, which thus *necessitates* the introduction of a scalar (compensator) field ϕ that is nonzero everywhere (except perhaps at infinity). If the Dirac field ψ represents only the test particle, then one need solve only the coupled equations of motion of the metric $g^{\mu\nu}$ and scalar field ϕ : the former is given by $(T_G)_{\mu\nu} + \nu(T_\phi)_{\mu\nu} = 0$ using (6) and (9), and the latter by (11) with $\psi = 0$.

The solutions of these equations for a static spherically symmetric spacetime are investigated in [65]. In general, the nonzero energy-momentum of the scalar field means that the metric does not have the MK form. A special analytical solution is identified, however, for which the scalar field is everywhere nonzero and finite, given by

$$\phi(r) = \phi_0 \left(1 + \frac{r}{a}\right)^{-1}, \quad (18)$$

where ϕ_0 and a are finite positive constants, but its entire energy-momentum tensor nonetheless vanishes identically, $(T_\phi)_{\mu\nu} = 0$ [65–67]. In this case, the metric then clearly still satisfies the CG vacuum field equations

$(T_G)_{\mu\nu} = 0$ and so can be written in the form of the MK line-element (12)–(13), in which case $a = (2 - 3\beta\gamma)/\gamma$ in (18). One should also note that the scalar field equation of motion (11), with $\psi = 0$, introduces an *additional* constraint relative to the vacuum case, which imposes the following relationship between the integration constants in the MK metric:

$$k + \frac{\gamma^2(1 - \beta\gamma)}{(2 - 3\beta\gamma)^2} = -2\lambda\varphi_0^2. \quad (19)$$

Thus, the constant a in (18) is expressible wholly in terms of the coefficients in the MK metric (12), and the overall normalization φ_0 in (18) may also be expressed in terms of these coefficients and the constant λ appearing the scalar field potential energy term in (5). Consequently, one is *not* free to assume different values for the constants a and φ_0 , as is done in some of the CG literature, where it is assumed that the φ field in which a test particle (star) moves is somehow generated by the test particle itself, rather than being an “ambient” field that permeates the vacuum region [55].

For metrics of the special form (12), one may show that (18) is the most general static, spherically symmetric form for φ that is a so-called “ghost solution,” i.e., a nonzero matter field configuration that solves the equations of motion but has vanishing energy-momentum.⁷ Thus, if the line-element has the general MK form (12)–(13), the energy-momentum tensor (9) of the scalar field vanishes *if and only if* $\varphi(r)$ has the form (18). In particular, it does *not* vanish for a constant scalar field $\varphi(r) = \varphi_0$. Hence, irrespective of the assumed physical nature of the scalar field, provided the matter action has the form (5), φ *cannot* have a constant value if the spacetime geometry is described by the general MK metric, since this combination is prohibited by the vacuum field equations. If one sets $\varphi(r) = \varphi_0$, the scalar field energy-momentum tensor vanishes only if $G_{\mu\nu} + 6\lambda\varphi_0^2 g_{\mu\nu} = 0$, so that the only vacuum metric allowed has the SdS form (15) with $k' = -2\lambda\varphi_0^2$. These considerations cast doubt on much of the CG literature devoted to the fitting of galaxy rotation curves [52–57].

As we discuss in the next section, the form (18) is determined directly by the required form of the transformation to the Einstein gauge $\varphi(r) = \varphi_0$ for metrics of the special form (12). Since the scalar field energy-momentum vanishes straightforwardly in the Einstein gauge, one may view the form (18) as merely an artefact of solving the equations of motion in a gauge in which the metric is assumed to have the special form (12).

⁷It is worth noting that other ghost solutions exist, for example for the Dirac field in both Einstein–Weyl and Einstein–Cartan gravity for certain systems [70–72].

In any case, the immediate consequence of (18) is that a (fermionic) test particle has a dynamically induced mass $m = \mu\varphi$ that varies with coordinate radius and hence it does not follow a timelike geodesic, which violates the key assumption made in deriving the rotation curve (16) in the MK frame.

VI. GALAXY ROTATION CURVES IN THE EINSTEIN FRAME

Rather than including the effect of the radially dependent scalar field on the massive test particle trajectory directly in the MK frame, we first consider the approach used in [66], where one takes advantage of the conformal invariance of the theory and performs a conformal transformation to the Einstein frame, in which the scalar field has the constant value φ_0 and so the test particle has the same mass $m = \mu\varphi_0$ everywhere and hence does follow a timelike geodesic.

Instead of considering the MK metric directly, however, it is more informative to illustrate a conformal transformation to the Einstein gauge for a *general* static, spherically symmetric metric of the form

$$ds^2 = A(r)dt^2 - \frac{dr^2}{B(r)} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (20)$$

which clearly coincides with the special form (12) when $A(r) = B(r)$; we also consider a general form for $\varphi(r)$. Since φ has Weyl weight $w = -1$, under a (radial) conformal transformation one has $\varphi(r) \rightarrow \tilde{\varphi}(r) = \varphi(r)/\Omega(r)$, and so the required conformal transformation to achieve $\tilde{\varphi}(r) = \varphi_0$ is simply $\Omega(r) = \varphi(r)/\varphi_0$, and the metric becomes $\tilde{g}_{\mu\nu}(x) = \Omega^2(r)g_{\mu\nu}(x)$. To bring the angular part of the metric back into the standard form in (20), but expressed in terms of a new radial coordinate r' , one must then perform the (radial) coordinate transformation $r' = r\Omega(r)$ to obtain $\tilde{g}'_{\mu\nu}(x') = X^\rho{}_\mu X^\sigma{}_\nu \tilde{g}_{\rho\sigma}(x(x'))$. In so doing, one finds that the resulting line-element again has the form (20), but expressed in terms of the new radial coordinate r' and the metric functions

$$\tilde{A}'(r') = \Omega^2(r(r'))A(r(r')), \quad \tilde{B}'(r') = f^2(r(r'))B(r(r')), \quad (21)$$

where we have defined the function $f(r) \equiv 1 + r \frac{d \ln \Omega(r)}{dr}$. Thus, as one might expect, even if (20) has the special form (12) in which $A(r) = B(r)$, this form is *not* preserved in general by these transformations. Indeed, this is achieved *only* if $f^2(r) = \Omega^2(r)$, which is readily solved on demanding that $\Omega(r)$ is finite everywhere to obtain $\Omega(r) = \varphi(r)/\varphi_0$, where $\varphi(r)$ is given by (18) with a arbitrary. Thus, *independently* of the considerations in [65], the general form (18) for the scalar field is picked

out *uniquely* as that for which the corresponding (finite) conformal transformation to the Einstein frame preserves the special form (12) of the metric. Equally, starting from a metric of the special form (12), if $\varphi(r)$ does not have the form (18), where a may be arbitrary, then the resulting transformed line-element in the Einstein frame does *not* also have this special form.

Adopting the form (18) for the scalar field and applying the above approach to the MK metric, for which $B(r)$ is given by (13) and $a = (2 - 3\beta\gamma)/\gamma$, the required conformal transformation to the Einstein gauge is simply

$$\Omega(r) = \left(1 + \frac{\gamma}{2 - 3\beta\gamma} r\right)^{-1}, \quad (22)$$

and one finds that $\tilde{B}'(r')$ is again given precisely by the SdS form (15). As might be expected by comparing the forms of the conformal transformations (14) and (22), the coordinate r' has the same form as that originally used in [14,67,69] to obtain (15), which is given by $r' = r(1 + \frac{\gamma}{2-3\beta\gamma}r)^{-1}$. Note that this tends to the finite value $r' \rightarrow (2 - 3\beta\gamma)/\gamma$ as $r \rightarrow \infty$. Indeed, the only difference in the two approaches is that the conformal and (radial) coordinate transformations are performed in opposite orders.

In any case, since a test particle has a constant mass in the Einstein gauge, and thus follows a timelike geodesic, one thus identifies (17) as the rotation curve that is physically realized. Moreover, one should note from (15) and (19) that one now requires $k' = -2\lambda\varphi_0^2$ in (17). If one assumes that φ_0 in (18) may be system dependent, then there remains the possibility that exists in the vacuum case of attempting to model some (typically rising) rotation curves by fitting for k' separately for each galaxy, as in [73]. Such an assumption seems questionable when viewed in the Einstein gauge, however, where φ_0 is more naturally interpreted as a system-independent quantity that leads to a ‘global’ cosmological constant $\Lambda = -6\lambda\varphi_0^2$. In this case, one may therefore no longer fit for k' separately for each galaxy, or at all if one considers Λ to be fixed by cosmological observations. It is also worth noting that, to obtain a positive cosmological constant Λ , one must have $\lambda < 0$, which thus requires a negative scalar field vacuum energy $\lambda\varphi_0^4$, at least with the usual sign conventions adopted in the matter action (5).

VII. GALAXY ROTATION CURVES IN THE MK FRAME

We now close the loop in our considerations by instead including the effect of a radially dependent scalar field on massive particle trajectories directly in the MK frame. In the interests of generality, however, we will first present our results for an arbitrary static, spherically symmetric metric of the special form (12) and an arbitrary radial scalar field

$\varphi(r)$, before explicitly considering the case of the MK metric (13) and the scalar field configuration (18).⁸

We begin by again assuming that a matter test particle is represented by a Dirac field, and construct an appropriate action from which its equation of motion can be derived. The construction of the action for a spin-1/2 point particle and the subsequent transition to the full classical approximation in which the particle spin is then neglected is discussed in [81]. In the presence of a Yukawa coupling to a scalar compensator field, this yields

$$S_p = - \int d\zeta \left[p_a u^a - \frac{1}{2} e (p_a p^a - \mu^2 \varphi^2) \right], \quad (23)$$

where the dynamical variables are the tetrad components of the particle 4-momentum $p_a(\zeta) = e_a^\mu p_\mu(\zeta)$ and 4-velocity $u^a(\zeta) = e^a_\mu dx^\mu(\zeta)/d\zeta$, and the einbein $e(\zeta)$ along the worldline $x^\mu(\zeta)$, which is parametrized by ζ .

As also shown in [81], in order that $u^a u_a = u^\mu u_\mu = 1$ for a massive particle, the einbein must take the form $e = 1/(\mu\varphi)$. In this case, the Weyl weights of the quantities appearing in (23) are $w(p_a) = -1$, $w(u^a) = 0$, $w(e) = 1$, $w(\zeta) = 1$ and $w(\varphi) = -1$, so that the action is indeed scale-invariant. On varying the action with respect to the dynamical variables p_a , x^μ and e , one finds that the particle equation of motion may then be written in the coordinate frame as

$$u^\nu \nabla_\nu u^\mu = (g^{\mu\nu} - u^\mu u^\nu) \partial_\nu \ln \varphi. \quad (24)$$

Thus, as expected, it is only when φ is constant that the particle follows a timelike geodesic. It is worth noting that one may also arrive at the equation of motion (24) in a more heuristic manner by simply positing the action of a particle with position-dependent mass $m(x) = \mu\varphi(x)$ to be $S_p = - \int m(x) ds = -\mu \int \varphi(x) ds$, which is a straightforward generalization of the usual particle action assumed in GR [34], and identifying the parameters ζ and s (although we shall draw a distinction between ζ and proper time below).

Assuming a static, spherically symmetric system with $\varphi = \varphi(r)$ and a line-element in the special form (12), one finds that for a massive particle worldline $x^\mu(\zeta)$ in the equatorial plane $\theta = \pi/2$, the t - and ϕ -equations of motion are

⁸It is, in fact, straightforward to perform the calculation for a metric of the *general* form (20) and an arbitrary radial scalar field $\varphi(r)$, but we present here only the results for $A(r) = B(r)$ for the sake of brevity. We discuss the wider implications of the gauge choice $A(r) = B(r)$ in [80], and in particular describe the manner in which it distorts the scaling properties of variables, thereby making it extremely difficult to identify intrinsic φ -independent quantities that may be used for performing all calculations, including the derivation of the geodesic equations.

$$B\Omega \frac{dt}{d\zeta} = \mathcal{k}, \quad r^2\Omega \frac{d\phi}{d\zeta} = \mathcal{h}, \quad (25)$$

where \mathcal{k} and \mathcal{h} are constants, and we may replace the r -equation of motion with the much simpler first integral $u^\mu u_\mu = 1$, which reads

$$B \left(\frac{dt}{d\zeta} \right)^2 - B^{-1} \left(\frac{dr}{d\zeta} \right)^2 - r^2 \left(\frac{d\phi}{d\zeta} \right)^2 = 1, \quad (26)$$

where $\Omega(r) \equiv \varphi(r)/\varphi_0$ and the constants \mathcal{k} and \mathcal{h} are defined such that one recovers the familiar timelike geodesic equations in GR for an affine parameter ζ if $\varphi(r) = \varphi_0$ and so $\Omega = 1$.

As discussed in [3], however, the parameter ζ cannot be interpreted as the proper time of the particle, since it has Weyl weight $w(\zeta) = 1$ and so it is not invariant under conformal transformations. Rather, the proper time interval measured by some (atomic) clock moving with the particle is instead given by $d\tau \propto \varphi d\zeta$, which is correctly invariant under conformal transformations. Without loss of generality, one may choose the constant of proportionality such that $d\tau = (\varphi/\varphi_0)d\zeta = \Omega d\zeta$, and so $d\tau$ and $d\zeta$ coincide if $\varphi(r) = \varphi_0$. Thus, when expressed in terms of the proper time τ of the particle, and denoting $d/d\tau$ by an overdot, the equations of motion (25)–(26) become

$$B\Omega^2 \dot{t} = \mathcal{k}, \quad r^2\Omega^2 \dot{\phi} = \mathcal{h}, \quad B\dot{t}^2 - B^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = \Omega^{-2}. \quad (27)$$

On substituting the first two equations into the third, one straightforwardly obtains the energy equation for massive particle trajectories,

$$r^2\Omega^4 + \left(\Omega^2 + \frac{\mathcal{h}^2}{r^2} \right) B = \mathcal{k}^2. \quad (28)$$

Then substituting $\dot{r} = \dot{\phi} dr/d\phi = (\mathcal{h}/r^2\Omega^2) dr/d\phi$, defining the reciprocal radial variable $u \equiv 1/r$ and differentiating with respect to ϕ , one obtains the shape (or orbit) equation for massive particle trajectories,

$$\frac{d^2u}{d\phi^2} + \frac{1}{2} \frac{d}{du} \left[\left(u^2 + \frac{\hat{\Omega}^2}{\mathcal{h}^2} \right) \hat{B} \right] = 0, \quad (29)$$

where we have defined the functions $\hat{B}(u) \equiv B(1/u)$ and $\hat{\Omega}(u) \equiv \Omega(1/u)$. As expected, if $\Omega = 1$ the equations (27)–(29) reduce to the familiar equations for a timelike geodesic in the equatorial plane $\theta = \pi/2$ of a static, spherically symmetric system with line-element of the special form (12) [82].

We now specialize to the case where the scalar field $\varphi(r)$ has the form (18), with a arbitrary, for which $(r\Omega) = i\Omega^2$. In this case, on defining the new radial coordinate $r' = r\Omega(r)$

and metric function $\tilde{B}'(r') = B(r(r'))\Omega^2(r(r'))$, a dramatic simplification takes place whereby the equations of motion (27) become

$$\tilde{B}'\dot{t} = \mathcal{k}, \quad r'^2\dot{\phi} = \mathcal{h}, \quad \tilde{B}'\dot{t}^2 - \tilde{B}'^{-1}\dot{r}'^2 - r'^2\dot{\phi}^2 = 1, \quad (30)$$

and the energy and shape equations (28) and (29), respectively, have the forms

$$\dot{r}'^2 + \left(1 + \frac{\mathcal{h}^2}{r'^2} \right) \tilde{B}' = \mathcal{k}^2, \quad (31)$$

$$\frac{d^2u'}{d\phi^2} + \frac{1}{2} \frac{d}{du'} \left[\left(u'^2 + \frac{1}{\mathcal{h}^2} \right) \hat{B}' \right] = 0, \quad (32)$$

where $u' = 1/r'$ and $\hat{B}'(u') = \tilde{B}'(1/u')$. These equations have precisely the same form in terms of the particle proper time as the equations for a timelike geodesic in the equatorial plane $\theta = \pi/2$ of a static, spherically symmetric spacetime with line-element of the special form (12), but in terms of the radial variable r' and the metric function $\tilde{B}'(r')$.

Specialising further to the case where $B(r)$ has the MK form (13) and $a = (2 - 3\beta\gamma)/\gamma$, so that $\Omega(r)$ is given by (22), we know from Sec. VI that $\tilde{B}'(r')$, as defined above, has the SdS form (15) with $k' = -2\lambda\varphi_0^2$. Thus, by explicitly taking into account the presence of the radially dependent scalar field in the MK frame, we have arrived directly at the same conclusion that we reached previously in Sec. VI by transforming to the Einstein frame, namely that in terms of the radial coordinate r' it is the rotation curve (17) that is physically realized, which does not have the flat region observed in galaxies. This thereby eliminates, as it must, the ambiguity discussed in Sec. IV, where the physical predictions appeared to depend on the conformal frame in which the calculation was performed.

VIII. GRAVITATIONAL LENSING

As discussed in Sec. I, the literature concerning gravitational lensing in the MK metric remains inconclusive, with considerable disagreement between different studies [35–37, 40–47]. It is uncontroversial, however, that null geodesics are unaffected by conformal transformations. Thus, on performing a joint conformal and (radial) coordinate transformation, such as those considered above, if $r = r(\phi)$ is the original orbit equation for a massless particle in the equatorial plane $\theta = \pi/2$, then it is transformed simply into $r' = r'(r(\phi))$. Although this, of course, leads to local changes in the trajectory, it has been suggested previously that the global behavior is unaffected and, in particular, that the range of ϕ in the orbit equation remains unchanged, and hence so too does the scattering angle or deflection [83]. This is valid, however, *only* if $r' \rightarrow r$ as $r \rightarrow \infty$, which does not hold for the transformation from the MK frame to the Einstein frame

discussed in Sec. VI, for which $r' \rightarrow (2 - 3\beta\gamma)/\gamma$ as $r \rightarrow \infty$. Consequently, the scattering angle or deflection will, in general, *differ* between the two frames. This does not correspond to any *physical* difference, however, but is merely a consequence of the fact that r' is finite as $r \rightarrow \infty$ for the two radial coordinates used.⁹

Indeed, it is instructive in this context to revisit the calculation of particle trajectories in the MK frame, but for the case of massless particles. As discussed in [81], one may deduce the motion of photons by directly considering the dynamics of the electromagnetic field in the gravitational background, but one may also arrive at the same conclusions by reconsidering the particle action (23), which is immediately applicable to massless fermions (such as a neutrino) by setting $\mu = 0$. In this case, one finds that $u^\mu u_\mu = 0$ irrespective of the form of the einbein e , which one is therefore free to choose in the most convenient manner. Here we take $e = 1/\varphi$, so that the weight of each variable in the action matches that in the massive case discussed in Sec. VII.

One then finds that the equation of motion (24) is replaced by $u^\mu \nabla_\nu u^\mu = -u^\mu u^\nu \partial_\nu \ln \varphi$, but this nonetheless leaves the t - and ϕ -equations of motion (25) unchanged and the first integral (26) differs only in that the right-hand side is zero. Once again, one cannot use ζ to parametrize the particle trajectory since it has Weyl weight $w(\zeta) = 1$ and so is not invariant under conformal transformations. As previously, the appropriate invariant measure is $d\tau = (\varphi/\varphi_0)d\zeta = \Omega d\zeta$, although τ cannot be interpreted as a proper time in this case, since the worldline is null.¹⁰ Following through an analogous calculation to that performed in Sec. VII, one finds that the energy and shape equations corresponding to (31) and (32) are

$$\dot{\nu}^2 + \frac{\hbar^2}{r'^2} \tilde{B}' = \mathcal{E}^2, \quad (33)$$

$$\frac{d^2 u'}{d\phi^2} + \frac{1}{2} \frac{d}{du'} (u'^2 \hat{B}') = 0, \quad (34)$$

which have the same form as the equations in terms of an affine parameter for a null geodesic in the equatorial plane $\theta = \pi/2$ of a spacetime with line-element of the special form (12), but in terms of the new radial coordinate $r' = 1/u' = r\Omega(r)$ and $\tilde{B}'(r') = B(r(r'))\Omega^2(r(r'))$.

As in Sec. VII, if one now specializes to the case where the original metric function $B(r)$ has the MK form (13) and $a = (2 - 3\beta\gamma)/\gamma$, so that $\Omega(r)$ is given by (22), then the function $\tilde{B}'(r')$ has the SdS form (15) with $k' = -2\lambda\varphi_0^2$.

⁹For the same reason, the scattering angle or deflection of *massive* particle trajectories will also differ between the MK and Einstein frames, even after taking into account the effect of the scalar field.

¹⁰For arbitrary e , the invariant interval is proportional to $\varphi^2 e d\zeta$.

Thus, in terms of the radial coordinate r' , the trajectories of massless particles in the MK frame follow null geodesics of the SdS metric. This therefore resolves the uncertainty in the literature regarding gravitational lensing in the MK frame, since the SdS metric lacks the linear term that has prompted so much debate in the CG literature, and the consequences of the quadratic cosmological constant term have recently been properly determined [51]. Hence, one may arrive at unambiguous predictions for gravitational lensing that can then be easily recast in terms of the original radial coordinate r used in the MK form for $B(r)$ in (13), if desired.

Finally, as shown in Sec. VI, we note again here that one may only reach another metric having the special form (12) from the MK metric by a (finite) conformal transformation (and subsequent radial coordinate transformation) if $\Omega(r) = \varphi(r)/\varphi_0$, where $\varphi(r)$ is given by (18) with a arbitrary. This provides some insight into previous work seeking to use gauge transformations in CG to make matter attractive to null geodesics in the MK metric irrespective of the sign of its linear term [67,83]. In particular, setting $a = 1/\gamma$ in (18) and performing a joint conformal and (radial) coordinate transformation as outlined in Sec. VI, one finds on neglecting any products of β and/or γ that $\tilde{B}'(r') \approx 1 - 2\beta/r' - \gamma r' - k r'^2$, which has the same form as the MK metric function (13) at this level of approximation, but with the sign of the linear term reversed. Nonetheless, this result must be treated with some caution, since one finds that the exact expression for $\tilde{B}'(r')$, without making any such approximations, does *not* have the same form as (13) with merely a linear term of opposite sign.

IX. CONCLUSIONS

We have revisited the most celebrated phenomenological consequence of CG, namely that the MK vacuum solution for a static, spherically symmetric system predicts flat galaxy rotation curves, without the need for dark matter [4,52–57]. This prediction is based on the assumption that massive (test) particles have fixed rest masses and follow timelike geodesics in the MK metric. The conformal equivalence of the MK and SdS metrics raises concerns, however, that this prediction may be a gauge artefact, since performing a similar analysis in the SdS metric yields rotation curves without any flat region, as is well known. Since CG is (by construction) invariant to such transformations, they should not change the observable consequences of the theory, unless the conformal symmetry is broken in some way, either dynamically or by imposing boundary conditions. Indeed, if boundary conditions are involved, interesting physics can arise quite generally from differences between solutions that are gauge transformations of each other, an obvious example being the Aharanov–Bohm effect [84]. Moreover, some care must clearly be taken regarding boundary conditions at infinity for both the MK and SdS metric, since neither is

asymptotically flat, although the presence of both a constant term differing from unity and a linear term in the MK metric exacerbates this difficulty relative to the SdS metric. It is therefore interesting that imposing a simple Neumann boundary condition on the metric selects the vacuum solutions of GR with a cosmological constant from the wider set of vacuum solutions of CG [78], and hence selects the SdS metric rather than the MK metric for static, spherically symmetric systems.

If the conformal symmetry is unbroken, however, the key to resolving the question of which rotation curves are physically realized is to recognize that massive (test) particles constitute some form of ordinary matter, typically represented by a Dirac field, which must generate its mass dynamically through interaction with a scalar field. The consequent necessity for a scalar field that is nonzero everywhere means that, in general, the spacetime outside a static, spherically symmetric matter source in CG is not described by the MK vacuum solution, as demonstrated in [65]. Nonetheless, a special solution is identified in [65] for which the metric retains the MK form, since the scalar field energy-momentum vanishes, despite the field being nonzero everywhere. Indeed, such ghost solutions are found in other physical contexts [70–72].

Despite having no effect on the geometry, ghost solutions can have important dynamical effects, and so are not trivial, as claimed in [67]. This is especially true for the scalar field in the special solution obtained in [65], since it facilitates dynamical mass generation through its interaction with the Dirac field that represents ordinary matter. Since the scalar field is radially dependent in the MK frame, so too are the masses of test particles, which therefore do not follow timelike geodesics. In particular, on making a conformal transformation to the Einstein gauge, in which the scalar field takes a constant value everywhere and so massive particles do follow timelike geodesics, the resulting metric is equivalent merely to the standard SdS form, which lacks the linear term in the MK metric that is key to the successful fitting of flat galaxy rotation curves [66]. Moreover, by considering directly the motion of a Dirac particle in the presence of a nonuniform scalar field, we further show that massive particles in the MK frame also follow timelike geodesics of the SdS metric, as they must

for conformal invariance to hold. This therefore resolves the apparent dependence of the physical predictions on the frame in which the calculation is performed. More importantly, this unambiguously identifies the rotation curves of the SdS metric, rather than the MK metric, as those that are physically realized, which have no flat region and hence do not fit observations of galaxies. We further show that the scalar field equation of motion introduces an additional constraint relative to the vacuum case, such that the coefficient of the quadratic term in the SdS metric is interpreted most naturally as proportional to a global cosmological constant; this therefore also precludes the modeling of rising rotation curves by fitting this coefficient separately for each galaxy, as performed in [73], albeit in the context of Weyl–Dirac gravity.

In addition, independently of the considerations of [65], we show that the general form of the conformal transformation linking MK and Einstein frames is picked out uniquely as that which preserves the general form of any diagonal static, spherically symmetric metric of the form (12), namely with a radial coefficient that is (minus) the reciprocal of its temporal one.

Finally, we briefly discussed the consequences of our analysis for the study of gravitational lensing in the MK metric, which has caused considerable confusion in the literature, with many analyses producing contradictory results [35–37,40–47]. One may straightforwardly resolve these disagreements by instead performing calculations in the SdS frame, for which previous uncertainties regarding the effects of the cosmological constant term have now been clarified [51]. One can then perform a conformal transformation and accompanying radial coordinate transformation to the MK frame, if desired. We also comment on the limited validity of previous work seeking to make matter attractive to null geodesics in the MK metric, irrespective of the sign of the linear term, by an appropriate choice of conformal gauge [67,83].

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