Calculation of multipole moments of axistationary electrovacuum spacetimes

Gyula Fodor,[1] Etevaldo dos Santos Costa Filho,[2] and Betti Hartmann[2,3,4]

[1]Wigner Research Centre for Physics, 1525 Budapest 114, P.O. Box 49, Hungary
[2]Instituto de Física de São Carlos, Universidade de São Paulo, São Carlos, São Paulo 13560-970, Brazil
[3]Institut für Physik, Carl-von-Ossietzky Universität Oldenburg, 26111 Oldenburg, Germany
[4]Department of Physics and Earth Sciences, Jacobs University Bremen, 28759 Bremen, Germany

(Received 23 May 2021; accepted 30 July 2021; published 7 September 2021)

The multipole moments of stationary axially symmetric vacuum or electrovacuum spacetimes can be expressed in terms of the power series expansion coefficients of the Ernst potential on the axis. In this paper we present a simpler, more efficient calculation of the multipole moments, applying methods introduced by Bäckdahl and Herberthson. For the nonvacuum electromagnetic case, our results for the octupole and higher moments differ from the results already published in the literature. The reason for this difference is that we correct an earlier unnoticed mistake in the power series solution of the Ernst equations. We also apply the presented method to directly calculate the multipole moments of a five-parameter charged magnetized generalization of the Kerr and Tomimatsu-Sato exact solutions.

DOI: 10.1103/PhysRevD.104.064012

I. INTRODUCTION

The purpose of this paper is the presentation of an improved, more efficient method for the calculation of the multipole moments of stationary axially symmetric spacetimes, when the moments are expressed in terms of the power series expansion coefficients of the Ernst potential on the symmetry axis. For the case when electromagnetic fields are also allowed, we give the correct expressions for the octupole and higher moments, correcting a mistake in the literature. As an application of the method, we calculate the multipole moments of a charged magnetized generalization of the Kerr and Tomimatsu-Sato solutions.

Multipole moment tensors for asymptotically flat static vacuum spacetimes were introduced in 1970 by Geroch, in a coordinate system independent way [1]. The generalization of the definition for stationary spacetimes has been given by Hansen [2]. In the stationary case there are two sets of multipole tensors, the mass moments and the angular momentum (or mass-current) moments, which can be unified into a set of complex valued quantities. Alternative but equivalent definitions in terms of specific coordinate systems have been proposed by Thorne [3,4], and also by Simon and Beig [5]. A good review of early results on the topic can be found in [6].

The concrete physical applications of multipole moments in general relativity have been pioneered by the work of Fintan D. Ryan [7]. The gravitational radiation emitted by a compact object orbiting around a much larger central object can be used to determine the multipole moments of the central body. In case of extreme-mass-ratio inspirals, one is expected to be able to determine the first few gravitational multipole moments by the proposed space-based gravitational wave detector LISA [8–11]. Since the multipole moments of the Kerr black hole are uniquely determined by the mass and the angular momentum, this should provide a practical way of testing the no-hair theorem [12–14]. The multipole moments of so-called bumpy black holes and the gravitational radiation of test bodies orbiting around them have been studied in [15–17]. We expect that astrophysical observations will provide a way in the near future to test general relativity in the strong field regime [18–20]. In addition to gravitational waves, the observation of large compact objects and their multipole moments may be achieved by other methods, such as measuring the motion of stars or pulsars around them, study of accretion disks, or the observation black hole shadows by the event horizon telescope [21–24].

Further, highly relativistic physical systems where multipole moments are important are neutron stars. In that case, the observation of the spacetime structure outside the star is expected to give information about the equation of state of the matter in the interior [25,26]. The innermost stable circular orbit marks the inner edge of accretion disks. Its properties, in terms of the multipole moments, has been calculated in [27–29]. It is possible to find certain universal relations between the multipole moments of a neutron star, establishing a no-hair property for these objects. The three-hair relations determine the higher multipole moments by the mass, angular momentum, and quadrupole moment, in an approximately equation of state independent way [30–32]. Exact solutions for the vacuum exterior region have played important role in the establishment of these universal relations [33–39].

Multipole moments can also be defined in various alternative gravitational theories. The motion of compact bodies around massive black holes has been studied in dynamical Chern-Simons gravity, applying a generalization of Thorne's moments [40]. The universality properties of black holes and neutron stars have been investigated in the case of dilatonic Einstein-Gauss-Bonet theory [41,42]. In case of scalar-tensor theory of gravity, an additional set of real moments is associated to the massless scalar field [43,44]. Universal relations between multipole moments of scalarized neutron stars have been considered in [45]. Multipole moments for an arbitrary theory of gravity have been defined in terms of canonical Noether charges in [46]. Six sets of real multipole moment tensors are expected to arise in a generic theory, in contrast to the two sets in vacuum general relativity [46].

The above physical applications have been made possible by strict mathematical results establishing the theory of multipole moments for the nonlinear Einstein equations in the stationary case. Multipole moments are defined for any asymptotically flat stationary spacetime, and if two vacuum spacetimes have the same multipole moments, then they agree at least in a neighborhood of conformal infinity [47–50].

Most astrophysically relevant nonradiating spacetimes are expected to be axially symmetric. In the axisymmetric case the multipole moment tensor of order n can be represented by a single scalar moment, called P_n by Hansen [2]. These scalar moments can be expressed in terms of the power series expansion coefficients of the Ernst potential on the symmetry axis [51]. An algorithm for this calculation has been published in 1989 by Fodor, Hoenselaers, and Perjés [52]. Since the calculation of the Ernst potential on the axis is relatively easy, the results presented in [52] became a standard tool for obtaining the multipole moments. The method has been applied not only for exact solutions, but also for gravitational radiation [7], innermost circular orbits [27], and neutron stars [31]. Multipole moments of a rigidly rotating disk of dust have been calculated in [53]. If the axistationary spacetime is reflection symmetric with respect to the equatorial plane, then for even n the moments P_n are real, while for odd n they are purely imaginary [54,55].

Further important developments on the mathematical theory of gravitational multipole moments in the vacuum case can be found in a series of papers published by Bäckdahl and Herberthson [56–61]. A major aim in their considerations is the proof of a long-standing conjecture by Geroch [1]. This conjecture claims that if one chooses any set of multipole moments that satisfy some appropriate convergence conditions, then there always exists a spacetime having precisely those moments. This has been proven first for the static axially symmetric case [56,57], and then for stationary axially symmetric spacetimes [58,59]. The stationary case without the assumption of axisymmetry has been considered in [60], with a proof of the necessary part of the conjecture. Finally, a proof for the general static case has been given in [61].

Bäckdahl and Herberthson also introduce some very useful tools that make the calculation of multipole moments considerably simpler. They define a complex null vector field, which makes the operation of taking the symmetric and trace-free part of tensors simple and trivial. They also introduce the concept of the leading order part of functions, which allows the use of functions depending on only one variable instead of two. As far as we know, these tools have not been used yet for the calculation of multipole moments of exact solutions, apart from the Kerr case. In this paper we will recalculate the results given in [52], where the moments are given in terms of the Ernst potential on the axis, using the methods of Bäckdahl and Herberthson. We will also look at the electromagnetic generalization of this procedure.

Multipole moments for stationary Einstein-Maxwell fields have been defined by Simon [62]. In this case there are two sets of complex multipole moment tensors, and there are also two complex Ernst potentials. Conditions on equatorial symmetry or antisymmetry for stationary axisymmetric electrovacuum spacetimes have been discussed in [63,64].

For the axially symmetric electrovacuum case, the procedure for calculating the multipoles in terms of the axis coefficients of the Ernst potentials have been published first by Hoenselaers and Perjés [65]. Unfortunately, there have been two mistakes in that paper, which also affected the end results for the multipole moments. The first mistake has been found and corrected by Sotiriou and Apostolatos [66]. However, a second mistake was pointed out by Perjés one year earlier in a conference proceedings article [67], without giving the resulting change in the multipole moments. Unfortunately, this remained unnoticed in all subsequent papers. The article [66] still contains expressions for the moments which are incorrect in the electrovacuum case. In the present paper we again calculate the moments, now using a simpler, more efficient method based on the results of Bäckdahl and Herberthson. We give the correct expressions for the scalar gravitational multipole moments P_n and electromagnetic moments Q_n up to order $n = 6$. Long expressions for higher order moments (even up to $n = 18$) can be easily obtained by the *Mathematica* or *Maple* files provided as Supplemental Material [68].

Denoting the expansion coefficients along the axis of the gravitational and electromagnetic Ernst potentials by m_n and q_n, respectively, for the first three moments we obtain the expected result:

$$P_n = m_n, \qquad Q_n = q_n \quad \text{for } n = 0, 1, 2. \tag{1}$$

For the vacuum case it is known that $P_3 = m_3$, but if there are electromagnetic fields, then we obtain that generally, for the octupole moments $P_3 \neq m_3$ and $Q_3 \neq q_3$. This is a clear difference from earlier results published in [65,66], where the differences started only from $n = 4$. Actually, the mistake, realized first in [67], has not been in the calculation of the moments, which is the same procedure for the vacuum and electrovacuum case, but in the power series solution of the Ernst equations.

We find it important to publish the correct expressions for the power series solution and for the moments in the electromagnetic case, since these results have been used in several subsequent papers. The most important application is the calculation of the gravitational waves emitted by a small body orbiting around a massive compact object [69]. The incorrect expressions for the power series solution of the field equations have been reprinted in some papers [28,70,71], unaware of the mistake pointed out in [67]. Fixing the error may be useful, since in the not too far future, astrophysical observations may become precise enough to make the octupole moment a measurable quantity. Another application of the results in [65,66] is the characterization of new exact solutions in an invariant way by multipole moments. More than a dozen papers present the values of octupole or even higher moments for electrovacuum solutions, which are necessarily affected by our corrections, and we list just a few here [33,72–77].

In the last section of the paper we apply the earlier discussed methods for the calculation of the multipole moments of a five-parameter exact solution presented in [35], which is a charged magnetized generalization of both the Kerr and the $\delta = 2$ Tomimatsu-Sato solutions. The solution is general enough to describe both subextreme and hyperextreme configurations, and the expressions that we obtain for the multipole moments are valid for both cases. We print the moments up to order $n = 5$, but we provide algebraic manipulation software code as Supplemental Material [68], to allow higher order calculations. We stress that this metric is not a useless exact solution without physical applications. The five real parameters correspond to the mass, angular momentum, quadrupole moment, charge, and magnetic dipole moment of the solution, and by appropriate choice of the parameters these can be set to any required values.

The structure of the paper is the following. In Sec. II we first present the field equations for general stationary spacetimes when electromagnetic fields are present, and review the definition of multipole moments. In Sec. III we specialize to axially symmetric solutions, and present the theory needed for the definition of the scalar multipole moments using Weyl coordinates. Here we also discuss those tools and methods introduced by Bäckdahl and Herberthson which are useful for the calculation of the moments of exact solutions. In Sec. IV we calculate the gravitational and electromagnetic multipole moments in terms of the expansion coefficients of the Ernst potentials on the axis. By listing the results, we correct a mistake that remained unnoticed in the literature for quite many years. Finally, in Sec. V we directly apply the Bäckdahl-Herberthson method to calculate the multipole moments of an exact solution, which is general enough to approximate well the exterior region of rotating neutron stars [34,35,78].

II. STATIONARY ELECTROVACUUM SPACETIMES

A. Ernst equations

Since we consider stationary spacetimes, we use tensorial quantities defined on the three-manifold $\mathcal{M}$ of the trajectories of the timelike Killing vector ξ^μ [79]. Denoting the spacetime metric by $g_{\mu\nu}$, and the norm of the Killing vector by $f = -\xi^\mu \xi_\mu > 0$, on the trajectories of ξ^μ we define the rescaled induced metric as $h_{\mu\nu} = f g_{\mu\nu} + \xi_\mu \xi_\nu$. For the three-dimensional tensors we use Latin indices, which are raised and lowered by the metric h_{ab}. The derivative operator belonging to h_{ab} is denoted by ∇_a. Using a coordinate system adapted to the timelike Killing vector, the spacetime metric can be written as

$$\mathrm{d}s^2 = -f(\mathrm{d}t + \omega_a \mathrm{d}x^a)^2 + \frac{1}{f} h_{ab} \mathrm{d}x^a \mathrm{d}x^b, \tag{2}$$

where f, ω_a, and h_{ab} are independent of t. For stationary electromagnetic fields the complex electromagnetic Ernst potential Φ can be defined in terms of the four-dimensional vector potential $A_\mu = (A_t, A_a)$ as [62,80]

$$\Phi = A_t + iA', \tag{3}$$

where the real scalar A' is determined by

$$\nabla_a A' = f \epsilon_{abc} (\nabla^b A^c + \omega^b \nabla^c A_t), \tag{4}$$

$\epsilon_{abc} = \sqrt{h} \varepsilon_{abc}$ is the three-dimensional Levi-Civita tensor, and the spatial indices of A_μ have been raised by h^{ab}. For the electromagnetic case the complex Ernst potential is defined as [62,80]

$$\mathcal{E} = f + i\chi - \Phi\bar{\Phi}, \tag{5}$$

where

$$\nabla_a \chi = f^2 \epsilon_{abc} \nabla^b \omega^c + i(\bar{\Phi} \nabla_a \Phi - \Phi \nabla_a \bar{\Phi}), \tag{6}$$

and the overline denotes complex conjugation.

As alternatives for $\mathcal{E}$ and Φ, we introduce the complex potentials

$$\xi = \frac{1 - \mathcal{E}}{1 + \mathcal{E}}, \qquad q = \frac{2\Phi}{1 + \mathcal{E}}. \tag{7}$$

Then the Einstein and Maxwell equations are equivalent to the following equations [80–82]:

$$\Theta\Delta\xi = 2(\bar{\xi}\nabla^a\xi - \bar{q}\nabla^a q)\nabla_a\xi, \tag{8}$$

$$\Theta\Delta q = 2(\bar{\xi}\nabla^a\xi - \bar{q}\nabla^a q)\nabla_a q, \tag{9}$$

$$\Theta^2 R_{ab} = 2\mathrm{Re}(\nabla_a\xi\nabla_b\bar{\xi} - \nabla_a q\nabla_b\bar{q} + s_a\bar{s}_b), \tag{10}$$

where $\Delta = \nabla^a\nabla_a$ is the Laplacian, R_{ab} is the Ricci tensor belonging to three-dimensional metric h_{ab}, and

$$\Theta = \xi\bar{\xi} - q\bar{q} - 1, \tag{11}$$

$$s_a = \xi\nabla_a q - q\nabla_a\xi. \tag{12}$$

B. Asymptotic flatness

According to the definition of Penrose and Geroch [1,83], a three-dimensional manifold $\mathcal{M}$ with positive definite metric h_{ab} is *asymptotically flat* if the following conditions hold:

(1) There exists a manifold $\tilde{\mathcal{M}}$ with metric $\tilde{h}_{\tilde{a}\tilde{b}}$ and a diffeomorphism $\psi:\mathcal{M}\to\tilde{\mathcal{M}}\backslash\Lambda$, where Λ is a single point in $\tilde{\mathcal{M}}$, such that ψ is a conformal isometry with conformal factor Ω, i.e., $\tilde{h}_{\tilde{a}\tilde{b}} = \Omega^2(\psi^* h)_{\tilde{a}\tilde{b}}$.[1]

(2) The function Ω can be extended as a C^2 scalar to the point Λ corresponding to spatial infinity, such that

$$\Omega|_\Lambda = 0, \qquad \tilde{\nabla}_{\tilde{a}}\Omega|_\Lambda = 0, \qquad \tilde{\nabla}_{\tilde{a}}\tilde{\nabla}_{\tilde{b}}\Omega|_\Lambda = 2\tilde{h}_{\tilde{a}\tilde{b}}|_\Lambda, \tag{13}$$

where $\tilde{\nabla}_{\tilde{a}}$ is the derivative operator on $\tilde{\mathcal{M}}$ belonging to $\tilde{h}_{\tilde{a}\tilde{b}}$.

We use a tilde on the coordinate indices to indicate that these are tensors on $\tilde{\mathcal{M}}$, and the coordinate system used on that manifold is generally different from the mapped version of the original coordinates on $\mathcal{M}$.

C. Multipole moments

Let us choose a scalar field ϕ on $\mathcal{M}$ and assume that $\tilde{\phi} = \Omega^{-1/2}\phi$ can be smoothly extended to the point Λ. We define a set of tensor fields recursively [1,2,62],

$$\mathcal{P}^{(0)} = \tilde{\phi}, \tag{14}$$

$$\mathcal{P}^{(1)}_{\tilde{a}} = \tilde{\nabla}_{\tilde{a}}\mathcal{P}^{(0)}, \tag{15}$$

$$\mathcal{P}^{(n+1)}_{\tilde{a}_1\ldots\tilde{a}_{n+1}} = \mathcal{C}\left[\tilde{\nabla}_{\tilde{a}_{n+1}}\mathcal{P}^{(n)}_{\tilde{a}_1\ldots\tilde{a}_n} - \frac{1}{2}n(2n-1)\tilde{R}_{\tilde{a}_1\tilde{a}_2}\mathcal{P}^{(n-1)}_{\tilde{a}_3\ldots\tilde{a}_{n+1}}\right], \tag{16}$$

where $\tilde{R}_{\tilde{a}\tilde{b}}$ is the Ricci tensor belonging to $\tilde{h}_{\tilde{a}\tilde{b}}$, and $\mathcal{C}$ denotes the operation of taking the symmetric trace-free part. For details on how to perform the operation $\mathcal{C}$ see, e.g., [3,84] or the Appendix of [52]. The multipole moment tensors are defined as the values of these tensor fields at infinity,

$$M^{(n)}_{\tilde{a}_1\ldots\tilde{a}_n} = \mathcal{P}^{(n)}_{\tilde{a}_1\ldots\tilde{a}_n}|_\Lambda. \tag{17}$$

The choice $\phi = \xi$ gives the gravitational moment tensors $P^{(n)}_{\tilde{a}_1\ldots\tilde{a}_n} \equiv M^{(n)}_{\tilde{a}_1\ldots\tilde{a}_n}$, while the choice $\phi = q$ yields the electromagnetic moments $Q^{(n)}_{\tilde{a}_1\ldots\tilde{a}_n} \equiv M^{(n)}_{\tilde{a}_1\ldots\tilde{a}_n}$. The real and imaginary parts of $P^{(n)}_{\tilde{a}_1\ldots\tilde{a}_n}$ are the mass and angular momentum moments, respectively, while $Q^{(n)}_{\tilde{a}_1\ldots\tilde{a}_n}$ provides the electric and magnetic moments. The imaginary parts of $P^{(n)}_{\tilde{a}_1\ldots\tilde{a}_n}$ are called current or mass-current moments in papers related to gravitational radiation [3,7,69], and this became the standard in recent literature.

III. AXISYMMETRIC ELECTROVACUUM

A. Spacetime metric

In the axially symmetric case, specializing in (2), we write the metric into the Weyl-Lewis-Papapetrou form,

$$\mathrm{d}s^2 = -f(\mathrm{d}t - \omega\mathrm{d}\varphi)^2 + \frac{1}{f}[e^{2\gamma}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \rho^2\mathrm{d}\varphi^2], \tag{18}$$

where f, ω, and γ are functions of the coordinates ρ and z. In terms of the spatial coordinates $x^a = (\rho, z, \varphi)$ the metric on $\mathcal{M}$ is

$$h_{ab} = \begin{pmatrix} e^{2\gamma} & 0 & 0 \\ 0 & e^{2\gamma} & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}. \tag{19}$$

The only nonvanishing component of ω_a in (2) is now $\omega_\varphi \equiv -\omega$. At the rotation axis $\rho = 0$ is necessarily $\gamma = 0$, because of the absence of conical singularity.

In axistationary spacetimes the vector potential has only two components, A_t and A_φ, which depend on the coordinates ρ and z. The potential Φ is defined by (3), and Eq. (4) for A' can be written as [85]

$$\partial_\rho A' = \frac{f}{\rho}(\partial_z A_\varphi + \omega\partial_z A_t), \qquad \partial_z A' = -\frac{f}{\rho}(\partial_\rho A_\varphi + \omega\partial_\rho A_t). \tag{20}$$

[1]This can be alternatively written as $(\psi_*\tilde{h})_{\tilde{a}\tilde{b}} = \Omega^2 h_{\tilde{a}\tilde{b}}$, as an equation on $\mathcal{M}$. The important point is that the factor Ω^2 multiplies the physical metric $h_{\tilde{a}\tilde{b}}$, in order to allow the unphysical metric be regular at Λ.

The Ernst potential $\mathcal{E}$ is defined by (5), where now (6) takes the form

$$\partial_\rho \chi = -\frac{1}{\rho} f^2 \partial_z \omega + i(\bar{\Phi}\partial_\rho \Phi - \Phi \partial_\rho \bar{\Phi}),$$
$$\partial_z \chi = \frac{1}{\rho} f^2 \partial_\rho \omega + i(\bar{\Phi}\partial_z \Phi - \Phi \partial_z \bar{\Phi}). \tag{21}$$

For arbitrary axially symmetric functions f and g

$$\Delta f = e^{-2\gamma}\left(\partial_\rho^2 f + \frac{1}{\rho}\partial_\rho f + \partial_z^2 f\right). \tag{22}$$

$$\nabla^a f \nabla_a g = e^{-2\gamma}(\partial_\rho f \partial_\rho g + \partial_z f \partial_z g). \tag{23}$$

This shows that the factor $e^{-2\gamma}$ drops out from the Ernst equations, (8) and (9), giving a coupled system of equations for ξ and q. The components of the Ricci tensor in Eq. (10) are now

$$R_{ab} = \begin{pmatrix} -\partial_\rho^2\gamma + \frac{1}{\rho}\partial_\rho\gamma - \partial_z^2\gamma & \frac{1}{\rho}\partial_z\gamma & 0 \\ \frac{1}{\rho}\partial_z\gamma & -\partial_\rho^2\gamma - \frac{1}{\rho}\partial_\rho\gamma - \partial_z^2\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

B. Asymptotic coordinates

In order to describe the region near infinity, on the manifold $\mathcal{M}$ we introduce the coordinates

$$\tilde{\rho} = \frac{\rho}{r^2}, \qquad \tilde{z} = \frac{z}{r^2}, \tag{25}$$

where $r^2 = \rho^2 + z^2$. In terms of the coordinates $\tilde{x}^{\tilde{a}} = (\tilde{\rho}, \tilde{z}, \varphi)$ the three-dimensional metric becomes

$$h_{\tilde{a}\tilde{b}} = \frac{1}{\tilde{r}^4}\begin{pmatrix} e^{2\gamma} & 0 & 0 \\ 0 & e^{2\gamma} & 0 \\ 0 & 0 & \tilde{\rho}^2 \end{pmatrix}, \tag{26}$$

where $\tilde{r}^2 = \tilde{\rho}^2 + \tilde{z}^2 = r^{-2}$. The field equations, (8)–(10), are still valid in this coordinate system, where now

$$\Delta f = \tilde{r}^4 e^{-2\gamma}\left(\partial_{\tilde{\rho}}^2 f + \frac{1}{\tilde{\rho}}\partial_{\tilde{\rho}} f + \partial_{\tilde{z}}^2 f - \frac{2\tilde{\rho}}{\tilde{r}^2}\partial_{\tilde{\rho}} f - \frac{2\tilde{z}}{\tilde{r}^2}\partial_{\tilde{z}} f\right), \tag{27}$$

$$\nabla^{\tilde{a}} f \nabla_{\tilde{a}} g = \tilde{r}^4 e^{-2\gamma}(\partial_{\tilde{\rho}} f \partial_{\tilde{\rho}} g + \partial_{\tilde{z}} f \partial_{\tilde{z}} g). \tag{28}$$

The nonvanishing components of the Ricci tensor are

$$R_{\tilde{\rho}\tilde{\rho}} = -\partial_{\tilde{\rho}}^2\gamma + \frac{1}{\tilde{\rho}}\partial_{\tilde{\rho}}\gamma - \partial_{\tilde{z}}^2\gamma - \frac{2\tilde{\rho}}{\tilde{r}^2}\partial_{\tilde{\rho}}\gamma + \frac{2\tilde{z}}{\tilde{r}^2}\partial_{\tilde{z}}\gamma, \tag{29}$$

$$R_{\tilde{z}\tilde{z}} = -\partial_{\tilde{\rho}}^2\gamma - \frac{1}{\tilde{\rho}}\partial_{\tilde{\rho}}\gamma - \partial_{\tilde{z}}^2\gamma + \frac{2\tilde{\rho}}{\tilde{r}^2}\partial_{\tilde{\rho}}\gamma - \frac{2\tilde{z}}{\tilde{r}^2}\partial_{\tilde{z}}\gamma, \tag{30}$$

$$R_{\tilde{\rho}\tilde{z}} = \frac{1}{\tilde{\rho}}\partial_{\tilde{z}}\gamma - \frac{2\tilde{z}}{\tilde{r}^2}\partial_{\tilde{\rho}}\gamma - \frac{2\tilde{\rho}}{\tilde{r}^2}\partial_{\tilde{z}}\gamma. \tag{31}$$

Since $\partial_{\tilde{\rho}}\tilde{r} = \frac{\tilde{\rho}}{\tilde{r}}$ and $\partial_{\tilde{z}}\tilde{r} = \frac{\tilde{z}}{\tilde{r}}$ the Laplacian (27) can also be written as

$$\Delta f = \tilde{r}^5 e^{-2\gamma}\left(\partial_{\tilde{\rho}}^2 \frac{f}{\tilde{r}} + \frac{1}{\tilde{\rho}}\partial_{\tilde{\rho}}\frac{f}{\tilde{r}} + \partial_{\tilde{z}}^2\frac{f}{\tilde{r}}\right). \tag{32}$$

C. Conformal mapping

We use a specific conformal factor $\Omega = \tilde{r}^2$ to define the metric $\tilde{h}_{\tilde{a}\tilde{b}} = \Omega^2 h_{\tilde{a}\tilde{b}}$. Using the $\tilde{x}^{\tilde{a}} = (\tilde{\rho}, \tilde{z}, \varphi)$ coordinates this metric has the form

$$\tilde{h}_{\tilde{a}\tilde{b}} = \begin{pmatrix} e^{2\gamma} & 0 & 0 \\ 0 & e^{2\gamma} & 0 \\ 0 & 0 & \tilde{\rho}^2 \end{pmatrix}, \tag{33}$$

which obviously can be smoothly extended to the point $\tilde{\rho} = \tilde{z} = 0$, so it is a metric on $\tilde{\mathcal{M}}$. This point, denoted by Λ, corresponds to spatial conformal infinity. Since on the axis $\gamma = 0$, the choice $\Omega = \tilde{r}^2$ obviously satisfies the conditions of asymptotic flatness given in (13).

The metric in (33) has the same structure as in (19). Hence, the Laplacian $\tilde{\Delta} f$ and the product $\tilde{\nabla}^{\tilde{a}} f \tilde{\nabla}_{\tilde{a}} g$ belonging to this new metric have the same form as in (22) and (23), respectively, in terms of the tilded coordinates. We define the rescaled potentials as

$$\tilde{\xi} = \Omega^{-1/2}\xi = \frac{\xi}{\tilde{r}}, \qquad \tilde{q} = \Omega^{-1/2} q = \frac{q}{\tilde{r}}. \tag{34}$$

We introduce the operators $\tilde{D}_{\tilde{a}}$ defined in [65,66], such that

$$\tilde{D}_{\tilde{\rho}}\tilde{\xi} = \tilde{z}\partial_{\tilde{\rho}}\tilde{\xi} - \tilde{\rho}\partial_{\tilde{z}}\tilde{\xi}, \quad \tilde{D}_{\tilde{z}}\tilde{\xi} = \tilde{\rho}\partial_{\tilde{\rho}}\tilde{\xi} + \tilde{z}\partial_{\tilde{z}}\tilde{\xi} + \tilde{\xi}, \quad \tilde{D}_\varphi\tilde{\xi} = 0, \tag{35}$$

$$\tilde{D}_{\tilde{\rho}}\tilde{q} = \tilde{z}\partial_{\tilde{\rho}}\tilde{q} - \tilde{\rho}\partial_{\tilde{z}}\tilde{q}, \quad \tilde{D}_{\tilde{z}}\tilde{q} = \tilde{\rho}\partial_{\tilde{\rho}}\tilde{q} + \tilde{z}\partial_{\tilde{z}}\tilde{q} + \tilde{q}, \quad \tilde{D}_\varphi\tilde{\xi} = 0. \tag{36}$$

Using (28) and (32), the Ernst equations, (8) and (9), can be written formally into the same structure as earlier,

$$\Theta\tilde{\Delta}\,\tilde{\xi} = 2(\bar{\tilde{\xi}}\tilde{D}^{\tilde{a}}\tilde{\xi} - \bar{\tilde{q}}\tilde{D}^{\tilde{a}}\tilde{q})\tilde{D}_{\tilde{a}}\tilde{\xi}, \tag{37}$$

$$\Theta\tilde{\Delta}\,\tilde{q} = 2(\bar{\tilde{\xi}}\tilde{D}^{\tilde{a}}\tilde{\xi} - \bar{\tilde{q}}\tilde{D}^{\tilde{a}}\tilde{q})\tilde{D}_{\tilde{a}}\tilde{q}, \tag{38}$$

where $\Theta = \tilde{r}^2\tilde{\xi}\,\bar{\tilde{\xi}} - \tilde{r}^2\tilde{q}\,\bar{\tilde{q}} - 1$, and the indices are raised and lowered using the metric $\tilde{h}_{\tilde{a}\tilde{b}}$. These equations can be used to solve for $\tilde{\xi}$ and $\tilde{q}$ even if γ is not known yet.

The Ricci tensor $\tilde{R}_{\tilde{a}\tilde{b}}$ belonging to the new unphysical metric $\tilde{h}_{\tilde{a}\tilde{b}}$ has the same form as in (24),

$$\tilde{R}_{\tilde{a}\tilde{b}} = \begin{pmatrix} -\partial^2_{\tilde{\rho}}\gamma + \frac{1}{\tilde{\rho}}\partial_{\tilde{\rho}}\gamma - \partial^2_{\tilde{z}}\gamma & \frac{1}{\rho}\partial_{\tilde{z}}\gamma & 0 \\ \frac{1}{\tilde{\rho}}\partial_{\tilde{z}}\gamma & -\partial^2_{\tilde{\rho}}\gamma - \frac{1}{\tilde{\rho}}\partial_{\tilde{\rho}}\gamma - \partial^2_{\tilde{z}}\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{39}$$

However, the Einstein equation (10) is only valid for the physical metric $h_{\tilde{a}\tilde{b}}$, with Ricci tensor components given in (29)–(31). From the linear combination of the equations containing $R_{\tilde{\rho}\tilde{\rho}} + R_{\tilde{z}\tilde{z}}$, we can express $\partial^2_{\tilde{\rho}}\gamma + \partial^2_{\tilde{z}}\gamma$ in terms of ξ and q. Similarly, using the equations containing $R_{\tilde{\rho}\tilde{\rho}} - R_{\tilde{z}\tilde{z}}$ and $R_{\tilde{\rho}\tilde{z}}$ we can solve for the first derivatives $\partial_{\tilde{\rho}}\gamma$ and $\partial_{\tilde{z}}\gamma$. Although the resulting expressions are rather long in terms of $\tilde{\xi}$ and $\tilde{q}$, using the operator $\tilde{D}_{\tilde{a}}$ given in (35)–(36) and defining

$$\tilde{s}_{\tilde{a}} = \tilde{r}(\bar{\tilde{\xi}}\tilde{D}_{\tilde{a}}\tilde{q} - \tilde{q}\tilde{D}_{\tilde{a}}\bar{\tilde{\xi}}) \tag{40}$$

as in [66], the field equation containing the Ricci tensor can be written into the form

$$\Theta^2\tilde{R}_{\tilde{a}\tilde{b}} = 2\mathrm{Re}(\tilde{D}_{\tilde{a}}\tilde{\xi}\tilde{D}_{\tilde{b}}\bar{\tilde{\xi}} - \tilde{D}_{\tilde{a}}\tilde{q}\tilde{D}_{\tilde{b}}\bar{\tilde{q}} + \tilde{s}_{\tilde{a}}\bar{\tilde{s}}_{\tilde{b}}), \tag{41}$$

which has formally the same structure as the original equation (10). We note that in [65] the factor $\tilde{r}$ was missed in the definition (40) of $\tilde{s}_{\tilde{a}}$, which caused some errors in the final expressions of the multipole moments.

D. Multipole moments

In the axially symmetric case each $M^{(n)}_{\tilde{a}_1\ldots\tilde{a}_n}$ multipole moment tensor is necessarily proportional to the tensor $\mathcal{C}(n_{\tilde{a}_1}\ldots n_{\tilde{a}_n})$, where $n^{\tilde{a}}$ is the unit vector at Λ parallel to the rotational axis [1,2]. Since this has been claimed without proof in the literature, we provide a proof in Appendix. It follows that each moment tensor is determined by a single scalar, i.e.,

$$M^{(n)}_{\tilde{a}_1\ldots\tilde{a}_n} = \hat{M}_n\mathcal{C}(n_{\tilde{a}_1}\ldots n_{\tilde{a}_n})|_\Lambda \tag{42}$$

for some constants $\hat{M}_n$. Since at the point Λ necessarily $\gamma = 0$, the components of the vector are $n^{\tilde{a}} = (0,1,0)$ and $n_{\tilde{a}} = (0,1,0)$. The scalar moments for axial symmetry are defined as [2]

$$M_n = \frac{1}{n!}M^{(n)}_{\tilde{a}_1\ldots\tilde{a}_n}n^{\tilde{a}_1}\ldots n^{\tilde{a}_n}|_\Lambda \equiv \frac{1}{n!}M^{(n)}_{\tilde{z}\ldots\tilde{z}}. \tag{43}$$

In particular, the scalar gravitational moments are $P_n = P^{(n)}_{\tilde{z}\ldots\tilde{z}}/n!$, and the scalar electromagnetic moments are $Q_n = Q^{(n)}_{\tilde{z}\ldots\tilde{z}}/n!$. The mass and electric charge of the system has to be real, and given by $P_0 \equiv M$ and $Q_0 \equiv Q$, respectively. Setting a center of mass reference system, the gravitational dipole moment is pure imaginary, and gives the angular momentum of the configuration by $P_1 = iJ$.

It can be shown that (see, e.g., Appendix of [52])

$$n^{\tilde{a}_1}\ldots n^{\tilde{a}_n}\mathcal{C}(n_{\tilde{a}_1}\ldots n_{\tilde{a}_n})|_\Lambda = \frac{n!}{(2n-1)!!} = \frac{2^n(n!)^2}{(2n)!}. \tag{44}$$

Hence, by substituting (42) into (43) we obtain that

$$M_n = \frac{1}{(2n-1)!!}\hat{M}_n = \frac{2^n n!}{(2n)!}\hat{M}_n. \tag{45}$$

Some authors define the scalar moments as $\hat{M}_n$ instead of M_n [58,59].

In the general stationary case, the transformation formula for the multipole moments under a change of the conformal factor $\tilde{\Omega} = \tilde{\omega}\Omega$ has been obtained by Beig in [86]. For axial symmetry, the scalar multipole moments transform as [87]

$$\tilde{M}_n = \sum_{k=0}^{n}\binom{n}{k}\left(-\frac{1}{2}(\partial_{\tilde{z}}\tilde{\omega})_\Lambda\right)^{n-k}M_k. \tag{46}$$

This transformation depends only on the single number $(\partial_{\tilde{z}}\tilde{\omega})_\Lambda$, which corresponds to a translation of the configuration along the axis direction.

E. Complex null vector

Using the coordinates $\tilde{x}^{\tilde{a}} = (\tilde{\rho},\tilde{z},\varphi)$, Bäckdahl and Herberthson [58,59] define the complex vector

$$\boldsymbol{\eta} = \frac{\partial}{\partial\tilde{z}} - i\frac{\partial}{\partial\tilde{\rho}}, \tag{47}$$

which has the components $\eta^{\tilde{a}} = (-i,1,0)$. From the metric form (33) it is easy to see that it is a null vector, $\tilde{h}_{\tilde{a}\tilde{b}}\eta^{\tilde{a}}\eta^{\tilde{b}} = 0$. For the covariant derivative in the direction $\boldsymbol{\eta}$ we use the notation $\tilde{\nabla}_{\boldsymbol{\eta}} = \eta^{\tilde{a}}\tilde{\nabla}_{\tilde{a}}$. The following important property can be checked by direct calculation:

$$\tilde{\nabla}_{\boldsymbol{\eta}}\eta^{\tilde{a}} = 2\eta^{\tilde{a}}\partial_{\boldsymbol{\eta}}\gamma. \tag{48}$$

If we multiply the recursion definition (16) at all indices by $\eta^{\tilde{a}_i}$, we obviously do not have to symmetrize. Furthermore, since $\eta^{\tilde{a}_i}$ is a null vector, the terms that must be added to make the tensor trace-free do not contribute either. The use of the complex null vector $\eta^{\tilde a}$ completely eliminates the problem of taking the symmetric trace-free part. Introducing

$$f_n = \eta^{\tilde a_1}\ldots\eta^{\tilde a_n}\mathcal{P}^{(n)}_{\tilde a_1\ldots\tilde a_n}, \tag{49}$$

the first term on the right-hand side of (16) can be written as

$$\begin{aligned}\eta^{\tilde a_1}\ldots\eta^{\tilde a_{n+1}}\tilde\nabla_{\tilde a_{n+1}}\mathcal{P}^{(n)}_{\tilde a_1\ldots\tilde a_n} &= \tilde\nabla_{\boldsymbol\eta} f_n - \mathcal{P}^{(n)}_{\tilde a_1\ldots\tilde a_n}\tilde\nabla_{\boldsymbol\eta}(\eta^{\tilde a_1}\ldots\eta^{\tilde a_n})\\ &= \partial_{\boldsymbol\eta} f_n - 2nf_n\partial_{\boldsymbol\eta}\gamma,\end{aligned} \tag{50}$$

where (48) has been used to get the second line. This way, (14)–(16) provide the recursion formula for f_n, which has been given first in [58],

$$f_0 = \tilde\phi, \tag{51}$$

$$f_1 = \partial_{\boldsymbol\eta} f_0, \tag{52}$$

$$f_{n+1} = \partial_{\boldsymbol\eta} f_n - 2nf_n\partial_{\boldsymbol\eta}\gamma - \frac{1}{2}n(2n-1)\eta^{\tilde a}\eta^{\tilde b}\tilde R_{\tilde a\tilde b}f_{n-1}. \tag{53}$$

We note that our function γ has been denoted by β in [58,59]. In those papers there is also a further conformal transformation, specified by a function κ. Our choice (33) of the three-metric corresponds to $\kappa = \beta$ there.

Multiplying (42) at all indices by $\eta^{\tilde a_i}$, since $\eta^{\tilde a}n_{\tilde a} = 1$, we get $\hat M_n = f_n|_\Lambda$. From (45) it can be seen that the scalar moments can be obtained by calculating the values of the functions f_n at conformal infinity,

$$M_n = \frac{1}{(2n-1)!!}f_n|_\Lambda = \frac{2^n n!}{(2n)!}f_n|_\Lambda. \tag{54}$$

Using the form (39) of the Ricci tensor, it is easy to see that

$$\eta^{\tilde a}\eta^{\tilde b}\tilde R_{\tilde a\tilde b} = -\frac{2i}{\rho}\partial_{\boldsymbol\eta}\gamma. \tag{55}$$

On the other hand, from (41) we obtain

$$\Theta^2\eta^{\tilde a}\eta^{\tilde b}\tilde R_{\tilde a\tilde b} = 2\eta^{\tilde a}\eta^{\tilde b}(\tilde D_{\tilde a}\tilde\xi\tilde D_{\tilde b}\bar{\tilde\xi} - \tilde D_{\tilde a}\tilde q\tilde D_{\tilde b}\bar{\tilde q} + \tilde s_{\tilde a}\bar{\tilde s}_{\tilde b}). \tag{56}$$

F. Leading order function

Since the tensors $\mathcal{P}^{(n)}_{\tilde a_1\ldots\tilde a_n}$ are symmetric in their indices, they have, in general, $(n+1)(n+2)/2$ independent components. Adding trace terms, the quantities $S^{(n)}_{\tilde a}$ for $0 \le \tilde a \le n$ have been introduced in [52], which decreased the necessary components to $n+1$. The introduction of the functions f_n in [58] reduces the number of components to one at each order n, which significantly simplifies the calculation of the moments.

A further big simplification arises from the fact that all derivatives are taken in the $\boldsymbol\eta$ direction. Hence, for each f_n we can define an associated function that depends only on a single variable instead of two. In order to make use of this idea, the concept of leading order functions has been introduced in [58,59]. The naming comes from the leading order terms of Legendre polynomials used initially for the static case in [56,57].

Let us assume that f is a complex valued axially symmetric analytic function of the coordinates $(\tilde x = \tilde\rho\cos\phi, \tilde y = \tilde\rho\sin\phi, \tilde z)$ in a neighborhood of Λ, the point corresponding to conformal infinity. [We note that f is a general function here, not related to the norm of the Killing vector in (2)]. In this paper we only use analyticity in the "real" sense, for functions that depend on real variables and are locally given by convergent power series.

We can also consider f as an analytic function $f(\tilde\rho, \tilde z)$ on the plane $\mathbb{R}^2$, satisfying $f(-\tilde\rho, \tilde z) = f(\tilde\rho, \tilde z)$. The power series expansion of f can be written as

$$f(\tilde\rho, \tilde z) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_{kl}\tilde\rho^k\tilde z^l = \sum_{N=0}^{\infty}\sum_{k=0}^{N} a_{k,N-k}\tilde\rho^k\tilde z^{N-k}, \tag{57}$$

where a_{kl} are complex constants. Consider then the function g, depending on two real variables ϑ and ζ, defined by

$$g(\vartheta, \zeta) = \sum_{N=0}^{\infty}\sum_{k=0}^{N}(-i)^k a_{k,N-k}\vartheta^k\zeta^{N-k}, \tag{58}$$

where a_{kl} are the same as in Eq. (57). The function g can be considered as a transformed version of f under $\tilde z \to \zeta$ and $\tilde\rho \to -i\vartheta$. Notice that if f converges for $\tilde\rho^2 + \tilde z^2 < r_0^2$, then g converges for $\vartheta^2 + \zeta^2 < r_0^2$. Moreover, this map defines a bijection between g and f. Now, we can define the *leading order part* of f as the function $f_L(\zeta) = g(\zeta, \zeta)$, which depends only on a single real variable. From (58) it follows that

$$f_L(\zeta) = \sum_{N=0}^{\infty}\tilde a_N\zeta^N, \qquad \tilde a_N = \sum_{k=0}^{N}(-i)^k a_{k,N-k}, \tag{59}$$

which obviously converges for $\zeta\sqrt{2} < r_0$, showing that $f_L(\zeta)$ is real analytic. In other words, although $f(\tilde\rho, \tilde z)$ is not complex analytic as a function depending on the complex variables $\tilde\rho$ and $\tilde z$, effectively we just substitute $\tilde z = \zeta$ and $\tilde\rho = -i\zeta$ to obtain

$$f_L(\zeta) = f(-i\zeta, \zeta). \tag{60}$$

(Note that we do not claim or use that any of f, g, or f_L are analytic functions of the complex variables $\tilde\rho + i\tilde z$ or $\tilde\rho - i\tilde z$.)

It can be easily checked that for any function f

$$(\tilde{\nabla}_{\boldsymbol{\eta}} f)_L(\zeta) = f_L{}'(\zeta), \tag{61}$$

where the prime denotes differentiation with respect to ζ. We do not write out the argument (ζ) of the leading order parts from now on.

If f represents an axially symmetric function on the spacetime which is regular on the axis, then $f(-\tilde{\rho},\tilde{z}) = f(\tilde{\rho},\tilde{z})$ and $a_{kl} = 0$ for odd k. For this type of function, complex conjugation has the property $(\bar{f})_L = \overline{(f_L)}$. In this case we also have

$$\overline{(\tilde{\nabla}_{\boldsymbol{\eta}} f)_L} = (\tilde{\nabla}_{\boldsymbol{\eta}} \bar{f})_L, \qquad \overline{f_L{}'} = \bar{f}_L{}'. \tag{62}$$

This also shows that the leading order part of regular real functions is real, in particular, γ_L and $\gamma_L{}'$ are real.

The leading order part of sums, products or quotients of functions is equal to the sum, product or quotient of the leading order parts, respectively. Introducing the notation

$$\tilde{R}_L = (\eta^{\tilde{a}}\eta^{\tilde{b}}\tilde{R}_{\tilde{a}\tilde{b}})_L, \tag{63}$$

from (55) we get

$$\gamma_L{}' = \frac{\zeta}{2}\tilde{R}_L. \tag{64}$$

This shows that $\tilde{R}_L$ also has to be real. Since the leading order part of $\tilde{r}^2$ is zero, and $\Theta = \tilde{r}^2\tilde{\xi}\bar{\tilde{\xi}} - \tilde{r}^2\tilde{q}\bar{\tilde{q}} - 1$, it follows that $\Theta_L = -1$. In order to calculate the leading order part of (56) we can first check that

$$(\eta^{\tilde{a}}\tilde{D}_{\tilde{a}} f)_L = 2\zeta f_L{}' + f_L = 2\sqrt{\zeta}(\sqrt{\zeta} f_L)'. \tag{65}$$

Furthermore, since there is a factor $\tilde{r}$ in the definition (40) of $\tilde{s}_{\tilde{a}}$, it follows that $(\eta^{\tilde{a}}\tilde{s}_{\tilde{a}})_L = 0$. Hence, the leading order part of (56) can be written as

$$\tilde{R}_L = 2|2\zeta\tilde{\xi}_L{}' + \tilde{\xi}_L|^2 - 2|2\zeta\tilde{q}_L{}' + \tilde{q}_L|^2. \tag{66}$$

As we have seen in (54), the multipole moments are given by the values of the functions f_n at the point Λ. To simplify the appearance of the expressions we introduce the notation

$$y_n = (f_n)_L, \tag{67}$$

where y_n are functions of ζ. Then the scalar moments for the axially symmetric case can be calculated as

$$M_n = \frac{1}{(2n-1)!!} y_n\bigg|_{\zeta=0} = \frac{2^n n!}{(2n)!} y_n\bigg|_{\zeta=0}. \tag{68}$$

From (51)–(53) we get the recursive definition of y_n [58],

$$y_0 = \tilde{\phi}_L, \tag{69}$$

$$y_1 = y_0{}', \tag{70}$$

$$y_{n+1} = y_n{}' - 2n y_n \gamma_L{}' - \frac{1}{2}n(2n-1)\tilde{R}_L y_{n-1}. \tag{71}$$

If $\tilde{\xi}$ and $\tilde{q}$ is known, then $\tilde{\xi}_L$ and $\tilde{q}_L$ corresponding to $\tilde{\phi}_L$ can be calculated easily by (59) or (60). Then $\tilde{R}_L$ is given by (66) and $\gamma_L{}'$ by (64). Using (68), the gravitational moments $P_n \equiv M_n$ are obtained from the choice $\tilde{\phi} = \tilde{\xi}$, and the electromagnetic moments $Q_n \equiv M_n$ from $\tilde{\phi} = \tilde{q}$.

IV. EXPANSION AND MULTIPOLE MOMENTS

A. Expansion along the axis

The conformally rescaled potentials $\tilde{\xi}$ and $\tilde{q}$ have been defined in (34). If the value of these potentials on the rotation axis is known, then the Ernst equations determine them on the whole space. Because of the asymptotic flatness, the functions $\tilde{\xi}$ and $\tilde{q}$ must be smooth at the point Λ. Hence, we specify their axis values by the series expansion coefficients m_n and q_n,

$$\tilde{\xi} = \sum_{n=0}^{\infty} m_n \tilde{z}^n, \qquad \tilde{q} = \sum_{n=0}^{\infty} q_n \tilde{z}^n. \tag{72}$$

According to (25), on the axis $\tilde{z} = 1/z$, where z is the axial coordinate in (18). Since we use the conformal factor $\Omega = \tilde{r}^2$, and along the axis $\tilde{r} = |\tilde{z}| = 1/|z|$, it follows that the expansions of the original Ernst potentials on the axis are

$$\xi = \frac{1}{|z|}\sum_{n=0}^{\infty}\frac{m_n}{z^n}, \qquad q = \frac{1}{|z|}\sum_{n=0}^{\infty}\frac{q_n}{z^n}. \tag{73}$$

We note that this definition of the coefficients m_n and q_n is not as coordinate system specific as it may appear at first sight. In (73) the coordinate z acts as a parametrization along the line representing the symmetry axis, and its value is unimportant elsewhere. This parametrization satisfies $h_{ab}(\frac{\partial}{\partial z})^a(\frac{\partial}{\partial z})^b = 1$ on the axis where $\gamma = 0$, and h_{ab} is given by (19). Generally, it is easy to find such parametrization even if the metric is not given in the Weyl-Lewis-Papapetrou form.

However, a freedom of constant shift remains in z. Setting $z = \hat{z} - z_0$, where z_0 is a constant, the potentials can be also expanded in terms of $\hat{z}$,

$$\xi = \frac{1}{|\hat{z}|}\sum_{n=0}^{\infty}\frac{\hat{m}_n}{\hat{z}^n}, \qquad q = \frac{1}{|\hat{z}|}\sum_{n=0}^{\infty}\frac{\hat{q}_n}{\hat{z}^n}, \tag{74}$$

where

$$\hat{m}_n = \sum_{k=0}^{n}\binom{n}{k} z_0^{n-k} m_k, \qquad \hat{q}_n = \sum_{k=0}^{n}\binom{n}{k} z_0^{n-k} q_k. \tag{75}$$

This shows that the coefficients m_n and q_n transform in the same way as the scalar multipole moments in (46). In order to make m_n and q_n unique, we use this transformation to make the real part of the gravitational dipole moment m_1 zero, choosing a center of mass system.

B. Expansion of the Ernst equations

We look for the solution of the Ernst equations, (37) and (38), in the power series form

$$\tilde{\xi} = \sum_{\substack{k=0\\l=0}}^{\infty} a_{kl}\tilde{\rho}^k\tilde{z}^l, \qquad \tilde{q} = \sum_{\substack{k=0\\l=0}}^{\infty} b_{kl}\tilde{\rho}^k\tilde{z}^l. \tag{76}$$

The potentials $\tilde{\xi}$ and $\tilde{q}$ have to be smooth and regular on the rotation axis, which implies that the expansion contains only even powers of $\tilde{\rho}$. Hence, for odd k necessarily $a_{kl} = b_{kl} = 0$. Obviously, $a_{0l} = m_l$ and $b_{0l} = q_l$. We intend to calculate the coefficients a_{kl} and b_{kl} in terms of the expansion coefficients m_l and q_l.

Since there has been a mistake in the equations for a_{kl} and b_{kl} published in the literature, we give a more detailed presentation here. Expanding out Eq. (37), we keep the linear terms on the left-hand side, and on the other side in curly brackets we group those terms together which have the same behavior in powers of $\tilde{\rho}$ and $\tilde{z}$,

$$\begin{aligned}
\tilde{\xi}_{,\tilde{\rho}\tilde{\rho}} + \frac{1}{\tilde{\rho}}\tilde{\xi}_{,\tilde{\rho}} + \tilde{\xi}_{,\tilde{z}\tilde{z}} = {}& \{(\tilde{\xi}\bar{\tilde{\xi}} - \tilde{q}\bar{\tilde{q}})(\tilde{\rho}^2\tilde{\xi}_{,\tilde{\rho}\tilde{\rho}} + \tilde{\rho}\tilde{\xi}_{,\tilde{\rho}} + \tilde{z}^2\tilde{\xi}_{,\tilde{z}\tilde{z}}) \\
& + 2\tilde{\rho}\,\tilde{z}\,\tilde{\xi}_{,\tilde{z}}(\bar{\tilde{\xi}}\tilde{\xi}_{,\tilde{\rho}} - \bar{\tilde{q}}\tilde{q}_{,\tilde{\rho}}) + 2\tilde{\rho}\,\tilde{z}\,\tilde{\xi}_{,\tilde{\rho}}(\bar{\tilde{\xi}}\tilde{\xi}_{,\tilde{z}} - \bar{\tilde{q}}\tilde{q}_{,\tilde{z}}) \\
& - 2(\tilde{\rho}\tilde{\xi}_{,\tilde{\rho}} + \tilde{z}\tilde{\xi}_{,\tilde{z}} + \tilde{\xi})[\bar{\tilde{\xi}}(\tilde{\rho}\tilde{\xi}_{,\tilde{\rho}} + \tilde{z}\tilde{\xi}_{,\tilde{z}} + \tilde{\xi}) - \bar{\tilde{q}}(\tilde{\rho}\tilde{q}_{,\tilde{\rho}} + \tilde{z}\tilde{q}_{,\tilde{z}} + \tilde{q})]\} \\
& + \left\{\tilde{z}^2(\tilde{\xi}\bar{\tilde{\xi}} - \tilde{q}\bar{\tilde{q}})\left(\tilde{\xi}_{,\tilde{\rho}\tilde{\rho}} + \frac{1}{\tilde{\rho}}\tilde{\xi}_{,\tilde{\rho}}\right) - 2\tilde{z}^2\tilde{\xi}_{,\tilde{\rho}}(\bar{\tilde{\xi}}\tilde{\xi}_{,\tilde{\rho}} - \bar{\tilde{q}}\tilde{q}_{,\tilde{\rho}})\right\} \\
& + \{\tilde{\rho}^2(\tilde{\xi}\bar{\tilde{\xi}} - \tilde{q}\bar{\tilde{q}})\tilde{\xi}_{,\tilde{z}\tilde{z}} - 2\tilde{\rho}^2\tilde{\xi}_{,\tilde{z}}(\bar{\tilde{\xi}}\tilde{\xi}_{,\tilde{z}} - \bar{\tilde{q}}\tilde{q}_{,\tilde{z}})\}.
\end{aligned} \tag{77}$$

Substituting the expansions of $\tilde{\xi}$ and $\tilde{q}$, the right-hand side is a sum of terms

$$\begin{aligned}
&(a_{kl}\bar{a}_{mn} - b_{kl}\bar{b}_{mn})a_{pq}\{(p^2 + q^2 - 2pk - 2ql - 2p - 3q - 2k - 2l - 2)\tilde{\rho}^{k+m+p}\tilde{z}^{l+n+q} \\
&\quad + p(p-2k)\tilde{\rho}^{k+m+p-2}\tilde{z}^{l+n+q+2} + q(q-1-2l)\tilde{\rho}^{k+m+p+2}\tilde{z}^{l+n+q-2}\}
\end{aligned} \tag{78}$$

for all k, l, m, n, p, q non-negative integers. The three kinds of terms correspond to the three curly brackets in (77). The integer q here is obviously different from the Ernst potential q in (73); however, we keep this notation to make comparisons with earlier papers easier. Equating the $\tilde{\rho}^r\tilde{z}^s$ terms for non-negative integers r and s, we obtain the recursion relation for the components of $\tilde{\xi}$,

$$\begin{aligned}
(r+2)^2 a_{r+2,s} = {}& -(s+2)(s+1)a_{r,s+2} + \sum_{\substack{k+m+p=r\\l+n+q=s}} (a_{kl}\bar{a}_{mn} - b_{kl}\bar{b}_{mn}) \\
& \times [a_{pq}(p^2 + q^2 - 2p - 3q - 2k - 2l - 2pk - 2ql - 2) \\
& + a_{p+2,q-2}(p+2)(p+2-2k) + a_{p-2,q+2}(q+2)(q+1-2l)].
\end{aligned} \tag{79}$$

Equation (38) can be obtained from (37) by exchanging $\tilde{\xi}$ and $\tilde{q}$, followed by reversing the signature of the cubic terms (including those coming from multiplication with Θ). The recursion for the components of $\tilde{q}$ can be written as

$$\begin{aligned}
(r+2)^2 b_{r+2,s} = {}& -(s+2)(s+1)b_{r,s+2} + \sum_{\substack{k+m+p=r\\l+n+q=s}} (a_{kl}\bar{a}_{mn} - b_{kl}\bar{b}_{mn}) \\
& \times [b_{pq}(p^2 + q^2 - 2p - 3q - 2k - 2l - 2pk - 2ql - 2) \\
& + b_{p+2,q-2}(p+2)(p+2-2k) + b_{p-2,q+2}(q+2)(q+1-2l)].
\end{aligned} \tag{80}$$

In papers [65,66] in the third line of the equations corresponding to (79) and (80) we can find $-4p-5q$ instead of the correct $-2p-3q-2k-2l$ terms. The mistake has been corrected in the proceedings paper [67], but this remained unnoticed in all subsequent papers. For the purely gravitational case, when $b_{kl}=0$, the two expressions are equivalent because of the symmetry of the $a_{kl}a_{pq}$ product. This shows that the corresponding expression in [52] is still correct. Calculating a_{rs} and b_{rs} up to some order in $r+s$, forming $\tilde{\xi}$ and $\tilde{q}$, then substituting back to (37) and (38), we have checked that the equations are satisfied up to order $r+s-2$. The same procedure with the originally published recursion formula fails, showing that the two expressions are clearly not equivalent. As we will see, the values of the multipole moments P_n and Q_n expressed in terms of m_l and q_l will also be influenced.

The sums in (79) and (80) are essentially sums for four integer variables instead of six, since two integers are determined by the given value of s and r. We have to be careful with the start of the summations in p and q, because of the shift by ± 2 of the indices in the second and third term of (78). A natural way to calculate the sums in (79) is

$$\begin{aligned}(r+2)^2 a_{r+2,s} = &-(s+2)(s+1)a_{r,s+2}\\ &+\sum_{\substack{p=0\\ \text{even}}}^{r}\sum_{q=0}^{s}\sum_{\substack{k=0\\ \text{even}}}^{r-p}\sum_{l=0}^{s-q}(a_{kl}\bar{a}_{mn}-b_{kl}\bar{b}_{mn})\\ &\times a_{pq}(p^2+q^2-2p-3q-2k-2l-2pk-2ql-2)\\ &+\sum_{\substack{p=-2\\ \text{even}}}^{r}\sum_{q=2}^{s}\sum_{\substack{k=0\\ \text{even}}}^{r-p}\sum_{l=0}^{s-q}(a_{kl}\bar{a}_{mn}-b_{kl}\bar{b}_{mn})a_{p+2,q-2}(p+2)(p+2-2k)\\ &+\sum_{\substack{p=2\\ \text{even}}}^{r}\sum_{q=-2}^{s}\sum_{\substack{k=0\\ \text{even}}}^{r-p}\sum_{l=0}^{s-q}(a_{kl}\bar{a}_{mn}-b_{kl}\bar{b}_{mn})a_{p-2,q+2}(q+2)(q+1-2l),\end{aligned} \tag{81}$$

and similarly for (80). In each term we have to substitute $m=r-p-k$ and $n=s-q-l$. Obviously, r can be assumed to be even, and we have to include only even values of p and k in the sums. Because of the $p+2$ factor, it would not be really necessary to include the terms with $p=-2$ in the second sum. However, the third sum would give an incorrect value if we started the summation from $q=0$, since the terms corresponding to $q=-1$ and nonzero l are nonvanishing.

Our aim with (79) and (80) is to express all a_{kl} and b_{kl} coefficients in terms of $a_{0l}=m_l$ and $b_{0l}=q_l$. While the two linear terms contain coefficients with $k+l=r+s+2$, the cubic summation terms only have coefficients with $k+l\le r+s$. Denoting $r+s=N$, we can proceed by increasing the order in N one by one. First, for $N=0$ we calculate a_{20}, then for $N=1$ we get a_{21}. Continuing with $N=2$, first for $r=0$ we get a_{22} and then for $r=2$ we obtain a_{40}. Proceeding further in this way, parallel for both a_{kl} and b_{kl}, we can express the new coefficients by those that have been already calculated earlier. As Supplemental Material [68], we attach a *Mathematica* and an equivalent *Maple* file (named "moments-general"), where we implement this solution procedure, and validate it by substituting back into the Ernst equations.

C. Expressions for the scalar moments

Using (57) and (59), the expansion of the leading order functions can be calculated as

$$\tilde{\xi}_L=\sum_{N=0}^{\infty}\tilde{a}_N\zeta^N,\qquad \tilde{a}_N=\sum_{\substack{k=0\\ \text{even}}}^{N}a_{k,N-k}(-1)^{k/2}, \tag{82}$$

$$\tilde{q}_L=\sum_{N=0}^{\infty}\tilde{b}_N\zeta^N,\qquad \tilde{b}_N=\sum_{\substack{k=0\\ \text{even}}}^{N}b_{k,N-k}(-1)^{k/2}. \tag{83}$$

In order to calculate the multipole moments up to order $N_{\max}$, we need to obtain first the values of a_{kl} and b_{kl} up to order $k+l=N_{\max}$. Then by (82) and (83) we can get the leading order parts $\tilde{\xi}_L$ and $\tilde{q}_L$, also up to order $N_{\max}$ in ζ. After this, $\tilde{R}_L$ can be calculated by (66), and $\gamma_{L'}$ by (64). Since they will be used only from the second multipole moment, it is enough to calculate them up to order $N_{\max}-2$. Then we can use the recursive formula, (69)–(71), to calculate the functions y_n. For the electromagnetic case we need to calculate two sets of moments, and hence two sets of y_n functions for $n\le N_{\max}$. The gravitational moments $P_n\equiv M_n$ are obtained by choosing $\tilde{\phi}=\tilde{\xi}$, and the electromagnetic moments $Q_n\equiv M_n$ by setting $\tilde{\phi}=\tilde{q}$. Each y_n has to be calculated up to order $N_{\max}-n$ in the variable ζ. The two sets of scalar moments can be obtained by taking the values at $\zeta=0$, according to (68).

In order to make the final expressions for the multipole moments simpler, we use the following notations introduced in [52,65,66]:

$$M_{ij} = m_i m_j - m_{i-1} m_{j+1}, \qquad S_{ij} = m_i q_j - m_{i-1} q_{j+1}, \tag{84}$$

$$Q_{ij} = q_i q_j - q_{i-1} q_{j+1}, \qquad H_{ij} = q_i m_j - q_{i-1} m_{j+1}, \tag{85}$$

for $i > j \geq 0$ integers. For the first seven gravitational moments we obtain the results

$$P_0 = m_0, \tag{86}$$

$$P_1 = m_1, \tag{87}$$

$$P_2 = m_2, \tag{88}$$

$$P_3 = m_3 + \frac{1}{5}\bar{q}_0 S_{10}, \tag{89}$$

$$P_4 = m_4 - \frac{1}{7}\bar{m}_0 M_{20} + \frac{3}{35}\bar{q}_1 S_{10} + \frac{1}{7}\bar{q}_0(3S_{20} - 2H_{20}), \tag{90}$$

$$\begin{aligned} P_5 = m_5 &- \frac{1}{21}\bar{m}_1 M_{20} - \frac{1}{3}\bar{m}_0 M_{30} + \frac{1}{21}\bar{q}_2 S_{10} + \frac{1}{21}\bar{q}_1(4S_{20} - 3H_{20}) \\ &+ \frac{1}{21}\bar{q}_0(\bar{q}_0 q_0 S_{10} - \bar{m}_0 m_0 S_{10} + 14S_{30} + 13S_{21} - 7H_{30}), \end{aligned} \tag{91}$$

$$\begin{aligned} P_6 = m_6 &- \frac{5}{231}\bar{m}_2 M_{20} - \frac{4}{33}\bar{m}_1 M_{30} + \frac{1}{33}\bar{m}_0^2 m_0 M_{20} - \frac{1}{33}\bar{m}_0(18M_{40} + 8M_{31}) \\ &+ \frac{1}{33}\bar{q}_3 S_{10} + \frac{1}{231}\bar{q}_2(25S_{20} - 20H_{20}) + \frac{2}{231}\bar{q}_1(35S_{30} + 37S_{21} - 21H_{30}) \\ &- \frac{1}{1155}(37\bar{q}_1\bar{m}_0 + 13\bar{q}_0\bar{m}_1)m_0 S_{10} + \frac{1}{33}\bar{q}_0^2(5q_0 S_{20} - 4m_0 Q_{20} + 3q_1 S_{10}) \\ &+ \frac{10}{231}\bar{q}_1\bar{q}_0 q_0 S_{10} + \frac{2}{33}\bar{q}_0\bar{m}_0(2m_0 H_{20} - 3q_0 M_{20} - 2m_1 S_{10}) \\ &+ \frac{1}{33}\bar{q}_0(30S_{40} + 32S_{31} - 24H_{31} - 12H_{40}). \end{aligned} \tag{92}$$

The electromagnetic moments are

$$Q_0 = q_0, \tag{93}$$

$$Q_1 = q_1, \tag{94}$$

$$Q_2 = q_2, \tag{95}$$

$$Q_3 = q_3 - \frac{1}{5}\bar{m}_0 H_{10}, \tag{96}$$

$$Q_4 = q_4 + \frac{1}{7}\bar{q}_0 Q_{20} - \frac{3}{35}\bar{m}_1 H_{10} - \frac{1}{7}\bar{m}_0(3H_{20} - 2S_{20}), \tag{97}$$

$$\begin{aligned} Q_5 = q_5 &+ \frac{1}{21}\bar{q}_1 Q_{20} + \frac{1}{3}\bar{q}_0 Q_{30} - \frac{1}{21}\bar{m}_2 H_{10} - \frac{1}{21}\bar{m}_1(4H_{20} - 3S_{20}) \\ &+ \frac{1}{21}\bar{m}_0(\bar{m}_0 m_0 H_{10} - \bar{q}_0 q_0 H_{10} - 14H_{30} - 13H_{21} + 7S_{30}), \end{aligned} \tag{98}$$

$$
\begin{aligned}
Q_6 = q_6 &+ \frac{5}{231}\bar{q}_2 Q_{20} + \frac{4}{33}\bar{q}_1 Q_{30} + \frac{1}{33}\bar{q}_0^2 q_0 Q_{20} + \frac{1}{33}\bar{q}_0(18Q_{40} + 8Q_{31}) \\
&- \frac{1}{33}\bar{m}_3 H_{10} - \frac{1}{231}\bar{m}_2(25H_{20} - 20S_{20}) - \frac{2}{231}\bar{m}_1(35H_{30} + 37H_{21} - 21S_{30}) \\
&- \frac{1}{1155}(37\bar{m}_1\bar{q}_0 + 13\bar{m}_0\bar{q}_1)q_0 H_{10} + \frac{1}{33}\bar{m}_0^2(5m_0 H_{20} - 4q_0 M_{20} + 3m_1 H_{10}) \\
&+ \frac{10}{231}\bar{m}_1\bar{m}_0 m_0 H_{10} + \frac{2}{33}\bar{m}_0\bar{q}_0(2q_0 S_{20} - 3m_0 Q_{20} - 2q_1 H_{10}) \\
&- \frac{1}{33}\bar{m}_0(30H_{40} + 32H_{31} - 24S_{31} - 12S_{40}).
\end{aligned}
\tag{99}
$$

These expressions can be checked and higher order results can be obtained by the *Mathematica* or *Maple* file (named "moments-general") attached as Supplemental Material [68].

For $n \geq 3$ the expressions for P_n and Q_n are clearly different from the results published earlier in the literature [65,66]. The reason for the difference from the more recent result is the application of the proper version of Eqs. (79) and (80). If in our calculations we use the incorrect version of (79) and (80), then we obtain exactly the expressions published for the multipole moments in [66], which makes us confident in the correctness of our results.

The most striking difference from the earlier results is that in the correct expressions generally $P_3 \neq m_3$ and $Q_3 \neq q_3$ in the electromagnetic case. Although the multipole moments shown above have been calculated by the simpler method using the leading order functions, we have checked that the earlier method using the quantities $S_{\tilde{a}}^{(n)}$ introduced in [52] lead to identical results.

The structure of the expressions for P_n and Q_n are very similar. In fact, it is easy to obtain the result for Q_n by taking the expression for P_n and make the exchanges $m_n \leftrightarrow q_n$ and $\bar{m}_n \leftrightarrow -\bar{q}_n$, with a minus sign in the conjugated quantities. This also implies that we have to exchange $M_{ij} \leftrightarrow Q_{ij}$ and $S_{ij} \leftrightarrow H_{ij}$. This property follows from an analogous formal symmetry of the Ernst equations, (8) and (9), for the exchanges $\xi \leftrightarrow q$ and $\bar{\xi} \leftrightarrow -\bar{q}$.

V. MULTIPOLE MOMENTS OF EXACT SOLUTIONS

In this last section of the paper we will consider a five-parameter stationary axially symmetric exact solution published in 2000 by Manko *et al.* [35]. These are part of a wider class of two-soliton [33] and N-soliton solutions [88], having the special property that they can be written in terms of prolate or oblate spheroidal coordinates. As special cases, this five-parameter solution includes the Kerr-Newman and the charged $\delta = 2$ Tomimatsu-Sato solutions.

The Tomimatsu-Sato solutions [89,90] are families of stationary axially symmetric vacuum solutions labeled by a positive integer parameter δ. The $\delta = 1$ family corresponds to the Kerr solution. The electrically charged generalizations have been constructed by Ernst [91] and Yamazaki [92]. A further generalization of the $\delta = 2$ solution by adding a magnetic dipole parameter has been published by Manko *et al.* in 1998 [72,93]. This solution has been generalized further in [35] by adding a fifth parameter, allowing the specification of the quadrupole moment and the inclusion of the Kerr-Newman solution. The results for the moments of the $\delta = 2$, 3, 4 vacuum Tomimatsu-Sato solutions have been presented up to order P_{12} in [87].

The five-parameter solution of Manko *et al.* [35] has important physical applications. This solution and its direct generalizations have been used for the description of the exterior region of neutron stars [29,38,78,94,95], black holes [96,97], and has been cited in several reviews [25,98,99].

A. Prolate and oblate spheroidal coordinates

The metric functions of the Tomimatsu-Sato solutions and their generalizations have been presented in [35,89,90] using prolate spheroidal coordinates x and y, which are related to the Weyl-Lewis-Papapetrou coordinates in (18) by

$$
\rho = \kappa\sqrt{x^2 - 1}\sqrt{1 - y^2}, \qquad z = \kappa x y, \tag{100}
$$

where κ is a positive constant. The range of the coordinates is $x > 1$ and $1 \geq y \geq -1$. A simple form of the inverse relation is [100]

$$
x = \frac{r_+ + r_-}{2\kappa}, \qquad y = \frac{r_+ - r_-}{2\kappa}, \qquad r_\pm = \sqrt{\rho^2 + (z \pm \kappa)^2}. \tag{101}
$$

The metric in the prolate spheroidal case is

$$
\begin{aligned}
\mathrm{d}s^2 = &-f(\mathrm{d}t - \omega \mathrm{d}\varphi)^2 \\
&+ \frac{\kappa^2}{f}\left[e^{2\gamma}(x^2 - y^2)\left(\frac{\mathrm{d}x^2}{x^2 - 1} + \frac{\mathrm{d}y^2}{1 - y^2}\right)\right. \\
&\left. + (x^2 - 1)(1 - y^2)\mathrm{d}\varphi^2\right].
\end{aligned}
\tag{102}
$$

The value of the constant κ is fixed by the freely specifiable parameters of the solution. If the angular momentum, charge, or dipole moment is large enough, the solution may become hyperextreme, and κ^2 turns out to be negative. It has been already realized by Tomimatsu and Sato in [90] that the hyperextreme extension of their solutions can be obtained by the complex transformation $\kappa \to -i\kappa$, $x \to ix$. The same is true for the electromagnetic generalizations. This corresponds to using oblate spheroidal coordinates

$$\rho = \kappa\sqrt{x^2+1}\sqrt{1-y^2}, \qquad z = \kappa xy \tag{103}$$

with real $\kappa > 0$. The spacetime metric is then

$$\mathrm{d}s^2 = -f(\mathrm{d}t - \omega \mathrm{d}\varphi)^2 + \frac{\kappa^2}{f}\left[e^{2\gamma}(x^2+y^2)\left(\frac{\mathrm{d}x^2}{x^2+1} + \frac{\mathrm{d}y^2}{1-y^2}\right) + (x^2+1)(1-y^2)\mathrm{d}\varphi^2\right]. \tag{104}$$

Hyperextreme Tomimatsu-Sato solutions have been already discussed in [101–104]. Although hyperextreme solutions always contain naked singularities, these solutions can still describe well the exterior region of rotating bodies. Actually, it is known that the exterior region of relatively small mass rotating objects often correspond to hyperextreme spacetimes. Defining the angular momentum per unit mass as $a = \frac{J}{M}$, in the absence of electromagnetic fields the configuration is hyperextreme if $\frac{a}{M} > 1$. For example, for the Earth $\frac{a}{M} \approx 740$, and for a vinyl LP record spinning on a turntable $\frac{a}{M} \approx 10^{18}$ [105,106]. The exterior region of a rotating disk of dust can also be hyperextreme [107,108]. For main sequence stars, $\frac{a}{M}$ may be as large as 100, which has to decrease to $\frac{a}{M} \ll 1$ if they evolve into neutron stars [109]. The $\frac{a}{M}$ ratio can also be as large as 500 for the bulge of spiral galaxies [110].

B. Five-parameter solution

The solution introduced by Manko *et al.* [35] depends on five parameters: m, $\hat{q}$, a, μ, and b. We use a hat on the constant q to distinguish it from the potential in (7). In the absence of NUT (Newman, Unti and Tamburino) charges and magnetic monopoles, m and $\hat{q}$ are real and correspond to the mass and electric charge, respectively. By shifting to a center of mass system, the mass dipole moment can be made zero. Then the parameter a is real and gives the angular momentum per unit mass of the solution. The parameters μ and b are related, but not equal, to the magnetic dipole moment and mass-quadrupole moment.

We define the constants c and ν by

$$c = a - b, \qquad \mu = \hat{q}\nu, \tag{105}$$

and will use them at most places instead of a and μ in the following. To simplify the equations, Manko *et al.* have introduced the following combinations of the constants:

$$\delta = \frac{\hat{q}^2\nu^2 - m^2b^2}{m^2 - c^2 - \hat{q}^2}, \qquad d = \frac{1}{4}(m^2 - c^2 - \hat{q}^2). \tag{106}$$

The parameters b, μ, and consequently $c = a - b$ may be complex, but the solution in x, y spheroidal coordinates only exists if the combination $d + \delta$ is real. If $d + \delta > 0$ is real, then the solution is subextreme, and the spacetime metric is described by (102). In the hyperextreme case $d + \delta < 0$ is real, and the metric is given by (104). In both cases, we have a solution of the Einstein equations if

$$\kappa = \sqrt{|d+\delta|}. \tag{107}$$

The complex Ernst potentials are given in [35] by the expressions

$$\mathcal{E} = \frac{A - 2mB}{A + 2mB}, \qquad \Phi = \frac{2C}{A + 2mB}, \tag{108}$$

where in the subextreme case

$$A = 4[(\kappa^2x^2 - \delta y^2)^2 - d^2 - i\kappa^3 xyc(x^2-1)] - (1-y^2)[c(d-\delta) - m^2b + \hat{q}^2\nu][c(y^2+1) + 4i\kappa xy], \tag{109}$$

$$B = \kappa x[2\kappa^2(x^2-1) + (bc + 2\delta)(1-y^2)] + iy[2\kappa^2 b(x^2-1) - (\kappa^2 c - m^2 b + \hat{q}^2\nu - 2a\delta)(1-y^2)], \tag{110}$$

$$C = 2\kappa^2\hat{q}(x^2-1)(\kappa x + i\nu y) + \hat{q}(1-y^2)\{\kappa x(2\delta + \nu c) - iy[c(d-\delta) - m^2 b + \hat{q}^2\nu - 2\nu\delta]\}. \tag{111}$$

The expressions for A, B, and C in the hyperextreme case can be obtained by the substitution $\kappa \to -i\kappa$, $x \to ix$.

C. Axis expansion

Along the upper part of the rotation axis $y = 1$, and the axial Weyl coordinate can be given there by $z = \kappa x$. The factor κ ensures that z is a parametrization satisfying $h_{ab}(\frac{\partial}{\partial z})^a(\frac{\partial}{\partial z})^b = 1$, as required in Sec. IV A. The complex potentials ξ and q are defined according to (7). To shorten the resulting expressions we introduce one more notation for the following combination of the constants:

$$v = d - \delta. \tag{112}$$

For both the subextreme and the hyperextreme cases, on the upper part of the axis we obtain

$$\xi = \frac{m(z+ib)}{z^2 - icz + v}, \qquad q = \frac{\hat{q}(z+i\nu)}{z^2 - icz + v}. \tag{113}$$

According to (73), the first few expansion coefficients turn out to be

$$m_0 = m, \tag{114}$$

$$m_1 = im(c+b), \tag{115}$$

$$m_2 = -m(c^2 + v + bc), \tag{116}$$

$$m_3 = -im[c^3 + 2cv + b(c^2 + v)], \tag{117}$$

$$m_4 = m[c^4 + 3c^2v + v^2 + b(c^3 + 2cv)], \tag{118}$$

$$m_5 = im[c^5 + 4c^3v + 3cv^2 + b(c^4 + 3c^2v + v^2)], \tag{119}$$

and

$$q_0 = \hat{q}, \tag{120}$$

$$q_1 = i\hat{q}(c+\nu), \tag{121}$$

$$q_2 = -\hat{q}(c^2 + v + \nu c), \tag{122}$$

$$q_3 = -i\hat{q}[c^3 + 2cv + \nu(c^2 + v)], \tag{123}$$

$$q_4 = \hat{q}[c^4 + 3c^2v + v^2 + \nu(c^3 + 2cv)], \tag{124}$$

$$q_5 = i\hat{q}[c^5 + 4c^3v + 3cv^2 + \nu(c^4 + 3c^2v + v^2)]. \tag{125}$$

It is possible to give the general expression for the coefficients. Defining

$$s = \sqrt{c^2 + 4v}, \qquad f_k = \frac{i^k}{2^{k+1}s}[(c+s)^{k+1} - (c-s)^{k+1}], \tag{126}$$

for all $k \geq 0$ integers we have

$$m_k = m(f_k + ibf_{k-1}), \qquad q_k = \hat{q}(f_k + i\nu f_{k-1}). \tag{127}$$

D. Lower order moments

At this stage we can use the general expressions, (86)–(99), to calculate the multipole moments P_n and Q_n up to order 6. Clearly $P_n = m_n$ and $Q_n = q_n$ only for $n = 0, 1, 2$. It can be seen that for the five-parameter solution generally $P_3 \neq m_3$ and $Q_3 \neq q_3$, since $S_{10} = -H_{10} = im\hat{q}(b-\nu)$. From the general expression of multipole moments, it follows that the magnetic dipole moment is $m_d = -iP_1 = \hat{q}(a - b + \nu)$, and the mass-quadrupole moment is $P_2 = -m(d - \delta + a^2 - ab)$.

According to the results worked out in the papers [54,55,63,64], if the solution is reflection symmetric with respect to the equatorial plane, then m_n is real for even n, and purely imaginary for odd n. The behavior of the constants q_n is either the same as that of m_n, or just the opposite, q_n is purely imaginary for even n, and real for odd n. Assuming that $\hat{q}$ is nonzero, the expression of the magnetic dipole moment shows that $\nu - b$ must be real for reflection symmetry. Using (106), the quadrupole moment can be written into the alternative form $P_2 = \frac{m}{2}(m^2 + a^2 - \hat{q}^2 - b^2 - 2\kappa^2)$. From this it follows that reflection symmetry is possible only if b is either real or pure imaginary.

The multipole moments of the Kerr-Newman metric have been already presented in [66] as $P_n = m_n = m(ia)^n$ and $Q_n = q_n = \hat{q}(ia)^n$. From the expression (121) for q_1 it follows that the five-parameter solution can reduce to the Kerr-Newman metric only if $\nu = b$. Furthermore, comparing with the quadrupole moment m_2 it follows that the solution becomes the Kerr-Newman metric only if $b^2 = a^2 + \hat{q}^2 - m^2$. This shows that for the subextreme Kerr-Newman solution the parameters $b = \nu$ are purely imaginary.

Setting $b = 0$, we obtain the charged magnetized generalization of the $\delta = 2$ Tomimatsu-Sato solution published in [72,93]. The original vacuum $\delta = 2$ Tomimatsu-Sato solution can be obtained by setting $b = 0$, $\hat{q} = 0$, and $\mu = 0$. The two parameters in the vacuum Tomimatsu-Sato solution, satisfying $p_v^2 + q_v^2 = 1$, are related to the constants in the five-parameter solution by $q_v = a/m$. Furthermore, $\kappa = mp_v/2$.

Instead of giving here the expressions for the first few multipole moments applying (86)–(99), we continue by a direct calculation of the multipole moments, using a generalization of the method used in [2,58] to calculate the moments of the Kerr metric. That way we obtain a fast and efficient method by which one can obtain the multipole moments of the five-parameter solution up to any desired order by a relatively simple algorithm. Since the expansion coefficients m_n and q_n have the same form for the subextreme and the hyperextreme cases, it follows that the multipole moments are also the same for the two cases. For this reason it is sufficient to consider the subextreme case in the following.

E. Conformal mapping

The first step is the introduction of asymptotic coordinates $\hat{x}^{\hat{a}} = (R, \theta, \phi)$ by

$$x = \frac{1}{\kappa R} + \frac{\kappa R}{4} \tag{128}$$

and $y = \cos\theta$. In this case $R = 0$ corresponds to conformal infinity Λ, and the nonzero components of the metric are

$$h_{RR} = \frac{e^{2\gamma}}{16R^4}[16 + \kappa^4 R^4 - 8\kappa^2 R^2 \cos(2\theta)], \tag{129}$$

$$h_{\theta\theta} = \frac{e^{2\gamma}}{16R^2}[16 + \kappa^4 R^4 - 8\kappa^2 R^2 \cos(2\theta)], \tag{130}$$

$$h_{\phi\phi} = \frac{1}{16R^2}(4 - \kappa^2 R^2)^2 \sin^2\theta. \tag{131}$$

It follows that an appropriate choice of conformal factor is

$$\Omega = \frac{4R^2}{4 - \kappa^2 R^2}. \tag{132}$$

Introducing cylindrical coordinates $\tilde{x}^{\tilde{a}} = (\tilde{\rho}, \tilde{z}, \phi)$ by

$$R^2 = \tilde{\rho}^2 + \tilde{z}^2, \qquad \cos\theta = \frac{\tilde{z}}{R}, \tag{133}$$

the metric takes the familiar form

$$\tilde{h}_{\tilde{a}\tilde{b}} = \begin{pmatrix} e^{2\tilde{\gamma}} & 0 & 0 \\ 0 & e^{2\tilde{\gamma}} & 0 \\ 0 & 0 & \tilde{\rho}^2 \end{pmatrix}, \tag{134}$$

where

$$e^{2\tilde{\gamma}} = e^{2\gamma}\left\{1 + \frac{16\kappa^2\tilde{\rho}^2}{[4 - \kappa^2(\tilde{\rho}^2 + \tilde{z}^2)]^2}\right\}. \tag{135}$$

Comparing with (33), it might be surprising at first sight that a transformed version of the function γ appears in the metric. The reason for this is that the coordinates $\tilde{\rho}$ and $\tilde{z}$ are clearly different now from the Weyl coordinates used in (25).

The metric function γ for the five-parameter solution is given in [35] as

$$e^{2\gamma} = \frac{E}{16\kappa^8(x^2 - y^2)^4}, \tag{136}$$

where

$$\begin{aligned} E = \{&4[\kappa^2(x^2 - 1) + \delta(1 - y^2)]^2 \\ &+ c[c(d - \delta) - m^2 b + \hat{q}^2\nu](1 - y^2)^2\}^2 \\ &- 16\kappa^2(x^2 - 1)(1 - y^2)\{c[\kappa^2(x^2 - y^2) + 2\delta y^2] \\ &+ (m^2 b - \hat{q}^2\nu)y^2\}^2. \end{aligned} \tag{137}$$

F. Calculation of the multipole moments

We intend to calculate the necessary leading order functions and use the recursive formulas, (69)–(71) to calculate the multipole moments. We calculate the Ernst potentials ξ and q using their definition (7), from the complex potentials $\mathcal{E}$ and Φ given by (108)–(111). Here we need to substitute x from (128) and $y = \tilde{z}/R$, where $R = \sqrt{\tilde{\rho}^2 + \tilde{z}^2}$. Since we will take the leading order part, and since $R_L = 0$, for the coordinate x we can substitute the simpler expression $x = 1/(\kappa R)$ without influencing the final result. According to (60), the leading order part of a function can be obtained by substituting $\tilde{z} = \zeta$ and $\tilde{\rho} = -i\zeta$.

In order to start the recursion (69)–(71), for the gravitational moments we need to calculate the leading order part of $\tilde{\xi} = \Omega^{-1/2}\xi$, and for the electromagnetic moments we need the leading order part of $\tilde{q} = \Omega^{-1/2}q$. Here we have to use the conformal factor given in (132). However, since we will take the leading order part, we can drop the $\kappa^2 R^2$ term from the denominator of Ω, and simply use $\Omega = R^2$ instead. The result for the leading order part of the conformally transformed complex potentials turns out to be

$$\tilde{\xi}_L = \frac{\alpha}{\beta}, \qquad \tilde{q}_L = \frac{\lambda}{\beta}, \tag{138}$$

where

$$\alpha = 2m[2 + (2i - c\zeta)b\zeta - 2\delta\zeta^2 + i\zeta^3(w - 2b\delta)], \tag{139}$$

$$\beta = w(c\zeta + 4i)\zeta^3 - 4ic\zeta + 4(1 - \delta\zeta^2)^2, \tag{140}$$

$$\lambda = 2\hat{q}[2 + (2i - c\zeta)\nu\zeta - 2\delta\zeta^2 + i\zeta^3(w - 2\nu\delta)], \tag{141}$$

and

$$w = vc + \hat{q}^2\nu - m^2 b, \qquad v = d - \delta. \tag{142}$$

Since the leading order part of (135) is simply $e^{2\tilde{\gamma}_L} = e^{2\gamma_L}(1 - \kappa^2\zeta^2)$, using (136) and (137) we obtain

$$e^{2\tilde{\gamma}_L} = \frac{16(c\zeta - w\zeta^3)^2 + (cw\zeta^4 + 4(1 - \delta\zeta^2)^2)^2}{16[1 - (2\delta + v)\zeta^2]^3}. \tag{143}$$

We are now ready to define the functions y_n according to (69)–(71). For the gravitational moments we need to take $y_0 = \tilde{\xi}_L$, and for the electromagnetic moments $y_0 = \tilde{q}_L$. The function y_1 can be obtained by taking the ζ derivative of y_0, i.e., $y_1 = y_0{\prime}$. For the further y_n functions we need to replace γ by $\tilde{\gamma}$ in (71),

$$y_{n+1} = y_n{\prime} - 2ny_n\tilde{\gamma}_L{\prime} - \frac{1}{2}n(2n - 1)\tilde{R}_L y_{n-1}. \tag{144}$$

The ζ derivative of $\tilde{\gamma}_L$ can be obtained from (143). Since it only depends on the structure of the three-metric, Eq. (64) is valid now in terms of $\tilde{\gamma}$, giving $\tilde{R}_L = \frac{2}{\zeta}\tilde{\gamma}_L{\prime}$. The multipole moments can be obtained by taking the values of the y_n functions at $\zeta = 0$, according to (68).

We present the results for the first few scalar moments by writing out the terms that must be added to the axis coefficients m_n and q_n. These coefficients can be easily calculated for general n according to (127), and their values up to $n = 5$ are listed in (114)–(125). For the first six scalar multipole moments we obtain

$$P_0 = m_0, \tag{145}$$

$$P_1 = m_1, \tag{146}$$

$$P_2 = m_2, \tag{147}$$

$$P_3 = m_3 + \frac{im\hat{q}^2}{5}(b - \nu), \tag{148}$$

$$P_4 = m_4 - \frac{m^3}{7}(b^2 + bc - v) - \frac{m\hat{q}^2}{35}(7cb - 12c\nu - 8b\nu + 5v + 3\nu^2), \tag{149}$$

$$P_5 = m_5 + \frac{im}{21}\{\hat{q}^2(\hat{q}^2 - m^2)(b - \nu) + m^2(b - 6c)(b^2 + bc - v) + \hat{q}^2[c^2(11\nu - 5b) - 6cv + c\nu(8b - 3\nu) + v(10\nu - 9b) - b\nu^2]\}, \tag{150}$$

and

$$Q_0 = q_0, \tag{151}$$

$$Q_1 = q_1, \tag{152}$$

$$Q_2 = q_2, \tag{153}$$

$$Q_3 = q_3 - \frac{i\hat{q}m^2}{5}(\nu - b), \tag{154}$$

$$Q_4 = q_4 + \frac{\hat{q}^3}{7}(\nu^2 + \nu c - v) + \frac{\hat{q}m^2}{35}(7c\nu - 12cb - 8\nu b + 5v + 3b^2) \tag{155}$$

$$Q_5 = q_5 + \frac{i\hat{q}}{21}\{m^2(m^2 - \hat{q}^2)(\nu - b) - \hat{q}^2(\nu - 6c)(\nu^2 + \nu c - v) - m^2[c^2(11b - 5\nu) - 6cv + cb(8\nu - 3b) + v(10b - 9\nu) - \nu b^2]\}. \tag{156}$$

We have checked that we obtain the same results by substituting (114)–(125) into the general expressions (86)–(99) for the multipole moments, which strongly supports the correctness of our calculations.

Since the variable v is not independent from the constants m, $\hat{q}$, b, ν, and c, the expressions for P_n and Q_n can be written in various equivalent forms. Using $v = d - \delta$ and (106) we can get

$$v(m^2 - c^2 - \hat{q}^2) - \frac{1}{4}(m^2 - c^2 - \hat{q}^2)^2 - m^2b^2 + \hat{q}^2\nu^2 = 0. \tag{157}$$

In the above expressions we have used this to eliminate terms containing c^2v.

If one is interested in the moments up to order N, then for efficiency it is advisable to take first the power series expansion at $\zeta = 0$ of the functions $\tilde{\xi}_L$, $\tilde{q}_L$ up to order N, and apply only then the recursion formula (144). As Supplemental Material [68], we provide a *Mathematica* and an equivalent *Maple* file (named "five-par-sol"), which can be used for the calculation of the moments of the five-parameter solution up to higher orders.

ACKNOWLEDGMENTS

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior—Brasil (CAPES)-Finance Code 001. B. H. would like to thank Fundação de Amparo à Pesquisa do Estado de São Paulo for financial support under Grant No. 2019/01511-5 as well as the Deutsche Forschungsgemeinschaft Research Training Group 1620 Models of Gravity for financial support. G. F. acknowledges the support of the Nemzeti Kutatási Fejlesztési és Innovációs Hivatal (NKFIH) through the OTKA Grant No. K138277.

APPENDIX: INVARIANT SYMMETRIC TRACELESS TENSORS IN THE AXIS

In this Appendix we present a simple construction type proof for the statement that in the axially symmetric case each $M^{(n)}_{\tilde{a}_1 \ldots \tilde{a}_n}$ multipole moment tensor is necessarily proportional to the tensor $\mathcal{C}(n_{\tilde{a}_1} \ldots n_{\tilde{a}_n})$, where $n^{\tilde{a}}$ is the unit vector parallel to the rotational axis. This result made it possible for Hansen [2] to define the scalar moments M_n by Eq. (43). The claim in [2], that the only tensors invariant under the action of the axial Killing vector are outer products of the metric and the axis vector, is correct, but only if one considers tensors that are symmetric in their indices.

The only properties of the multipole tensors $M^{(n)}_{\tilde{a}_1 \ldots \tilde{a}_n}$ that we use is that they are rotation invariant, traceless, and symmetric in their indices. After the conformal mapping, using the coordinates $\tilde{x}^{\tilde{a}} = (\tilde{\rho}, \tilde{z}, \varphi)$, the unphysical metric $\tilde{h}_{\tilde{a}\tilde{b}}$ takes the form (33), where on the axis, and hence also at the point Λ necessarily $\gamma = 0$. The Killing vector field that induces rotation around the axis has the components $\xi^{\tilde{a}} = (0, 0, 1)$. Using the associated Cartesian coordinates $x^{\hat{a}} = (\tilde{x}, \tilde{y}, \tilde{z})$ defined by $\tilde{x} = \tilde{\rho}\cos\phi$ and $\tilde{y} = \tilde{\rho}\sin\phi$, the Killing vector has the components $\xi^{\hat{a}} = (-\tilde{y}, \tilde{x}, 0)$. If a tensor is invariant under rotation, then it must have zero Lie derivative along the vector field $\xi^{\hat{a}}$, using the Cartesian coordinate system which is regular on the axis.

We only consider tensors at the point Λ, where the metric is $\tilde{h}_{\hat{a}\hat{b}} = \mathrm{diag}(1,1,1)$. We investigate tensors that are symmetric in their n lower indices, and introduce the notation

$$M^{(n)}_{\underbrace{\tilde{x}\ldots\tilde{x}}_{a}\underbrace{\tilde{y}\ldots\tilde{y}}_{b}\underbrace{\tilde{z}\ldots\tilde{z}}_{n-a-b}} = S^{(n)}_{a,b}. \quad (A1)$$

Expressing the Lie derivative using the standard expression in terms of partial derivatives, since on the axis $\xi^{\hat{a}}$ vanishes, only the terms proportional to the derivatives of the Killing vector remain. The partial derivative tensor of $\xi^{\hat{b}}$ has only two nonzero components,

$$\partial_{\hat{a}}\xi^{\hat{b}} = \delta^{\tilde{x}}_{\hat{a}}\delta^{\hat{b}}_{\tilde{y}} - \delta^{\tilde{y}}_{\hat{a}}\delta^{\hat{b}}_{\tilde{x}}. \quad (A2)$$

Using this, it is easy to show that the Lie derivative of a symmetric tensor at Λ can be calculated as

$$\mathcal{L}_\xi S^{(n)}_{a,b} = aS^{(n)}_{a-1,b+1} - bS^{(n)}_{a+1,b-1} \equiv 0. \quad (A3)$$

This is only valid for $a \geq 1, b \geq 1$, but because of the factors a and b it also turns out to be correct for $a = 0$ or $b = 0$ as well, when only one term remains. Assuming the vanishing of the Lie derivative, from this immediately follows that $S^{(n)}_{1,b} = 0$ and $S^{(n)}_{a,1} = 0$ for any a and b. Using the relation (A3) multiple times we can see that $S^{(n)}_{a,b} = 0$ when either a or b is an odd number. It also follows that the nonzero components, which have even a and b, are uniquely determined by the numbers $S^{(n)}_{0,b}$.

The trace-free condition at Λ can be written as

$$S^{(n)}_{a,b} + S^{(n)}_{a+2,b} + S^{(n)}_{a,b+2} = 0. \quad (A4)$$

Using this together with (A3), it is easy to see that all nonzero components of $S^{(n)}_{a,b}$ are proportional to $S^{(n)}_{0,0}$. Hence, rotationally invariant symmetric trace-free tensors at Λ are uniquely determined by a single parameter. Since the symmetric trace-free part of the exterior products of the axis vector $n_{\tilde{a}}$ is obviously nonvanishing, the tensor $\mathcal{C}(n_{\tilde{a}_1}\ldots n_{\tilde{a}_n})$ is necessarily proportional to $M^{(n)}_{\tilde{a}_1\ldots\tilde{a}_n}$.

We note that if we relax the condition of symmetry in the indices, there are more general invariant tensors. For example for two indices, the two-form $\mathrm{d}\tilde{x} \wedge \mathrm{d}\tilde{y}$ is clearly rotationally invariant.

[1] R. Geroch, Multipole moments. II. Curved space, J. Math. Phys. (N.Y.) **11**, 2580 (1970).
[2] R. O. Hansen, Multipole moments of stationary spacetimes, J. Math. Phys. (N.Y.) **15**, 46 (1974).
[3] K. S. Thorne, Multipole expansions of gravitational radiation, Rev. Mod. Phys. **52**, 299 (1980).
[4] Y. Gürsel, Multipole moments for stationary systems: The equivalence of the Geroch-Hansen formulation and the Thorne formulation, Gen. Relativ. Gravit. **15**, 737 (1983).
[5] W. Simon and R. Beig, The multipole structure of stationary space-times, J. Math. Phys. (N.Y.) **24**, 1163 (1983).
[6] H. Quevedo, Multipole moments in general relativity—static and stationary vacuum solutions—, Fortschr. Phys. Prog. Phys. **38**, 733 (1990).
[7] F. D. Ryan, Gravitational waves from the inspiral of a compact object into a massive, axisymmetric body with arbitrary multipole moments, Phys. Rev. D **52**, 5707 (1995).
[8] F. D. Ryan, Accuracy of estimating the multipole moments of a massive body from the gravitational waves of a binary inspiral, Phys. Rev. D **56**, 1845 (1997).
[9] C. Li and G. Lovelace, Generalization of Ryan's theorem: Probing tidal coupling with gravitational waves from nearly circular, nearly equatorial, extreme-mass-ratio inspirals, Phys. Rev. D **77**, 064022 (2008).
[10] L. Barack and C. Cutler, Using LISA extreme-mass-ratio inspiral sources to test off-Kerr deviations in the geometry of massive black holes, Phys. Rev. D **75**, 042003 (2007).
[11] S. Babak, J. Gair, A. Sesana, E. Barausse, C. F. Sopuerta, C. P. L. Berry, E. Berti, P. Amaro-Seoane, A. Petiteau, and A. Klein, Science with the space-based interferometer LISA. V. Extreme mass-ratio inspirals, Phys. Rev. D **95**, 103012 (2017).
[12] P. T. Chruściel, J. L. Costa, and M. Heusler, Stationary black holes: Uniqueness and beyond, Living Rev. Relativity **15**, 7 (2012).
[13] V. Cardoso and L. Gualtieri, Testing the black hole 'no-hair' hypothesis, Classical Quant. Grav. **33**, 174001 (2016).
[14] V. Cardoso and P. Pani, Testing the nature of dark compact objects: A status report, Living Rev. Relativity **22**, 4 (2019).
[15] N. A. Collins and S. A. Hughes, Towards a formalism for mapping the spacetimes of massive compact objects: Bumpy black holes and their orbits, Phys. Rev. D **69**, 124022 (2004).
[16] K. Glampedakis and S. Babak, Mapping spacetimes with LISA: Inspiral of a test body in a 'quasi-Kerr' field, Classical Quant. Grav. **23**, 4167 (2006).
[17] S. J. Vigeland, Multipole moments of bumpy black holes, Phys. Rev. D **82**, 104041 (2010).

[18] J. R. Gair, M. Vallisneri, S. L. Larson, and J. G. Baker, Testing general relativity with low-frequency, space-based gravitational-wave detectors, Living Rev. Relativity **16**, 7 (2013).
[19] E. Berti *et al.*, Testing general relativity with present and future astrophysical observations, Classical Quant. Grav. **32**, 243001 (2015).
[20] K. Yagi and L. C. Stein, Black hole based tests of general relativity, Classical Quant. Grav. **33**, 054001 (2016).
[21] C. M. Will, Testing the general relativistic "no-hair" theorems using the galactic center black hole Sagittarius A^*, Astrophys. J. **674**, L25 (2008).
[22] A. E. Broderick, T. Johannsen, A. Loeb, and D. Psaltis, Testing the no-hair theorem with event horizon telescope observations of Sagittarius A^*, Astrophys. J. **784**, 7 (2014).
[23] A. G. Suvorov and A. Melatos, Testing modified gravity and no-hair relations for the Kerr-Newman metric through quasiperiodic oscillations of galactic microquasars, Phys. Rev. D **93**, 024004 (2016).
[24] D. Psaltis, N. Wex, and M. Kramer, A quantitative test of the no-hair theorem with Sgr A^* using stars, pulsars, and the event horizon telescope, Astrophys. J. **818**, 121 (2016).
[25] V. Paschalidis and N. Stergioulas, Rotating stars in relativity, Living Rev. Relativity **20**, 7 (2017).
[26] A. Maselli, G. Pappas, P. Pani, L. Gualtieri, S. Motta, V. Ferrari, and L. Stella, A new method to constrain neutron star structure from quasi-periodic oscillations, Astrophys. J. **899**, 139 (2020).
[27] M. Shibata and M. Sasaki, Innermost stable circular orbits around relativistic rotating stars, Phys. Rev. D **58**, 104011 (1998).
[28] J. D. Sanabria-Gómez, J. L. Hernández-Pastora, and F. L. Dubeibe, Innermost stable circular orbits around magnetized rotating massive stars, Phys. Rev. D **82**, 124014 (2010).
[29] E. Berti and N. Stergioulas, Approximate matching of analytic and numerical solutions for rapidly rotating neutron stars, Mon. Not. R. Astron. Soc. **350**, 1416 (2004).
[30] G. Pappas and T. A. Apostolatos, Effectively Universal Behavior of Rotating Neutron Stars in General Relativity Makes them Even Simpler than their Newtonian Counterparts, Phys. Rev. Lett. **112**, 121101 (2014).
[31] K. Yagi, K. Kyutoku, G. Pappas, N. Yunes, and T. A. Apostolatos, Effective no-hair relations for neutron stars and quark stars: Relativistic results, Phys. Rev. D **89**, 124013 (2014).
[32] K. Yagi and N. Yunes, Approximate universal relations for neutron stars and quark stars, Phys. Rep. **681**, 1 (2017).
[33] V. S. Manko, J. Martín, and E. Ruiz, Six-parameter solution of the Einstein–Maxwell equations possessing equatorial symmetry, J. Math. Phys. (N.Y.) **36**, 3063 (1995).
[34] V. S. Manko, E. W. Mielke, and J. D. Sanabria-Gómez, Exact solution for the exterior field of a rotating neutron star, Phys. Rev. D **61**, 081501(R) (2000).
[35] V. S. Manko, J. D. Sanabria-Gómez, and O. V. Manko, Nine-parameter electrovac metric involving rational functions, Phys. Rev. D **62**, 044048 (2000).
[36] C. Teichmüller, M. B. Fröb, and F. Maucher, Analytical approximation of the exterior gravitational field of rotating neutron stars, Classical Quant. Grav. **28**, 155015 (2011).
[37] L. A. Pachón, J. A. Rueda, and J. D. Sanabria-Gómez, Realistic exact solution for the exterior field of a rotating neutron star, Phys. Rev. D **73**, 104038 (2006).
[38] G. Pappas and T. A. Apostolatos, An all-purpose metric for the exterior of any kind of rotating neutron star, Mon. Not. R. Astron. Soc. **429**, 3007 (2013).
[39] V. S. Manko and E. Ruiz, Exterior field of slowly and rapidly rotating neutron stars: Rehabilitating spacetime metrics involving hyperextreme objects, Phys. Rev. D **93**, 104051 (2016).
[40] C. F. Sopuerta and N. Yunes, Extreme- and intermediate-mass ratio inspirals in dynamical Chern-Simons modified gravity, Phys. Rev. D **80**, 064006 (2009).
[41] B. Kleihaus, J. Kunz, and S. Mojica, Quadrupole moments of rapidly rotating compact objects in dilatonic Einstein-Gauss-Bonnet theory, Phys. Rev. D **90**, 061501(R) (2014).
[42] B. Kleihaus, J. Kunz, S. Mojica, and M. Zagermann, Rapidly rotating neutron stars in dilatonic Einstein-Gauss-Bonnet theory, Phys. Rev. D **93**, 064077 (2016).
[43] G. Pappas and T. P. Sotiriou, Multipole moments in scalar-tensor theory of gravity, Phys. Rev. D **91**, 044011 (2015).
[44] G. Pappas and T. P. Sotiriou, Geodesic properties in terms of multipole moments in scalar–tensor theories of gravity, Mon. Not. R. Astron. Soc. **453**, 2862 (2015).
[45] G. Pappas, D. D. Doneva, T. P. Sotiriou, S. S. Yazadjiev, and K. D. Kokkotas, Multipole moments and universal relations for scalarized neutron stars, Phys. Rev. D **99**, 104014 (2019).
[46] G. Compère, R. Oliveri, and A. Seraj, Gravitational multipole moments from Noether charges, J. High Energy Phys. 18 (2018) 54.
[47] B. C. Xanthopoulos, Multipole moments in general relativity, J. Phys. A **12**, 1025 (1979).
[48] R. Beig and W. Simon, Proof of a multipole conjecture due to Geroch, Commun. Math. Phys. **78**, 75 (1980).
[49] P. Kundu, On the analyticity of stationary gravitational fields at spatial infinity, J. Math. Phys. (N.Y.) **22**, 2006 (1981).
[50] R. Beig and W. Simon, On the multipole expansion for stationary space-times, Proc. R. Soc. A **376**, 333 (1981).
[51] C. Hoenselaers, On multipole moments in general relativity, in *Gravitational Collapse and Relativity: Proceedings of Yamada Conference XIV*, edited by H. Sato and T. Nakamura (World Scientific, Singapore, 1986), pp. 176–184.
[52] G. Fodor, C. Hoenselaers, and Z. Perjés, Multipole moments of axisymmetric systems in relativity, J. Math. Phys. (N.Y.) **30**, 2252 (1989).
[53] A. Kleinwächter, R. Meinel, and G. Neugebauer, The multipole moments of the rigidly rotating disk of dust in general relativity, Phys. Lett. A **200**, 82 (1995).
[54] P. Kordas, Reflection-symmetric, asymptotically flat solutions of the vacuum axistationary Einstein equations, Classical Quant. Grav. **12**, 2037 (1995).
[55] R. Meinel and G. Neugebauer, Asymptotically flat solutions to the Ernst equation with reflection symmetry, Classical Quant. Grav. **12**, 2045 (1995).
[56] M. Herberthson, The gravitational dipole and explicit multipole moments of static axisymmetric spacetimes, Classical Quant. Grav. **21**, 5121 (2004).

[57] T. Bäckdahl and M. Herberthson, Static axisymmetric spacetimes with prescribed multipole moments, Classical Quant. Grav. **22**, 1607 (2005).

[58] T. Bäckdahl and M. Herberthson, Explicit multipole moments of stationary axisymmetric spacetimes, Classical Quant. Grav. **22**, 3585 (2005).

[59] T. Bäckdahl, Axisymmetric stationary solutions with arbitrary multipole moments, Classical Quant. Grav. **24**, 2205 (2007).

[60] T. Bäckdahl and M. Herberthson, Calculation of, and bounds for, the multipole moments of stationary spacetimes, Classical Quant. Grav. **23**, 5997 (2006).

[61] M. Herberthson, Static spacetimes with prescribed multipole moments: A proof of a conjecture by Geroch, Classical Quant. Grav. **26**, 215009 (2009).

[62] W. Simon, The multipole expansion of stationary Einstein–Maxwell fields, J. Math. Phys. (N.Y.) **25**, 1035 (1984).

[63] L. A. Pachón and J. D. Sanabria-Gómez, Note on reflection symmetry in stationary axisymmetric electrovacuum spacetimes, Classical Quant. Grav. **23**, 3251 (2006).

[64] F. J. Ernst, V. S. Manko, and E. Ruiz, Equatorial symmetry/antisymmetry of stationary axisymmetric electrovac spacetimes, Classical Quant. Grav. **23**, 4945 (2006).

[65] C. Hoenselaers and Z. Perjés, Multipole moments of axisymmetric electrovacuum spacetimes, Classical Quant. Grav. **7**, 1819 (1990).

[66] T. P. Sotiriou and T. A. Apostolatos, Corrections and comments on the multipole moments of axisymmetric electrovacuum spacetimes, Classical Quant. Grav. **21**, 5727 (2004).

[67] Z. Perjés, Gravitational multipole moments, in *The Tenth Marcel Grossmann Meeting (2003)*, edited by M. Novello, S. P. Bergliaffa, and R. Ruffini (World Scientific, Singapore, 2006), pp. 59–69.

[68] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.104.064012 for there are two pairs of files attached as Supplemental Material to this article. The codes can be executed by the *Mathematica* or *Maple* computer algebra software. The files moments-general.nb and moments-general.mw are for the calculation of multipole moments of axisymmetric electrovacuum spacetimes in terms of the expansion coecients of the Ernst potential. The files five-par-sol.nb and five-par-sol.mw can be used for the calculation of multipole moments of the 5-parameter solution.

[69] T. P. Sotiriou and T. A. Apostolatos, Tracing the geometry around a massive, axisymmetric body to measure, through gravitational waves, its mass moments and electromagnetic moments, Phys. Rev. D **71**, 044005 (2005).

[70] L. A. Pachón and F. L. Dubeibe, The influence of the Lande g-factor in the classical general relativistic description of atomic and subatomic systems, Classical Quant. Grav. **28**, 055002 (2011).

[71] F. Dubeibe, F. Lora-Clavijo, and G. A. González, Pseudo-Newtonian planar circular restricted 3-body problem, Phys. Lett. A **381**, 563 (2017).

[72] O. V. Manko, V. S. Manko, and J. D. Sanabria-Gómez, Remarks on the charged, magnetized Tomimatsu-Sato $\delta = 2$ solution, Gen. Relativ. Gravit. **31**, 1539 (1999).

[73] L. Herrera, G. A. González, L. A. Pachón, and J. A. Rueda, Frame dragging, vorticity and electromagnetic fields in axially symmetric stationary spacetimes, Classical Quant. Grav. **23**, 2395 (2006).

[74] J. Sod-Hoffs and E. D. Rodchenko, On the properties of the Ernst–Manko–Ruiz equatorially antisymmetric solutions, Classical Quant. Grav. **24**, 4617 (2007).

[75] A. F. Gutiérrez-Ruiz and L. A. Pachón, Electromagnetically induced frame dragging around astrophysical objects, Phys. Rev. D **91**, 124047 (2015).

[76] V. S. Manko, A note on magnetic generalizations of the Kerr and Kerr–Newman solutions, Classical Quant. Grav. **34**, 177002 (2017).

[77] V. S. Manko and E. Ruiz, Simple metric for a magnetized, spinning, deformed mass, Phys. Rev. D **97**, 104016 (2018).

[78] E. Berti, F. White, A. Maniopoulou, and M. Bruni, Rotating neutron stars: An invariant comparison of approximate and numerical space-time models, Mon. Not. R. Astron. Soc. **358**, 923 (2005).

[79] R. Geroch, A method for generating solutions of Einstein's equations, J. Math. Phys. (N.Y.) **12**, 918 (1971).

[80] W. Israel and G. A. Wilson, A class of stationary electromagnetic vacuum fields, J. Math. Phys. (N.Y.) **13**, 865 (1972).

[81] W. Kinnersley, Generation of stationary Einstein-Maxwell fields, J. Math. Phys. (N.Y.) **14**, 651 (1973).

[82] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations*, 2nd ed., Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2003).

[83] R. Penrose, Zero rest-mass fields including gravitation: Asymptotic behaviour, Proc. R. Soc. A **284**, 159 (1965).

[84] A. Trautman, Introduction to gravitational radiation theory, in *Lectures on General Relativity*, Brandeis Summer Institute in Theoretical Physics, edited by H. Bondi, A. Trautman, and F. A. E. Pirani (Prentice-Hall, NJ, 1965), pp. 249–373.

[85] F. J. Ernst, New formulation of the axially symmetric gravitational field problem. II, Phys. Rev. **168**, 1415 (1968).

[86] R. Beig, The multipole expansion in general relativity, Acta Phys. Austriaca **53**, 249 (1981).

[87] G. Fodor, Gravitational multipole moments, in *Relativity Today*, edited by Z. Perjés (Nova Science Publishers, New York, 1991), pp. 121–131; *Proceedings of the Third Hungarian Workshop, 1989 Tihany*, ISBN 156072028X, https://wigner.hu/~gfodor/Fodor92.pdf.

[88] E. Ruiz, V. S. Manko, and J. Martín, Extended N-soliton solution of the Einstein-Maxwell equations, Phys. Rev. D **51**, 4192 (1995).

[89] A. Tomimatsu and H. Sato, New Exact Solution for the Gravitational Field of a Spinning Mass, Phys. Rev. Lett. **29**, 1344 (1972).

[90] A. Tomimatsu and H. Sato, New series of exact solutions for gravitational fields of spinning masses, Prog. Theor. Phys. **50**, 95 (1973).

[91] F. J. Ernst, Charged version of Tomimatsu-Sato spinning-mass field, Phys. Rev. D **7**, 2520 (1973).

[92] M. Yamazaki, On the charged Kerr–Tomimatsu–Sato family of solutions, J. Math. Phys. (N.Y.) **19**, 1376 (1978).

[93] O. V. Manko, V. S. Manko, and J. D. Sanabria-Gómez, Charged, magnetized Tomimatsu-Sato $\delta = 2$ solution, Prog. Theor. Phys. **100**, 671 (1998).

[94] M. Stute and M. Camenzind, Towards a self-consistent relativistic model of the exterior gravitational field of rapidly rotating neutron stars, Mon. Not. R. Astron. Soc. **336**, 831 (2002).

[95] F. L. Dubeibe, L. A. Pachón, and J. D. Sanabria-Gómez, Chaotic dynamics around astrophysical objects with non-isotropic stresses, Phys. Rev. D **75**, 023008 (2007).

[96] C. Bambi, Evolution of the spin parameter of accreting compact objects with non-Kerr quadrupole moment, J. Cosmol. Astropart. Phys. 05 (2011) 009.

[97] C. Bambi, Towards the use of the most massive black hole candidates in active galactic nuclei to test the Kerr paradigm, Phys. Rev. D **85**, 043001 (2012).

[98] N. Stergioulas, Rotating stars in relativity, Living Rev. Relativity **6**, 3 (2003).

[99] M. Camenzind, *Compact Objects in Astrophysics: White Dwarfs, Neutron Stars and Black Holes* (Springer Berlin Heidelberg, Berlin, Heidelberg, 2007).

[100] D. Kramer and G. Neugebauer, The superposition of two Kerr solutions, Phys. Lett. **75A**, 259 (1980).

[101] W. Kinnersley and E. F. Kelley, Limits of the Tomimatsu-Sato gravitational field, J. Math. Phys. (N.Y.) **15**, 2121 (1974).

[102] Y. Tanabe, Multipole moments in general relativity, Prog. Theor. Phys. **55**, 106 (1976).

[103] V. S. Manko and C. Moreno, Extension of the parameter space in the Tomimatsu–Sato solutions, Mod. Phys. Lett. A **12**, 613 (1997).

[104] V. Manko and E. Ruiz, Singularities in the Kerr–Newman and charged $\delta = 2$ Tomimatsu–Sato spacetimes endowed with negative mass, Prog. Theor. Exp. Phys. **2013**, 103E01 (2013).

[105] K. Rosquist, Some physical consequences of the multipole structure of the Kerr and Kerr-Newman solutions, in *The Eleventh Marcel Grossmann Meeting*, edited by H. Kleinert, R. T. Jantzen, and R. Ruffini (World Scientific, Singapore, 2008), pp. 2294–2298.

[106] W. Dietz and C. Hoenselaers, Two mass solutions of Einstein's vacuum equations: The double Kerr solution, Ann. Phys. (N.Y.) **165**, 319 (1985).

[107] A. Kleinwächter, H. Labranche, and R. Meinel, On the black hole limit of rotating discs and rings, Gen. Relativ. Gravit. **43**, 1469 (2011).

[108] M. Breithaupt, Y.-C. Liu, R. Meinel, and S. Palenta, On the black hole limit of rotating discs of charged dust, Classical Quant. Grav. **32**, 135022 (2015).

[109] F. de Felice and Y. Yunqiang, Stellar rotation and gravitational collapse: The a/m issue, J. Phys. A **15**, 3341 (1982).

[110] M. Bradley, A. Curir, and F. de Felice, Massive black holes in the nuclei of spiral galaxies: M31 and the problem of the initial conditions, Astrophys. J. **381**, 72 (1991).