Adiabatic duality: Duality of cosmological models with varying slow-roll parameter and sound speed

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There have been thousands of cosmological models for our early Universe proposed in the literature, and many of them claimed to be able to give rise to scale-invariant power spectrum as was favored by the observational data. It is thus interesting to think about whether there are some relations among them, e.g., the duality relation. In this paper, we investigate duality relations between cosmological models in framework of general relativity, not only with varying slow-roll parameter ϵ but also with sound speed c_s, which can then be understood as *adiabatic duality*. Several duality relationships are formulated analytically and verified numerically. We show that models with varying ϵ and constant c_s can be dual in the scalar spectral index, but not the tensor one. On the other hand, allowing both ϵ and c_s to vary can make models dual in both scalar and tensor spectral indices.

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I. INTRODUCTION

It is always interesting to ask what our Universe was like at its earliest stage. A most acceptable answer might be that it experienced a period of an inflation era [1–3], for it provides a solution to many big bang problems. Nevertheless, due to its incapability of solving another issue of the big bang scenario known as the singularity problem [4,5], many other scenarios/models are coming out constantly as its complementary/alternative candidates, such as bounce [6], ekpyrotic [7], slow expansion [8], and so on. These scenarios/models bring our early Universe with full of possibilities.

For these models, what is among the most important things is the necessity of being consistent with the observational data. Especially, there has been precise evidence that the scalar perturbations are nearly scale invariant, with only a few percent level of deviation [9]. Decades before, it had been thought that there were only two possibilities that could obtain nearly scale invariance, namely, slow-roll inflation and matter bounce (bounce with matterlike contracting phase before the Universe's expansion) [10,11], while ekpyrotic and slow expansion models suffered from the blue power spectrum [12,13]. However, it was later realized that, by requiring a varying slow-roll parameter ϵ [14] in these models scale invariance can be recovered again [15–17]. The reason that varying ϵ can promote scale invariance of the power spectrum is simply due to the fact that it can get involved in the perturbation action and change the behavior of the perturbation equations, which is also known as the "adiabatic mechanism" (see Ref. [15] and also Refs. [18,19] for a debate). Further study shows that, for models constructed under GR, scale invariance will be obtained as long as the condition

$$\frac{(a\sqrt{|\epsilon|})''}{a\sqrt{|\epsilon|}} = \frac{2}{|\tau|^2} \tag{1}$$

is satisfied, where a is the scale factor and τ is the conformal time, $\tau \equiv \int a^{-1}(t)dt$. With the varying behavior of ϵ, the constraint on a by this condition gets loosened, making more cosmic evolutions allowable.

The adiabatic mechanism can also be applied to the inflation model itself. Recently, there has been a model attracting people's eyes called the ultra-slow-roll inflation [20,21]. It possesses an "exact" flat potential, namely, $dV/d\phi = 0$, which further results in a decreasing ϵ, namely, $\epsilon \sim a^{-6}$. Although as an inflation model it is not necessary to have varying ϵ, this interestingly makes the behavior of its perturbations like those of the *matter contraction phase* in Refs. [10,11] or *slow expansion phase* in Ref. [8], which is dominated by its growing mode, rather than the constant one. It implies some links between inflation and other cosmological models.

Other than ϵ, the behavior of the power spectrum can also be affected by the sound speed c_s. The sound speed is a factor in front of the spatial derivative of the perturbations in the equation of motion; therefore, different from ϵ which

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modifies the background evolution, the sound speed modifies the effective horizon as well as the values when perturbations exit and reenter the horizon. For perturbations that are not conserved outside the horizon, such a modification will correspondingly affect the scale variance of the power spectrum. Therefore, if we further allow c_s to vary, we may have even more models with a scale-invariant power spectrum [22–26].

Although the current work focuses on the framework of GR, as a side remark, let us also mention that when the modified gravity is taken into account the possibility of getting scale invariance will also be enlarged, due to the fact that the scale factor can be corrected by a conformal factor F, namely, $a \to a\sqrt{F}$. For relevant works, see Refs. [27–30].

Given the more-than-enough models that can meet with the current observational data, as an extension, we would ask if there can be more links between those models. Especially, among the models with varying ϵ and c_s, will they have some relations such as dualities? Actually, there have been many papers discussing dualities between early Universe models; for example, Ref. [11] showed us the duality between slow-roll inflation and matterlike contraction, Ref. [31] discussed the dualities of the primordial perturbation spectra from various expanding/contracting phases with constant ϵ, while Ref. [32] presented that the duality between ekpyrosis with varying ϵ is dual to inflation with constant ϵ. In Refs. [33–35], there are also debates on whether there is duality between slow-roll inflation and ultra-slow-roll inflation models. In this paper, we try to investigate as a whole the duality among varying ϵ and c_s models, in order to see whether such nontrivial parameters will bring us anything new about the duality relations. Since these duality relations are based on the aforementioned adiabatic mechanism, they can be called a kind of "adiabatic duality," in contrast to the "conformal duality" studied in Refs. [27–30].

The rest of the paper is arranged as follows. In Sec. II, we show the formulation of perturbations from a single-field cosmological model in the general case. In Sec. III, we focus on the duality for varying ϵ and constant c_s models, while in Sec. IV, we extend our discussion to the case where both ϵ and c_s are varying. In Sec. V, we check our analysis by performing numerical calculation of the perturbation equations. In Sec. VI, we discuss the effects of primordial non-Gaussianities on the duality. Section VII is the final conclusions and discussions.

II. PERTURBATIONS FROM A SINGLE-FIELD MODEL

We will consider the linear perturbations generated in the early Universe, which are described by the action

$$S = \frac{1}{2}\int d^4x\sqrt{-g}[M_p^2 R + 2P(\phi, X)], \tag{2}$$

where ϕ is a scalar field while $X \equiv -\partial_\mu\phi\partial^\mu\phi/2$. Hereafter, we choose the unit $M_p^2 = 1$. As there is only one scalar degree of freedom in this kind of model, the scalar perturbations are purely adiabatic. A tedious but conventional calculation shows that such adiabatic perturbations obey the perturbation equation,

$$u'' + \left(c_s^2 k^2 - \frac{z''}{z}\right)u = 0, \tag{3}$$

where we define the perturbation variable $u \equiv z\zeta$, with ζ denoting the curvature perturbation, and $z \equiv a\sqrt{|\epsilon|}/c_s$. The slow-roll parameter ϵ is defined as $\epsilon \equiv -\dot{H}/H^2$, where H is the Hubble parameter, and the sound speed squared c_s^2 is defined as

$$c_s^2 \equiv \frac{P_{,X}}{(2XP_{,X} - P)_{,X}}. \tag{4}$$

Moreover, the prime in Eq. (3) means a derivative with respect to conformal time τ.

In the usual case where $z(\tau)$ can be parametrized as a power-law form of τ, one in general has $z''/z \propto |\tau|^{-2}$. Therefore, it is reasonable to set

$$\frac{z''}{z} = \frac{4\nu_z^2 - 1}{4|\tau|^2}, \tag{5}$$

where ν_z is a parameter. Moreover, we assume that the c_s also has a power-law form of τ, namely, $c_s \sim (-\tau)^s$ with s the power index; then, Eq. (3) has the Hankel-function solution,

$$u \simeq \sqrt{|\tau|}\left[c_1 H_\nu\left(\int c_s k d\tau\right) + c_2 H_{-\nu}\left(\int c_s k d\tau\right)\right], \tag{6}$$

where $\nu \equiv \nu_z/(s+1)$. Here, $s + 1 > 0$ is required in order to ensure that the fluctuation modes can exit the sound horizon. Note that, in general, the index of the Hankel function ν is different from ν_z; however, for the constant c_s case where $s = 0$, the two indices coincide with each other. Moreover, comparing with the initial condition solution [36]

$$u_{\mathrm{ini}} = \frac{1}{\sqrt{2c_s k}} e^{i\int c_s k d\tau}, \tag{7}$$

which is obtained from Eq. (3) in the $k \to \infty$ limit, one can fix the coefficients $c_1 = c_2 = \sqrt{\pi/(s+1)}/2$. Therefore, the power spectrum can be obtained as

$$P_S \equiv \frac{k^3}{2\pi^2}\left|\frac{u}{a\sqrt{|\epsilon|}/c_s}\right|^2 \sim \frac{(s+1)^2 H_*^2}{8\pi^2|\epsilon_*|c_{s*}}\left(\frac{\tau}{\tau_*}\right)^{-(3-2\nu)(s+1)}|c_s k\tau|^{3-2|\nu|}, \quad (8)$$

with the spectral index

$$n_S - 1 \equiv \frac{d\ln P_S}{d\ln k} = 3 - 2|\nu|, \quad (9)$$

where $*$ means values taken at some pivot time point $\tau = \tau_*$. From the expression, one can easily see that both ν and $-\nu$ can give rise to the same spectral index. Moreover, to get the scale-invariant power spectrum which is favored by the observational data, we need to have $|\nu| = 3/2$. In the case where ϵ and c_s are constants, this requires either $a \sim (-\tau)^{-1}$ or $a \sim (-\tau)^2$ [11]. In the former case, the perturbations are dominated by their constant mode, which makes their behavior like those in slow-roll inflation regime, while in the latter case, those are dominated by their growing model, like a matter-dominated contraction. However, as we will see below, for varying ϵ and c_s, the case may be different.

We also consider the tensor perturbation generated by model (2), which is important as it provides the primordial gravitational waves that we are searching for. The tensor perturbation equation can be derived from the action (2) as

$$v'' + \left(k^2 - \frac{a''}{a}\right)v = 0, \qquad \frac{a''}{a} = \frac{4\nu_T^2 - 1}{4|\tau|^2}, \quad (10)$$

where $v \equiv ah/2$ and h is the tensor mode of the metric perturbation. Note that, since we restrict ourselves in the case of general relativity (GR), the sound speed of tensor perturbation is unity. Similar calculation shows that the power spectrum for tensor perturbation is

$$P_T \equiv \frac{k^3}{\pi^2}\left|\frac{v}{a/2}\right|^2 \sim \frac{2H_*^2}{\pi^2}\left(\frac{\tau}{\tau_*}\right)^{-(3-2\nu_T)}|k\tau|^{3-2|\nu_T|}, \quad (11)$$

with the tensor spectral index

$$n_T \equiv \frac{d\ln P_T}{d\ln k} = 3 - 2|\nu_T|. \quad (12)$$

In practical analysis and observations, people are used to expressing the tensor spectrum in terms of the tensor-scalar ratio, which is

$$r \equiv \frac{P_T}{P_S} = \frac{16\epsilon_*|c_{s*}|}{(s+1)^2}\left(\frac{\tau}{\tau_*}\right)^{\eta+s}. \quad (13)$$

III. COSMIC DUALITY FOR VARYING ϵ AND CONSTANT c_s

As a first step, we now consider the case where the slow-roll parameters are varying while the sound speed remains constant. From the very definition of the slow-roll parameter, one thus derives the expression of ϵ in terms of conformal time τ as

$$\epsilon = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} = 1 + \left(\frac{1}{\mathcal{H}}\right)', \quad (14)$$

where $\mathcal{H}$ is conformal Hubble parameter, $\mathcal{H} \equiv aH$. The conformal time τ will be negative, with its absolute value $|\tau| = -\tau$ decreasing. Assuming $\epsilon(\tau) = \epsilon_0(-\tau)^\eta$, one can solve (14) to get

$$\frac{1}{\mathcal{H}(\tau)} \simeq \frac{(-\tau)(1+\eta-\epsilon)}{1+\eta}, \qquad \mathcal{H}(\tau) \simeq \frac{1+\eta}{(-\tau)(1+\eta-\epsilon)}. \quad (15)$$

Here, the approximation in computing $\mathcal{H}(\tau)$ is due to the fact that the integration constant has been neglected. This approximation is acceptable as we will verify our final analytical result of the spectral index by numerical simulations later. Moreover, according to $\mathcal{H} \equiv a'(\tau)/a(\tau)$, one also has

$$a(\tau) = e^{\int \mathcal{H}d\tau} \simeq (-\tau)^{-1}|1+\eta-\epsilon|^{1/\eta}. \quad (16)$$

Substituting the expressions of $a(\tau)$ and $\epsilon(\tau)$ into the expression of z (where we set $c_s = 1$), one has

$$z \simeq \sqrt{|\epsilon_0|}(-\tau)^{\frac{\eta}{2}-1}|1+\eta-\epsilon_0(-\tau)^\eta|^{1/\eta},$$
$$\frac{z''}{z} = \frac{1}{4}(1-\epsilon+\eta)^{-2}(-\tau)^{-2}[(1+\eta)^2(\eta-2)(\eta-4) - 2\epsilon(2-\eta^2+\eta^3) + \epsilon^2\eta(\eta-2)], \quad (17)$$

and from (5), we get

$$\nu = \nu_z = \pm\frac{1}{2}\left(\frac{\epsilon - 3(1+\eta)}{\epsilon - (1+\eta)} - \eta\right) \times \sqrt{1 - \frac{4\epsilon\eta(\eta+1)}{[\epsilon(1-\eta) + (\eta+1)(\eta-3)]^2}} \simeq \pm\frac{1}{2}\left(\frac{\epsilon - 3(1+\eta)}{\epsilon - (1+\eta)} - \eta\right). \quad (18)$$

In deriving the second step, notice that since we only consider the zeroth-order approximation for varying ϵ, the last term in the square root can be ignored safely either the value of epsilon becomes large or small.

Note also that when $\eta = 0$, ϵ becomes constant and the result recovers the usual one of $2\nu = \pm(\epsilon - 3)/(\epsilon - 1)$.

Reference [11] points out that any two scenarios giving opposite ν will become dual to each other, for they give rise to the same power spectrum. Here, we revisit this remark for scenarios with varying ϵ. For two scenarios with

$$\nu = \frac{1}{2}\left(\frac{\epsilon - 3(1+\eta)}{\epsilon - (1+\eta)} - \eta\right), \quad \tilde{\nu} = \frac{1}{2}\left(\frac{\tilde{\epsilon} - 3(1+\tilde{\eta})}{\tilde{\epsilon} - (1+\tilde{\eta})} - \tilde{\eta}\right), \tag{19}$$

a dual relation between the two is $|\nu| = |\tilde{\nu}|$, namely,

$$\frac{\epsilon - 3(1+\eta)}{\epsilon - (1+\eta)} - \eta = \pm\left(\frac{\tilde{\epsilon} - 3(1+\tilde{\eta})}{\tilde{\epsilon} - (1+\tilde{\eta})} - \tilde{\eta}\right). \tag{20}$$

We first consider the case where "$\pm$" → "$-$" in Eq. (20). Since now both ϵ and $\tilde{\epsilon}$ are varying, an interesting case is that they approach different directions. For $\epsilon(\tau) \to \pm\infty$ and $\tilde{\epsilon}(\tilde{\tau}) \to 0$, or vice versa, one has

$$\eta + \tilde{\eta} = 4, \tag{21}$$

which is a duality relation between η and $\tilde{\eta}$. Considering the constraint of scale invariance of the power spectrum, namely, $3 - 2|\nu| = 0$, we have the following possibilities:

(i) $\nu = -\tilde{\nu} = 3/2$, which leads to $\eta = 4$, $\tilde{\eta} = 0$.

(ii) $\nu = -\tilde{\nu} = -3/2$, which leads to $\eta = -2$, $\tilde{\eta} = 6$.

There are also nontrivial possibilities for ϵ and $\tilde{\epsilon}$ approaching the same direction. For example, for both ϵ and $\tilde{\epsilon}$ approaching to $\pm\infty$, one has

$$\eta + \tilde{\eta} = 2, \tag{22}$$

and considering the constraint of scale invariance, we have $\eta = -2$ and $\tilde{\eta} = 4$. For both ϵ and $\tilde{\epsilon}$ approaching 0, one has

$$\eta + \tilde{\eta} = 6, \tag{23}$$

and considering the constraint of scale invariance, we have $\eta = 0$ and $\tilde{\eta} = 6$.

Another duality relation arises for $\pm \to +$ in Eq. (20). Note that this becomes trivial for constant ϵ and will give $\epsilon = \tilde{\epsilon}$ only. However, for varying ϵ, by requiring ϵ and $\tilde{\epsilon}$ approaching different directions [$\epsilon(\tau) \to \pm\infty$ and $\tilde{\epsilon}(\tilde{\tau}) \to 0$, or vice versa], one has

$$|\eta - \tilde{\eta}| = 2. \tag{24}$$

Considering the scale invariance, we have the following possibilities:

(i) $\nu = \tilde{\nu} = 3/2$, which leads to $\eta = 4$, $\tilde{\eta} = 6$.

(ii) $\nu = \tilde{\nu} = -3/2$, which leads to $\eta = -2$, $\tilde{\eta} = 0$.

Moreover, if ϵ and $\tilde{\epsilon}$ approach the same direction, it gives a trivial result as well.

From above, one can see that, requiring the scalar spectral index to be identical, we can in total get four kinds of duality relations of cosmological models with varying slow-roll parameter ϵ. Moreover, taking into account the observational constraint that the scalar spectrum is scale invariant, we can actually reduce to four representative models, which, under different relations, are dual to each other: $\epsilon \to 0$, $\eta \to 0$ [slow-roll inflation (SR)], $|\epsilon| \to \infty$, $\eta \to -2$ [slow-evolving universe I (SE1)], $|\epsilon| \to \infty$, $\eta \to 4$ [slow-evolving universe II (SE2)], and $\epsilon \to 0$, $\eta \to 6$ [ultra-slow-roll inflation (USR)]. It is clearer to draw a sketch plot to express these models under the duality relation, as shown in Fig. 1.

As a side remark, we mention that, in principle, one can also use Eq. (18) to make up duality relations for models with constant ϵ, such as inflation or matter bounce, just as is done in Ref. [11]. However, in those cases, the approximations of $(\epsilon - 3)/(\epsilon - 1)$ will be dependent on specific values of ϵ. Therefore, our duality relations will not apply. We will not bring these cases into the current discussion. On the other hand, we can also discuss the duality relation given by tensor perturbation. According to Eq. (16), it is straightforward to get

$$\frac{a''}{a} = (1 - \epsilon + \eta)^{-2}(-\tau)^{-2}(1+\eta)^2(2-\epsilon), \tag{25}$$

and from Eq. (10), one has

$$\nu_T = \pm\frac{1}{2}\frac{\epsilon - 3(1+\eta)}{\epsilon - (1+\eta)}\sqrt{1 - \frac{4\epsilon\eta(\eta+1)}{[\epsilon - 3(1+\eta)]^2}} \simeq \pm\frac{1}{2}\frac{\epsilon - 3(1+\eta)}{\epsilon - (1+\eta)}. \tag{26}$$

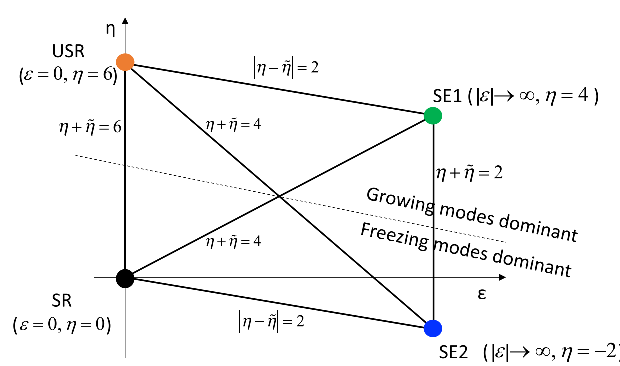

FIG. 1. The sketch plot that demonstrates the dual relationship of models for varying ϵ and constant c_s in the (ϵ, η) plane. The colored points represent models, and the black lines between them denote various relationships. Moreover, the dashed line divides the whole region into two parts. In the upper parts, the perturbations are dominated by the growing modes, while in the lower parts, the perturbations are dominated by the freezing (constant) modes.

Similarly, we are using the zeroth approximation for varying ϵ and when $\eta = 0$, and the result covers the usual case of $2\nu_T = \pm(\epsilon - 3)/(\epsilon - 1)$.

One can see that requiring the duality relation to be maintained also for for tensor spectral index, $|\nu_T| = |\tilde{\nu}_T|$, namely, to have

$$\frac{\epsilon - 3(1+\eta)}{\epsilon - (1+\eta)} = \pm\frac{\tilde{\epsilon} - 3(1+\tilde{\eta})}{\tilde{\epsilon} - (1+\tilde{\eta})}, \tag{27}$$

results in the fact that ϵ and $\tilde{\epsilon}$ must be approaching to the same direction, and $\pm$ can only be $+$. This means that the duality relations (21) and (24) will be broken, while only (22) and (23) remain. Therefore, if we detect the tensor spectral index, the dual symmetry among these models will get reduced.

As is well known, the tensor perturbations contribute to the primordial gravitational waves. Note that recently more and more programs detecting gravitational waves are coming out, among which there are not only those aiming at medium-/low-frequency gravitational waves (GWs) (mainly generated by compact binary systems), such as FAST [37], LISA [38], LIGO [39], SKA [40], TianQin [41], Taiji [42], GECAM [43], NANOGrav [44], and so on but also those aiming at the primordial GW program (mainly via polarizations of cosmic microwave background photons), such as AliCPT [45], ACT [46], POLARBEAR [47], SPT [48], BICEP [49], LiteBIRD [50], and so on. These programs can make the detections of tensor spectrum (in terms of tensor/scalar ratio r), and even tensor spectral index, possible in the future. This will break the duality relation between those models and thus can differentiate different models of the early Universe.

IV. COSMIC DUALITY FOR BOTH VARYING ϵ AND c_s

In the following, we will extend our consideration to include the case where c_s is also varying. Assuming that $c_s = c_{s0}(-\tau)^s$, Eq. (17) will be modified as

$$\begin{aligned} z &\simeq \sqrt{|\epsilon_0|}c_{s0}^{-2}(-\tau)^{\frac{\eta}{2}-1-s}|1+\eta-\epsilon_0(-\tau)^\eta|^{1/\eta}, \\ \frac{z''}{z} &= \frac{1}{4}(1-\epsilon+\eta)^{-2}(-\tau)^{-2}[(1+\eta)^2(\eta-2-2s)(\eta-4-2s) \\ &\quad -2\epsilon(\eta+1)(\eta^2-2\eta-4s\eta+8s+4s^2+2) \\ &\quad +\epsilon^2(\eta-2s)(\eta-2-2s)], \end{aligned} \tag{28}$$

and from (5), we get

$$\nu = \pm\frac{1}{2(s+1)}\left(\frac{\epsilon-3(1+\eta)}{\epsilon-(1+\eta)} - \eta + 2s\right) \times\sqrt{1-\frac{4\epsilon\eta(\eta+1)}{[\epsilon(1-\eta+2s)+(\eta+1)(\eta-3-2s)]^2}}. \tag{29}$$

In the limit of small ϵ and large ϵ, we have

$$\nu = \begin{cases} \pm\frac{1}{2}(2-\frac{\eta+1}{s+1}), & |\epsilon| \gg 1, \\ \pm\frac{1}{2}(2-\frac{\eta-1}{s+1}), & |\epsilon| \ll 1. \end{cases} \tag{30}$$

As shown in the last section, taking into account the tensor spectral index, the two models to be dual must have the same approximate behavior of ϵ; therefore, for $\epsilon, \tilde{\epsilon} \to \pm\infty$, the duality relation for ν is

$$\left(2-\frac{\eta+1}{s+1}\right) = \pm\left(2-\frac{\tilde{\eta}+1}{\tilde{s}+1}\right). \tag{31}$$

When $\pm \to -$, the above relation reduces to

$$\frac{\eta+1}{s+1}+\frac{\tilde{\eta}+1}{\tilde{s}+1} = 4. \tag{32}$$

Note that if we set $s = \tilde{s} = 0$, Eq. (32) will further reduce to Eq. (22). In other words, Eq. (32) will be the generalized version of (22) by taking into account the varying of sound speed. Moreover, for the case $\pm \to +$, which is trivial in the absence of $s, \tilde{s}$, we can also get a somehow nontrivial relation, namely,

$$\frac{\eta+1}{s+1} = \frac{\tilde{\eta}+1}{\tilde{s}+1}. \tag{33}$$

Furthermore, we consider the constraint of scale invariance of the power spectrum, $3 - 2|\nu| = 0$. For $\pm \to -$, we have $\nu = -\tilde{\nu} = 3/2$, which leads to $\eta = -2 - s$, $\tilde{\eta} = 4 + 5\tilde{s}$.

One could see that the duality between two model points ($\eta = -2$, $\tilde{\eta} = -4$) on the η axis (one dimensions) in the last section has been extended to that of two lines on the (η, s) plane (two dimensions). For $\nu = -\tilde{\nu}$, the two models dual to each other lie on the two lines separately, while for $\nu = \tilde{\nu}$, as is the case of $\pm \to +$, both models will lie on the same line, and which line depends on whether $\nu/\tilde{\nu}$ is positive or not. Therefore, models presented by either two points lying on those two lines can be dual to each other. To illustrate this, we plot the two lines in the (η, s) plane in Fig. 2. The solid line represents the relation $\eta = -s - 2$, while the dashed line represents the another relation $\tilde{\eta} = 4 + 5\tilde{s}$. We also point out SE1 and SE2 scenarios when $s = 0$ by the orange point and blue point, respectively.

We can also do the same thing for $\epsilon, \tilde{\epsilon} \to 0$. In this case, the duality relation for ν is

$$\left(2-\frac{\eta-1}{s+1}\right) = \pm\left(2-\frac{\tilde{\eta}-1}{\tilde{s}+1}\right). \tag{34}$$

When $\pm \to -$, the above relation reduces to

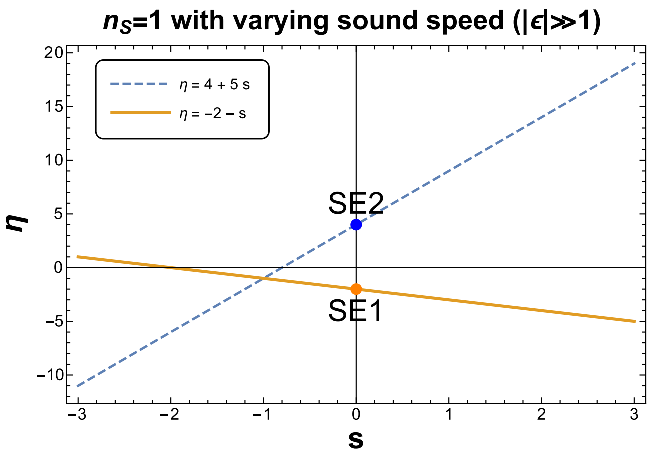

FIG. 2. The sketch plot that demonstrates the dual relationship of models for both varying ϵ and c_s, provided that $\epsilon, \tilde{\epsilon} \to \pm\infty$. The yellow solid line and the blue dashed line represent models with $\eta = -2 - s$ and $\tilde{\eta} = 4 + 5\tilde{s}$, respectively. The intersection points of the two lines with the vertical axis denotes models with $s = 0$ (constant c_s), namely, the slow-evolving models defined in the context.

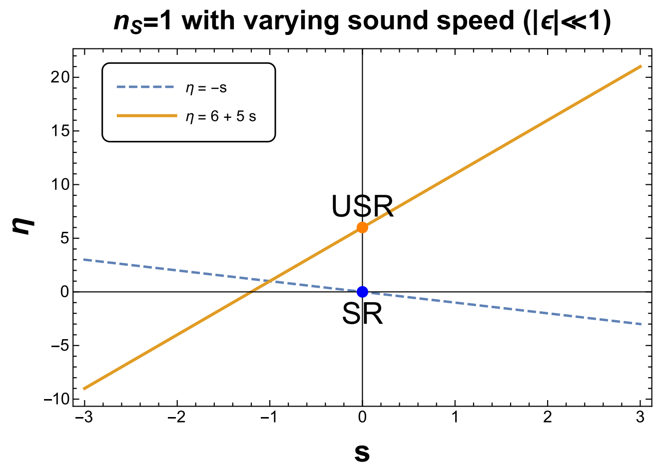

FIG. 3. The sketch plot that demonstrates the dual relationship of models for both varying ϵ and c_s, provided that $\epsilon, \tilde{\epsilon} \to 0$. The yellow solid line and the blue dashed line represent models with $\eta = 6 + 5s$ and $\tilde{\eta} = -\tilde{s}$, respectively. The intersection points of the two lines with the vertical axis denote models with $s = 0$ (constant c_s), namely, the ultra-slow-roll and usual slow-roll inflation models.

$$\frac{\eta - 1}{s + 1} + \frac{\tilde{\eta} - 1}{\tilde{s} + 1} = 4, \tag{35}$$

and by setting $s = \tilde{s} = 0$, Eq. (35) will further reduce to Eq. (23); namely, (35) will be the generalized version of (23) by taking into account the varying of sound speed. Moreover, for the case $\pm \to +$, which is trivial in the absence of $s, \tilde{s}$, we can also get a somehow nontrivial relation, namely,

$$\frac{\eta + 1}{s + 1} = \frac{\tilde{\eta} + 1}{\tilde{s} + 1}. \tag{36}$$

We also consider the constraint of scale invariance of the power spectrum, $3 - 2|\nu| = 0$. For $\pm \to -$, we have $\nu = -\tilde{\nu} = 3/2$, which leads to $\eta = -s$, $\tilde{\eta} = 6 + 5\tilde{s}$.

In a similar manner as above, models presented by either of the two points lying on those two lines can be dual to each other. To illustrate this, we plot these two lines in Fig. 3. The solid line represents the relation $\tilde{\eta} = 6 + 5\tilde{s}$, while the dashed line represents the another relation $\eta = -s$. The two points where $s = 0$ correspond to SR and USR scenarios, respectively.

V. NUMERICAL VERIFICATION

In the above section, we finished the theoretical analysis of which cosmological models with parametrized slow-roll parameter and sound of speed can give rise to spectral indices that can be dual to each other. The analysis is, however, semianalytical, and several approximations have been used. To verify our results, in this section, we calculate numerically the equation of motion, Eqs. (3) as well as (10) with different behaviors of $a(\tau)$, $\epsilon(\tau)$, and $c_s(\tau)$, to see how their tensor and scalar spectra (and their indices) will behave.

We plot our numerical results for scalar and tensor power spectra for constant and varying sound speed models in Figs. 4–7, respectively. The lines in the figures represent the spectrum for each model, while their slopes reflect the information of the spectral indices. One can see from Fig. 4 that, for the trivial sound speed case ($s = 0$), the spectrum of the four models will have the same behavior at least around the observable range, namely, $k \simeq 0.05\ \mathrm{Mpc}^{-1}$. Moreover, this is not only for indices of the spectra (slope of each line) but for amplitudes as well. The coincidence of the amplitudes can be done by setting proper initial conditions of background quantities such as $a(\tau)$, $H(\tau)$, and $\epsilon(\tau)$. For the smaller k region, however, there might be some differences; for example, the slow-evolution models present an oscillating behavior, which might be due to the features in the earlier time that possibly break down some of the approximations in our analytical study. On the other hand, as shown in Fig. 5, neither the amplitude nor slope of the tensor spectra coincides with the other. The reason for the slope has already been shown by calculation in the last section, while the reason for the amplitude is also understandable; since $P_T/P_S = 16|\epsilon|$ and those models have different ϵ, it is impossible to have both P_S and P_T coincide. That means, in the $s = 0$ case, we can only have the scalar power spectra dual to each other, but not tensor ones.

For the $s \neq 0$ case, however, things become different. First of all, as there is one more degree of freedom, the models dual to each other become richer. In Fig. 6,

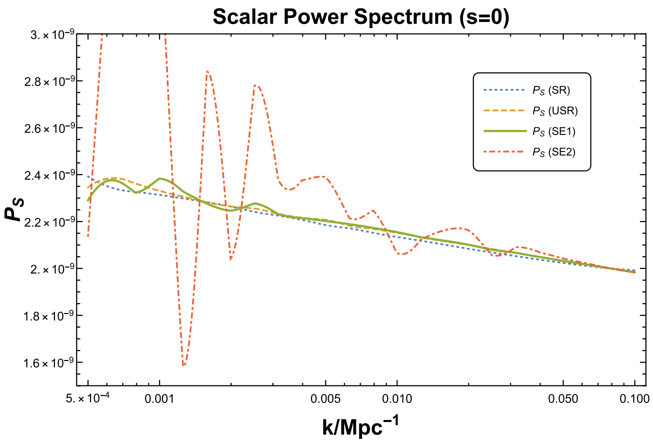

FIG. 4. The scalar power spectrum with constant sound speed ($s = 0$). The nearly scale-invariant (i.e., $n_S \simeq 0.965$ at the pivot scale $k_* = 0.05\ \mathrm{Mpc}^{-1}$) scalar power spectra against wave number k are shown by four lines (blue dotted, orange dashed, green solid, and red dot-dashed) corresponding to four different scenarios SR, USR, SE1, and SE2, respectively. Here, we set $\epsilon < 0$, $H > 0$ for slow-evolving models and $\epsilon > 0$, $H > 0$ for inflation models.

we show that for several choices of s, as long as the relationship $\eta = -s$, $\tilde{\eta} = 6 + 5\tilde{s}$ (upper panel) or $\eta = -s - 2$, $\tilde{\eta} = 4 + 5\tilde{s}$ (lower panel) is satisfied, the amplitude and slope of each line will coincide with each other (note that in the analytical study we approximate the spectral index to be unity but the realistic observation favors $n_S \simeq 0.965$, so the numerical values will be slightly deviated from the analytical formulas). Moreover, for the lower panel for the slow-evolution case, one can see that, while the duality happens around the observable range, $k \simeq 0.05\ \mathrm{Mpc}^{-1}$, it

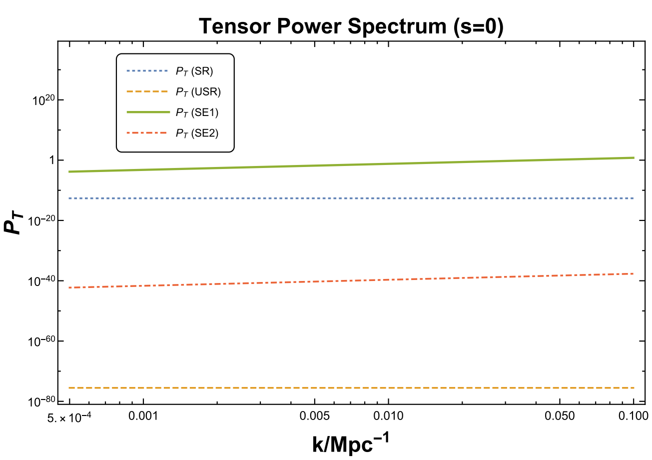

FIG. 5. The tensor power spectrum with constant sound speed ($s = 0$). The tensor power spectra against wave number k are shown by four lines (blue dotted, orange dashed, green solid, and red dot-dashed) corresponding to four different scenarios SR, USR, SE1, and SE2, respectively. Here, we set $\epsilon < 0$, $H > 0$ for slow-evolving models and $\epsilon > 0$, $H > 0$ for inflation models.

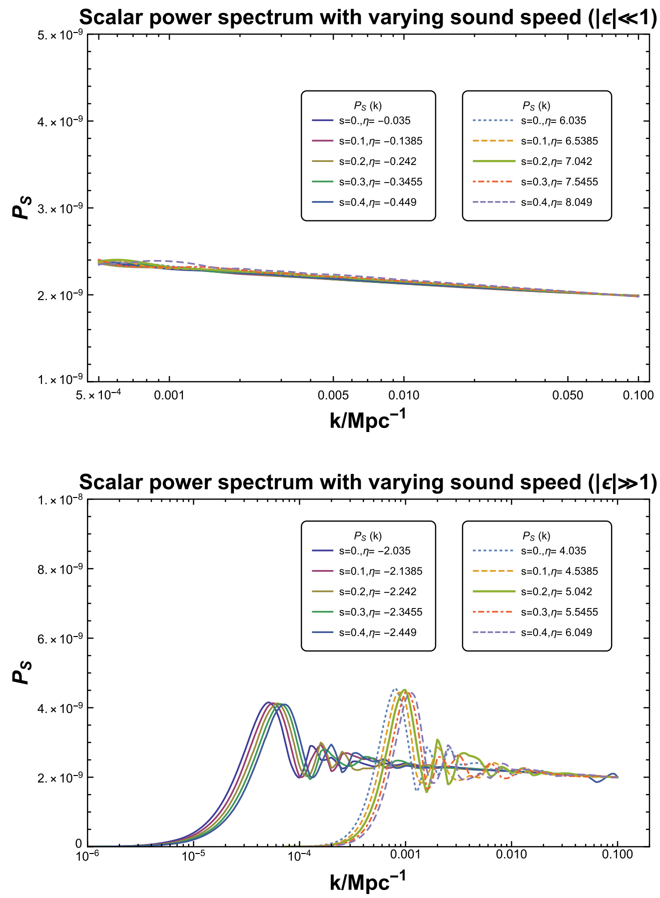

FIG. 6. The scalar power spectrum with varying sound speed. Upper panel: the nearly scale-invariant (i.e., $n_S \simeq 0.965$ at the pivot scale $k_* = 0.05\ \mathrm{Mpc}^{-1}$) scalar power spectra against wave number k are shown for SR and USR models ($s = 0$) and their variations ($s = 0.1, 0.2, 0.3, 0.4$). Lower panel: the nearly scale-invariant scalar power spectra against wave number k (i.e., $n_S \simeq 0.965$ at the pivot scale $k_* = 0.05\ \mathrm{Mpc}^{-1}$) are shown for SE1 and SE2 models ($s = 0$) and their variations ($s = 0.1, 0.2, 0.3, 0.4$). Here, we set $\epsilon < 0$, $H > 0$ for slow-evolving models and $\epsilon > 0$, $H > 0$ for inflation models.

may not do so for smaller k region, for the same reason as in the $s = 0$ case.

For tensor spectra, one can see from Fig. 7 that the slopes of each line now get identical, indicating that, different from the $s \neq 0$ case, the tensor spectral indices can be dual to each other. However, the amplitude of the tensor spectrum still cannot be the same because these models have quite different ϵ (although in this case $P_T/P_S = 16|\epsilon|c_s$, where c_s can also help to do the modulation, since c_s is constrained to be between [0, 1], the modulation is not efficient enough). Therefore, with varying sound speed taken into account, only the spectral index of the scalar and tensor spectra can be made dual to each other.

The nonduality in the amplitude of tensor perturbations, due to the discrepancy of ϵ, has also been realized in

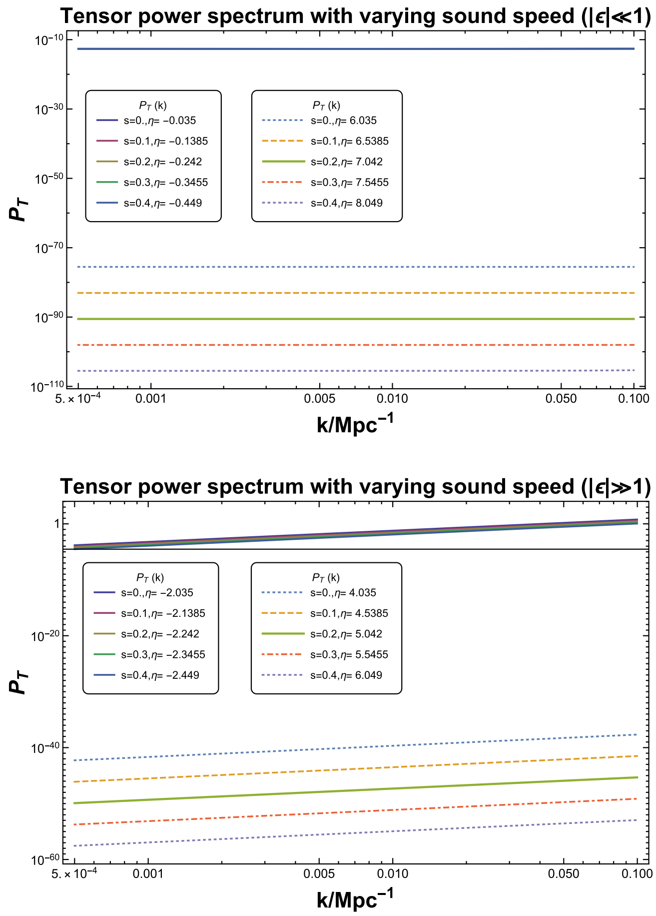

FIG. 7. The tensor power spectrum with varying sound speed. Upper panel: the tensor power spectra against wave number k are shown for SR and USR models ($s = 0$) and their variations ($s = 0.1, 0.2, 0.3, 0.4$), where the tensor spectral index $n_T \simeq 0$. Lower panel: the tensor power spectra against wave number k are shown for SE1 and SE2 models ($s = 0$) and their variations ($s = 0.1, 0.2, 0.3, 0.4$), where the tensor spectral index $n_T \simeq 2$. Here, we set $\epsilon < 0$, $H > 0$ for slow-evolving models and $\epsilon > 0$, $H > 0$ for inflation models.

Ref. [34], although they have been considering such a problem in the case of scalar perturbations. Can we have the amplitude of tensor spectrum coincide as well, namely, to have full duality of all the quadratic perturbations for cosmological models? Fortunately, the answer maybe "yes," but some delicate mechanisms may be needed. For example, in Ref. [35], the authors suggested that in the case of ultra-slow-roll inflation the ultra-slow-roll region is not an attractor solution but only a transient phase, which would eventually evolve into the slow-roll phase. Therefore, in this model, the perturbations produced will be totally the same as that of the slow-roll inflation models, and there will be fully duality. However, such a mechanism seems model dependent; namely, according to each specific model, the details might be different. Since in this paper we are only trying to discuss the general features without going into details of each model, such mechanisms are somehow beyond the scope of our discussion.

VI. DISCUSSION: NON-GAUSSIANITIES

In previous sections, we discussed the duality relations in linear perturbation theories. However, they will be broken when the higher-order perturbations (i.e., the non-Gaussianities) are taken into account. To see this, let us first write down the third-order perturbation action for a general single field [51,52],

$$\begin{aligned} S_3 = \int dt d^3x \Bigg\{ & -a^3\left[\Sigma\left(1-\frac{1}{c_s^2}\right)+2\lambda\right]\frac{\dot{\zeta}^3}{H^3} \\ & + \frac{a^3\epsilon}{c_s^4}(\epsilon-3+3c_s^2)\zeta\dot{\zeta}^2 + \frac{a\epsilon}{c_s^2}(\epsilon-2s+1-c_s^2)\zeta(\partial\zeta)^2 \\ & - 2a\frac{\epsilon}{c_s^2}\dot{\zeta}(\partial\zeta)(\partial\chi) + \frac{a^3\epsilon}{2c_s^2}\frac{d}{dt}\left(\frac{\eta}{c_s^2}\right)\zeta^2\dot{\zeta} \\ & + \frac{\epsilon}{2a}(\partial\zeta)(\partial\chi)\partial^2\chi + \frac{\epsilon}{4a}(\partial^2\zeta)(\partial\chi)^2 \Bigg\}, \end{aligned} \tag{37}$$

where $\Sigma \equiv XP_{,X} + 2X^2P_{,XX}$, $\lambda \equiv X^2P_{,XX} + 2X^3P_{,XXX}/3$, $\partial^2\chi \equiv a^2\epsilon\dot{\zeta}$. According to the "in-in" formalism [53], the three-point correlation function is described as

$$\begin{aligned} & \langle|\zeta(\tau,\vec{k}_1)\zeta(\tau,\vec{k}_2)\zeta(\tau,\vec{k}_3)|\rangle \\ & = -i\mathcal{T}\int_{t_0}^{t} dt' \langle|[\zeta(t,\vec{k}_1)\zeta(t,\vec{k}_2)\zeta(t,\vec{k}_3), H_{\rm int}(t')]|\rangle, \end{aligned} \tag{38}$$

where $H_{\rm int} = -S_3$ is the third-order interaction Hamiltonian and $\mathcal{T}$ is the time-ordering operator. On the other hand, it is useful to define the bispectrum of the three-point correlation function,

$$\begin{aligned} & \langle|\zeta(\tau,\vec{k}_1)\zeta(\tau,\vec{k}_2)\zeta(\tau,\vec{k}_3)|\rangle \\ & \quad = (2\pi)^3\delta^3\left(\sum_i \vec{k}_i\right)\mathcal{B}(k_1,k_2,k_3), \end{aligned} \tag{39}$$

and the non-Gaussianity estimator is defined as

$$f_{nl} \equiv \frac{5}{6}\frac{\mathcal{B}(k_1,k_2,k_3)}{(2\pi^2/k_1^3)(2\pi^2/k_2^3)P_S(k_1)P_S(k_2) + 2\,\text{perms.}}, \tag{40}$$

which can be constrained by the observational data [54].

From the action (37), one can see that the varying ϵ and c_s will also affect the non-Gaussianity. The effects are at least in two aspects:

(1) The time dependence of ϵ and c_s will be inherited to f_{nl} through the action (37). For example, the equilateral non-Gaussianity is calculated when all the fluctuation modes exit the horizon; therefore, the f_{nl} will be dependent on the total wave number

$K \equiv k_1 + k_2 + k_3$. For large K, it also brings the danger of breaking the perturbation theory [32].

(2) The dominant term will be different. From (37), we can see that each term has a different power of ϵ; therefore, for $\epsilon \gg 1$ and $\epsilon \ll 1$, the dominant term will be different, which will give rise to different bispectra $\mathcal{B}(k_1, k_2, k_3)$. Moreover, while the usual slow-roll inflation mainly generates equilateral non-Gaussianity, for nonattractor inflation with $c_s \neq 1$, large local non-Gaussianity will also be generated, dominating over the equilateral one [51]. Therefore, the detection of the non-Gaussianities will be a good probe into the early Universe models.

VII. CONCLUSIONS

In this paper, we discussed cosmological models of the early Universe, in the framework of GR, but relaxed other parameters such as the slow-roll parameter and sound speed to be varying quantities. It was found that, provided those parameters behave under certain relations, the model will give the same spectral index. Based on the adiabatic mechanism of perturbation production, those relations can be viewed as the adiabatic duality relationship that links different models together.

For models in which only slow-roll parameter ϵ is varying, we found that there are four possible duality relationships between the parameter η, which is the power-law index of ϵ (also known as the second slow-roll parameter), namely, $\eta + \tilde{\eta} = 6$, $\eta + \tilde{\eta} = 4$, $\eta + \tilde{\eta} = 2$ as well as $|\eta - \tilde{\eta}| = 2$, depending on the evolution trend of ϵ. However, considering the requirement that the scalar power spectrum must be nearly scale invariant, we found that only four kinds of models as well as the matter bounce model could dual to each other. Moreover, when the tensor power spectrum and spectral index are taken into account, the duality relation will be broken.

We also extended the discussion to the wider case, where the sound speed c_s is varying as well. In this case, there are two duality relationships, namely, $\eta = -s$, $\tilde{\eta} = 5\tilde{s} + 6$ and $\eta = -s - 2$, $\tilde{\eta} = 5\tilde{s} + 4$, but the models dual to each other get enlarged; even scale invariance of scalar perturbation is still required. Moreover, the spectral index of tensor spectrum can also be dual, although the amplitude of the tensor spectrum cannot. Therefore, in contrast to conformal duality, the adiabatic duality might not be a full duality of early-time cosmological models. Although all the models can be made within the current bound of tensor perturbations, the future GW detectors may have the power to differentiate these models on the observational side. We checked all the above results by performing numerical calculations. We also mentioned that, via some specific mechanisms, we may have a chance to have full duality among cosmological models. However, these mechanisms seem to be model dependent and have to be studied case by case.

We would like to demonstrate that our analysis has been from a very general ansatz solution, namely, $\epsilon \sim (-\tau)^\eta$. As long as this condition is not deviated too much, our results can be applied. In this sense, the duality relations are somehow insensitive to the specific initial conditions for the scalar field and its velocity. However, the test of non-Gaussianities will be a good filter to the early Universe models, for it can break the duality in various aspects. It will be interesting to investigate in a more thorough way in future work.

Before ending, let us also mention an interesting extension of our study, which is to the multiple-field models. For multiple-field models, the primordial perturbations consist of not only adiabatic modes but also isocurvature modes. Therefore, the scale-invariant power spectrum of curvature perturbations is sourced by the isocurvature ones, which is known as the "isocurvature mechanism" [55,56]. Currently, we still do not know whether in this case there is still duality among different scenarios, and even if it does exist, the relationship must be very different, which we can dub as "isocurvature duality" as a counterpart of the current duality. We will leave the related discussions for a separate work.

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