Multiplicity distribution of dipoles in QCD from the Le-Mueller-Munier equation

Eugene Levin[®]

Department of Particle Physics, School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Science, Tel Aviv University, Tel Aviv 69978, Israel and Departamento de Física, Universidad Técnica Federico Santa María and Centro Científico-Tecnológico de Valparaíso, Casilla 110-V, Valparaiso, Chile

(Received 5 July 2021; accepted 1 September 2021; published 28 September 2021)

In this paper we derived in QCD the Balitsky-Fadin-Kuraev-Lipatov (BFKL) linear, inhomogeneous equation for the factorial moments of multiplicity distribution (M_k) from Le-Mueller-Munier equation. In particular, the equation for the average multiplicity of the color-singlet dipoles (N) turns out to be the homogeneous BFKL while $M_k \propto N^k$ at small x. Second, using the diffusion approximation for the BFKL kernel we show that the factorial moments are equal to $M_k = k!N(N-1)^{k-1}$ which leads to the multiplicity distribution $\frac{\sigma_n}{\sigma_{in}} = \frac{1}{N} (\frac{N-1}{N})^{n-1}$. We also suggest a procedure for finding corrections to this multiplicity distribution which will be useful for descriptions of the experimental data.

DOI: 10.1103/PhysRevD.104.056025

I. INTRODUCTION

During the past several years a robust relation between the principle features of high energy scattering and entanglement properties of the hadronic wave function have been in focus of the high energy and nuclear physics communities [1-19]. In this paper, we continue to explore the relation between the entropy in the parton approach [20–23] and the entropy of entanglement in a proton wave function [5]. In Ref. [5], it was proposed that parton distributions can be defined in terms of the entropy of entanglement between the spatial region probed by deep inelastic scattering (DIS) and the rest of the proton. This approach leads to a simple relation $S = \ln N$ between the average number of color-singlet dipoles and the entropy of the produced hadronic state S. This simple relation shows that a proton becomes a maximally entangled state in the region of small Bjorken x. All these conclusions were made from estimates in the simple, even naive model for QCD cascade of color-singlet dipoles. However, it has been demonstrated in Refs. [6,10,11,16,19] that these ideas are in qualitative and, partly, in quantitative agreement with the available experimental data. Actually, it is shown in Ref. [5] that the simple cascade of color-singlet dipoles leads to the multiplicity distribution:

$$\frac{\sigma_n}{\sigma_{\rm in}} = \frac{1}{N} \left(\frac{N-1}{N} \right)^{n-1} \tag{1}$$

where N is the average number of dipoles.

The goal of this paper is to study the multiplicity distribution and the entanglement entropy in the effective theory for QCD at high energies (see Ref. [24] for a general review). We have approached this problem in Refs. [5,18] and have demonstrated that Eq. (1) arises in QCD cascades. In this paper we analyze the multiplicity distribution for Balitsky-Kovchegov (BK) cascade [25] in which one dipole at low energy generates a large number of color-singlet dipoles at high energy. The equations for such a cascade are known (see Refs. [24,26–28]) and the first try to solve them have been undertaken in Ref. [18]. However, in this paper we return to this problem and study the multiplicity distribution using the new equation [Le-Mueller-Munier (LMM) equation] for the probability generating function that has been derived in Ref. [29].

In the next section we derive the LMM equation from the equation for the BK parton cascade. In the rest of the paper we discuss the equations for the factorial moments that follow from the LMM equation. We show that every factorial moment satisfies the linear but inhomogeneous equation with the Balitsky-Fadin-Kuraev-Lipatov (BFKL) kernel [30,31]. We attempt to solve these equations and demonstrate that in the diffusion approximation to the BFKL kernel factorial moments are equal to

$$M_k = k! N(N-1)^{k-1}.$$
 (2)

We show that Eq. (2) leads to Eq. (1).

^{*}leving@tauex.tau.ac.il, eugeny.levin@usm.cl

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Funded by SCOAP³.

In Sec. VI we suggest an approach to go beyond diffusion approximation, which cannot give a reliable description of the experimental data even in the leading order of perturbative QCD. In this approach we propose to solve exactly the equations for the factorial moments and using the difference between the exact solution and Eq. (2) $[\Delta M_k = M_k(\text{exact}) - M_k(\text{Eq. (2)})]$ we develop the way to estimate the multiplicity distributions beyond diffusion approximation. In the conclusion section we summarize our results.

II. GENERAL FEATURES OF THE CASCADE OF COLOR-SINGLET DIPOLES IN QCD

In QCD at large number of colors N_c ($N_c \gg 1$) the color-singlet dipoles play the role of partons (see Ref. [24] for review). As discussed in Refs. [24,26–28] for them we can write the following equations:

$$\frac{\partial P_n(Y, \boldsymbol{r}, \boldsymbol{b}; \boldsymbol{r}_1, \boldsymbol{b}_1, \boldsymbol{r}_2, \boldsymbol{b}_2 \dots \boldsymbol{r}_i, \boldsymbol{b}_i, \dots, \boldsymbol{r}_n, \boldsymbol{b}_n)}{\partial Y} = -\sum_{i=1}^n \omega_G(r_i) P_n(Y, \boldsymbol{r}, \boldsymbol{b}; \boldsymbol{r}_1, \boldsymbol{b}_1, \boldsymbol{r}_2, \boldsymbol{b}_2 \dots \boldsymbol{r}_i, \boldsymbol{b}_i, \dots, \boldsymbol{r}_n, \boldsymbol{b}_n) + \bar{\alpha}_S \sum_{i=1}^{n-1} \frac{(\boldsymbol{r}_i + \boldsymbol{r}_n)^2}{(2\pi)r_i^2 r_n^2} P_{n-1}(Y, \boldsymbol{r}, \boldsymbol{b}; \boldsymbol{r}_1, \boldsymbol{b}_1, \dots, (\boldsymbol{r}_i + \boldsymbol{r}_n), \boldsymbol{b}_{in}, \dots, \boldsymbol{r}_{n-1}, \boldsymbol{b}_n)$$
(3)

where $P_n(Y; \{r_i, b_i\})$ is the probability to have *n* dipoles of size r_i , at impact parameter b_i and at rapidity *Y*.¹ \boldsymbol{b}_{in} in Eq. (3) is equal to $\boldsymbol{b}_{in} = \boldsymbol{b}_i + \frac{1}{2}\boldsymbol{r}_i = \boldsymbol{b}_n - \frac{1}{2}\boldsymbol{r}_i$. $\omega_G(r)$ is defined below in Eq. (8b).

Equation (3) is a typical cascade equation in which the first term describes the reduction of the probability to find n dipoles due to the possibility that one of n dipoles can decay into two dipoles of arbitrary sizes, while the second term describes the growth due to the splitting of (n - 1) dipoles into n dipoles.

The initial condition for the DIS scattering is

$$P_1(Y = 0, \boldsymbol{r}, \boldsymbol{b}; \boldsymbol{r}_1, \boldsymbol{b}_1) = \delta^{(2)}(\boldsymbol{r} - \boldsymbol{r}_1)\delta^{(2)}(\boldsymbol{b} - \boldsymbol{b}_1); \qquad P_{n>1}(Y = 0; \{r_i\}) = 0$$
(4)

which corresponds to the fact that we are discussing a dipole of definite size which develops the parton cascade. Since $P_n(Y; \{r_i\})$ is the probability to find dipoles $\{r_i\}$, we have the following sum rule:

$$\sum_{n=1}^{\infty} \int \prod_{i=1}^{n} d^2 r_i d^2 b_i P_n(Y; \{ \boldsymbol{r}_i \boldsymbol{b}_i \}) = 1,$$
(5)

i.e., the sum of all probabilities is equal to 1.

This QCD cascade leads to the Balitsky-Kovchegov (BK) equation [24,25] for the amplitude and gives the theoretical description of the DIS. We introduce the generating functional [26]

$$Z(Y, \boldsymbol{r}, \boldsymbol{b}; [\boldsymbol{u}_i]) = \sum_{n=1}^{\infty} \int P_n(Y, \boldsymbol{r}, \boldsymbol{b}; \{\boldsymbol{r}_i \boldsymbol{b}_i\}) \prod_{i=1}^n u(\boldsymbol{r}_i \boldsymbol{b}_i) d^2 r_i d^2 b_i$$
(6)

where $u(\mathbf{r}_i \mathbf{b}_i) \equiv u_i$ is an arbitrary function. The initial conditions of Eq. (4) and the sum rules of Eq. (5) take the following form for the functional *Z*:

$$Z(Y = 0, \boldsymbol{r}, \boldsymbol{b}; [\boldsymbol{u}_i]) = \boldsymbol{u}(\boldsymbol{r}, \boldsymbol{b});$$
(7a)

$$Z(Y, r, [u_i = 1]) = 1.$$
(7b)

Multiplying both parts of Eq. (3) by $\prod_{i=1}^{n} u(\mathbf{r}_i \mathbf{b}_i)$ and integrating over r_i and b_i we obtain the following linear functional equation [28];

$$\frac{\partial Z(Y, \boldsymbol{r}, \boldsymbol{b}; [u_i])}{\partial y} = \int d^2 \boldsymbol{r}' K(\boldsymbol{r}', \boldsymbol{r} - \boldsymbol{r}' | \boldsymbol{r}) \left(-u(r, b) + u\left(\boldsymbol{r}', \boldsymbol{b} + \frac{1}{2}(\boldsymbol{r} - \boldsymbol{r}')\right) u\left(\boldsymbol{r} - \boldsymbol{r}', \boldsymbol{b} + \frac{1}{2}\boldsymbol{r}'\right) \right) \frac{\delta Z}{\delta u(r, b)}; \quad (8a)$$

¹In the lab frame rapidity Y is equal to $Y = y_{\text{dipole r}} - y_{\text{dipole r}}$, where $y_{\text{dipole r}}$ is the rapidity of the incoming fast dipole and $y_{\text{dipole r}}$ is the rapidity of dipoles r_i . Note that all rapidities of dipoles r_i are the same in Eq. (3).

$$K(\mathbf{r}', \mathbf{r} - \mathbf{r}'|\mathbf{r}) = \frac{1}{2\pi} \frac{r^2}{r'^2 (\mathbf{r} - \mathbf{r}')^2}; \qquad \omega_G(r) = \int d^2 r' K(\mathbf{r}', \mathbf{r} - \mathbf{r}'|\mathbf{r});$$
(8b)

where $y = \bar{\alpha}_S Y$.

Searching for the solution of the form $Z([u(r_i, b_i, Y)])$ for the initial conditions of Eqs. (7a) and (8a) can be rewritten as the nonlinear equation [26]:

$$\frac{\partial Z(Y, \boldsymbol{r}, \boldsymbol{b}; [u_i])}{\partial y} = \int d^2 \boldsymbol{r}' K(\boldsymbol{r}', \boldsymbol{r} - \boldsymbol{r}' | \boldsymbol{r}) \bigg\{ Z\bigg(\boldsymbol{r}', \boldsymbol{b} + \frac{1}{2}(\boldsymbol{r} - \boldsymbol{r}'); [u_i]\bigg) Z\bigg(\boldsymbol{r} - \boldsymbol{r}', \boldsymbol{b} + \frac{1}{2}\boldsymbol{r}'; [u_i]\bigg) - Z(Y, \boldsymbol{r}, \boldsymbol{b}; [u_i])\bigg\}.$$
(9)

Therefore, the QCD parton cascade of Eq. (3) takes into account nonlinear evolution.

III. DERIVATION OF THE LE-MUELLER-MUNIER EQUATION

In this section we derive the LMM equation which is proposed in Ref. [29]. First, we introduce the same notations as in Ref. [29]:

$$\tilde{w}_n(r, b, y) = \int \prod_{i=1}^n d^2 r_i d^2 b_i P_n(Y; \{r_i b_i\}).$$
(10)

One can see that $\tilde{w}_n(r, b, y)$ is the probability that the dipole with size *r* produces *n* dipoles with all possible sizes. Equation (7b) reads as

$$\sum_{n=1}^{\infty} \tilde{w}_n(r, b, y) = 1.$$
(11)

Taking all $u_i(r_i, b_i) = \lambda$ one can see that we can rewrite Eq. (6) in the form:

$$Z(Y, \boldsymbol{r}, \boldsymbol{b}; [u_i = \lambda]) \equiv \tilde{w}_{\lambda}(\boldsymbol{r}, \boldsymbol{b}, y) = \sum_{n=1}^{\infty} \lambda^n \tilde{w}_n(\boldsymbol{r}, \boldsymbol{b}, y).$$
(12)

Plugging Eq. (17) into Eq. (9) we obtain the LMM equation in the form:

$$\frac{\partial \tilde{w}_{\lambda}(\boldsymbol{r},\boldsymbol{b},\boldsymbol{y})}{\partial \boldsymbol{y}} = \int d^2 \boldsymbol{r}' \boldsymbol{K}(\boldsymbol{r}',\boldsymbol{r}-\boldsymbol{r}'|\boldsymbol{r}) \bigg\{ \tilde{w}_{\lambda} \bigg(\boldsymbol{r}',\boldsymbol{b}+\frac{1}{2}(\boldsymbol{r}-\boldsymbol{r}'),\boldsymbol{y}\bigg) \tilde{w}_{\lambda} \bigg(\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{b}+\frac{1}{2}\boldsymbol{r}',\boldsymbol{y}\bigg) - \tilde{w}_{\lambda}(\boldsymbol{r},\boldsymbol{b},\boldsymbol{y}) \bigg\}.$$
(13)

In addition, discussing multiplicity distribution $P_n = \frac{\sigma_n}{\sigma_{iot}}$, where σ_n is the cross section for production of *n* color-singlet dipoles, we need to integrate $w_n(r, b, y)$ over *b*. In this case the initial condition for the dipole cascade takes the form

$$P_1(Y=0, \mathbf{r}; \mathbf{r}_1) = \delta^{(2)}(\mathbf{r} - \mathbf{r}_1), \qquad P_{n>1}(Y=0; \{r_i\}) = 0$$
(14)

which leads to the probabilities that do not depend on impact parameters. Since in Eq. (3) *b* enters as a parameter, $P_n(Y; r_i)$, which does not depend on b_i , is also a solution to Eq. (3), which satisfies Eq. (14).

Equation [29] reduces to

$$\frac{\partial \tilde{w}_{\lambda}(\boldsymbol{r}, y)}{\partial y} = \int d^2 \boldsymbol{r}' K(\boldsymbol{r}', \boldsymbol{r} - \boldsymbol{r}' | \boldsymbol{r}) \{ \tilde{w}_{\lambda}(\boldsymbol{r}', y) \tilde{w}_{\lambda}(\boldsymbol{r} - \boldsymbol{r}', y) - \tilde{w}_{\lambda}(\boldsymbol{r}, y) \}$$
(15)

Equation (15) is a particular case of the general equation that has been derived in Ref. [29]. In this paper instead of Eq. (10) the more general form of this equation is proposed, viz.:

$$w_n(r, b, y, y_0) = \int \prod_{i=1}^n (d^2 r_i d^2 b_i S(r_i, b_i, y_0)) P_n(Y; \{\boldsymbol{r}_i \boldsymbol{b}_i\})$$
(16)

where $S(r_i, b_i, y_0)$ is the scattering *S*-matrix for elastic interaction of the dipole with size r_i at rapidity $Y_0(y_0 = \bar{\alpha}_S Y_0)$ and at impact parameter b_i with the target at Y = 0. Since *S* is a unitarity matrix, $w_n(r, b, y; y_0)$ is the probability that the dipole with size *r* produces *n* dipoles with all possible sizes, which interact with the target. Bearing this in mind, we see that Eq. (11) holds for $w_\lambda(r, b, y; y_0)$, which is defined as

$$Z(Y, \boldsymbol{r}, \boldsymbol{b}; [\boldsymbol{u}_i = \lambda S(\boldsymbol{r}_i, \boldsymbol{b}_i, \boldsymbol{y}_0)]) \equiv w_{\lambda}(\boldsymbol{r}, \boldsymbol{b}, \boldsymbol{y}, \boldsymbol{y}_0) = \sum_{n=1}^{\infty} \lambda^n w_n(\boldsymbol{r}, \boldsymbol{b}, \boldsymbol{y}, \boldsymbol{y}_0).$$
(17)

For the case of P_n which does not depend on b_i , inserting in Eq. (9) $u_i(r_i, b_i,) = \lambda S(r_i, b_i, y_0)$ we see that we obtain the LMM equation in its original form (see Ref. [29]):

$$\frac{\partial w_{\lambda}(\boldsymbol{r}, \boldsymbol{b}, \boldsymbol{y}, \boldsymbol{y}_{0})}{\partial \boldsymbol{y}} = \int d^{2}\boldsymbol{r}' \boldsymbol{K}(\boldsymbol{r}', \boldsymbol{r} - \boldsymbol{r}' | \boldsymbol{r}) \{ w_{\lambda}(\boldsymbol{r}', \boldsymbol{y}, \boldsymbol{y}_{0}) w_{\lambda}(\boldsymbol{r} - \boldsymbol{r}', \boldsymbol{y}, \boldsymbol{y}_{0}) - w_{\lambda}(\boldsymbol{r}, \boldsymbol{y}, \boldsymbol{y}_{0}) \}.$$
(18)

IV. AVERAGE NUMBER OF COLOR-SINGLET DIPOLES

The average number of dipoles can be calculated using the following formula:

$$N(\mathbf{r}, y, y_0) = \langle |n| \rangle = \sum_{n=1}^{\infty} n w_n(r, y, y_0) = \frac{\partial w_\lambda(\mathbf{r}, y, y_0)}{\partial \lambda} \Big|_{\lambda=1}.$$
(19)

Differentiating Eq. (13) with respect to λ we obtain that

$$\frac{\partial N(\mathbf{r}, y, y_0)}{\partial y} = \int d^2 \mathbf{r}' K(\mathbf{r}', \mathbf{r} - \mathbf{r}' | \mathbf{r}) \{ N(\mathbf{r}', y, y_0) + N(\mathbf{r} - \mathbf{r}', y, y_0) - N(\mathbf{r}, y, y_0) \}.$$
(20)

Equation (20) shows that the average number of dipoles satisfies the linear BFKL [30,31] equation and increases in the region of small x (large y). Therefore, we see that the general QCD cascade reproduces the main observation of Ref. [5] which was made in the oversimplified model for the QCD cascade. In this model the dependence on the size of the dipoles was neglected.

The general solution takes the following form:

$$N(\mathbf{r}, \mathbf{y}, \mathbf{y}_0) = \int_{e-i\infty}^{e+i\infty} \frac{d\gamma}{2\pi i} e^{\chi(\gamma)\mathbf{y} + \gamma\xi} n_{in}(\gamma, \mathbf{y}_0)$$
(21)

where $\xi = \ln(\frac{1}{r^2})$ and $\chi(\gamma)$ is the BFKL kernel:

$$\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma) = \begin{cases} \frac{1}{\gamma} & \text{for } \gamma \to 0 \leftarrow \text{double log approximation (DLA);} \\ \underbrace{4\ln 2}_{\omega_0} + \underbrace{14\zeta(3)}_{D} \left(\gamma - \frac{1}{2}\right)^2 & \text{for } \gamma \to \frac{1}{2} \leftarrow \text{diffusion approximation (DA);} \end{cases}$$
(22)

where $\psi(z)$ is the Euler ψ function (see Ref. [32] formula 8.36).

 $n_{in}(\gamma)$ has to be found from the initial condition $N(\mathbf{r}, \mathbf{b}, \gamma = 0, \gamma_0 = 0) = 1$ [see Eqs. (4) and (7a)]. It gives

$$n_{in}(\gamma) = \frac{1}{\gamma}.$$
(23)

Introducing multiplicity in the momentum representation,

$$N(k_T, y) = \int d^2 r e^{-ik_T \cdot r} \frac{N(r, y)}{r^2},$$
(24)

we can rewrite Eq. (20) in the form

$$\frac{\partial N(\boldsymbol{k}_T, \boldsymbol{y}, \boldsymbol{y}_0)}{\partial \boldsymbol{y}} = \int \frac{d^2 \boldsymbol{k}_T}{(2\pi)^2} K(\boldsymbol{k}_T, \boldsymbol{k}_T') N(\boldsymbol{k}_T', \boldsymbol{y}, \boldsymbol{y}_0)$$
(25)

where $K(k_T, k'_T)$ is the BFKL kernel in momentum representation:

$$K(\boldsymbol{k}_{T},\boldsymbol{k}_{T}')N(\boldsymbol{k}_{T}',y,y_{0}) = \frac{1}{(\boldsymbol{k}_{T}-\boldsymbol{k}_{T}')^{2}}N(\boldsymbol{k}_{T}',y,y_{0}) - \frac{k_{T}^{2}}{(\boldsymbol{k}_{T}-\boldsymbol{k}_{T}')^{2}((\boldsymbol{k}_{T}-\boldsymbol{k}_{T}')^{2}+\boldsymbol{k}_{T}'^{2})}N(\boldsymbol{k}_{T},y,y_{0}).$$
(26)

Solution to this equation has the same form of Eq. (21) but with the replacement of $\xi \to \xi' = \ln k_T^2$ and

$$n_{in}(\gamma) = \chi(\gamma)/\gamma. \tag{27}$$

Equation (27) reproduces the value of $N(k_T, y = 0, y_0 = 0)$, since Eq. (24) at y = 0 leads to $N(k_T, y) = \ln k_T^2 + O(1/k_T^2)$. It is worth mentioning that (i) in the double log approximation $n_{in}(\gamma) = 1/\gamma^2$, which leads to $N(k_T, y) = \ln k_T^2$ and (ii) in semiclassical approximation, which we will use below, we can neglect all corrections of the order of $1/k_T^2$.

V. EQUATIONS FOR MOMENTS OF THE MULTIPLICITY DISTRIBUTION

A. The second moment

1. Equation

We start a derivation of the evolution equation for the moments of the multiplicity distributions considering the second moment, which has the following form:

$$M_{2}(\mathbf{r}, y, y_{0}) = \langle |n(n-1)| \rangle = \sum_{n=1}^{\infty} n(n-1)w_{n}(r, y, y_{0}) = \frac{\partial^{2}w_{\lambda}(\mathbf{r}, y, y_{0})}{\partial\lambda^{2}} \Big|_{\lambda=1}.$$
(28)

Taking the second derivative with respect to λ from Eq. (18) we obtain the equation for $M_2(\mathbf{r}, y, y_0)$

$$\frac{\partial M_2(\mathbf{r}, y, y_0)}{\partial y} = \int d^2 \mathbf{r}' K(\mathbf{r}', \mathbf{r} - \mathbf{r}' | \mathbf{r}) \{ M_2(\mathbf{r}', y, y_0) + M_2(\mathbf{r} - \mathbf{r}', y, y_0) + 2N(\mathbf{r}', y, y_0)N(\mathbf{r} - \mathbf{r}', y, y_0) - M_2(\mathbf{r}, y, y_0) \}.$$
(29)

Equation (29) is a linear but inhomogeneous equation with the inhomogeneous term, which is determined by the multiplicity of the dipoles.

The initial condition [see Eqs. (4) and (7a)] for this equation is

$$M_2(\mathbf{r}, y = y_0, y_0) = 0. \tag{30}$$

First, lets us start to solve Eq. (29) making the first iteration at small $y = \Delta y$. For y = 0 N(r, y = 0) = 1 and $M_2(r, y = 0) = 0$, and hence

$$M_2^{(1)} = 2\Delta y \omega_G(r) \tag{31}$$

where ω is given by Eq. (8b). The first iteration of Eq. (20) leads to $N^{(1)} = 1 + \Delta y \omega_G(r)$. Comparing these two estimates one can see that the first iteration can be written as the expansion of the solution $M_2(r, y) = 2(N^2(r, y) - N(r, y))$ with respect to Δy . Hence, we see that at small Δy we obtain the simple expression for M_2 , which turns out to be the same as for the multiplicity distribution of Eq. (1) for the simple toy model [5]. We will try to prove this equation below, but we have succeeded only in the semiclassical approximation.

2. General solution

The general solution to Eq. (29) we can obtain going to momentum representation:

$$m_2(k_T, y) = \int \frac{d^2 r}{(2\pi)^2} e^{-ik_T \cdot r} \frac{M_2(r, y)}{r^2}.$$
(32)

Equation (29) takes the form

$$\frac{\partial m_2(\mathbf{k}_T, y, y_0)}{\partial y} = \int \frac{d^2 k_T}{(2\pi)^2} K(\mathbf{k}_T, \mathbf{k}_T') m_2(\mathbf{k}_T', y, y_0) + 2N^2(\mathbf{k}_T, y, y_0).$$
(33)

Taking the Mellin transform,

$$m_2(\mathbf{k}_T, y, y_0) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} e^{\gamma \xi'} \tilde{m}_2(\gamma, y, y_0) \quad \text{where } \xi' = \ln k_T^2$$
(34)

from both parts of Eq. (33) we obtain

$$\frac{\partial \tilde{m}_2(\gamma, y, y_0)}{\partial y} = \chi(\gamma) \tilde{m}_2(\gamma, y, y_0) + 2\tilde{n^2}$$
(35)

where $\chi(\gamma)$ is given by Eq. (8b) and $\tilde{n^2}$ denotes the Mellin image of $N^2(k_T, b, y, y_0)$.

Equation (35) has the following solution which satisfies the initial condition of Eq. (30):

$$\tilde{m}_{2}(\gamma, y, y_{0}) = e^{\chi(\gamma)y} \int_{0}^{y} dy' \, 2\tilde{n^{2}} e^{-\chi(\gamma)y'} + \tilde{m}_{2}^{\text{BFKL}}(\gamma, y, y_{0})$$
(36)

where $\tilde{m}_2^{\text{BFKL}}(\gamma, y, y_0)$ is a solution to the homogeneous linear BFKL equation with the initial condition of Eq. (30). In the following, we neglect the contribution of this term.

3. Semiclassical solution

For large y and ξ we can use the semiclassical approximation (SCA, see Refs. [24,33] and references therein) to take the integral over y' in Eq. (36). In this approximation we are searching for

$$N = e^{S_N} = e^{\omega(\xi', y)y + \gamma(y, \xi')\xi'}$$
(37)

where $\omega(\xi', y)$ and $\gamma(y, \xi')$ are smooth functions of y and $\xi': \partial \omega'(\xi', y)/\partial y \ll \omega^2(\xi', y), \partial \omega(\xi', y)/\partial \xi' \ll \omega(\xi', y)\gamma(y, \xi'),$ $\partial \gamma(\xi', y)/\partial \xi' \ll \gamma^2(\xi', y), \partial \gamma(\xi', y)/\partial \xi' \ll \omega(\xi', y)\gamma(y, \xi')$. Such form of N stems from Eq. (21) if we use the method of steepest descent for calculating the integral over γ . Indeed, using this method one can see that

$$\omega(\xi', y) = \chi(\gamma_{\rm SP}); \qquad \gamma(\xi', y) = \gamma_{\rm SP}; \qquad \text{equation for } \gamma_{\rm SP}: \left. \frac{d\chi(\gamma)}{d\gamma} \right|_{\gamma = \gamma_{\rm SP}} y = \xi'; \tag{38}$$

In the SCA the Mellin image of N^2 can be written as follows:

$$N^{2}(\xi', y, y_{0}) = e^{2S_{N}} = e^{2\omega(\xi', y)y + 2\gamma(y, \xi')\xi'} = \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\gamma}{2\pi i} e^{2\chi(\frac{1}{2}\gamma)y + \gamma\xi'} H(\gamma)$$
(39)

Indeed, taking the integral by the method of steepest descent we obtain the following equation for the saddle point $(\gamma_{SP}^{(2)})$:

$$2\frac{d\chi(\frac{1}{2}\gamma)}{d\gamma}\Big|_{\gamma=\gamma_{\rm SP}^{(2)}} y = \xi'; \tag{40}$$

with the solution $\gamma_{SP}^{(2)} = 2\gamma_{SP}$, where γ_{SP} is given by Eq. (38). Plugging this solution into Eq. (39) we see that

$$N^{2}(\xi', y, y_{0}) = e^{2\chi(\frac{1}{2}\gamma_{SP}^{(2)})y + \gamma_{SP}^{(2)}\xi'} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\Delta\gamma}{2\pi i} e^{\frac{d^{2}\chi(\frac{1}{2}\gamma_{SP}^{(2)})}{d\gamma}y(\Delta\gamma)^{2}} H(\gamma_{SP}^{(2)})$$
$$= e^{2\chi(\gamma_{SP})y + 2\gamma_{SP}\xi'} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\Delta\gamma}{2\pi i} e^{\frac{d^{2}\chi(\gamma_{SP})}{d\gamma}y(\Delta\gamma)^{2}} H(2\gamma_{SP}).$$
(41)

The integral over $\Delta \gamma$ leads to a smooth function, which in the SCA can be considered as a constant. Therefore, comparing Eq. (39) and Eq. (41) one can see that Eq. (39) is correct. We can derive Eq. (39) using a more general consideration. Actually, the expression for the Mellin transform of $N^2(\xi', y, y_0)$ is the convolution in γ of Mellin images of N, which has the following form:

$$\widetilde{n^2} = \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\gamma'}{2\pi i} n_{in}(\gamma') n_{in}(\gamma-\gamma') e^{(\chi(\gamma')+\chi(\gamma-\gamma'))y}.$$
(42)

Taking the integral over γ' using the method of steepest descent one can see that the equation for the saddle point has the following form:

$$\frac{d\chi(\gamma - \gamma')}{d\gamma'} + \frac{d\chi(\gamma')}{d\gamma'} = 0$$
(43)

with the solution $\gamma'_{SP} = \frac{1}{2}\gamma$. Plugging this solution in Eq. (42) we reduce it to

$$\widetilde{n^2} = e^{2\chi(\frac{1}{2}\gamma)y} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\Delta\gamma'}{2\pi i} n_{in}^2 \left(\frac{1}{2}\gamma\right) e^{\frac{d^2\chi(\gamma'_{SP})}{d\gamma'_{SP}}y(\Delta\gamma')^2}$$
(44)

which reproduces the Mellin transform of Eq. (39).

Plugging in Eq. (36) $\tilde{n^2} = e^{2\chi(\frac{1}{2}\gamma)y}H(\gamma)$ we can take the integral over y' and the solution has the form:

$$\tilde{m}_{2}(\gamma, y, y_{0}) = e^{\chi(\gamma)y}H(\gamma) \int_{0}^{y} dy' 2\tilde{n^{2}}e^{-\chi(\gamma)y'} = \frac{2}{2\chi(\frac{1}{2}\gamma) - \chi(\gamma)} \{e^{2\chi(\frac{1}{2}\gamma)y}H(\gamma) - e^{\chi(\gamma)y}H(\gamma)\}$$
(45)

Note, that in Eq. (45) we neglected the contribution of $\tilde{m}_2^{\text{BFKL}}(\gamma, y, y_0)$ in Eq. (36).

Before fixing $H(\gamma)$ we need to go back to coordinate representation. Indeed, in this representation we have simple initial conditions for M_2 of Eq. (30). Since all solutions are solutions of the linear equations and $\gamma_{SP} \ll 1$, we can replace $\xi' = \ln k_T^2$ by $\xi = \ln \frac{1}{r^2}$. Bearing this in mind, we can reduce the solution to the form

$$M_2(\xi, y, y_0) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} \frac{2\chi(\gamma)}{2\chi(\frac{1}{2}\gamma) - \chi(\gamma)} \left\{ e^{2\chi(\frac{1}{2}\gamma)y} - e^{\chi(\gamma)y} \right\} \frac{1}{\gamma} e^{\gamma\xi}.$$
(46)

First, we note that taking by the integral over γ using the method of steepest descent, we reproduce Eq. (45) with the particular choice of $H(\gamma) = \chi(\gamma)/\gamma$, which has been discussed in Eq. (27). Second, one can see that at $\gamma \to 0$ this solution coincides with Eq. (31).

In DLA this solution takes the form [see Eq. (22)]:

$$M_{2}^{\text{DLA}}(\xi', y, y_{0}) = \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\gamma}{2\pi i \, 3} \{e^{\frac{4}{\gamma}y} - e^{\frac{1}{\gamma}y}\} \frac{1}{\gamma} e^{\gamma\xi'}$$
(47a)

$$= \frac{2}{3} \{ N_{\text{SCA}}^2(r, y, y_0) - N_{\text{SCA}}(r, y, y_0) \} \leftarrow \text{with semiclassical accuracy.}$$
(47b)

However, it turns out that Eq. (47a) reproduces at least two terms of the expansion at small values of y of the following relation:



$$M_{2}[\text{Eq. } (47a)] = 2(N^{2}(r, y, y_{0}) - N(r, y, y_{0}))$$
$$= 2y\xi + \frac{3}{2}(y\xi)^{2} + \mathcal{O}((y\xi)^{3}).$$
(48)

Concluding, we see that Eq. (2) does not hold in the DLA, but Eq. (48) gives us some hope to find an approach in which it will be correct.

4. Diffusion approximation

Actually, the most adequate approach at high energies is the diffusion one [see Eq. (22)]. Plugging in Eq. (46) the BFKL kernel in the form $\chi(\gamma) = \omega_0 + D(\gamma - \frac{1}{2})^2$ and taking the integral using the method of steepest descent we obtain the saddle point

$$\gamma_{SP} = \frac{1}{2} + i \frac{\xi}{2Dy}; \qquad 2\chi \left(\frac{1}{2}\gamma_{SP}\right) = 2\omega_0 - \frac{\xi^2}{2Dy}; \qquad \chi(\gamma_{SP}) = \omega_0 - \frac{\xi^2}{4Dy}; \tag{49}$$

Considering $\frac{\xi}{2Dy} \ll 1$ we can neglect their contributions in the factor $\frac{2\chi(\gamma)}{2\chi(\frac{1}{2}\gamma)-\chi(\gamma)}$ reducing it to 2. Hence, Eq. (46) reads

$$M_2^{\rm DA}(r, y, y_0) = 2(N^2(r, y, y_0) - N(r, y, y_0)).$$
(50)

We can obtain the solution of Eq. (50) directly from the equation for M_2 [see Eq. (29)], if we note that the BFKL kernel has maxima at $r' \to 0$ and $|\mathbf{r} - \mathbf{r}'| \to 0$. In Fig. 1 we plot the term of Eq. (29), which is proportional to N^2 :

$$\int d^2 \mathbf{r}' K(\mathbf{r}', \mathbf{r} - \mathbf{r}' | \mathbf{r}) 2N(\mathbf{r}', y, y_0) N(\mathbf{r} - \mathbf{r}', y, y_0) \propto \int d^2 \mathbf{r}' K(\mathbf{r}', \mathbf{r} - \mathbf{r}' | \mathbf{r}) ((r'^2 (\mathbf{r} - \mathbf{r}')^2)^{\frac{1}{2}} = \int d\mathbf{r}' I\left(\tau = \frac{r'}{r}\right).$$
(51)

We can see from Fig. 1 that $I(\tau)$ has a maximum at $\tau = 1$. Note that in Eq. (51) we introduce $\gamma = \frac{1}{2}$, which corresponds to the DA, to estimates the value of this contribution.

Bearing this observation in mind, we can rewrite Eq. (29) in the following form:

$$\frac{\partial M_{2}(\mathbf{r}, y, y_{0})}{\partial y} = \int_{\mathbf{r}' < \mathbf{r}} d^{2}\mathbf{r}' K(\mathbf{r}', \mathbf{r} - \mathbf{r}'|\mathbf{r}) M_{2}(\mathbf{r}, y, y_{0}) + \int_{\mathbf{r}' < \mathbf{r}} d^{2}\mathbf{r}' K(\mathbf{r}', \mathbf{r} - \mathbf{r}'|\mathbf{r}) M_{2}(\mathbf{r}', y, y_{0}) \\
+ \int_{|\mathbf{r} - \mathbf{r}'| < \mathbf{r}} d^{2}\mathbf{r}' K(\mathbf{r}', \mathbf{r} - \mathbf{r}'|\mathbf{r}) M_{2}(\mathbf{r} - \mathbf{r}', y, y_{0}) + \int_{|\mathbf{r} - \mathbf{r}'| < \mathbf{r}} d^{2}\mathbf{r}' K(\mathbf{r}', \mathbf{r} - \mathbf{r}'|\mathbf{r}) M_{2}(\mathbf{r}, y, y_{0}) \\
+ 2 \int_{|\mathbf{r} - \mathbf{r}'| < \mathbf{r}} d^{2}\mathbf{r}' K(\mathbf{r}', \mathbf{r} - \mathbf{r}'|\mathbf{r}) N(\mathbf{r}, y, y_{0}) N(\mathbf{r} - \mathbf{r}', y, y_{0}) \\
+ 2 \int_{\mathbf{r}' < \mathbf{r}} d^{2}\mathbf{r}' K(\mathbf{r}', \mathbf{r} - \mathbf{r}'|\mathbf{r}) N(\mathbf{r}, y, y_{0}) N(\mathbf{r}', y, y_{0}) - \int d^{2}\mathbf{r} K(\mathbf{r}', \mathbf{r} - \mathbf{r}'|\mathbf{r}) M_{2}(\mathbf{r}, y, y_{0}). \tag{52}$$

One can see (i) that in Eq. (52) the terms, which are proportional to $M_2(\mathbf{r}, y, y_0)$, cancel each other out, and (ii) this equation can be presented in the form

$$\frac{\partial M_2(\mathbf{r}, y, y_0)}{\partial y} = \int d^2 \mathbf{r}' K(\mathbf{r}', \mathbf{r} - \mathbf{r}' | \mathbf{r}) \{ M_2(\mathbf{r}', y, y_0) + 2N(\mathbf{r}, y, y_0)N(\mathbf{r}', y, y_0) \}.$$
(53)

Substituting Eq. (50) into Eq. (53) we reduce it in the SCA to the equation

$$\frac{\partial 2(N^2 - N)}{\partial y} = 2\chi(\gamma_{SP})(2N^2 - N) = 2((\chi(2\gamma_{SP}) + \chi(\gamma_{SP}))N^2 - \chi(\gamma_{SP})N).$$
(54)

Hence, one can see that in the limit of small $\frac{\xi}{2Dy} \ll 1$, $\chi(2\gamma_{SP}) = \chi(\gamma_{SP})$ and Eq. (50) satisfies Eqs. (53) and (54). It is worth noting that Eq. (48) follows from the multiplicity distribution, which is given by Eq. (1). We will concentrate our efforts on DA in our presentation below.

B. The third moment

The third moment can be found from the generating function $\tilde{w}_{\lambda}(\mathbf{r}, \mathbf{b}, y, y_0)$ in the following way:

$$M_{3}(\mathbf{r}, y, y_{0}) = \langle |n(n-1)(n-2)| \rangle = \sum_{n=1}^{\infty} n(n-1)(n-2)w_{n}(r, y, y_{0}) = \frac{\partial^{3}w_{\lambda}(\mathbf{r}, y, y_{0})}{\partial\lambda^{3}} \Big|_{\lambda=1}.$$
(55)

Using Eq. (55) we obtain the equation for $M_3(\mathbf{r}, y, y_0)$ from Eq. (18):

$$\frac{\partial M_3(\mathbf{r}, y, y_0)}{\partial y} = \int d^2 \mathbf{r}' K(\mathbf{r}', \mathbf{r} - \mathbf{r}' | \mathbf{r}) \{ M_3(\mathbf{r}', y, y_0) + M_3(\mathbf{r} - \mathbf{r}', y, y_0) \} + 6M_2(\mathbf{r}', y, y_0)N(\mathbf{r} - \mathbf{r}', y, y_0) - M_3(\mathbf{r}, y, y_0) \}.$$

Rewriting Eq. (56) in momentum representation [see Eq. (32)] we reduce this equation to the form

$$\frac{\partial m_3(\mathbf{k}_T, y, y_0)}{\partial y} = \int \frac{d^2 \mathbf{k}_T'}{(2\pi)^2} K(\mathbf{k}_T, \mathbf{k}_T') m_3(\mathbf{k}_T', y, y_0) + 6m_2(\mathbf{k}_T, y, y_0) n(\mathbf{k}_T, y, y_0)$$
(56)

where we use lowercase letters denoting the moments in the momentum representation.

The Mellin image of Eq. (56) has the form

$$\frac{\partial \tilde{m}_3(\gamma, y, y_0)}{\partial y} = \chi(\gamma)\tilde{m}_3(\gamma, y, y_0) + \widetilde{6n^2n}$$
(57)

where $\widetilde{n^2 n}$ is the Mellin image of $m_2(\mathbf{k}_T, y, y_0)N(\mathbf{k}_T, y, y_0)$ which has the general form

$$\widetilde{n^2 n}(\gamma, y) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma'}{2\pi i} \widetilde{n^2}(\gamma', y) \widetilde{n}(\gamma - \gamma', y)$$
(58)

where $\widetilde{n^2}(\gamma, y)$ is determined by [see Eq. (46)]

$$\widetilde{n^2} = \frac{2\chi(\gamma)}{2\chi(\frac{1}{2}\gamma) - \chi(\gamma)} \left\{ e^{2\chi(\frac{1}{2}\gamma)y} - e^{\chi(\gamma)y} \right\} \frac{1}{\gamma}$$
(59)

Plugging Eqs. (21) and (59) into Eq. (58) one can see that $n^2 n(\gamma, y)$ is a sum of two terms. Taking the integral over γ' in each of them in the saddle point approximation [see Eqs. (42)–(44)] we obtain two equations for the saddle points:

$$2\frac{d\chi(\frac{1}{2}\gamma')}{d\gamma'} + \frac{d\chi(\gamma - \gamma')}{d\gamma'} = 0 \quad \text{with} \quad \gamma'_{SP} = \frac{2}{3}\gamma; \qquad \frac{d\chi(\gamma - \gamma')}{d\gamma'} + \frac{d\chi(\gamma')}{d\gamma'} = 0; \quad \text{with} \quad \gamma'_{SP} = \frac{1}{2}\gamma. \tag{60}$$

Hence, $\widetilde{n^2n}(\gamma, y)$ takes the form

$$\widetilde{n^2 n}(\gamma, y) = \frac{2\chi(\gamma)}{2\chi(\frac{1}{3}\gamma) - \chi(\frac{1}{2}\gamma)} \left\{ e^{3\chi(\frac{1}{3}\gamma)y} - e^{\chi(\frac{1}{2}\gamma)y} \right\} H_2(\gamma).$$
(61)

The solution does not depend on the form of $H_2(\gamma)$, which we will specify below.

Plugging this equation in Eq. (57) we obtain the solution

$$\tilde{m}_{3}(\gamma, y, y_{0}) = 6 \left(\frac{1}{3\chi(\frac{1}{3}\gamma) - \chi(\gamma)} \right) \left(\frac{2\chi(\gamma)}{2\chi(\frac{1}{3}\gamma) - \chi(\frac{1}{2}\gamma)} \right) \{ e^{3\chi(\frac{1}{3}\gamma)y} - e^{\chi(\gamma)y} \} \\ - \left(\frac{1}{2\chi(\frac{1}{3}\gamma) - \chi(\frac{1}{2}\gamma)} \right) \left(\frac{2\chi(\gamma)}{2\chi(\frac{1}{2}\gamma) - \chi(\gamma)} \right) \{ e^{2\chi(\frac{1}{3}\gamma)y} - e^{\chi(\gamma)y} \} H_{2}(\gamma).$$
(62)

Going to the coordinate representation as was discussed above and choosing $H_2(\gamma) = \chi(\gamma)/\gamma$, which reproduces the correct initial conditions we obtain

$$M_{3}(\xi, y, y_{0}) = 6 \int_{e-i\infty}^{e+i\infty} \frac{d\gamma}{2\pi i}, \left(\frac{\chi(\gamma)}{3\chi(\frac{1}{3}\gamma) - \chi(\gamma)}\right) \left(\frac{2\chi(\gamma)}{2\chi(\frac{1}{3}\gamma) - \chi(\frac{1}{2}\gamma)}\right) \{e^{3\chi(\frac{1}{3}\gamma)y} - e^{\chi(\gamma)y}\} - \left(\frac{\chi(\gamma)}{2\chi(\frac{1}{3}\gamma) - \chi(\frac{1}{2}\gamma)}\right) \left(\frac{2\chi(\gamma)}{2\chi(\frac{1}{2}\gamma) - \chi(\gamma)}\right) \{e^{2\chi(\frac{1}{2}\gamma)y} - e^{\chi(\gamma)y}\}\frac{1}{\gamma}e^{\gamma\xi}.$$
(63)

Using the method of steepest descent and neglecting contributions of the order of $\frac{\xi'}{2Dy}$ in all preexponential factors, we see that

$$M_3(\xi, y, y_0) = 6N(\xi, y, y_0)(N(\xi, y, y_0) - 1)^2,$$
(64)

which is the same as for the multiplicity distribution of Eq. (1).

C. General approach

The equation for a general moment

$$M_{k}(r, y, y_{0}) = \langle |n(n-1)\cdots(n-k+1)| \rangle = \sum_{n=1}^{\infty} (n(n-1)\cdots(n-k+1))w_{n}(r, y, y_{0}) = \frac{\partial^{k}w_{\lambda}(r, y, y_{0})}{\partial\lambda^{k}} \Big|_{\lambda=1}$$
(65)

we can obtain by differentiating Eq. (18) with respect to λ . It has the simple form in the momentum representation [see Eq. (32)]:

$$\frac{\partial}{\partial y}m_k(k_T, y, y_0) = \int \frac{d^2k_T}{(2\pi)^2}K(k_T, k_T')m_k(k_T', y, y_0) + \sum_{i=1}^{k-1}\frac{k!}{(k-i)!i!}m_{k-i}(k_T, y, y_0)m_i(k_T, y, y_0).$$
(66)

As we have discussed, we can go back to coordinate representation in Eq. (66), since m_k is the solution to the linear equations, and, therefore, has the Mellin image

$$m_k(\mathbf{k}_T, \mathbf{y}) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} e^{\omega(\gamma)\mathbf{y} + \gamma\xi'} m_{k0}(\gamma) \quad \text{with} \quad \xi' = \ln k_T^2.$$
(67)

In coordinate representation we have

$$M_{k}(r, y) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} e^{\omega(\gamma)y + \gamma\xi} m_{k0}(\gamma) H_{k}(\gamma) \quad \text{with} \quad \xi = \ln\left(\frac{1}{r^{2}}\right).$$
(68)

The coordinate image of $m_{k-i}(\mathbf{k}_T, y, y_0)m_k(\mathbf{k}_T, y, y_0)$ is $\int d^2 \mathbf{r}' K(\mathbf{r}', \mathbf{r} - \mathbf{r}'|\mathbf{r})M_{k-i}(\mathbf{r}', y, y_0)M_i(\mathbf{r} - \mathbf{r}', y, y_0)$. On the other hand, in SCA $m_{k-i}(\mathbf{k}_T, y, y_0)m_i(\mathbf{k}_T, y, y_0)$ has the image in γ representation, which is equal to Const

 $\exp(k\chi(\frac{1}{k}\gamma)y)m_{k-i,0}(\frac{\gamma}{k})m_{i,0}(\frac{\gamma}{k})\tilde{H}_{k}(\gamma)]$ see Eqs. (45) and (61)].² Using this image, one can see that the coordinate representation for $m_{k-i}(k_{T}, y, y_{0})m_{k}(k_{T}, y, y_{0})$ can be reduced to $\int d^{2}r'K(\mathbf{r}', \mathbf{r} - \mathbf{r}'|\mathbf{r})M_{k-i}(r', y, y_{0})M_{i}(r', y, y_{0})$, which means that $\tilde{H}_{k}(\gamma) = \chi(\gamma)$ as we expect from Eqs. (46) and (63).

Finally we can rewrite Eq. (66) in the coordinate representation in the form

$$\frac{\partial}{\partial y} M_k(r, y, y_0) = \int d^2 r' K(\mathbf{r}', \mathbf{r} - \mathbf{r}' | \mathbf{r}) \bigg\{ M_k(\mathbf{r}', y, y_0) + M_k(\mathbf{r} - \mathbf{r}', y, y_0) - M_k(\mathbf{r}, y, y_0) \\
+ \sum_{i=1}^{k-1} \frac{k!}{(k-i)!i!} M_{k-i}(r', y, y_0) M_i(r', y, y_0) \bigg\}.$$
(69)

Assuming that for all $i \le k - 1$

$$M_i(r, y, y_0) = i! N(r, y, y_0) (N(r, y, y_0) - 1)^{i-1},$$
(70)

which follows from Eq. (1), we will prove that for i = k we have the same expression.

Plugging Eq. (70) in Eq. (69) we get the inhomogeneous term in the form $k!(k-1)N^2(N-1)^{k-2}$. In the following we will use the fact that the Mellin image of $N^i(r, y, y_0)$ ($\tilde{n^i}$) is equal to

$$\widetilde{n^{i}} = e^{i\chi(\frac{1}{i}\gamma)y}\frac{1}{\gamma} = \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\omega}{2\pi i} e^{\omega y} \frac{1}{\omega - i\chi(\frac{1}{i}\gamma)}\frac{1}{\gamma}; \quad \text{double Mellin image } \widetilde{n^{i}} = \frac{1}{\omega - i\chi(\frac{1}{i}\gamma)}\frac{1}{\gamma}; \quad (71)$$

which can be derived using the method of steepest descent in the estimates of the integrals over γ 's [see Eq. (42) for example].

In the double Mellin transform Eq. (69) takes the form

$$(\omega - \chi(\gamma))\widetilde{\widetilde{M}_{k}} = k!(k-1)\chi(\gamma)\sum_{l=0}^{k-2} \frac{(-1)^{l}(k-2)!}{(k-2-l)!l!} \frac{1}{\omega - (k-l)\chi(\frac{1}{k-l}\gamma)}\frac{1}{\gamma}$$
(72)

Hence from Eq. (72) we have

$$M_{k}(r, y, y_{0}) = k!(k-1) \sum_{l=0}^{k-2} \frac{(-1)^{l}(k-2)!}{(k-2-l)!l!} \\ \times \int_{e-i\infty}^{e+i\infty} \frac{d\omega}{2\pi i} \int_{e-i\infty}^{e+i\infty} \frac{d\gamma}{2\pi i} e^{\omega y + \gamma \xi'} \frac{\chi(\gamma)}{(n-l)\chi(\frac{1}{k-l}\gamma) - \chi(\gamma)} \left\{ \frac{1}{\omega - (k-l)\chi(\frac{1}{k-l}\gamma)} - \frac{1}{\omega - \chi(\gamma)} \right\} \frac{1}{\gamma} \\ = k!(k-1) \sum_{l=0}^{n-2} \frac{(-1)^{l}(k-2)!}{(k-2-l)!l!} \int_{e-i\infty}^{e+i\infty} \frac{d\gamma}{2\pi i} e^{\gamma \xi'} \frac{\chi(\gamma)}{(k-l)\chi(\frac{1}{k-l}\gamma) - \chi(\gamma)} \left\{ e^{(k-l)\chi(\frac{1}{k-l}\gamma)y} - e^{\chi(\gamma)y} \right\} \frac{1}{\gamma}.$$
(73)

In Eq. (73) we calculate the integrals over ω closing the contour of integration on poles. Taking the integral over γ using the method of steepest descent in the diffusion approximation and neglecting the corrections of the order of ξ'/Dy (see discussions above) we reduce Eq. (73) to the following expression:

$$M_{k}(r, y, y_{0}) = k! \sum_{l=0}^{k-2} \frac{(-1)^{l} (k-1)!}{(k-1-l)! l!} (N^{k-l}(r, y, y_{0}) - N(r, b, y, y_{0}))$$

$$= k! \sum_{l=0}^{k-1} \frac{(-1)^{l} (k-1)!}{(k-1-l)! l!} N^{k-l}(r, y, y_{0}) = k! N(r, y, y_{0}) (N(r, y, y_{0}) - 1)^{k-1}.$$
(74)

Since we have obtained Eq. (70) for i = 2 and i = 3, we prove this equation for any value of *i*.

²Actually, the γ image of m_k is a sum of the terms with different k [see Eqs. (45) and (61)], but this does not change the conclusion.

VI. FINDING CORRECTIONS AND COMPARISON WITH EXPERIMENTS

Equation (70) generates the multiplicity distribution of Eq. (1). However, several questions arise, when we wish to compare this distribution with the experimental data, let us say with DIS. The average multiplicity of the color-singlet dipoles is equal to the sea quark structure function $x\Sigma_{\text{sea}}(x, Q^2)$ [19]. On the other hand, in Eq. (70) the multiplicity enters in the coordinate representation. In diffraction approximation the momentum and coordinate representations are related by the replacement $\ln k_T^2 \rightarrow -\ln r^2$. Therefore, my suggestion is to use Eq. (1) with $N = x\Sigma_{\text{sea}}(x, Q^2)$ but to calculate the corrections to this distribution. The multiplicity distribution can generally be written, using the cumulant generating function $f(\lambda)$ as follows [34,35]:

$$\frac{\sigma_n}{\sigma_{\rm in}} = \oint \frac{d\lambda}{2\pi i} \frac{e^{f(\lambda)}}{\lambda^{n+1}} \tag{75}$$

where the contour of integration is the circle around the point $\lambda = 0$ and $f(\lambda)$ is the cumulant generating function, which is defined as

$$f(\lambda) = \sum_{n=1}^{\infty} \frac{\kappa_n}{n!} (\lambda - 1)^n \tag{76}$$

where κ_n are cumulants. Generally speaking, we have the following definition for the cumulants:

$$\kappa_1 = N; \quad \kappa_2 = M_2 - N^2; \quad \kappa_3 = M_3 - 3M_2N + 2N^3; \kappa_4 = M_4 - 4M_3N - 3M_2^2 + 12M_2N^2 - 6N^4;$$
(77)

where M_k are the factorial moments, that we have discussed above [see Eq. (70)].

In our case we can view $f(\lambda)$ as a sum of $f(\lambda) = f^{\text{Eq. (1)}}(\lambda) + \Delta f(\lambda)$. $f^{\text{Eq. (1)}}(\lambda)$ generates the multiplicity distributions of Eq. (1), which includes the most dominant contributions and, in particular, the average number of color-singlet dipoles is taken into account exactly. We suggest to introduce the function $\Delta f(\lambda)$ in the following way:

$$\Delta f(\lambda) = \sum_{n=1}^{\infty} \frac{\Delta \kappa_n}{n!} (\lambda - 1)^n \tag{78}$$

with

$$\Delta \kappa_1 = 0; \qquad \Delta \kappa_2 = M_2(k_T, y) - 2N(N-1); \Delta \kappa_3 = M_3(k_T, y) - 6N(N-1)^2 - 3\Delta \kappa_2 N;$$
(79)

where $M_2(k_T, y)$ and $M_3(k_T, y)$ are the exact solutions to Eqs. (33) and (56), respectively.

Introducing two multiplicity distributions,

$$P_n(N) = \oint \frac{d\lambda}{2\pi i} \frac{e^{f^{\text{Eq.}(1)}(\lambda)}}{\lambda^{n+1}}, \qquad \tilde{P}_n(N) = \oint \frac{d\lambda}{2\pi i} \frac{e^{\Delta f(\lambda)}}{\lambda^{n+1}}, \quad (80)$$

one can see that the resulting multiplicity distribution takes the form

$$\frac{\sigma_n}{\sigma_{\rm in}} = \sum_{k=0}^n \frac{n!}{(n-k)!k!} P_{n-k}(N) \tilde{P}_k(N).$$
(81)

In the case of $\Delta \kappa_2 \neq 0$ but $\Delta \kappa_n = 0$ for n > 2 the distribution $\tilde{P}_n(N)$ has been found in Ref. [34] and it has the following form in our notations:

$$\tilde{P}_n(\Delta\kappa_2) = e^{\frac{\Delta\kappa_2}{2}} \frac{\left(-i\sqrt{\frac{\Delta\kappa_2}{2}}\right)^n}{n!} H_n\left(i\sqrt{\frac{\Delta\kappa_2}{2}}\right)$$
(82)

where H_n is the Hermite polynomial (see formula 8.95 in Ref. [32]).

VII. CONCLUSIONS

This paper has two main results. First, we derived the BFKL linear, inhomogeneous equation for the factorial moments of multiplicity distribution (M_k) from the LMM equation. In particular, the equation for the average multiplicity of the color-singlet (*N*) turns out to be the homogeneous BFKL equation which leads to the powerlike growth in the region of small *x*. From these equations it follows that $M_k \propto N^k$ at small *x*.

Second, using the diffusion approximation for the BFKL kernel, which is generally considered to be responsible for the small x behavior, we show that the factorial moments satisfy Eq. (2), which reproduces the multiplicity distribution of Eq. (1). This result is in agreement with the attempts [18] to find solutions to the equations for the cascade of color-singlet dipoles [see Eq. (3)].

We also suggest a procedure for finding corrections to this multiplicity distribution, which, we believe, will be useful for descriptions of the experimental data.

In general, the multiplicity distribution, that has been discussed in the paper, confirms the result of Ref. [5], that the entropy of color-singlet dipoles is equal to $S = \ln N$ in the region of small *x*, and gives the regular procedure to estimate corrections to this formula.

It is worthwhile mentioning that both SCA and DA have been developed before and many technical issues that matter have been discussed³ (see, for example, Refs. [24,36–40] and references therein).

³We thank our referee who drew our attention to Ref. [36], where some technical details of our approach have been clarified.

ACKNOWLEDGMENTS

We thank our colleagues at Tel Aviv university and UTFSM for encouraging discussions. Our special thanks go to Michael Lublinsky for fruitful discussion of Ref. [29]. This research was supported by ANID PIA/APOYO AFB180002 (Chile) and Fondecyt (Chile) Grant No. 1180118.

- K. Kutak, Gluon saturation and entropy production in proton-proton collisions, Phys. Lett. B 705, 217 (2011).
- [2] R. Peschanski, Dynamical entropy of dense QCD states, Phys. Rev. D 87, 034042 (2013).
- [3] A. Kovner and M. Lublinsky, Entanglement entropy and entropy production in the color glass condensate framework, Phys. Rev. D 92, 034016 (2015).
- [4] R. Peschanski and S. Seki, Entanglement entropy of scattering particles, Phys. Lett. B 758, 89 (2016).
- [5] D. E. Kharzeev and E. M. Levin, Deep inelastic scattering as a probe of entanglement, Phys. Rev. D 95, 114008 (2017).
- [6] O. Baker and D. Kharzeev, Thermal radiation and entanglement in proton-proton collisions at energies available at the CERN Large Hadron Collider, Phys. Rev. D 98, 054007 (2018).
- [7] J. Berges, S. Floerchinger, and R. Venugopalan, Dynamics of entanglement in expanding quantum fields, J. High Energy Phys. 04 (2018) 145.
- [8] Y. Hagiwara, Y. Hatta, B. W. Xiao, and F. Yuan, Classical and quantum entropy of parton distributions, Phys. Rev. D 97, 094029 (2018).
- [9] N. Armesto, F. Dominguez, A. Kovner, M. Lublinsky, and V. Skokov, The color glass condensate density matrix: Lindblad evolution, entanglement entropy and Wigner functional, J. High Energy Phys. 05 (2019) 025.
- [10] E. Gotsman and E. Levin, Thermal radiation and inclusive production in the CGC/saturation approach at high energies, Eur. Phys. J. C 79, 415 (2019).
- [11] E. Gotsman and E. Levin, Thermal radiation and inclusive production in the Kharzeev-Levin-Nardi model for ion-ion collisions, Phys. Rev. D 100, 034013 (2019).
- [12] A. Kovner, M. Lublinsky, and M. Serino, Entanglement entropy, entropy production and time evolution in high energy QCD, Phys. Lett. B **792**, 4 (2019).
- [13] D. Neill and W. J. Waalewijn, Entropy of a Jet, Phys. Rev. Lett. **123**, 142001 (2019).
- [14] Y. Liu and I. Zahed, Entanglement in Regge scattering using the AdS/CFT correspondence, Phys. Rev. D 100, 046005 (2019).
- [15] X. Feal, C. Pajares, and R. Vazquez, Thermal behavior and entanglement in Pb-Pb and p-p collisions, Phys. Rev. C 99, 015205 (2019).
- [16] Z. Tu, D. E. Kharzeev, and T. Ullrich, Einstein-Podolsky-Rosen Paradox and Quantum Entanglement at Subnucleonic Scales, Phys. Rev. Lett. **124**, 062001 (2020).
- [17] H. Duan, C. Akkaya, A. Kovner, and V. V. Skokov, Entanglement, partial set of measurements, and diagonality of the density matrix in the parton model, Phys. Rev. D 101, 036017 (2020).

- [18] E. Gotsman and E. Levin, High energy QCD: Multiplicity distribution and entanglement entropy, Phys. Rev. D 102, 074008 (2020).
- [19] D. E. Kharzeev and E. Levin, Deep inelastic scattering as a probe of entanglement: Confronting experimental data, Phys. Rev. D 104, L031503 (2021).
- [20] J. D. Bjorken, Asymptotic sum rules at infinite momentum, Phys. Rev. 179, 1547 (1969).
- [21] R. P. Feynman, Very High-Energy Collisions of Hadrons, Phys. Rev. Lett. 23, 1415 (1969).
- [22] J. D. Bjorken and E. A. Paschos, Electron-proton and γ -proton scattering and the structure of the nucleon, Phys. Rev. **185**, 1975 (1969).
- [23] V. N. Gribov, Inelastic processes at super high energies and the problem of nuclear cross sections, Yad. Fiz. 9, 640 (1969) [Sov. J. Nucl. Phys. 9, 369 (1969)]; Space-time description of hadron interactions at high energies, Proc. ITEP School on Elementary Particle Phys. 1, 65 (1973).
- [24] Yuri V. Kovchegov and Eugene Levin, *Quantum Chromodynamics at High Energies*, Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology (Cambridge University Press, Cambridge, England, 2012).
- [25] I. Balitsky, Operator expansion for high-energy scattering, Nucl. Phys. B463, 99 (1996); Factorization and high-energy effective action, Phys. Rev. D 60, 014020 (1999); Y. V. Kovchegov, Small-*x F*₂ structure function of a nucleus including multiple Pomeron exchanges, Phys. Rev. D 60, 034008 (1999).
- [26] A. H. Mueller, Soft gluons in the infinite momentum wave function and the BFKL pomeron, Nucl. Phys. B415, 373 (1994); Unitarity and the BFKL pomeron, Nucl. Phys. B437 (1995) 107.
- [27] E. Levin and M. Lublinsky, Towards a symmetric approach to high energy evolution: Generating functional with pomeron loops, Nucl. Phys. A763, 172 (2005).
- [28] E. Levin and M. Lublinsky, Balitsky's hierarchy from Mueller's dipole model and more about target correlations, Phys. Lett. B 607, 131 (2005); A linear evolution for nonlinear dynamics and correlations in realistic nuclei, Nucl. Phys. A730, 191 (2004).
- [29] A. D. Le, A. H. Mueller, and S. Munier, Analytical asymptotics for hard diffraction, Phys. Rev. D 104, 034026 (2021).
- [30] V. S. Fadin, E. A. Kuraev, and L. N. Lipatov, On the pomeranchuk singularity in asymptotically free theories, Phys. Lett. 60B, 50 (1975); E. A. Kuraev, L. N. Lipatov, and V. S. Fadin, The pomeranchuk singularity in non-Abelian gauge theories, Zh. Eksp. Teor. Fiz. 72, 377 (1977) [Sov. Phys. J. Exp. Theor. Phys. 45, 199 (1977); The pomeranchuk singularity in quantum chromodynamics,

Yad. Fiz. 28, 1597 (1978) [Sov. J. Nucl. Phys. 28, 822 (1978)].

- [31] L. N. Lipatov, The bare pomeron in quantum chromodynamics, Zh. Eksp. Teor. Fiz. **90**, 1536 (1986) [Sov. Phys. JETP **63**, 904 (1986)].
- [32] I. Gradstein and I. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed. (Academic Press, London, 1994).
- [33] S. Bondarenko, M. Kozlov, and E. Levin, QCD saturation in the semi-classical approach, Nucl. Phys. A727, 139 (2003).
- [34] A. H. Mueller, Distributions in Regge pole dominated inclusive reactions, Phys. Rev. D 4, 150 (1971).
- [35] J. Beran and S. Ghosh, *Moment Generating Function*, edited by M. Lovric, International Encyclopedia of Statistical Science (Springer, Berlin, Heidelberg, 2011).

- [36] B. W. Xiao, On the anomalous dimensions of the multiple pomeron exchanges, Nucl. Phys. A798, 132 (2008).
- [37] E. M. Levin, M. G. Ryskin, and A. G. Shuvaev, Anomalous dimension of the twist four gluon operator and pomeron cuts in deep inelastic scattering, Nucl. Phys. B387, 589 (1992).
- [38] E. Laenen, E. Levin, and A. G. Shuvaev, Anomalous dimensions of high twist operators in QCD at $N \rightarrow 1$ and large Q^2 , Nucl. Phys. **B419**, 39 (1994).
- [39] M. A. Braun, Double scattering on the nucleus in the perturbative QCD, Eur. Phys. J. C 6, 343 (1999).
- [40] E. Levin and J. Miller, The BFKL pomeron calculus: Summing enhanced diagrams, Nucl. Phys. A884–885, 51 (2012).