Weaving the covariant three-point vertices efficiently

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An efficient algorithm is developed for compactly weaving all the Lorentz-covariant three-point vertices in relation to the decay of a massive particle X of mass m_X and spin J into two particles $M_{1,2}$ with equal mass m and spin s. The closely related equivalence between the helicity formalism and the covariant formulation is utilized so as to identify the basic building blocks for constructing the covariant three-point vertex corresponding to each helicity combination explicitly. The massless case with $m = 0$ is worked out straightforwardly and the (anti)symmetrization of the three-point vertex required by the spin statistics of identical particles is made systematically. We show that the off-shell electromagnetic photon coupling to the states M_1 and M_2 can be accommodated in this framework. The power of the algorithm is demonstrated with a few typical examples with specific J and s values.

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I. INTRODUCTION

The Standard Model (SM) [1–4] of particle physics has been firmly established by the discovery of the spin-0 resonance of about 125 GeV mass at the Large Hadron Collider (LHC) at CERN [5,6]. However, even though a lot of high-energy experiments have searched for new phenomena beyond the SM (BSM) and they have tested the SM with great precision for decades, none of any BSM particles and phenomena have been observed so far at the TeV scale (see Ref. [7] for a comprehensive summary of hypothetical particles and concepts). In this present situation, considered to be unnatural, one meaningful strategy that can be taken in new physics searches is to keep our theoretical studies as model-independent as possible and to search for new particles with even more exotic characteristics including *spins higher than unity.*

Recently, the theory and phenomenology of high-spin particles [8–18] have drawn considerable interest. Various high-spin particles, although composite, exist in hadron physics (see Ref. [7] for several high-spin hadrons) so that the solid theoretical calculations of all the rates and distributions involving such high-spin states are required for correctly interpreting all the relevant experimental results [8–10]. A popular spin-$3/2$ particle is the gravitino appearing as the supersymmetric partner of the massless spin-2 graviton in supergravity [19–23]. The discovery of gravitational waves [24–26] strongly indicates the existence of massless spin-2 gravitons at the quantum level. The massive spin-2 particles as the Kaluza-Klein (KK) excitations of the massless graviton have been studied in the context of extra dimensional scenarios [27–29]. In addition, whether the dark matter (DM) of the Universe is formed with high-spin particles has been addressed in various recent works [11–17]. The relic density of the high-spin DM particles and their low-energy interactions with the SM particles have to be evaluated precisely for the plausibility of their indirect and direct observations. *For studying all of these theoretical and phenomenological aspects, it is crucial to systematically investigate all the allowed effective interactions of high-spin particles as well as the SM spin-0*, $1/2$, *and 1 particles in a model-independent way.*

In the present work, we focus on developing an efficient algorithm for compactly weaving all the three-point vertices consistent with *Lorentz invariance and locality.*[1] Specifically, we consider the decay of an on-shell massive particle of mass m_X and spin J into two on-shell massive particles, M_1 and M_2, of equal mass m and spin s. This study is a natural extension of the previous work [30] having dealt with the massless ($m = 0$) case with the identical particle (IP) condition imposed, and an intermediate bridge to the general case where all three particles have different masses and spins. We note in passing that the Lorentz-covariant structure of P and T symmetric scalar ($J = 0$), vector ($J = 1$), and tensor ($J = 2$) currents of the on-shell massive

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[1] If necessary, any other symmetry principles like local gauge invariance and/or various discrete symmetries could be invoked.

and massless particles of any spin s, which can be applied even in the case of off-shell X particles, has been studied recently in Refs. [31,32].

We adopt the conventional description of the integer and half-integer wave tensors [33–39] and we effectively utilize the closely related equivalence between the helicity formalism in the Jacob-Wick (JW) convention [40,41] and the standard covariant formulation. Their one-to-one correspondence enables us to identify every basic building block for constructing the covariant three-point vertex corresponding to each helicity combination explicitly.[2] We show that the massless ($m = 0$) case treated previously [30] can be worked out straightforwardly and the (anti)symmetrization of the vertex required by the spin statistics of identical particles [43] can be made systematically. The extension of the algorithm to the case where all three particles have different masses and spins is briefly touched upon.

This paper is organized as follows. In Sec. II, we discuss the key aspects of the two-body decay process $X \to M_1 M_2$ in the helicity formalism which allows us to treat any massive and massless particles on an equal footing. Section III is devoted to explicitly deriving and characterizing the integer and half-integer spin-s wave tensors and the spin-J wave tensor based on the conventional combination of spin-1 wave vectors and spin-1/2 spinors. In particular, the wave vectors and spinors in the X rest frame (XRF) are presented explicitly in the JW convention. Utilizing the close interrelationship between the helicity formalism and the covariant formulation we derive all the covariant basic and composite three-point operators both in the bosonic and fermionic case in Sec. IV. In Sec. V, the composite operators are shown to enable us to explicitly write down all the helicity-specific three-point vertices through which the covariant three-point vertex for any spin J and spin s can be weaved efficiently both in the massive and massless cases. In addition, we show that the relation valid in the case with two identical particles in the final state is systematically and straightforwardly derived and that the off-shell electromagnetic photon coupling to the states M_1 and M_2 can be accommodated in this framework. In order to demonstrate the power of the developed algorithm for constructing the covariant three-point vertices, we work out in detail several examples with specific J and s values, which are expected to be useful for various phenomenological investigations, in Sec. VI. Finally, we summarize our findings and conclude the present work by mentioning a few topics under study in Sec. VII.

[2]Another convenient procedure for describing the three-point vertex of on-shell massive particles of any spin is to use a spinor formalism developed in Ref. [42].

II. CHARACTERIZATION IN THE HELICITY FORMALISM

The helicity formalism [40,41] allows us to efficiently describe the two-body decay of an on-shell spin-J particle X of mass m_X into two on-shell massive particles, M_1 and M_2, with equal mass m and spin s. For the sake of a transparent and straightforward analytic analysis, we describe the two-body decay $X \to M_1 M_2$ in the XRF,

$$X(p, \sigma) \to M_1(k_1, \lambda_1) + M_2(k_2, \lambda_2), \tag{2.1}$$

in terms of the momenta, $\{p, k_1, k_2\}$, and helicities, $\{\sigma, \lambda_1, \lambda_2\}$, of the particles, as depicted in Fig. 1.

The helicity amplitude of the decay $X \to M_1 M_2$ is decomposed in terms of the polar and azimuthal angles, θ and ϕ, defining the momentum direction of the particle M_1 in a fixed coordinate system as

$$\mathcal{M}^{X\to M_1 M_2}_{\sigma;\lambda_1,\lambda_2}(\theta,\phi) = \mathcal{C}^J_{\lambda_1,\lambda_2} d^J_{\sigma,\lambda_1-\lambda_2}(\theta) e^{i(\sigma-\lambda_1+\lambda_2)\phi} \quad \text{with} \quad |\lambda_1 - \lambda_2| \le J, \tag{2.2}$$

in the JW convention [40,41] (see Fig. 1 for the kinematic configuration). Here, the helicity σ of the spin-J massive particle X takes one of the $2J+1$ values between $-J$ and J. In contrast, each of the helicities, $\lambda_{1,2}$, can take one of the $2s+1$ values between $-s$ and s in the massive ($m \neq 0$) case but they can take just two values of $\pm s$ in the massless ($m = 0$) case, as only the maximal-magnitude helicity values identical to the spin in magnitude are allowed physically for a massless particle of any spin. The reduced helicity amplitudes $\mathcal{C}^J_{\lambda_1,\lambda_2}$ in Eq. (2.2) do not depend on the X helicity σ due to rotational invariance and the polar-angle dependence is fully encoded in the Wigner d function $d^J_{\sigma,\lambda_1-\lambda_2}(\theta)$ given in the convention of Rose [44].

If two particles, M_1 and M_2, are identical, the Bose or Fermi symmetry in the integer or half-integer case leads to the IP relation for the reduced helicity amplitudes,

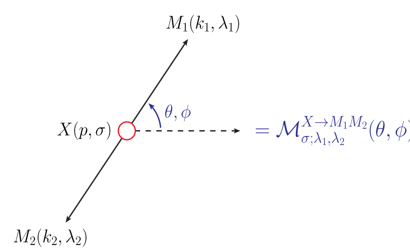

FIG. 1. Kinematic configuration for the helicity amplitude $\mathcal{M}^{X\to M_1 M_2}_{\sigma;\lambda_1,\lambda_2}(\theta,\phi)$ of the two-body decay $X \to M_1 M_2$ of a massive particle X into two massive particles, M_1 and M_2, in the XRF. The notations, $\{p, k_{1,2}\}$ and $\{\sigma, \lambda_{1,2}\}$, are the momenta and helicities of the decaying particle X and two particles, M_1 and M_2, respectively. The polar and azimuthal angles, θ and ϕ, are defined with respect to an appropriately chosen coordinate system.

$$\mathcal{C}^J_{\lambda_1,\lambda_2} = (-1)^J \mathcal{C}^J_{\lambda_2,\lambda_1} \quad \text{with} \quad |\lambda_1 - \lambda_2| \le J, \tag{2.3}$$

due to the (anti)symmetrization of the two identical final-state particles.[3]

First, *in the massive* ($m \neq 0$) *case*, the number $n[J,s]$ of independent reduced helicity amplitudes for specific J and s is given by

$$n[J,s] = \begin{cases} (2s+1)^2 & \text{for } J \ge 2s, \\ 2s+1+(4s+1)J - J^2 & \text{for } J < 2s, \end{cases} \tag{2.4}$$

valid for both integer and half-integer s. For example, we have $n[J,0]=1$, $n[1,1]=7$, and $n[2,1]=9$. We note that the expression (2.4) is consistent with that derived for the case with two arbitrary integer spin particles [37], if their spin values are set to be identical. On the other hand, the number of independent terms in the IP case reduces to

$$n[J,s]_{\rm IP} = \begin{cases} \frac{1}{2}(2s+1)[1+(-1)^J] + s(2s+1) & \text{for } J \ge 2s, \\ \frac{1}{2}(2s+1)[1+(-1)^J] + \frac{1}{2}[(4s+1)J - J^2] & \text{for } J < 2s, \end{cases} \tag{2.5}$$

which depends crucially on whether the X spin J is even or odd. For instance, $n[J,0]_{\rm IP} = [1+(-1)^J]/2$, $n[1,1]_{\rm IP} = 2$, and $n[2,1]_{\rm IP} = 6$. Therefore, any particle with an odd spin J cannot decay into two identical spinless particles.

In contrast, *in the massless* ($m=0$) *case*, the number $n[J,s]$ of independent reduced helicity amplitudes is reduced to

$$n[J,s] = \begin{cases} 4 & \text{for } J \ge 2s \\ 2 & \text{for } J < 2s \end{cases} \quad \text{for } s > 0 \quad \text{and} \quad n[J,0] = 1, \tag{2.6}$$

because only the maximal-magnitude helicity values of $\pm s$ are allowed for every massless particle of any spin s. The number of independent terms in the case with $s=0$ is one, irrespective of the X spin J. On the other hand the number of independent terms in the IP case is further reduced to

$$n[J,s]_{\rm IP} = \begin{cases} 2+(-1)^J & \text{for } J \ge 2s \\ 1+(-1)^J & \text{for } J < 2s \end{cases} \quad \text{for } s > 0 \quad \text{and} \quad n[J,0]_{\rm IP} = \frac{1}{2}[1+(-1)^J], \tag{2.7}$$

due to the Bose or Fermi symmetry. One immediate consequence is that any odd-J particle cannot decay into two identical massless particles of spin s larger than $J/2$ [30]. One well-known example is the decay of a spin-1 particle into two identical spin-1 massless particles like two photons [45,46].

III. SPIN-J AND SPIN-s WAVE TENSORS

Generically, the decay amplitude of one spin-J particle X of mass m_X into two particles, M_1 and M_2, of equal mass m and spin s, can be written in terms of the three-point vertex tensor Γ (see Fig. 2 for its diagrammatic description),

$$\mathcal{M}^{X\to M_1 M_2}_{\sigma;\lambda_1,\lambda_2} = \bar{u}_1^{\alpha_1\cdots\alpha_n}(k_1,\lambda_1)\Gamma^{\mu_1\cdots\mu_J}_{\alpha_1\cdots\alpha_n,\beta_1\cdots\beta_n}(p,k) v_2^{\beta_1\cdots\beta_n}(k_2,\lambda_2)\epsilon_{\mu_1\cdots\mu_J}(p,\sigma), \tag{3.1}$$

with the non-negative integer $n=s$ or $n=s-1/2$ in the integer or half-integer spin-s case, respectively. p and σ are the momentum and helicity of the particle X, and $k_{1,2}$ and $\lambda_{1,2}$ are the momenta and helicities of two particles, M_1 and M_2, respectively. Here, $p = k_1 + k_2$ and $k = k_1 - k_2$ are symmetric and antisymmetric under the interchange of two momenta, k_1 and k_2.

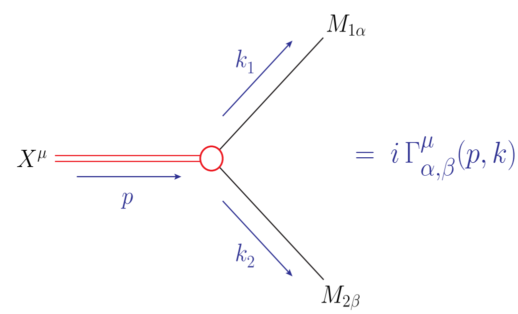

FIG. 2. Feynman rules $\Gamma^{\mu}_{\alpha,\beta}(p,k)$ for the general XM_1M_2 three-point vertex of a spin-J particle X of mass m_X and two massive particles, M_1 and M_2, of equal mass m and spin s. The indices, μ, α, and β, stand for the sequences of $\mu = \mu_1\cdots\mu_J$, $\alpha_1\cdots\alpha_n$, and $\beta_1\cdots\beta_n$ collectively with the non-negative integer $n=s$ or $n = s-1/2$ in the integer or half-integer spin-s case. The symmetric and antisymmetric momentum combinations, $p = k_1 + k_2$ and $k = k_1 - k_2$, are introduced for systematic classifications of the three-point vertex tensor in the following.

[3]Other discrete symmetries, like parity invariance, put their corresponding constraints on the reduced helicity amplitudes, although none of them are considered in the present work.

If the spin s is an integer, then the bosonic wave tensors $\bar{u}_1^{\alpha_1\cdots\alpha_s}(k_1,\lambda_1)$ and $v_2^{\beta_1\cdots\beta_s}(k_2,\lambda_2)$ for the particles with nonzero mass m are given by

$$\bar{u}_1^{\alpha_1\cdots\alpha_s}(k_1,\lambda_1) = \epsilon_1^{*\alpha_1\cdots\alpha_s}(k_1,\lambda_1), \tag{3.2}$$

$$v_2^{\beta_1\cdots\beta_s}(k_2,\lambda_2) = \epsilon_2^{*\beta_1\cdots\beta_s}(k_2,\lambda_2), \tag{3.3}$$

each of which is explicitly given by [33–39]

$$\epsilon_1^{*\alpha_1\cdots\alpha_s}(k_1,\lambda_1) = \sqrt{\frac{2^s(s+\lambda_1)!(s-\lambda_1)!}{(2s)!}}\sum_{\{\tau\}=-1}^{1}\delta_{\tau_1+\cdots+\tau_s,\lambda_1}\prod_{i=1}^{s}\frac{\epsilon_1^{*\alpha_i}(k_1,\tau_i)}{\sqrt{2}^{|\tau_i|}}, \tag{3.4}$$

$$\epsilon_2^{*\beta_1\cdots\beta_s}(k_2,\lambda_2) = \sqrt{\frac{2^s(s+\lambda_2)!(s-\lambda_2)!}{(2s)!}}\sum_{\{\tau\}=-1}^{1}\delta_{\tau_1+\cdots+\tau_s,\lambda_2}\prod_{i=1}^{s}\frac{\epsilon_2^{*\beta_i}(k_2,\tau_i)}{\sqrt{2}^{|\tau_i|}}, \tag{3.5}$$

with the convention $\{\tau\} = \tau_1,\ldots,\tau_s$ as a linear combination of products of s polarization vectors with appropriate Clebsch-Gordon coefficients. We note that the bosonic wave tensors are *totally symmetric, traceless, and divergence-free,*

$$\varepsilon_{\mu\nu\alpha_i\alpha_j}\epsilon_a^{\alpha_1\cdots\alpha_i\cdots\alpha_j\cdots\alpha_s}(k_a,\lambda_a) = 0, \tag{3.6}$$

$$g_{\alpha_i\alpha_j}\epsilon_a^{\alpha_1\cdots\alpha_i\cdots\alpha_j\cdots\alpha_s}(k_a,\lambda_a) = 0, \tag{3.7}$$

$$k_{\alpha_i}\epsilon_a^{\alpha_1\cdots\alpha_i\cdots\alpha_s}(k_a,\lambda_a) = 0, \tag{3.8}$$

with $a = 1$ and 2, and both of them satisfy the same on-shell wave equation $(k_{1,2}^2 - m^2)\epsilon_{1,2}^{\alpha_1\cdots\alpha_s}(k_{1,2},\lambda_{1,2}) = 0$ for any helicity value of $\lambda_{1,2}$ taking an integer between $-s$ and s. In contrast, if $m = 0$, the wave tensors with two allowed helicities $\pm s$ are given simply by a direct product of s spin-1 polarization vectors, each of which carry the same helicity of ± 1, as

$$\bar{u}_1^{\alpha}(k_1,\pm s) = \epsilon_1^{*\alpha_1\cdots\alpha_s}(k_1,\pm s) = \epsilon_1^{*\alpha_1}(k_1,\pm 1)\cdots\epsilon_1^{*\alpha_s}(k_1,\pm 1), \tag{3.9}$$

$$v_2^{\beta}(k_2,\pm s) = \epsilon_2^{*\beta_1\cdots\beta_s}(k_2,\pm s) = \epsilon_2^{*\beta_1}(k_2,\pm 1)\cdots\epsilon_2^{*\beta_s}(k_2,\pm 1), \tag{3.10}$$

which are totally symmetric, traceless, and divergence-free in the four-vector indices, $\alpha = \alpha_1\cdots\alpha_s$ and $\beta = \beta_1\cdots\beta_s$, as well.

On the other hand, for a half-integer $s = n + 1/2$ with a non-negative integer n, the fermionic wave tensors for the particles with nonzero mass are given by [33–36,38,39]

$$\bar{u}_1^{\alpha_1\cdots\alpha_n}(k_1,\lambda_1) = \sum_{\tau=\pm\frac{1}{2}}\sqrt{\frac{s+2\tau\lambda_1}{2s}}\epsilon_1^{*\alpha_1\cdots\alpha_n}(k_1,\lambda_1-\tau)\bar{u}_1(k_1,\tau), \tag{3.11}$$

$$v_2^{\beta_1\cdots\beta_n}(k_2,\lambda_2) = \sum_{\tau=\pm\frac{1}{2}}\sqrt{\frac{s+2\tau\lambda_2}{2s}}\epsilon_2^{*\beta_1\cdots\beta_n}(k_2,\lambda_2-\tau)v_2(k_2,\tau), \tag{3.12}$$

where $\bar{u}_1(k_1,\pm\frac{1}{2}) = u_1^\dagger(k_1,\pm\frac{1}{2})\gamma^0$ with the spin-1/2 particle spinor $u_1(k_1,\pm\frac{1}{2})$, and $v_2(k_2,\pm\frac{1}{2})$ is the spin-1/2 antiparticle spinor. The spin-1/2 spinors satisfy their own on-shell equations, $(\not{k}_1 - m)u_1(k_1,\pm\frac{1}{2}) = 0$ and $(\not{k}_2 + m)v_2(k_2,\pm\frac{1}{2}) = 0$. In contrast, if $m = 0$, the wave tensors with two allowed helicities $\pm s$ are given by a product of a u or v spinor and n spin-1 wave vectors with $n = s - 1/2$ as

$$\bar{u}_1^{\alpha}(k_1,\pm s) = \epsilon_1^{*\alpha_1}(k_1,\pm 1)\cdots\epsilon_1^{*\alpha_n}(k_1,\pm 1)\bar{u}_1\left(k_1,\pm\frac{1}{2}\right), \tag{3.13}$$

$$v_2^{\beta}(k_2,\pm s) = \epsilon_2^{*\beta_1}(k_2,\pm 1)\cdots\epsilon_2^{*\beta_n}(k_2,\pm 1)v_2\left(k_2,\pm\frac{1}{2}\right), \tag{3.14}$$

which are totally symmetric, traceless, and divergence-free in the four-vector indices, $\alpha = \alpha_1\cdots\alpha_n$ and $\beta = \beta_1\cdots\beta_n$, as well.

Similarly, the on-shell boson X of an integer spin J, mass m_X, momentum p, and helicity σ is represented by a totally symmetric, traceless, and divergence-free rank-J bosonic wave tensor, $\epsilon_{\mu_1\cdots\mu_J}(p,\sigma)$ [33–39]. The explicit form of the bosonic wave tensor is given by

$$\epsilon_{\mu_1\cdots\mu_J}(p,\sigma) = \sqrt{\frac{2^J(J+\sigma)!(J-\sigma)!}{(2J)!}}\sum_{\{\tau\}=-1}^{1}\delta_{\tau_1+\cdots+\tau_J,\sigma}\prod_{i=1}^{J}\frac{\epsilon_{\mu_i}(p,\tau_i)}{\sqrt{2}^{|\tau_i|}}, \tag{3.15}$$

with the convention $\{\tau\} = \tau_1,\ldots,\tau_s$, which satisfies the on-shell equation of motion $(p^2 - m_X^2)\epsilon_{\mu_1\cdots\mu_J}(p,\sigma) = 0$ for any helicity value of σ taking an integer value between $-J$ and J.

For calculating the reduced helicity amplitudes in the following, we show the explicit expressions for the wave vectors and spinors of the particle X and two particles $M_{1,2}$ in the XRF with the kinematic configuration as shown in Fig. 1. The JW convention of Refs. [40,41] is adopted for expressing the vectors and spinors. For the sake of notation, we introduce three unit vectors,

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{3.16}$$

$$\hat{\theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \tag{3.17}$$

$$\hat{\phi} = (-\sin\phi, \cos\phi, 0), \tag{3.18}$$

expressed in terms of the polar and azimuthal angles, θ and ϕ. The three unit vectors are mutually orthonormal, i.e., $\hat{n}\cdot\hat{\theta} = \hat{\theta}\cdot\hat{\phi} = \hat{\phi}\cdot\hat{n} = 0$ and $\hat{n}\cdot\hat{n} = \hat{\theta}\cdot\hat{\theta} = \hat{\phi}\cdot\hat{\phi} = 1$. The four-momentum sum $p = k_1 + k_2$ and the four-momentum difference $k = k_1 - k_2$ read

$$p = m_X\hat{p} = m_X(1, \vec{0}) \quad \text{and} \quad k = m_X\kappa\hat{k} = m_X\kappa(0, \hat{n}), \tag{3.19}$$

where $\kappa = \sqrt{1 - 4m^2/m_X^2}$ is the speed of the particles $M_{1,2}$ in the XRF. In the following, we use the normalized momenta $\hat{p} = (1, \vec{0})$ and $\hat{k} = (0, \hat{n})$ for calculating all the reduced helicity amplitudes in the XRF.

The spin-1 wave vectors for the particle X with momentum p and two particles $M_{1,2}$ with momenta $k_{1,2} = (p \pm k)/2 = m_X(1, \pm\kappa\hat{n})/2$ are given in the JW convention by

$$\epsilon(p, \pm 1) = \frac{1}{\sqrt{2}}(0, \mp 1, -i, 0), \qquad \epsilon(p, 0) = (0, 0, 0, 1), \tag{3.20}$$

$$\epsilon_1(k_1, \pm 1) = \frac{1}{\sqrt{2}}e^{\pm i\phi}(0, \mp\hat{\theta} - i\hat{\phi}),$$
$$\epsilon_1(k_1, 0) = \frac{m_X}{2m}(\kappa, \hat{n}), \tag{3.21}$$

$$\epsilon_2(k_2, \pm 1) = \frac{1}{\sqrt{2}}e^{\mp i\phi}(0, \pm\hat{\theta} - i\hat{\phi}),$$
$$\epsilon_2(k_2, 0) = \frac{m_X}{2m}(-\kappa, \hat{n}), \tag{3.22}$$

in the XRF, among which the transverse wave vectors satisfy the relation $\epsilon_2(k_2, \pm 1) = \epsilon_1(k_1, \mp 1) = -\epsilon_1^*(k_1, \pm 1) = -\epsilon_2^*(k_2, \mp 1)$ in the JW convention.

The spin-1/2 u_1 and v_2 spinors of the particles $M_{1,2}$ are given in the JW convention by

$$u_1\left(k_1, \pm\frac{1}{2}\right) = \sqrt{\frac{m_X}{2}}\begin{pmatrix} \sqrt{1 \mp \kappa}\,\chi_\pm(\hat{n}) \\ \sqrt{1 \pm \kappa}\,\chi_\pm(\hat{n}) \end{pmatrix} \quad \text{and}$$
$$v_2\left(k_2, \pm\frac{1}{2}\right) = \pm\sqrt{\frac{m_X}{2}}\begin{pmatrix} \sqrt{1 \pm \kappa}\,\chi_\pm(\hat{n}) \\ -\sqrt{1 \mp \kappa}\,\chi_\pm(\hat{n}) \end{pmatrix}, \tag{3.23}$$

where the 2-component spinors $\chi_\pm(\hat{n})$ are written in terms of the polar and azimuthal angles, θ and ϕ, as

$$\chi_+(\hat{n}) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\phi} \end{pmatrix} \quad \text{and} \quad \chi_-(\hat{n}) = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\phi} \\ \cos\frac{\theta}{2} \end{pmatrix}, \tag{3.24}$$

being mutually orthonormal, i.e., $\chi_a^\dagger(\hat{n})\chi_b(\hat{n}) = \delta_{a,b}$, with $a, b = \pm$, in the XRF.

IV. BASIC COVARIANT THREE-POINT VERTICES

In this section, we derive all the Lorentz-covariant operators corresponding to the reduced helicity amplitudes for three values of $J = 0$, 1, 2 and a fixed value of $s = 1$ and to those for two values of $J = 0$, 1 and a fixed value of $s = 1/2$. These covariant operators constitute the backbone for weaving the covariant three-point vertices for arbitrary J and s.

A. Bosonic vertex operators ($s = 1$)

First, we consider the decay of a spin-0 particle X into two spin-1 bosons, M_1 and M_2. The number of independent terms involving the $0 \to 1 + 1$ decay is $n[0, 1] = 3$, accounting for the three reduced helicity amplitudes, $\mathcal{C}^0_{\pm 1,\pm 1}$ and $\mathcal{C}^0_{0,0}$, in the XRF. After a little manipulation, we can find the three covariant three-point vertex operators defined as

$$S^\pm_{\alpha\beta} = \frac{1}{2}[g_{\perp\alpha\beta} \pm i\langle\alpha\beta\hat{p}\,\hat{k}\rangle] \leftrightarrow \mathcal{C}^0_{\pm 1,\pm 1} = 1, \tag{4.1}$$

$$S^0_{\alpha\beta} = \frac{4m^2}{m_X^2}\,\hat{p}_\alpha\hat{p}_\beta \leftrightarrow \mathcal{C}^0_{0,0} = -\kappa^2, \tag{4.2}$$

with the orthogonal tensor $g_{\perp\alpha\beta} = g_{\alpha\beta} - \hat{p}_\alpha\hat{p}_\beta + \hat{k}_\alpha\hat{k}_\beta$ and $\langle\alpha\beta\hat{p}\,\hat{k}\rangle = \varepsilon_{\alpha\beta\rho\sigma}\hat{p}^\rho\hat{k}^\sigma$ in terms of the totally antisymmetric Levi-Civita tensor with the sign convention $\varepsilon_{0123} = +1$. Each of the three covariant three-point vertices generates solely its corresponding reduced helicity amplitude in the JW convention, as shown in Eqs. (4.1) and (4.2).[4]

Second, there are in general $n[1, 1] = 7$ independent terms for the $1 \to 1 + 1$ decay mode, among which three generate the same helicity combinations, as in the case with

[4]In a convention different from the JW convention, every reduced helicity amplitude is modified simply by a proper phase.

$J = 0$. The corresponding covariant three-point vertices are simply $\hat{k}_\mu S^\pm_{\alpha\beta}$ and $\hat{k}_\mu S^0_{\alpha\beta}$ generating their corresponding reduced helicity amplitudes, $\mathcal{C}^1_{\pm1,\pm1} = -1$ and $\mathcal{C}^1_{0,0} = \kappa^2$, which are identical to $-\mathcal{C}^0_{\pm1,\pm1}$ and $-\mathcal{C}^0_{0,0}$, respectively. The remaining four covariant vertices and their corresponding reduced helicity amplitudes are given by

$$V^\pm_{1\alpha\beta;\mu} = \frac{m}{m_X}\hat{p}_\beta[g_{\perp\alpha\mu} \pm i\langle\alpha\mu\hat{p}\,\hat{k}\rangle] \leftrightarrow \mathcal{C}^1_{\pm1,0} = \kappa, \quad (4.3)$$

$$V^\pm_{2\alpha\beta;\mu} = \frac{m}{m_X}\hat{p}_\alpha[g_{\perp\beta\mu} \mp i\langle\beta\mu\hat{p}\,\hat{k}\rangle] \leftrightarrow \mathcal{C}^1_{0,\pm1} = -\kappa, \quad (4.4)$$

with the orthogonal tensors $g_{\perp\alpha\mu} = g_{\alpha\mu} - \hat{p}_\alpha\hat{p}_\mu + \hat{k}_\alpha\hat{k}_\mu$ and $g_{\perp\beta\mu} = g_{\beta\mu} - \hat{p}_\beta\hat{p}_\mu + \hat{k}_\beta\hat{k}_\mu$.[5]

Third, the decay of a spin-2 particle X into two spin-1 particles, M_1 and M_2, is in general described by $n[2,1] = 9$ independent terms. Seven of them can be constructed simply by multiplying the seven vertices participating in the spin-1 case by $\hat{k}$, i.e., $\hat{k}_{\mu_2}\hat{k}_{\mu_1}S^{\pm,0}_{\alpha\beta}$, $\hat{k}_{\mu_2}V^\pm_{1\alpha\beta;\mu_1}$, and $\hat{k}_{\mu_2}V^\pm_{2\alpha\beta;\mu_1}$, generating the reduced helicity amplitudes, $\mathcal{C}^2_{\pm1,\pm1} = 1$, $\mathcal{C}^2_{0,0} = -\kappa^2$, and $\mathcal{C}^2_{\pm1,0} = -\mathcal{C}^2_{0,\pm1} = -\kappa$, identical to $-\mathcal{C}^1_{\pm1,\pm1}$, $-\mathcal{C}^1_{0,0}$, $-\mathcal{C}^1_{\pm1,0}$, and $-\mathcal{C}^1_{0,\pm1}$, respectively. The two remaining covariant vertices and their corresponding reduced helicity amplitudes are

$$T^\pm_{\alpha\beta;\mu_1\mu_2} = \frac{1}{4}[g_{\perp\alpha\mu_1} \pm i\langle\alpha\mu_1\hat{p}\,\hat{k}\rangle][g_{\perp\beta\mu_2} \pm i\langle\beta\mu_2\hat{p}\,\hat{k}\rangle] \leftrightarrow \mathcal{C}^2_{\pm1,\mp1} = 1. \quad (4.5)$$

Note that the number of independent terms does not increase any more for J larger than 2, i.e., $n[J,1] = 9$ for $J \geq 2$. Generally, the covariant three-point vertex $\Gamma_{\alpha\beta;\mu_1\mu_2\cdots\mu_J} = \Gamma_{\alpha\beta;\mu_1\mu_2}\hat{k}_{\mu_3}\cdots\hat{k}_{\mu_J}$ for $J \geq 2$.

Scrutinizing the structure of all the scalar, vector, and tensor composite vertex operators carefully listed in Eqs. (4.1)–(4.5), we realize that any nontrivial helicity shifts in the XRF are generated essentially by two basic vector operators, $U^+_{1,2}$, and their complex conjugates, $U^-_{1,2}$, responsible for the positive and negative one-step transition of the M_1 and M_2 helicity states as

$$U^\pm_{1\alpha\mu} = \frac{1}{2}[g_{\perp\alpha\mu} \pm i\langle\alpha\mu\hat{p}\,\hat{k}\rangle] \Leftrightarrow [\lambda_1,\lambda_2] \to [\lambda_1 \pm 1, \lambda_2], \quad (4.6)$$

$$U^\pm_{2\beta\mu} = \frac{1}{2}[g_{\perp\beta\mu} \mp i\langle\beta\mu\hat{p}\,\hat{k}\rangle] \Leftrightarrow [\lambda_1,\lambda_2] \to [\lambda_1, \lambda_2 \pm 1], \quad (4.7)$$

respectively. In the operator form, the scalar and tensor composite three-point vertex operators in Eqs. (4.1) and (4.5) can be expressed in an inner product and an outer product of the operators $U^\pm_1$ and $U^\pm_2$, as

$$S^\pm_{\alpha\beta} = g^{\mu_1\mu_2}U^\pm_{1\alpha\mu_1}U^\pm_{2\beta\mu_2} \equiv [U^\pm_1 \cdot U^\pm_2]_{\alpha\beta} \Leftrightarrow [\lambda_1,\lambda_2] \to [\lambda_1 \pm 1, \lambda_2 \pm 1], \quad (4.8)$$

$$T^\pm_{\alpha\beta;\mu_1\mu_2} = U^\pm_{1\alpha\mu_1}U^\mp_{2\beta\mu_2} \equiv [U^\pm_1 \star U^\mp_2]_{\alpha\beta;\mu_1\mu_2} \Leftrightarrow [\lambda_1,\lambda_2] \to [\lambda_1 \pm 1, \lambda_2 \mp 1], \quad (4.9)$$

respectively. Furthermore, in addition to the normalized momentum $\hat{k}_\mu$, the momenta, $\hat{p}_{\alpha,\beta}$, can be used for matching the numbers of μ-, α-, and β-type indices for given J and s, while keeping the helicity values intact, and for defining the scalar and vector three-point vertices as

$$S^0_{\alpha\beta} = \frac{2m}{m_X}\hat{p}_\alpha\frac{2m}{m_X}\hat{p}_\beta, \qquad V^\pm_{1\alpha\beta;\mu} = \frac{2m}{m_X}\hat{p}_\beta U^\pm_{1\alpha\mu}, \quad \text{and} \quad V^\pm_{2\alpha\beta;\mu} = \frac{2m}{m_X}\hat{p}_\alpha U^\pm_{2\beta\mu}. \quad (4.10)$$

Clearly, these five composite operators in Eq. (4.10) do not contribute to the decay dynamics in the massless case with $m = 0$, consistent with the point that all the helicity-zero longitudinal modes are absent for any massless states.

In order to clarify the characteristics of the basic and composite operators, let us introduce an integer-helicity-lattice space consisting of $(2s+1)\times(2s+1)$ in order for each point $[\lambda_1,\lambda_2]$ to stand for its corresponding reduced helicity amplitude $\mathcal{C}^J_{\lambda_1,\lambda_2}$ existing only when $|\lambda_1 - \lambda_2| \leq J$ and $|\lambda_{1,2}| \leq s$. As shown in the left panel of Fig. 3, the one-step increasing horizontal and vertical transitions are dictated by the basic operators, U^+_1 and U^+_2, from the point $[\lambda_1,\lambda_2]$ to the point $[\lambda_1+1,\lambda_2]$ and the point $[\lambda_1,\lambda_2+1]$ in the helicity-lattice space, respectively. By combining the normalized momenta $\hat{p}$ and $\hat{k}$ and the basic transition operators properly, we can construct nine different transitions consisting of three scalar composite operators $S^{\pm,0}$, four vector composite operators $V^\pm_1$ and $V^\pm_2$, and two tensor composite operators $T^\pm$, as shown in the right panel of Fig. 3. Properly combining the nine composite operators enables us to reach every integer-helicity-lattice point. *To summarize, for any given integer J and integer s we can weave the covariant three-point vertex corresponding to every integer-helicity combination of $[\lambda_1,\lambda_2]$ efficiently and systematically.* The explicit form of every

[5]Making suitable use of Schouten identities, we can check that the set of seven three-point vertices listed above are equivalent to that of the seven VW^+W^- three-point vertex terms listed in Ref. [47].

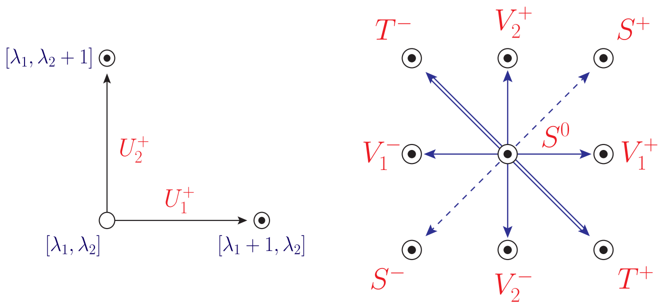

FIG. 3. (Left) A diagrammatic description of the basic operators, U_1^+ and U_2^+, moving the integer-helicity point $[\lambda_1, \lambda_2]$ to the integer-helicity point $[\lambda_1+1, \lambda_2]$ horizontally and to the integer-helicity point $[\lambda_1, \lambda_2+1]$ vertically in the helicity-lattice space, respectively. Although not presented, the transitions moving down the integer-helicity point by one step horizontally and vertically are dictated by the complex-conjugate operators, $U_1^- = (U_1^+)^*$ and $U_2^- = (U_2^+)^*$, respectively. (Right) A graphical description of the nine different transitions by the three scalar composite operators $S^\pm$ and S^0, four vector composite operators $V_1^\pm$ and $V_2^\pm$, and two tensor composite operators $T^\pm$.

covariant three-point vertex constructed by weaving the covariant composite operators is to be presented in Sec. V.

B. Fermionic vertex operators ($s = 1/2$)

First, we consider the decay of a spin-0 particle X into two spin-1/2 fermions, M_1 and M_2. The number of independent terms involving the $0 \to 1/2 + 1/2$ two-body decay is $n[0, 1/2] = 2$, accounting for the two reduced helicity amplitudes, $\mathcal{C}^0_{\pm 1/2, \pm 1/2}$. After a little manipulation, we can find the following two covariant three-point operators:

$$P^\pm = \frac{1}{2m_X}(1 \mp \kappa\gamma_5) \leftrightarrow \mathcal{C}^0_{\pm 1/2, \pm 1/2} = \kappa, \qquad (4.11)$$

with the $M_{1,2}$ speed $\kappa = \sqrt{1 - 4m^2/m_X^2}$ in the XRF.

Second, there are, in general, $n[1, 1/2] = 4$ independent terms for the $1 \to 1/2 + 1/2$ decay mode, among which two terms take the same helicity combinations as in the $J = 0$ case. The corresponding covariant three-point vertices are simply $\hat{k}_\mu P^\pm$ generating their corresponding reduced helicity amplitudes, $\mathcal{C}^1_{\pm 1/2, \pm 1/2} = -\kappa$, identical to $-\mathcal{C}^0_{\pm 1/2, \pm 1/2}$. The remaining two covariant vertices and their corresponding reduced helicity amplitudes are given by

$$W^\pm_\mu = \frac{1}{2\sqrt{2}m_X}(\pm\kappa\gamma_{\perp\mu} + \gamma_\mu\gamma_5) \leftrightarrow \mathcal{C}^1_{\pm 1/2, \mp 1/2} = \pm\kappa, \qquad (4.12)$$

with the orthogonal Dirac gamma matrix $\gamma_{\perp\mu} = \gamma_\mu + \frac{2m}{m_X \kappa}\hat{k}_\mu$.

Properly combining the fermionic transition operators in Eqs. (4.11) and (4.12), depicted diagrammatically in

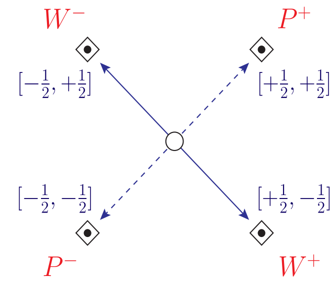

FIG. 4. A diagrammatic description of the half-step transitions by the four fermionic operators, $P^\pm$ and $W^\pm$, from the original point [0, 0] to the four half-integer-helicity points $[\pm\frac{1}{2}, \pm\frac{1}{2}]$ and $[\pm\frac{1}{2}, \mp\frac{1}{2}]$, respectively.

Fig. 4, and the bosonic transition operators in Eqs. (4.1)–(4.5) enables us to reach every half-integer-helicity-lattice point. *To summarize, for any given integer J and half-integer s, we can weave the covariant three-point vertex corresponding to every half-integer-helicity combination of $[\lambda_1, \lambda_2]$ efficiently and systematically.* The explicit form of the covariant fermionic three-point vertex constructed by weaving the fermionic as well as bosonic operators is to be presented in Sec. V.

V. WEAVING THE COVARIANT THREE-POINT VERTICES

Before building up all the covariant three-point vertices explicitly, we demonstrate that any covariant three-point vertex is indeed composed by the nine composite bosonic operators in the bosonic case and four basic fermionic operators in the fermionic case, as well as the normalized momentum $\hat{k}_\mu$, as worked out in Sec. IV. For the demonstration, it is crucial to take into account the following points, valid for the on-shell X and $M_{1,2}$ particles:

(i) All the bosonic wave tensors are totally symmetric, traceless, and divergence-free in their four-vector indices, and the fermionic spinors satisfy

$$\gamma_{\alpha_i} u_1^{\alpha_1 \cdots \alpha_i \cdots \alpha_n}(k_1, \lambda_1) = \gamma_{\beta_j} v_2^{\beta_1 \cdots \beta_j \cdots \beta_n}(k_2, \lambda_2) = 0, \quad (5.1)$$

with the non-negative integer $n = s - 1/2$. Therefore, any fermionic vertices involving γ_{α_i} or γ_{β_j} with $i, j = 1, \ldots, n$ can be effectively excluded. Any pair of μ, α, or β four-vector indices in any covariant three-point vertex can be shuffled freely due to the totally symmetric property of each wave tensor, and any term including p_{μ_i} for $i = 1, \cdots J$ can be excluded effectively due to the divergence-free condition of the wave tensor of the decaying particle X with momentum p. On the other hand, the divergence-free condition of the M_1 and M_2 wave tensors allows us to replace $k_{2\alpha_j}$ and $k_{1\beta_j}$ effectively by p_{α_j} and p_{β_j} for $j = 1, \cdots n$ with $n = s$ in the bosonic case and $n = s - 1/2$ in the fermionic case.

(ii) In principle, the two nonvanishing operators, $\langle\alpha\beta\mu\hat{p}\rangle$ and $\langle\alpha\beta\mu\hat{k}\rangle$, linking all the μ, α, and β four-vector indices, could be included for constructing in the covariant three-point vertices. However, it turns out that the operators can be replaced effectively by the three operators, $\hat{p}_\beta\langle\alpha\mu\hat{p}\,\hat{k}\rangle$, $\hat{p}_\alpha\langle\beta\mu\hat{p}\,\hat{k}\rangle$, and $\langle\alpha\beta\hat{p}\,\hat{k}\rangle$, through the relations

$$\langle\alpha\beta\mu\hat{p}\rangle = \hat{p}_\alpha\langle\beta\mu\hat{p}\,\hat{k}\rangle/\kappa + \hat{p}_\beta\langle\alpha\mu\hat{p}\,\hat{k}\rangle/\kappa - \langle\alpha\beta\hat{p}\,\hat{k}\rangle, \tag{5.2}$$

$$\langle\alpha\beta\mu\hat{k}\rangle = \hat{p}_\alpha\langle\beta\mu\hat{p}\,\hat{k}\rangle - \hat{p}_\beta\langle\alpha\mu\hat{p}\,\hat{k}\rangle, \tag{5.3}$$

derived by use of the Schouten identity reflecting that no rank-5 completely antisymmetric tensor exists in four dimensions.

(iii) Any product of more than two gamma matrices can be reduced to a linear combination of products with two or less numbers of gamma matrices by consecutively using the following relation:

$$\gamma_\lambda\gamma_\mu\gamma_\nu = g_{\lambda\mu}\gamma_\nu + g_{\mu\nu}\gamma_\lambda - g_{\lambda\nu}\gamma_\mu - i\epsilon_{\lambda\mu\nu\rho}\gamma^\rho\gamma_5. \tag{5.4}$$

The totally symmetric and traceless properties of the X wave tensor enable us to exclude the product of two μ-indexed gamma matrices with the relation $\gamma_{\mu_i}\gamma_{\mu_j} + \gamma_{\mu_j}\gamma_{\mu_i} = 2g_{\mu_i\mu_j}$. Moreover, the Gordon identities for the on-shell M_1 and M_2 spinors,

$$\bar{u}(k_1,\lambda_1)[i\sigma_{\mu\nu}\hat{p}^\nu]v(k_2,\lambda_2) = \frac{2m}{m_X}\bar{u}(k_1,\lambda_1)\left[\gamma_{\perp\mu} - \frac{m_X}{2m\kappa}\hat{k}_\mu\right]v(k_2,\lambda_2), \tag{5.5}$$

$$\bar{u}(k_1,\lambda_1)[i\sigma_{\mu\nu}\hat{k}^\nu]v(k_2,\lambda_2) = 0, \tag{5.6}$$

$$\bar{u}(k_1,\lambda_1)[i\sigma_{\mu\nu}\hat{p}^\nu\gamma_5]v(k_2,\lambda_2) = -\frac{\hat{k}_\mu}{\kappa}\bar{u}(k_1,\lambda_1)[\gamma_5]v(k_2,\lambda_2), \tag{5.7}$$

$$\bar{u}(k_1,\lambda_1)[i\sigma_{\mu\nu}\hat{k}^\nu\gamma_5]v(k_2,\lambda_2) = \frac{2m}{m_X\kappa}\bar{u}(k_1,\lambda_1)[\gamma_\mu\gamma_5]v(k_2,\lambda_2), \tag{5.8}$$

guarantee exploiting the gamma matrix set $\mathcal{D} = \{1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5\}$ consisting of two scalar operators, 1 and γ_5, and two vector operators, γ_μ and $\gamma_\mu\gamma_5$, for constructing the fermionic three-point vertices effectively and completely.

With all the effective replacement and exclusion procedures described above, we end up with the conclusion that the independent fundamental operators appearing in the covariant three-point vertices simply consist of the following three scalar-type operators:

$$\hat{p}_\alpha\hat{p}_\beta, \qquad g_{\alpha\beta}, \qquad \langle\alpha\beta\hat{p}\,\hat{k}\rangle, \tag{5.9}$$

and the following four vector-type operators:

$$\hat{p}_\alpha g_{\beta\mu}, \quad \hat{p}_\beta g_{\alpha\mu}, \quad \hat{p}_\alpha\langle\beta\mu\hat{p}\,\hat{k}\rangle, \quad \hat{p}_\beta\langle\beta\mu\hat{p}\,\hat{k}\rangle, \tag{5.10}$$

with the normalized momentum $\hat{k}_\mu$ used in increasing solely the number of the μ indices. Since all the seven fundamental bosonic operators and the four fermionic operators of the gamma matrix set $\mathcal{D}$ are expressed as linear combinations of the three scalar and four vector bosonic operators in Eqs. (4.1)–(4.4) and the four basic fermionic operators in Eqs. (4.11) and (4.12), it is guaranteed that no further operators are required for constructing the covariant three-point vertices.

As many indices of different types are involved in expressing a covariant three-point vertex, especially for high-spin particles, we introduce the following compact square-bracket notations:

$$[\hat{k}]^n \to (\hat{k}^n)_{\mu_1\cdots\mu_n} = \hat{k}_{\mu_1}\cdots\hat{k}_{\mu_n}, \tag{5.11}$$

$$[S^0]^n \to (S^0)^n_{\alpha_1\cdots\alpha_n,\beta_1\cdots\beta_n} = S^0_{\alpha_1\beta_1}\cdots S^0_{\alpha_n\beta_n}, \tag{5.12}$$

$$[S^\pm]^n \to (S^\pm)^n_{\alpha_1\cdots\alpha_n,\beta_1\cdots\beta_n} = S^\pm_{\alpha_1\beta_1}\cdots S^\pm_{\alpha_n\beta_n}, \tag{5.13}$$

$$[V^\pm_a]^n \to (V^\pm_a)^n_{\alpha_1\cdots\alpha_n,\beta_1\cdots\beta_n;\mu_1\cdots\mu_n} = V^\pm_{a\alpha_1\beta_1;\mu_1}\cdots V^\pm_{a\alpha_n\beta_n;\mu_n} \quad \text{with} \quad a = 1,2, \tag{5.14}$$

$$[T^\pm]^n \to (T^\pm)^n_{\alpha_1\cdots\alpha_n,\beta_1\cdots\beta_n;\mu_1\mu_2\cdots\mu_{2n-1}\mu_{2n}} = T^\pm_{\alpha_1\beta_1;\mu_1\mu_2}\cdots T^\pm_{\alpha_n\beta_n;\mu_{2n-1}\mu_{2n}}, \tag{5.15}$$

for a non-negative integer n. Obviously, the zeroth power ($n = 0$) of any operator or normalized four momentum is set to be 1. We emphasize once more that any permutation of the α, β, and μ four-vector indices can be regarded as equivalent as eventually the vertex operators are to be coupled with the X and $M_{1,2}$ wave tensors totally symmetric in the four-vector indices.

A. Bosonic three-point vertices

In the integer s case, any helicity-lattice point where both λ_1 and λ_2 are even or odd can be reached through a sequence of diagonal transitions by the scalar composite operators $S^\pm$ and S^0 and the tensor composite operators $T^\pm$. In fact, a little algebraic manipulation leads to the following expression for the covariant three-point vertex

$$[\mathcal{H}^{J,s}_{ii[\lambda_1,\lambda_2]}] = [\hat{k}]^{J-2|\lambda_-|}[S^0]^{s-|\lambda_+|-|\lambda_-|}[S^{\hat{\lambda}_+}]^{|\lambda_+|}[T^{\hat{\lambda}_-}]^{|\lambda_-|}, \tag{5.16}$$

in an operator form with two helicity combinations, $\lambda_+ = (\lambda_1 + \lambda_2)/2$ and $\lambda_- = (\lambda_1 - \lambda_2)/2$, and their signs,

$\hat{\lambda}_+ = \mathrm{sign}(\lambda_+)$ and $\hat{\lambda}_- = \mathrm{sign}(\lambda_-)$. We note that both λ_+ and λ_- take integer values. The subscript ii implies that both λ_+ and λ_- take integer values.

On the other hand, any helicity-lattice point where λ_1 and λ_2 are even and odd and vice versa can be reached through a sequence of diagonal transitions by the scalar composite operators $S^\pm$ and S^0 and the tensor composite operators $T^\pm$ followed by a proper vertical or horizontal transition with the vector composite operators $V_1^\pm$ or $V_2^\pm$. Explicitly, we have the following expression with half-integer λ_+ and λ_- for the three-point vertex

$$[\mathcal{H}^{J,s}_{hh[\lambda_1,\lambda_2]}] = [\hat{k}]^{J-2|\lambda_-|}[S^0]^{s-|\lambda_+|-|\lambda_-|}[\delta_{\hat{\lambda}_+,\hat{\lambda}_-}V_1^{\hat{\lambda}_+} + \delta_{\hat{\lambda}_+,-\hat{\lambda}_-}V_2^{\hat{\lambda}_+}] \times [S^{\hat{\lambda}_+}]^{|\lambda_+|-1/2}[T^{\hat{\lambda}_-}]^{|\lambda_-|-1/2}, \tag{5.17}$$

with two non-negative integer values of $|\lambda_\pm| - 1/2$ in this case. The subscript hh implies that both λ_+ and λ_- take half-integer values.

B. Fermionic three-point vertices

In the half-integer s case with two fermions M_1 and M_2, the helicity combinations can be categorized into two classes. One is when the helicity sum $[\lambda_1 + \lambda_2]$ and the helicity difference $[\lambda_1 - \lambda_2]$ are odd and even with a half-integer λ_+ and an integer λ_-, to be denoted by the notation hi. The other is when the helicity sum and difference are even and odd with an integer λ_+ and a half-integer λ_-, to be denoted by the notation ih. Taking a proper half-integer-helicity transition as described in Fig. 4 followed by a sequence of integer-helicity transitions, we can obtain the following expression of the fermionic three-point vertex with the helicities λ_1 and λ_2,

$$[\mathcal{H}^{J,s}_{hi[\lambda_1,\lambda_2]}] = [\hat{k}]^{J-2|\lambda_-|}[S^0]^{s-|\lambda_+|-|\lambda_-|}[P^{\hat{\lambda}_+}][S^{\hat{\lambda}_+}]^{|\lambda_+|-1/2}[T^{\hat{\lambda}_-}]^{|\lambda_-|}, \tag{5.18}$$

involving a fermionic scalar operator $P^\pm$ for half-integer λ_+ and integer λ_- and

$$[\mathcal{H}^{J,s}_{ih[\lambda_1,\lambda_2]}] = [\hat{k}]^{J-2|\lambda_-|}[S^0]^{s-|\lambda_+|-|\lambda_-|}[W^{\hat{\lambda}_-}][S^{\hat{\lambda}_+}]^{|\lambda_+|}[T^{\hat{\lambda}_-}]^{|\lambda_-|-1/2}, \tag{5.19}$$

with a fermionic vector operator $W^\pm$ for integer λ_+ and half-integer λ_-. The subscript hi (ih) implies that λ_+ takes a half-integer (integer) value and λ_- an integer (half-integer) value.

C. General form of three-point vertices

Wrapping up all the previous results in both the integer and half-integer spin-s cases, the general form of any covariant three-point vertex $\Gamma_{\alpha\beta;\mu}$ for given J and s can be expressed as a linear combination of all the allowed helicity-specific three-point vertices. The succinct operator form of the covariant three-point vertex is given by

$$[\Gamma] = \begin{cases} \sum_{\lambda_{1,2}=-s}^{s}[c^{J,s}_{ii[\lambda_1,\lambda_2]}[\mathcal{H}^{J,s}_{ii[\lambda_1,\lambda_2]}] + c^{J,s}_{hh[\lambda_1,\lambda_2]}[\mathcal{H}^{J,s}_{hh[\lambda_1,\lambda_2]}]] & \text{for an integer } s, \\ \sum_{\lambda_{1,2}=-s}^{s}[c^{J,s}_{hi[\lambda_1,\lambda_2]}[\mathcal{H}^{J,s}_{hi[\lambda_1,\lambda_2]}] + c^{J,s}_{ih[\lambda_1,\lambda_2]}[\mathcal{H}^{J,s}_{ih[\lambda_1,\lambda_2]}]] & \text{for a half-integer } s, \end{cases} \tag{5.20}$$

with the constraint $|\lambda_1 - \lambda_2| \le J$, i.e., $J - 2|\lambda_-| \ge 0$, where the helicity-specific coefficients, $c^{J,s}_{x[\lambda_1,\lambda_2]}$ with $x = ii, hh, hi, hi$, depend only on the X and $M_{1,2}$ masses. We claim that the expression (5.20), along with the helicity-specific vertices in Eqs. (5.16)–(5.19), is the key result of the present work. Although they are originally deduced from the comparison with the helicity amplitudes in the XRF, the Lorentz-covariant three-point vertices can be used in every reference frame because in any given reference frame the totally symmetric, traceless, and divergence-free, as well as on-shell conditions of the spin-J and spin-s wave functions, are valid, and also every helicity amplitude is expressed completely as a linear combination of the helicity amplitudes in the XRF through the so-called Wick helicity rotation [48–50].

D. Massless case

In the case with massless $M_{1,2}$ of spin s, the physically allowed helicity values are $\pm s$. Furthermore, the five scalar and vector operators S^0 and $V^\pm_{1,2}$ are vanishing as they are proportional to m^2 and m, respectively. Therefore, the helicity-specific vertices could survive only when $\lambda_1 = \lambda_2 = \pm s$ or $\lambda_1 = -\lambda_2 = \pm s$. These combinations satisfy $s - |\lambda_+| + |\lambda_-| = 0$. In addition, in the opposite helicity case, the X spin J cannot be smaller than $2s$. Consequently, in the $m = 0$ case, there could exist at most four independent terms in both the bosonic and fermionic cases, as counted in Eq. (2.6).

In the bosonic case with an integer s, the bosonic covariant three-point vertex is given in an operator form by

$$[\Gamma] = c^{J,s}_{ii[+s,+s]}[k]^J[S^+]^s + c^{J,s}_{ii[-s,-s]}[k]^J[S^-]^s + \theta(J-2s)\{c^{J,s}_{ii[+s,-s]}[k]^{J-2s}[T^+]^s + c^{J,s}_{ii[-s,+s]}[k]^{J-2s}[T^-]^s\}, \quad (5.21)$$

where the last two terms survive only when $J \geq 2s$, as denoted by the step function $\theta(J-2s) = 1$ for $J \geq 2s$ and 0 for $J < 2s$. On the other hand, in the fermionic case with a half-integer s, the fermionic covariant three-point vertex is given in an operator form as

$$[\Gamma] = c^{J,s}_{hi[+s,+s]}[k]^J[P^+][S^+]^{s-1/2} + c^{J,s}_{hi[-s,-s]}[k]^J[P^-][S^-]^{s-1/2} + \theta(J-2s)\{c^{J,s}_{ih[+s,-s]}[k]^{J-2s}[W^+][T^+]^{s-1/2} + c^{J,s}_{ih[-s,+s]}[k]^{J-2s}[W^-][T^-]^{s-1/2}\}. \quad (5.22)$$

The results in Eqs. (5.21) and (5.22) are consistent with those derived in Ref. [30].

E. Identical particle relation: Bose or Fermi symmetry

If two particles M_1 and M_2 are identical, the state of the two-particle system must be symmetric or antisymmetric under the interchange of two integer or half-integer spin particles. In this subsection we work out the constraints on the covariant three-point vertex imposed by the Bose or Fermi symmetry.

In the bosonic case with two identical particles of an integer spin s, the covariant three-point vertex tensor $\Gamma_{\alpha\beta;\mu}$ must be symmetric under the interchange of M_1 and M_2 due to Bose symmetry as

$$\Gamma_{\alpha_1\cdots\alpha_s,\beta_1\cdots\beta_s;\mu_1\cdots\mu_J}(p,k) = \Gamma_{\beta_1\cdots\beta_s,\alpha_1\cdots\alpha_s;\mu_1\cdots\mu_J}(p,-k), \quad (5.23)$$

under the transformations

$$\alpha_i \leftrightarrow \beta_j \quad \text{and} \quad k_1 \leftrightarrow k_2, \quad (5.24)$$

for any pair of $i, j = 1, \ldots, s$, leaving $p = k_1 + k_2$ invariant but changing the sign of k as $k \to -k$. Combining the index and momentum interchanges with the interchange of the M_1 and M_2 helicities, the helicity-specific three-point vertices transform under the Bose symmetrization as

$$[\mathcal{H}^{J,s}_{ii[\lambda_1,\lambda_2]}] \leftrightarrow (-1)^J[\mathcal{H}^{J,s}_{ii[\lambda_2,\lambda_1]}] \quad \text{and} \quad [\mathcal{H}^{J,s}_{hh[\lambda_1,\lambda_2]}] \leftrightarrow -(-1)^J[\mathcal{H}^{J,s}_{hh[\lambda_2,\lambda_1]}], \quad (5.25)$$

leading to the constraints on the helicity-specific coefficients as

$$c^{J,s}_{ii[\lambda_1,\lambda_2]} = (-1)^J c^{J,s}_{ii[\lambda_2,\lambda_1]} \quad \text{and} \quad c^{J,s}_{hh[\lambda_1,\lambda_2]} = -(-1)^J c^{J,s}_{hh[\lambda_2,\lambda_1]}. \quad (5.26)$$

One observation is that the diagonal ii and hh elements with $\lambda_1 = \lambda_2$ vanish for odd J and even J, respectively. Another observation is that any spin-1 ($J = 1$) particle cannot decay into two identical massless spin-1 particles, as the coefficient of the only allowed terms $c^{1,1}_{ii[\lambda,\lambda]}$ vanish, as proven more than seventy years ago [45,46]. We note that the so-called Landau-Yang theorem is generalized to the case with any values of J and s [30].

In the fermionic case with two identical fermions of a half-integer spin s, interchanging two identical massless fermions, i.e., taking the opposite fermion flow line [51,52], we can rewrite the helicity amplitude of the decay $X \to ff$ with a massless fermion $M = f$ as

$$\tilde{\mathcal{M}}^{X\to ff}_{\sigma;\lambda_1,\lambda_2} = \bar{u}^\beta_2(k_2,\lambda_2)\Gamma^\mu_{\beta,\alpha}(p,-k)v^\alpha_1(k_1,\lambda_1)\epsilon_\mu(p,\sigma) = v^{\alpha T}_1(k_1,\lambda_1)\Gamma^{\mu T}_{\beta,\alpha}(p,-k)\bar{u}^{\beta T}_2(k_2,\lambda_2)\epsilon_\mu(p,\sigma), \quad (5.27)$$

with the superscript T denoting the transpose of the matrix. Introducing the charge-conjugation operator C satisfying $C^\dagger = C^{-1}$ and $C^T = -C$ and relating the v spinor to the u spinor as

$$v^\alpha(k,\lambda) = C\bar{u}^{\alpha T}(k,\lambda), \quad (5.28)$$

with $\bar{u} = u^\dagger\gamma^0$, we can rewrite the amplitude as

$$\tilde{\mathcal{M}}^{X\to ff}_{\sigma;\lambda_1,\lambda_2} = -\bar{u}^\beta_1(k_1,\lambda_1)C\Gamma^{\mu T}_{\beta,\alpha}(p,-k)C^{-1}v^\alpha_2(k_2,\lambda_2)\epsilon_\mu(p,\sigma). \quad (5.29)$$

Since Fermi statistics requires $\tilde{\mathcal{M}} = -\mathcal{M}$, the three-point vertex tensor must satisfy the relation

$$C\Gamma^{\mu T}_{\beta,\alpha}(p,-k)C^{-1} = \Gamma^\mu_{\alpha,\beta}(p,k), \quad (5.30)$$

which enables us to classify all the allowed terms systematically [43,53,54].

The basic relation for the charge-conjugation invariance of the Dirac equation is $C\gamma^T_\mu C^{-1} = -\gamma_\mu$ with a unitary matrix C. Repeatedly using the basic relation, we can derive

$$\Gamma^c \equiv C\Gamma^T C^{-1} = \epsilon_C\Gamma \quad \text{with} \quad \epsilon_C = \begin{cases} +1 & \text{for } \Gamma = 1, \gamma_5, \gamma_\mu\gamma_5, \\ -1 & \text{for } \Gamma = \gamma_\mu. \end{cases} \quad (5.31)$$

There are no further independent terms as any other operator can be replaced by a linear combination of 1, γ_5, γ_μ, and $\gamma_\mu\gamma_5$ by use of the so-called Gordon identities, when coupled to the u and v spinors as shown in Eqs. (5.5)–(5.8).

Consequently, according to all the transformation properties of covariant three-point vertex operators worked out above, the helicity-specific three-point vertices transform under the Fermi symmetrization as

$$\mathcal{H}^{J,s}_{hi[\lambda_1,\lambda_2]} \leftrightarrow (-1)^J \mathcal{H}^{J,s}_{hi[\lambda_2,\lambda_1]} \text{ and } \mathcal{H}^{J,s}_{ih[\lambda_1,\lambda_2]} \leftrightarrow -(-1)^J \mathcal{H}^{J,s}_{ih[\lambda_2,\lambda_1]}, \tag{5.32}$$

leading to the constraints on the helicity-specific coefficients as

$$c^{J,s}_{hi[\lambda_1,\lambda_2]} = (-1)^J c^{J,s}_{hi[\lambda_2,\lambda_1]} \quad \text{and} \quad c^{J,s}_{ih[\lambda_1,\lambda_2]} = -(-1)^J c^{J,s}_{ih[\lambda_2,\lambda_1]}. \tag{5.33}$$

One observation is that the diagonal hi and ih elements with $\lambda_1 = \lambda_2$ vanish for odd J and even J, respectively, as in the bosonic case.

F. Off-shell electromagnetic gauge-invariant vertices

Due to the electromagnetic (EM) gauge invariance, any off-shell photon couples to a conserved current. Therefore, in any timelike photon exchange process involving the $\gamma^* M_1 M_2$ vertex, the off-shell photon can be treated as a spin-1 particle of mass $m_X = \sqrt{p^2}$. Moreover, the covariant three-point $\gamma^* M_1 M_2$ vertex can be cast into a manifestly EM gauge-invariant form [36,43] as

$$\Gamma^\mu_{\mathrm{EM}\alpha,\beta} = p^2 \Gamma^\mu_{\alpha,\beta} - (p\cdot\Gamma_{\alpha,\beta}) p^\mu, \tag{5.34}$$

automatically satisfying the current conservation condition $p_\mu \Gamma^\mu_{\mathrm{EM}\alpha,\beta} = p\cdot\Gamma_{\mathrm{EM}\alpha,\beta} = 0$.

The IP condition on the redefined EM gauge-invariant three-point vertex is identical to that on the original three-point vertex as the momentum p is invariant under Bose or Fermi symmetry. Note that the case with an off-shell spin-J particle coupled to a conserved tensor current can be treated in a similar manner as in the off-shell photon case.

VI. VARIOUS SPECIFIC EXAMPLES

In order to check the validity of our algorithm for weaving the three-point covariant vertices explicitly, we investigate the helicity-specific three-point vertices for three spin values $J = 0, 1, 2$, which can be directly employed for performing model-independent analyses in the context of various phenomenological aspects in this section.

A. $J = 0$ three-point vertices

We consider the decay of a spin-0 particle into two spin-s particles. In this case, the restriction $|\lambda_1 - \lambda_2| \le J = 0$ forces $\lambda_1 = \lambda_2 = \lambda$ to be satisfied so that the three-point vertex consists of the $n[0,s] = 2s+1$ independent terms of the form

$$[\mathcal{H}^{0,s}_{ii[\lambda,\lambda]}] = [S^0]^{s-|\lambda|}[S^{\hat\lambda}]^{|\lambda|}, \tag{6.1}$$

with λ varying from $-s$ to s and $\hat\lambda = \mathrm{sign}(\lambda)$ in the bosonic case with a non-negative integer s, and

$$[\mathcal{H}^{0,s}_{hi[\lambda,\lambda]}] = [S^0]^{s-|\lambda|}[P^{\hat\lambda}][S^{\hat\lambda}]^{|\lambda|-1/2}, \tag{6.2}$$

in the fermionic case with a positive half-integer s. Note that the three-point vertices do not change in number and form even in the IP case with $J = 0$, as all the scalar composite operators, S^0, $S^\pm$, and $P^\pm$, are symmetric under Bose and Fermi symmetry transformations. We check that, if P and T symmetries are imposed, the covariant form of the $J = 0$ scalar three-point vertices is perfectly consistent with that of the scalar current of massive as well as massless on-shell states of any spin presented in Refs. [31,32].

B. $J = 1$ three-point vertices

Let us now consider the case with $J = 1$. There exist $n[1,s] = 6s+1$ independent terms decomposed into two classes. One class with $2s+1$ terms is when two helicities are identical, i.e., $\lambda_1 = \lambda_2$. The corresponding helicity-specific vertex is given by

$$[\mathcal{H}^{1,s}_{ii[\lambda,\lambda]}] = [\hat k][S^0]^{s-|\lambda|}[S^{\hat\lambda}]^{|\lambda|} = [\hat k][\mathcal{H}^{0,s}_{ii[\lambda,\lambda]}] \tag{6.3}$$

in the bosonic case, and

$$[\mathcal{H}^{1,s}_{hi[\lambda,\lambda]}] = [\hat k][S^0]^{s-|\lambda|}[P^{\hat\lambda}][S^{\hat\lambda}]^{|\lambda|-1/2} = [\hat k][\mathcal{H}^{0,s}_{hi[\lambda,\lambda]}] \tag{6.4}$$

in the fermionic case. It is noteworthy that all the helicity-specific vertices in Eqs. (6.3) and (6.4) vanish in the IP case, as the operator $[\hat k]$ is antisymmetric under Bose and Fermi symmetrization. The other class with $4s$ independent terms is when the difference of two helicities are ± 1, i.e., $\lambda_1 = \lambda_2 \pm 1$. The corresponding helicity-specific bosonic vertex is given by

$$[\mathcal{H}^{1,s}_{hh[\lambda,\lambda\mp1]}] = [S^0]^{s-|\lambda_+|-1/2}[\delta_{\hat\lambda_+,\pm} V_1^\pm + \delta_{\hat\lambda_+,\mp} V_2^\mp][S^{\hat\lambda_+}]^{|\lambda_+|-1/2}, \tag{6.5}$$

with $\lambda_+ = \lambda \mp 1/2$ and the constraint $|\lambda_+| \le s - 1/2$, and the helicity-specific fermionic vertex by

$$[\mathcal{H}^{1,s}_{ih[\lambda,\lambda\mp1]}] = [S^0]^{s-|\lambda_+|-1/2}[W^\pm][S^{\hat\lambda_+}]^{|\lambda_+|}, \tag{6.6}$$

with $\lambda_+ = \lambda \mp 1/2$ and the constraint $|\lambda_+| \le s - 1/2$. In the IP case with two identical particles ($M_1 = M_2$), the covariant three-point vertex in the bosonic case is

$$[\mathcal{H}^{1,s}_{hh[\lambda,\lambda\mp 1]}]_{\rm IP} = \frac{1}{2}[S^0]^{s-|\lambda_+|-1/2}[\delta_{\hat{\lambda}_+,\pm}(V_1^\pm + V_2^\pm) + \delta_{\hat{\lambda}_+,\mp}(V_1^\mp + V_2^\mp)][S^{\hat{\lambda}_+}]^{|\lambda_+|-1/2}, \quad (6.7)$$

with $(V_1^\pm + V_2^\pm)_{\alpha,\beta;\mu} = (m/m_X)[(g_{\perp\alpha\mu}\hat{p}_\beta + g_{\perp\beta\mu}\hat{p}_\alpha) \pm i(\langle\alpha\mu\hat{p}\,\hat{k}\rangle\hat{p}_\beta - \langle\beta\mu\hat{p}\,\hat{k}\rangle\hat{p}_\alpha)]$, and the covariant three-point vertex in the fermionic case is

$$[\mathcal{H}^{1,s}_{ih[\lambda,\lambda\mp 1]}]_{\rm IP} = \frac{1}{2}[S^0]^{s-|\lambda_+|-1/2}[W^+ + W^-][S^{\hat{\lambda}_+}]^{|\lambda_+|}, \quad (6.8)$$

with $(W^+ + W^-)_\mu = \gamma_\mu\gamma_5/\sqrt{2}m_X$, which is of a typical axial-vector type. These results are consistent with those in Ref. [43]. Explicitly, for $J = 1$ and $s = 1/2$, the surviving three-point vertex is simply proportional to $\gamma_\mu\gamma_5$.

For $J = 1$ and $s = 1$, the three-point vertex, which can be applied to the model-independent description of the anomalous VZZ vertices with virtual $V = \gamma$, Z or on-shell Z' [47,55–58], is composed of two independent terms, $(V_1^\pm + V_2^\pm)_{\alpha\beta;\mu}$, proportional to the mass m so that it is vanishing in the massless limit. Again, it turns out to be that, with the P and T symmetries imposed, the covariant form of the $J = 1$ vector three-point vertices is perfectly consistent with that of the vector current of massive as well as massless on-shell states of any spin presented in Refs. [31,32], when the current is assumed to be coupled to an on-shell spin-1 vector particle.

C. $J = 2$ three-point vertices

In this subsection we consider the case with $J = 2$. The number of independent terms are $n[2,0] = 1$ for $s = 0$ and $n[2,s] = 10s - 1$ for $s > 0$, which are decomposed into two classes. The first class with $6s + 1$ terms is when two helicities are identical, i.e., $\lambda_1 = \lambda_2$ and when the difference of two helicities is 1. The corresponding helicity-specific vertices are given by

$$[\mathcal{H}^{2,s}_{ii[\lambda,\lambda]}] = [\hat{k}]^2[\mathcal{H}^{0,s}_{ii[\lambda,\lambda]}] \quad \text{and} \quad [\mathcal{H}^{2,s}_{hh[\lambda,\lambda\pm 1]}] = [\hat{k}][\mathcal{H}^{1,s}_{hh[\lambda,\lambda\pm 1]}] \quad (6.9)$$

in the bosonic case, and

$$[\mathcal{H}^{2,s}_{hi[\lambda,\lambda]}] = [\hat{k}]^2[\mathcal{H}^{0,s}_{hi[\lambda,\lambda]}] \quad \text{and} \quad [\mathcal{H}^{2,s}_{ih[\lambda,\lambda\pm 1]}] = [\hat{k}][\mathcal{H}^{1,s}_{ih[\lambda,\lambda\pm 1]}] \quad (6.10)$$

in the fermionic case. They constitute $2s + 1$ and $4s$ independent covariant three-point vertices both in the bosonic and fermion cases. The second class with $2(2s - 1)$ independent terms appear for the helicity difference of $\lambda_1 - \lambda_2 = \pm 2$. The corresponding helicity-specific vertices are given by

$$[\mathcal{H}^{2,s}_{ii[\lambda,\lambda\mp 2]}] = [S^0]^{s-|\lambda_+|-1}[S^{\hat{\lambda}_+}|^{|\lambda_+|}[T^\pm] \quad (6.11)$$

in the bosonic case with $\lambda_+ = \lambda \mp 1$, and

TABLE I. Specific examples in the integer spin-s case. Listed are the number of independent terms $n[J,s]$, the allowed helicity assignments (λ_1,λ_2), the helicity-specific covariant three-point vertices $[H^{J,s}_{[\lambda_1,\lambda_2]}]$ in an operator form and their corresponding reduced helicity amplitudes $\mathcal{C}^J_{\lambda_1,\lambda_2}$, the helicity-specific covariant three-point vertex $[H^{J,s}_{{\rm IP}[\lambda_1,\lambda_2]}]$ in an operator form, and the number of independent terms $n[J,s]_{\rm IP}$ in the IP case for a few integer J and integer s assignments. This table is presented for demonstrating the effectiveness of the algorithm for weaving and characterizing the covariant triple vertices.

Integer spin-s case						
(J,s)	$n[J,s]$	(λ_1,λ_2)	$[H^{J,s}_{[\lambda_1,\lambda_2]}]$	$\mathcal{C}^J_{\lambda_1,\lambda_2}$	$[H^{J,s}_{{\rm IP}[\lambda_1,\lambda_2]}]$	$n[J,s]_{\rm IP}$
(0,0)	1	(0,0)	[1]	1	[1]	1
(0,1)	3	$(\pm1,\pm1)$	$[S^\pm]$	1	$[S^\pm]$	3
		(0,0)	$[S^0]$	$-\kappa^2$	$[S^0]$	
(1,1)	7	$(\pm1,\pm1)$	$[\hat{k}][S^\pm]$	-1	$\cdots$	2
		(0,0)	$[\hat{k}][S^0]$	κ^2	$\cdots$	
		$(\pm1,0)$	$[V_1^\mp]$	κ	$\frac{1}{2}[V_1^\pm + V_2^\pm]$	
		$(0,\pm1)$	$[V_2^\pm]$	$-\kappa$		
(2,1)	9	$(\pm1,\pm1)$	$[\hat{k}]^2[S^\pm]$	1	$[\hat{k}]^2[S^\pm]$	6
		(0,0)	$[\hat{k}]^2[S^0]$	$-\kappa^2$	$[\hat{k}]^2[S^0]$	
		$(\pm1,0)$	$[\hat{k}][V_1^\mp]$	$-\kappa$	$\frac{1}{2}[\hat{k}][V_1^\pm - V_2^\pm]$	
		$(0,\pm1)$	$[\hat{k}][V_2^\pm]$	κ		
		$(\pm1,\mp1)$	$[T^\pm]$	1	$\frac{1}{2}[T^+ + T^-]$	

TABLE II. Specific examples in the half-integer spin-s case. Listed are the number of independent terms $n[J,s]$, the allowed helicity assignments (λ_1,λ_2), the helicity-specific covariant triple vertices $[H^{J,s}_{[\lambda_1,\lambda_2]}]$ in an operator form and their corresponding reduced helicity amplitudes $\mathcal{C}^J_{\lambda_1,\lambda_2}$, the helicity-specific covariant three-point vertex $[H^{J,s}_{\mathrm{IP}[\lambda_1,\lambda_2]}]$ in an operator form, and the number of independent terms $n[J,s]_{\mathrm{IP}}$ in the IP case for a few integer J and half-integer s assignments.

Half-integer spin-s case						
(J,s)	$n[J,s]$	(λ_1,λ_2)	$[H^{J,s}_{[\lambda_1,\lambda_2]}]$	$\mathcal{C}^J_{\lambda_1,\lambda_2}$	$[\mathcal{H}^{J,s}_{\mathrm{IP}[\lambda_1,\lambda_2]}]$	$n[J,s]_{\mathrm{IP}}$
$(0,1/2)$	2	$(\pm 1/2,\pm 1/2)$	$[P^\pm]$	κ	$[P^\pm]$	2
$(1,1/2)$	4	$(\pm 1/2,\pm 1/2)$	$[\hat{k}][P^+]$	$-\kappa$	$\cdots$	1
		$(\pm 1/2,\mp 1/2)$	$[W^\pm]$	$\pm\kappa$	$[W^+ + W^-]$	
$(2,1/2)$	4	$(\pm 1/2,\pm 1/2)$	$[\hat{k}]^2[P^\pm]$	κ	$[\hat{k}]^2[P^\pm]$	3
		$(\pm 1/2,\mp 1/2)$	$[\hat{k}][W^\pm]$	$\mp\kappa$	$[\hat{k}][W^+ - W^-]$	

$$[\mathcal{H}^{2,s}_{hi[\lambda,\lambda\mp 2]}] = [S^0]^{s-|\lambda_+|-1}[P^{\hat{\lambda}_+}][S^{\hat{\lambda}_+}]^{|\lambda_+|-1/2}[T^\pm] \qquad (6.12)$$

in the fermionic case with $\lambda_+ = \lambda \mp 1$.

The $J=2$ covariant three-point vertices can be adopted for studying the massive KK graviton or off-shell graviton interactions with two spin-s particles of equal mass m. Again, with the P and T symmetries imposed, the covariant form of the $J=2$ tensor three-point vertices is perfectly consistent with that of the spin-2 tensor currents of massive as well as massless on-shell states of any spin presented in Refs. [31,32], when the tensor current is coupled to an on-shell spin-2 particle.

D. Specific decay modes

Now, we demonstrate the power of the algorithm by explicitly working out all the characteristic features of four specific decay modes with the $[J,s]$ values of [0, 0], [0, 1], [1, 1], and [2, 1] in the integer spin-s case. We fully write down the number of independent terms $n[J,s]$, the allowed helicity assignments (λ_1,λ_2), the helicity-specific covariant three-point vertices $[\mathcal{H}^{J,s}_{[\lambda_1,\lambda_2]}]$ in an operator form and their corresponding reduced helicity amplitudes $\mathcal{C}^J_{\lambda_1,\lambda_2}$, as well as the helicity-specific covariant three-point vertices $[\mathcal{H}^{J,s}_{\mathrm{IP}[\lambda_1,\lambda_2]}]$ in an operator form, and the number of independent terms $n[J,s]_{\mathrm{IP}}$ in the IP case. The results are summarized succinctly in Table I.

The algorithm for weaving the general covariant three-point vertices in the half-integer spin-s case is demonstrated by explicitly evaluating three specific decay modes with the $[J,s]$ values of $[0,1/2]$, $[1,1/2]$, and $[2,1/2]$. The results are summarized in Table II.

VII. CONCLUSIONS

We have developed an efficient algorithm for compactly weaving all the covariant three-point vertices for the decay of an on-shell spin-J particle X of mass m_X into two on-shell particles, $M_{1,2}$, with equal mass m and spin s. For this development, we have made good use of the closely related equivalence between the helicity formalism and the covariant formulation for identifying the basic building blocks and composite three-point vertex operators for constructing all the effective covariant three-point vertices. All the helicity-specific covariant three-point vertices are presented in an operator form in Eqs. (5.16) and (5.17) in the bosonic case and in Eqs. (5.18) and (5.19) in the fermionic case, respectively. The massless ($m=0$) case could be worked out straightforwardly and the (anti) symmetrization of the covariant three-point vertices required by Bose or Fermi spin statistics of two identical final-state particles could be made systematically in the context of this efficient algorithm.

This general algorithm for constructing the effective covariant three-point vertices is expected to be useful in studying various phenomenological aspects such as the indirect and direct searches of high-spin DM particles and the pair production of high-spin particles at present and future high-energy colliders.

Naturally, it will be valuable to extend our algorithm for dealing with the general case when all three particles have different masses and spins. It is also an interesting question as to whether the bosonic and fermionic cases can be synthesized in a unified framework, covering various forms of wave tensors for particles of any spin. These generalization and synthesis are presently under study and the results will be reported separately.

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