NNLO soft function for threshold single inclusive jet production

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(Received 28 June 2021; accepted 2 September 2021; published 27 September 2021)

We present a computation of the global soft function for single inclusive hadronic jet production at next-to-next-to-leading order in the strong coupling constant involving four lightlike Wilson lines. Our soft function belongs to $\mathrm{SCET_I}$ observables, which are not sensitive the rapidity divergences and obey the non-Abelian exponentiation theorem. We provide the calculation of the soft function involving four collinear light-cone directions for all $2 \to 2$ processes extending the previous result. We perform the threshold resummation for these processes at the next-to-next-to-leading logarithmic accuracy and present the corresponding numerical results.

DOI: 10.1103/PhysRevD.104.054032

I. INTRODUCTION

The new data obtained with increasingly high precision at the Large Hadron Collider (LHC) and other experiments provide an opportunity to search for new physics from a small mismatch between the Standard Model (SM) theoretical predictions and experimental measurements. The only available theoretical tool for making such predictions from the first principles is the Quantum Chromodynamics (QCD) perturbative expansion in the strong coupling constant α_S or $\alpha_S L$ if the large logarithm L of the fraction of scales is present. To reliably interpret the collider data, the precision of QCD theoretical predictions has to match the small experimental errors. Such calculations are carried out at fixed-order in QCD perturbation theory and may include the resummation of large logarithms in particular regions of phase space by analytic resummation or through the use of parton-shower simulations. These days, considerable efforts have been made towards the development of theoretical predictions for physical processes at colliders with complete control over the final-state kinematics beyond the next-to-leading order (NLO) accuracy. With regards to fixed-order calculations, over the past few years significant results have been achieved. These include the calculations of fully differential gluon-gluon fusion Higgs production [1,2], the single jet production in deep-inelastic scattering [3] at N^3LO in the strong coupling constant, the next-to-next-to-leading order (NNLO) calculations of the single inclusive jet or dijet production [4], $V/N + j$ processes [5–11], Higgs production through the vector boson fusion [12,13], single top [14–16] and $t\bar{t}$ production [17,18] at the LHC, the jet productions in deep-inelastic scattering [19,20], and the heavy flavor production in deep-inelastic neutrino scattering [21]. The developments in the higher order calculations rely on the new higher loop calculations as well as on the new theoretical schemes for treating the singularities in the real emissions. The current status of the loop calculations for the collider processes can be illustrated by the calculation of the analytic two-loop five-point amplitude [22–24], as well as by the new ideas that appear [25–28].

Jets are crucial at high-energy particle and nuclear physics frontiers serving as a tool for precision tests of the SM as well as for the searches for new physics at the TeV scale at the LHC. As demonstrated by the measurements at the LHC and the Relativistic Heavy Ion Collider, they are also unique probes of nonperturbative dynamics, such as collinear parton distribution functions (PDFs) [29] and transverse momentum dependent parton distribution functions [30], the intrinsic spin of the nucleon [31–33], and the hot medium effects in the quark-gluon plasma [34–36]. The experimental advances are supplemented by the progress in the development of new jet substructure techniques [37,38], accurate extractions of the SM parameters [39], as well as the three-dimensional tomographic images of nucleons [40–45] out of jets. These developments will receive additional impetus

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at the future Electron-Ion Collider [46,47], additionally enabling the possibility to consider the polarization degrees of freedom.

The double differential cross section for the $pp \to$ jet $+ X$ process has the following form [48]:

$$\frac{p_T^2 \mathrm{d}^2\sigma}{\mathrm{d}p_T^2 \mathrm{d}\eta} = \sum_{i_1,i_2} \int_0^{V(1-W)} \mathrm{d}z \int_{\frac{VW}{1-z}}^{1-\frac{1-V}{1-z}} \mathrm{d}v\, x_1^2 f_{i_1}(x_1) x_2^2 f_{i_2}(x_2) \times \frac{\mathrm{d}^2\hat{\sigma}_{i_1 i_2}}{\mathrm{d}v\mathrm{d}z}(v, z, p_T, R), \tag{1}$$

with p_T being the transverse momentum and η the rapidity of the signal jet, and where $V = 1 - \frac{p_T e^{-\eta}}{\sqrt{S}}$, $VW = \frac{p_T e^{-\eta}}{\sqrt{S}}$, and the hadronic center-of-mass energy is $\sqrt{S}$. The PDFs f_i depend on the momentum fractions $x_1 = \frac{VW}{v(1-z)}$ and $x_2 = \frac{1-V}{(1-v)(1-z)}$, and the sum is taken over all partonic channels, initiating the process with the cross sections given by $\hat{\sigma}_{i_1 i_2}$. The kinematic variables are defined as $t = (p_1 - p_3)^2$ and $u = (p_2 - p_3)^2$, with $p_{1,2}$ being the momenta of the two incoming partons, p_3 being the momentum of the parton which initiates the signal jet, and p_4 being the momentum of the recoiling system against the signal jet. The partonic cross sections $\hat{\sigma}_{i_1 i_2}$ are functions of p_T and the partonic kinematic variables $s = x_1 x_2 S$, $v = \frac{u}{u+t}$, and z. They can be further factorized in the $R \to 0$ and $z \to 0$ limit:

$$\begin{aligned}\frac{\mathrm{d}^2\hat{\sigma}_{i_1 i_2}}{\mathrm{d}v\mathrm{d}z} &= s \int \mathrm{d}s_X \mathrm{d}s_c \mathrm{d}s_G \delta(zs - s_X - s_G - s_c) \\ &\times \mathrm{Tr}[H_{i_1 i_2}(v, p_T, \mu_h, \mu) S_G(s_G, \mu_{s_G}, \mu)] \\ &\times J_X(s_X, \mu_X, \mu) \sum_m \mathrm{Tr}[J_m(p_T R, \mu_J, \mu) \\ &\otimes_\Omega S_{c,m}(s_c R, \mu_{sc}, \mu)],\end{aligned} \tag{2}$$

with the traces taken in color space. The sum is taken over all collinear splittings with m being the energetic collinear parton multiplicity, and with the associated angular integrals denoted by $\otimes_\Omega$ [49]. The jet is constructed by the anti-k_T algorithm [50], and the finite mass of the signal jet is allowed. Resummation is performed by solving the renormalization group (RG) equation in Soft-Collinear Effective Theory (SCET) [51–56]. The hard function H describes the effect of hard radiation and is known to NNLO for numerous phenomenologically interesting processes containing final-state jets [57–63]. The jet function describes the final jet collinear radiation, and the recoil jet function J_X is known to three loops both for the quark and gluon jets [64,65] while the signal jet function J_m is known to two loops for the quark jet [66]. The global soft function S_G is the focus of our paper. It accounts for the wide-angle soft radiation which cannot resolve the small radius R of the jet.

Threshold logarithms play a role near the exclusive phase space boundary when the production of the signal jet becomes possible and the invariant mass $\sqrt{s_4}$ of the recoiling jet vanishes. The signal jet, however, has a finite invariant mass at the threshold, and radiation inside the jet cone is possible. After the cancellation of infrared divergences, the logarithms of the form $\alpha_S^n(\ln^k(z)/z)_+$ remain, with $k \le 2n-1$, $z = s_4/s$, and s being the partonic center-of-mass energy. These terms are large in the threshold limit $z \to 0$ and have to be resummed to all orders in order to obtain reliable perturbative results. Since the parton luminosity functions are steeply falling, these terms dominate over a wide range of the jet-p_T even when far from the hadronic threshold as shown in [67].

The global soft function S_G, which is the focus of our paper, is defined in terms of Wilson lines along the directions $n_i^\mu = \frac{p_i^\mu}{E_i}$ of the partons participating in the hard scattering. With the use of the color-space formalism [68,69], the Wilson line for a particle in a color representation with generators T_i^a is defined as a path ordered exponential:

$$S_i(x) = \mathcal{P} \exp\left(i g_S \int_{-\infty}^0 \mathrm{d}t\, n_i \cdot A^a(x + t n_i) T_i^a\right). \tag{3}$$

For a gluon carrying a color index c, the generator is $(T^a)_{bc} = -i f_{abc}$. For an outgoing quark (or incoming antiquark) with index ν, the generator is $(T^a)_{\mu\nu} = t^a_{\mu\nu}$ and for an incoming quark (or outgoing antiquark), the generator is $(T^a)_{\mu\nu} = -t^a_{\nu\mu}$, respectively. The soft function measures the probability of the soft emissions from N Wilson lines with the momentum component $\omega = n_N \cdot p_X$ along the recoiling jet direction and is defined as

$$\begin{aligned}S &= \sum_X \delta\left(\omega - \sum_i n_N \cdot k_i\right) \\ &\times \langle 0|(S_1 S_2 \cdots S_N)^\dagger |X\rangle \langle X|(S_1 S_2 \cdots S_N)|0\rangle,\end{aligned} \tag{4}$$

where $S_i \equiv S_i(0)$. Our soft function belongs to $\mathrm{SCET_I}$ observables which obey the non-Abelian exponentiation theorem [70,71]. At NNLO for the case of three collinear directions, $N = 3$, the soft function was calculated in [72] and recently at next-to-next-to-next-to-leading order [73]. The aim of our paper is to extend this approach for the case of four collinear directions, $N = 4$, and calculate the global soft function for a $p_1 p_2 \to p_3 p_4$ process with $p_i^2 = 0$ to the next-to-next-to-leading order in α_s. For the case of four collinear directions, the color structure is not diagonal as for the case of three collinear directions. For each color configuration, we calculate the corresponding matrix elements and perform the resummation of the large logarithms. Our manuscript is organized as follows. In Sec. II, we provide the derivation and the analytical results on the soft function involving four light-cone directions. In Sec. III, we perform renormalization and resummation. In Sec. IV, we provide the corresponding numerical results, and we conclude in Sec. V. Auxiliary relations are provided in Appendix.

II. DERIVATION

Unlike [72], instead of calculating the soft function diagram-by-diagram, we use the corresponding matrix elements. The formal form of the global soft function for i real emissions is given by

$$S_G(\omega) = \int \prod_i \frac{\mathrm{d}^d k_i}{(2\pi)^{d-1}} \delta(k_i^2)\theta(k_i^0)|\mathcal{M}|^2_{eik}[\{k_i\}] \times \delta\left(\omega - \sum_i n_4 \cdot k_i\right), \tag{5}$$

where n_4 is the lightlike vector along the momentum p_4, $p_4 = E_4 n_4$, and we have assumed that the eikonal matrix element, $|\mathcal{M}|^2_{eik}$, has already included the virtual corrections.

For later use, we employ the notation

$$[\mathrm{d}k_i] \equiv \frac{\mathrm{d}^d k_i}{(2\pi)^{d-1}} \delta(k_i^2)\theta(k_i^0). \tag{6}$$

Our situation is similar to [72], with the only difference being that in [72] there are only three eikonal directions but in our case we have four eikonal directions, n_1 to n_4, which complicates the color structure in $|\mathcal{M}|^2_{eik}$ quite a lot. The leading order soft function is $S_G^{(0)}(\omega) = \delta(\omega)$. At NLO, we only have one emission and the matrix element is given by

$$|\mathcal{M}|^2_{eik} = -g_s^2 \sum_{i,j=1,i\neq j}^{4} T_i \cdot T_j \frac{n_i \cdot n_j}{n_i \cdot k n_j \cdot k}, \tag{7}$$

and therefore:

$$S_G^{(1)}(\omega) = -g_s^2 \sum_{i,j=1,i\neq j}^{4} T_i \cdot T_j \int \frac{\mathrm{d}^d k_i}{(2\pi)^{d-1}} \delta(k^2)\theta(k^0) \times \frac{n_i \cdot n_j}{n_i \cdot k n_j \cdot k} \delta\left(\omega - \sum_i n_4 \cdot k\right) \tag{8}$$

(see [74] for all the notations, especially the color factor T_i).

For the terms in the sum corresponding to $i = 4$ or $j = 4$, the integration will be scaleless after we use the δ function to remove the integral over $n_4 \cdot k$, and hence these terms are zero. For $i \neq 4$ and $j \neq 4$, following [72], we get

$$S_G^{(1)}(\omega) = \sum_{i,j\neq 4,i\neq j} T_i \cdot T_j \frac{g_s^2}{(2\pi)^{d-1}} \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} \frac{1}{\epsilon} \left(\frac{n_{i4} n_{j4}}{2 n_{ij}}\right)^{\epsilon} \times \frac{1}{\mu}\left(\frac{\omega}{\mu}\right)^{-1-2\epsilon}, \tag{9}$$

and here we have defined $n_{ij} = n_i \cdot n_j$.

The NNLO contribution contains the double-real emission of $q\bar{q}$ and gg, the real-virtual contribution, and the exponentiation of the NLO. We consider each of these separately and summarize the results. First consider the double-real emission. In this case we have

$$S_{G,r}^{(2)}(\omega) = \int \prod_{i=1}^{2} \frac{\mathrm{d}^d k_i}{(2\pi)^{d-1}} \delta(k_i^2)\theta(k_i^0)|\mathcal{M}|^2_{eik}[\{k_i\}] \times \delta(\omega - n_4 \cdot (k_1 + k_2)). \tag{10}$$

The matrix element for $q\bar{q}$ pair is given by Eqs. (95)–(97) in [74] and for the gluon pair the matrix element is given by Eqs. (108)–(110) in [74]. As a first step we calculate the $\mathcal{I}_{ij}$ and $\mathcal{S}_{ij}$ defined in [74] by decomposing them into the functions in [72]. For the $q\bar{q}$ pair we have one term that comes from

$$\begin{aligned} S_{G,r}^{(2)}(\omega) &= (4\pi\mu^{2\epsilon}\alpha_s)^2 T_R \sum_{i,j=1}^{4} \int \prod_{i=1}^{2} \frac{\mathrm{d}^d k_i}{(2\pi)^{d-1}} \delta(k_i^2)\theta(k_i^0)\mathcal{I}_{ij}(k_1,k_2)\delta(\omega - n_4 \cdot (k_1 + k_2)) \\ &= (4\pi\mu^{2\epsilon}\alpha_s)^2 T_R \sum_{i,j=1}^{4} \int \frac{\mathrm{d}^d q}{(2\pi)^d} \theta(q^0)\theta(q^2)\delta(\omega - n_4 \cdot q)\tilde{\mathcal{I}}_{ij}(q) \\ &\quad \times \int \prod_{i=1}^{2} \frac{\mathrm{d}^d k_i}{(2\pi)^{d-1}} \delta(k_i^2)\theta(k_i^0)(2\pi)^d \delta^{(d)}(q - k_1 - k_2). \end{aligned} \tag{11}$$

Here,

$$\tilde{\mathcal{I}}_{ij}(q) = -\frac{2 n_{ij}}{n_i \cdot q n_j \cdot q (q^2)}, \tag{12}$$

where we have neglected the asymmetric part in k_1 and k_2 which will also contribute. The third line is the 2-body phase space volume for a particle with momentum q decaying into k_1 and k_2, which is

$$P_2 = 2^{-3+2\epsilon}\pi^{-1+\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}(q^2)^{-\epsilon} \tag{13}$$

[see [75], Eq. (A.1) and Eq. (A.9)].

Therefore, we have

$$\begin{aligned} S^{(2)}_{G,r}(\omega) &= -(4\pi\mu^{2\epsilon}\alpha_s)^2 T_R 2^{-2+2\epsilon}\pi^{-1+\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\frac{1}{(2\pi)^d} \\ &\times \sum_{i\neq j=1}^{4}\int \mathrm{d}^d q\theta(q^0)\theta(q^2)\delta(\omega - n_4\cdot q) \\ &\times \frac{n_{ij}}{n_i\cdot q n_j\cdot q(q^2)^{1+\epsilon}}, \end{aligned} \tag{14}$$

where the second line is I_2 of Eq. (41) in [72]. With regards to the real-virtual contribution, we have one real emission and one loop which leads to

$$\begin{aligned} S^{(2)}_{G,v}(\omega) = \int \frac{\mathrm{d}^d k_1}{(2\pi)^{d-1}}\delta(k_1^2)\theta(k_1^0)|\mathcal{M}|^2_{eik,v}[k_1] \\ \times \delta(\omega - n_4\cdot k_1), \end{aligned} \tag{15}$$

where the matrix element can be found in Eq. (26) of [76]. Once we figure out the matrix element, the integration is straightforward as NLO. For the soft quark pair emission, the matrix element is given by [74]:

$$(4\pi\alpha_s\mu^{2\epsilon})^2 T_R \sum_{i,j=1}^{n}\mathcal{I}_{ij}|\mathcal{M}_{ij}|^2, \tag{16}$$

where

$$\mathcal{I}_{ij} = -\frac{2n_{ij}k_1\cdot k_2 + n_i\cdot(k_1-k_2)n_j\cdot(k_1-k_2)}{2(k_1\cdot k_2)^2 n_i\cdot(k_1+k_2)n_j\cdot(k_1+k_2)}. \tag{17}$$

The non-Abelian piece of the double-gluon matrix element is given by [74]:

$$-C_A(4\pi\alpha_s\mu^{2\epsilon})^2\sum_{i,j=1}^{n}T_i\cdot T_j\mathcal{S}_{ij}|\mathcal{M}_{ij}|^2, \tag{18}$$

where $\mathcal{S}_{ij}$ can be written as

$$\begin{aligned} \mathcal{S}_{ij} &= \mathcal{S}^{s.o.}_{ij} - \frac{1}{2}\frac{n_i\cdot k_1 n_j\cdot k_2 + n_j\cdot k_1 n_i\cdot k_2}{n_i\cdot(k_1+k_2)n_j\cdot(k_1+k_2)}\mathcal{S}^{s.o.}_{ij} \\ &+ (1-\epsilon)\mathcal{I}_{ij} - (1+\epsilon)\frac{n_{ij}}{k_1\cdot k_2 n_i\cdot(k_1+k_2)n_j\cdot(k_1+k_2)} \end{aligned} \tag{19}$$

with the strong-ordering limit being

$$\begin{aligned} \mathcal{S}^{s.o.}_{ij} &= \frac{n_{ij}}{k_1\cdot k_2}\left(\frac{1}{n_i\cdot k_1 n_j\cdot k_2} + i\leftrightarrow j\right) \\ &- \frac{n_{ij}^2}{n_i\cdot k_1 n_j\cdot k_1 n_i\cdot k_2 n_j\cdot k_2}. \end{aligned} \tag{20}$$

With regards to the double-gluon emission, we first focus on the term

$$\mathcal{S}^{s.o.}_{ij} - \frac{1}{2}\frac{n_i\cdot k_1 n_j\cdot k_2 + n_j\cdot k_1 n_i\cdot k_2}{n_i\cdot(k_1+k_2)n_j\cdot(k_1+k_2)}\mathcal{S}^{s.o.}_{ij}, \tag{21}$$

which can be decomposed to

$$\frac{n_{ij}}{q^2}\times\left(\frac{n_j\cdot(k_1+2k_2)}{n_j\cdot q n_i\cdot k_1 n_j\cdot k_2} + \frac{n_j\cdot(k_1-k_2)}{n_i\cdot q n_j\cdot q n_j\cdot k_2}\right) + i\leftrightarrow j \tag{22}$$

and

$$-\frac{n_{ij}^2}{n_i\cdot k_1 n_j\cdot k_1 n_i\cdot k_2 n_j\cdot k_2} \tag{23}$$

and

$$2\frac{n_{ij}}{2n_i\cdot q n_j\cdot q}\left(\frac{n_{ij}}{2n_i\cdot k_1 n_j\cdot k_2} + i\leftrightarrow j\right), \tag{24}$$

where we have defined $q = k_1 + k_2$.

Therefore we have

$$\boxed{S^{gg}_{G,ij,1}(\omega) = \left(\frac{1}{2\pi}\right)^{2d-2}(2I_{7,2} + 2I_6 - I_4) + 2I_3^{s.o.}} \tag{25}$$

where $I_{7,2}$, I_6, and I_4 are given in [72] and $I_3^{s.o.}$ is given by

$$\begin{aligned} I_3^{s.o.} &= 2\int \mathrm{d}^d q\delta(\omega - n_4\cdot q)\frac{n_{ij}}{2n_i\cdot q n_j\cdot q} \\ &\times \int [\mathrm{d}k_1][\mathrm{d}k_2]\frac{n_{ij}}{2n_i\cdot k_1 n_j\cdot k_2}\delta^{(d)}(q-k_1-k_2), \end{aligned} \tag{26}$$

which is:

$$\begin{aligned} & 2\pi^{1-\epsilon}\left[\frac{1}{2\pi}\right]^{2d-2}\frac{\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Omega_{d-3}}{4}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+ n_4^-}\right)^{-2\epsilon} \\ & \times 4^{1+2\epsilon}\sqrt{\pi}\frac{\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(1-\epsilon)}\frac{1}{2}\sqrt{\pi}\frac{\Gamma(-2\epsilon)}{\Gamma(\frac{1}{2}-2\epsilon)} \\ & \times \int \mathrm{d}u^2\times(1-u^2)^{-\epsilon}(u^2)^{-1-2\epsilon} \\ & \times {}_2F_1(-\epsilon,-\epsilon,1-\epsilon,1-u^2)\,{}_2F_1(-2\epsilon,-2\epsilon,1-\epsilon,u^2). \end{aligned} \tag{27}$$

Expanding the hypergeometric function and performing the integration over u^2, we get

$$I_3^{s.o.} = \frac{1}{4\pi^2}\frac{c_R}{16\pi^2}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+ n_4^-}\right)^{-2\epsilon}\left(-\frac{1}{2\epsilon^3}+\frac{7\pi^2}{12\epsilon}+\frac{43}{3}\zeta_3+\frac{121\pi^4}{720}\epsilon\right). \tag{28}$$

Here we have defined $c_R = (4\pi)^{2\epsilon}e^{-2\epsilon\gamma_E}$. Therefore, we have

$$S^{gg}_{G,ij,1}(\omega) = \frac{1}{4\pi^2}\frac{c_R}{16\pi^2}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+ n_4^-}\right)^{-2\epsilon} \times\left[-\frac{2}{\epsilon^3}-\frac{2}{\epsilon^2}+\left(-4+\frac{4}{3}\pi^2\right)\frac{1}{\epsilon}+\left(-8+\pi^2+\frac{202}{3}\zeta_3\right)+\left(-16+2\pi^2+\frac{124}{3}\zeta_3+\frac{229}{180}\pi^4\right)\epsilon\right]. \tag{29}$$

Now we turn to the second line of $\mathcal{S}_{ij}$, which is

$$(1-\epsilon)\mathcal{I}_{ij} + (1+\epsilon)\frac{-2n_{ij}}{q^2 n_i\cdot q n_j\cdot q}. \tag{30}$$

The second term is given by

$$(1+\epsilon)\times\left(-\frac{2n_{ij}}{q^2 n_i\cdot q n_j\cdot q}\right) = (1+\epsilon)\times\tilde{\mathcal{I}}_{ij}(q). \tag{31}$$

Therefore we find

$$S^{gg}_{G,ij,2,+}(\omega) = -(1+\epsilon)\times 2^{-2+2\epsilon}\pi^{-1+\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\frac{1}{(2\pi)^d}\times I_2, \tag{32}$$

which is

$$S^{gg}_{G,ij,2,+}(\omega) = \frac{1}{4\pi^2}\frac{c_R}{16\pi^2}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+ n_4^-}\right)^{-2\epsilon}\left(-\frac{1}{\epsilon^2}-\frac{3}{\epsilon}-6+\frac{\pi^2}{2}+\left(-12+\frac{3\pi^2}{2}+\frac{62}{3}\zeta_3\right)\epsilon\right). \tag{33}$$

For the first term, it can be related to the $|\mathcal{M}|^2$ for producing a $q\bar{q}$ pair:

$$\int\frac{d^d q}{(2\pi)^d}\delta(\omega - n_4\cdot q)\int d\mathrm{PS}\mathcal{I}_{ij}(2\pi)^d\delta^{(d)}(q-k_1-k_2) = 2\int\frac{d^d q}{(2\pi)^d}\delta(\omega-n_4\cdot q)\mathrm{Im}\Pi_{\mu\nu}(q\to q)\frac{n_i^\mu}{n\cdot q}\frac{n_j^\nu}{n_j\cdot q}\frac{1}{(q^2)^2}. \tag{34}$$

The sign on the right-hand side of the equation is determined from the fact that the matrix element involving the virtual quark loop has an additional factor of $(-i)^2 i^2 = 1$ (vertices plus virtual photon propagators). This gives

$$\begin{aligned}
&2(-8)\frac{1}{(4\pi)^{2-\epsilon}}\Gamma(\epsilon)\frac{\Gamma^2(2-\epsilon)}{\Gamma(4-2\epsilon)}\mathrm{Im}[e^{i\pi\epsilon}]\int\frac{d^d q}{(2\pi)^d}\delta(\omega-n_4\cdot q)(q^2 g_{\mu\nu}-q_\mu q_\nu)(q^2)^{-\epsilon}\frac{n_i^\mu}{n\cdot q}\frac{n_j^\nu}{n_j\cdot q}\frac{1}{(q^2)^2}\\
&= 2(-8)\frac{1}{(4\pi)^{2-\epsilon}}\Gamma(\epsilon)\frac{\Gamma^2(2-\epsilon)}{\Gamma(4-2\epsilon)}\mathrm{Im}[e^{i\pi\epsilon}]\int\frac{d^d q}{(2\pi)^d}\delta(\omega-n_4\cdot q)\frac{n_{ij}}{n\cdot q}\frac{1}{n_j\cdot q}\frac{1}{(q^2)^{1+\epsilon}}\\
&= 2(-8)\frac{1}{(4\pi)^{2-\epsilon}}\Gamma(\epsilon)\frac{\Gamma^2(2-\epsilon)}{\Gamma(4-2\epsilon)}\sin[\pi\epsilon]\frac{1}{(2\pi)^d}\times I_2.
\end{aligned} \tag{35}$$

Therefore we have from the first $(1-\epsilon)\mathcal{I}_{ij}$ term:

$$S^{gg}_{G,ij,2,qq}(\omega) = \frac{1}{4\pi^2}\frac{c_R}{16\pi^2}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+ n_4^-}\right)^{-2\epsilon}\left(-\frac{2}{3\epsilon^2}-\frac{4}{9\epsilon}-\frac{26}{27}+\frac{\pi^2}{3}+\left[\frac{2\pi^2}{9}-\frac{160}{81}+\frac{124}{9}\zeta_3\right]\epsilon\right). \tag{36}$$

Regarding the $q\bar{q}$ pair emission, we have from the $\mathcal{I}_{ij}$ contribution:

$$S^{q\bar{q}}_{G,ij}(\omega) = \frac{1}{4\pi^2}\frac{c_R}{16\pi^2}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+ n_4^-}\right)^{-2\epsilon}\left(-\frac{2}{3\epsilon^2} - \frac{10}{9\epsilon} - \frac{56}{27} + \frac{\pi^2}{3} + \left[\frac{5\pi^2}{9} - \frac{328}{81} + \frac{124}{9}\zeta_3\right]\epsilon\right). \tag{37}$$

The real-virtual matrix element is given by

$$-(g_s\mu^\epsilon)^4 \times \left[-\frac{1}{4\pi^2}\frac{(4\pi)^\epsilon}{\epsilon^2}\frac{\Gamma^3(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)}(S_2+S_3)\right] = (4\pi\alpha_s\mu^{2\epsilon})^2 \times \left[\frac{1}{4\pi^2}\frac{(4\pi)^\epsilon}{\epsilon^2}\frac{\Gamma^3(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)}(S_2+S_3)\right], \tag{38}$$

where

$$S_2 = C_A \cos(\pi\epsilon)\sum_{i\neq j}[S_{ij}(q)]^{1+\epsilon}|\mathcal{M}_{i,j}(p)|^2 \tag{39}$$

and

$$S_3 = 2\,\sin(\pi\epsilon)\sum_{i\neq j\neq k} S_{ki}(q)[S_{ij}(q)]^\epsilon(\lambda_{ij} - \lambda_{iq} - \lambda_{jq})|\mathcal{M}_{k,i,j}(p)|^2. \tag{40}$$

Here,

$$S_{ij}(q) \equiv \frac{n_{ij}}{2n_i\cdot q n_j\cdot q}|\mathcal{M}_{k,i,j}(p)|^2 = f_{abc}\langle\mathcal{M}^0(p)|T_k^a T_i^b T_j^c|\mathcal{M}^0(p)\rangle \tag{41}$$

and $\lambda_{AB} = 1$ if A and B are both incoming or outgoing, otherwise $\lambda_{AB} = 0$. Integrating over the phase space gives

$$\begin{aligned}&\int[\mathrm{d}q]\delta(\omega - n_4\cdot q)S_2\\ &\quad= C_A\cos(\pi\epsilon)\int[\mathrm{d}q]\delta(\omega - n_4\cdot q)[S_{ij}(q)]^{1+\epsilon}\\ &\quad= C_A\cos(\pi\epsilon)\left(\frac{1}{2\pi}\right)^{d-1}\times I_{7,1}.\end{aligned} \tag{42}$$

Therefore, its contribution, except for the $(4\pi\alpha_s\mu^{2\epsilon})^2$ factor, is given by

$$\begin{aligned}S_{G,rv,2} = {}&\frac{C_A}{4\pi^2}\frac{c_R}{16\pi^2}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+n_4^-}\right)^{-2\epsilon}\\ &\times\left(-\frac{1}{\epsilon^3} + \frac{\pi^2}{2\epsilon} + \frac{80}{3}\zeta_3 + \frac{47\pi^4}{120}\epsilon\right).\end{aligned} \tag{43}$$

Now we turn to S_3 which only contributes to four or more external hard partons. S_3 vanishes since the origin of the S_3 term is the 3-gluon vertex and the antisymmetric part of the other pieces connected to three different eikonal lines. Finally, the coupling renormalization term is given by

$$-\frac{\alpha_s}{2\pi}\times\frac{11C_A - 2n_F}{6\epsilon}\times\mathrm{NLO}. \tag{44}$$

A. Summary of results

We summarize our results as follows. The NLO soft function is given by

$$S_G^{(1)}(\omega,\mu) = \sum_{i\neq j, i,j\neq 4} T_i\cdot T_j S_{G,ij}^{(1)}(\omega,\mu), \tag{45}$$

where

$$S_{G,ij}^{(1)}(\omega,\mu) = \frac{\alpha_s}{2\pi}L_{ij}^{(1)}\left(\frac{2}{\epsilon} - \frac{\pi^2}{6}\epsilon - \frac{14\zeta_3}{3}\epsilon^2 + O(\epsilon^3)\right) \tag{46}$$

with

$$L_{ij}^{(1)} = \left(\frac{n_{i4}n_{j4}}{2n_{ij}}\right)^\epsilon\frac{1}{\mu}\left(\frac{\omega}{\mu}\right)^{-1-2\epsilon}. \tag{47}$$

1. NNLO contribution

The NNLO soft function is given by

$$\begin{aligned}&S_G^{(2)}(\omega,\mu)\\ &\quad= \frac{1}{4}\sum_{i\neq j,i,j\neq 4}\sum_{k\neq l,k,l\neq 4}\{T_i\cdot T_j, T_k\cdot T_l\}S_{ij,kl}^{ab}(\omega,\mu)\\ &\qquad+\sum_{i\neq j,i,j\neq 4}T_i\cdot T_j(S_{ij}^{C_A}(\omega,\mu) + S_{ij}^{N_F}(\omega,\mu) + S_{ij}^{ren}(\omega,\mu)).\end{aligned} \tag{48}$$

The individual contributions are given by

$$S^{ab}_{ij,kl}(\omega,\mu) = \left(\frac{\alpha_s}{2\pi}\right)^2 L_{ij,kl}\left[-\frac{4}{\epsilon^3}+\frac{10\pi^2}{3\epsilon}+\frac{248\zeta_3}{3} + \frac{25\pi^4}{18}\epsilon + O(\epsilon^2)\right], \tag{49}$$

$$S^{C_A}_{ij}(\omega,\mu) = \frac{\alpha_s^2 C_A}{4\pi^2} L^{(2)}_{ij}\left[\frac{11}{6\epsilon^2}+\frac{1}{\epsilon}\left(\frac{67}{18}-\frac{\pi^2}{6}\right) + \frac{202}{27}-\frac{11\pi^2}{12}-7\zeta_3 + \left(\frac{1214}{81}-\frac{67\pi^2}{36}-\frac{11\pi^4}{45}-\frac{341}{9}\zeta_3\right)\epsilon + O(\epsilon^2)\right], \tag{50}$$

$$S^{N_F}_{ij}(\omega,\mu) = \frac{\alpha_s^2 T_R N_F}{4\pi^2} L^{(2)}_{ij}\left(-\frac{2}{3\epsilon^2}-\frac{10}{9\epsilon}-\frac{56}{27}+\frac{\pi^2}{3} + \left[\frac{5\pi^2}{9}-\frac{328}{81}+\frac{124}{9}\zeta_3\right]\epsilon + O(\epsilon^2)\right), \tag{51}$$

$$S^{ren}_{ij}(\omega,\mu) = \frac{\alpha_s^2}{4\pi^2}\frac{11C_A-2N_F}{3} \times L^{(1)}_{ij}\left(-\frac{1}{\epsilon^2}+\frac{\pi^2}{12}+\frac{7}{3}\zeta_3\epsilon + O(\epsilon^2)\right). \tag{52}$$

Here,

$$L^{(2)}_{ij} = \frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+ n_4^-}\frac{1}{\mu^2}\right)^{-2\epsilon} = \left(\frac{n_{i4}n_{j4}}{2n_{ij}}\right)^{2\epsilon}\frac{1}{\mu}\left(\frac{\omega}{\mu}\right)^{-1-4\epsilon} \tag{53}$$

and

$$L_{ij,kl} = \left(\frac{n_{i4}n_{j4}}{2n_{ij}}\right)^{\epsilon}\left(\frac{n_{k4}n_{l4}}{2n_{kl}}\right)^{\epsilon}\frac{1}{\mu}\left(\frac{\omega}{\mu}\right)^{-1-4\epsilon}. \tag{54}$$

B. Plus expansion of our results in ϵ

We perform the plus expansion of our results with the use of the formula

$$\frac{1}{x^{1+\alpha}} = -\frac{\delta(x)}{\alpha} + \sum_{n=0}^{\infty}(-\alpha)^n\left(\frac{\theta(x)\ln^n(x)}{x}\right)_+. \tag{55}$$

We use the conventions $(ij) = \ln(\frac{n_{i4}n_{j4}}{2n_{ij}})$ and $(ijkl) = \ln(\frac{n_{i4}n_{j4}n_{k4}n_{l4}}{4n_{ij}n_{kl}})$. We show that

$$S^{(1)}_G(\omega,\mu) = \sum_{i\neq j;i,j\neq 4}(T_i\cdot T_j)\frac{\alpha_S}{2\pi}\left[-\frac{1}{\epsilon^2}\delta(\omega)+\frac{1}{\epsilon}\left(2\left(\frac{\theta(\omega)}{\omega}\right)_+ - \delta(\omega)(ij)\right) + \delta(\omega)\left(\frac{\pi^2}{12}-\frac{1}{2}(ij)^2\right) + 2(ij)\left(\frac{\theta(\omega)}{\omega}\right)_+ - 4\left(\frac{\ln(\frac{\omega}{\mu})}{\omega}\right)_+\right]. \tag{56}$$

The divergent and finite parts are given by

$$S^{ab}_{ij,kl}|_{\text{Divergent}} = \frac{\alpha_S^2}{4\pi^2}\sum_{i\neq j,i,j\neq 4}^{4}\sum_{k\neq l,k,l\neq 4}^{4}\{T_i\cdot T_j, T_k\cdot T_l\}\left[\frac{\delta(\omega)}{2\epsilon^4}+\frac{1}{2\epsilon^3}\left(\delta(\omega)(ijkl)-4\left(\frac{\theta(\omega)}{\omega}\right)_+\right) + \frac{1}{\epsilon^2}\left(-2\left(\frac{\theta(\omega)}{\omega}\right)_+(ijkl)+\delta(\omega)\left(-\frac{5\pi^2}{12}+\frac{1}{4}(ijkl)^2\right)+8\left(\frac{\ln(\frac{\omega}{\mu})}{\omega}\right)_+\right) + \frac{1}{\epsilon}\left(\delta(\omega)\left(-\frac{31\zeta_3}{3}-\frac{5\pi^2(ijkl)}{12}+\frac{(ijkl)^3}{12}\right)+\left(\frac{\theta(\omega)}{\omega}\right)_+\left(\frac{5\pi^2}{3}-(ijkl)^2\right)\right) + \frac{1}{\epsilon}\left(8(ijkl)\left(\frac{\ln(\frac{\omega}{\mu})}{\omega}\right)_+ - 16\left(\frac{\ln^2(\frac{\omega}{\mu})}{\omega}\right)_+\right)\right], \tag{57}$$

$$S^{ab}_{ij,kl}|_{\text{Finite}} = \frac{\alpha_S^2}{4\pi^2}\sum_{i\neq j,i,j\neq 4}^{4}\sum_{k\neq l,k,l\neq 4}^{4}\{T_i\cdot T_j, T_k\cdot T_l\}\left[\delta(\omega)\left(\frac{(ijkl)^4}{48}-\frac{25\pi^4}{144}-\frac{31\zeta_3}{3}(ijkl)-\frac{5\pi^2}{24}(ijkl)^2\right) + \left(\frac{\theta(\omega)}{\omega}\right)_+\left(\frac{124\zeta_3}{3}+\frac{5\pi^2}{3}(ijkl)-\frac{(ijkl)^3}{3}\right)+\left(\frac{\ln(\frac{\omega}{\mu})}{\omega}\right)_+\left(4(ijkl)^2-\frac{20\pi^2}{3}\right) -16(ijkl)\left(\frac{\ln^2(\frac{\omega}{\mu})}{\omega}\right)_+ + \frac{64}{3}\left(\frac{\ln^3(\frac{\omega}{\mu})}{\omega}\right)_+\right], \tag{58}$$

$$S_{ij}^{C_A}|_{\text{Divergent}} = \frac{\alpha_S^2 C_A}{4\pi^2} \sum_{i\neq j;i,j\neq 4}^{4} (T_i \cdot T_j)\Bigg[-\frac{11\delta(\omega)}{24\epsilon^3} + \frac{\delta(\omega)}{\epsilon^2}\left(\frac{\pi^2}{24} - \frac{67}{72}\right) + \frac{\delta(\omega)}{\epsilon}\left(\frac{11\pi^2}{48} + \frac{7\zeta_3}{4} - \frac{101}{54} + \left(\frac{\pi^2}{12} - \frac{67}{36}\right)\cdot(ij) - \frac{11}{12}(ij)^2\right) + \frac{1}{\epsilon}\left(\frac{67}{18} - \frac{\pi^2}{6} + \frac{11}{3}(ij)\right)\left(\frac{\theta(\omega)}{\omega}\right)_+ - \frac{22}{3\epsilon}\left(\frac{\ln(\frac{\omega}{\mu})}{\omega}\right)_+\Bigg], \tag{59}$$

$$S_{ij}^{C_A}|_{\text{Finite}} = \frac{\alpha_S^2 C_A}{4\pi^2} \sum_{i\neq j;i,j\neq 4}^{4} (T_i \cdot T_j)\Bigg[\left(\frac{\theta(\omega)}{\omega}\right)_+ \left(\frac{202}{27} - \frac{11\pi^2}{12} - 7\zeta_3 + (ij)\left(\frac{67}{9} - \frac{\pi^2}{3}\right) + \frac{11}{3}(ij)^2\right) + \delta(\omega)\left(\frac{341\zeta_3}{36} + \frac{11\pi^4}{180} + \frac{67\pi^2}{144} - \frac{607}{162} + (ij)\left(\frac{7\zeta_3}{2} + \frac{11\pi^2}{24} - \frac{101}{27}\right) + (ij)^2\left(\frac{\pi^2}{12} - \frac{67}{36}\right) - \frac{11}{18}(ij)^3\right) + \left(\frac{\ln(\frac{\omega}{\mu})}{\omega}\right)_+ \left(\frac{2\pi^2}{3} - \frac{134}{9} - \frac{44}{3}(ij)\right) + \frac{44}{3}\left(\frac{\ln^2(\frac{\omega}{\mu})}{\omega}\right)_+\Bigg], \tag{60}$$

$$S_{ij}^{N_F}|_{\text{Divergent}} = \frac{\alpha_S^2 T_R N_F}{4\pi^2} \sum_{i\neq j;i,j\neq 4}^{4} (T_i \cdot T_j)\Bigg[\frac{\delta(\omega)}{6\epsilon^3} + \frac{1}{\epsilon^2}\left(\left[\frac{5}{18} + \frac{(ij)}{3}\right]\delta(\omega) - \frac{2}{3}\left(\frac{\theta(\omega)}{\omega}\right)_+\right) + \frac{1}{\epsilon}\left(\left(\frac{14}{27} - \frac{\pi^2}{12} + \frac{5}{9}(ij) + \frac{(ij)^2}{3}\right)\delta(\omega) - \left(\frac{10}{9} + \frac{4(ij)}{9}\right)\left(\frac{\theta(\omega)}{\omega}\right)_+ + \frac{8}{3}\left(\frac{\ln(\frac{\omega}{\mu})}{\omega}\right)_+\right)\Bigg], \tag{61}$$

$$S_{ij}^{N_F}|_{\text{Finite}} = \frac{\alpha_S^2 T_R N_F}{4\pi^2} \sum_{i\neq j;i,j\neq 4}^{4} (T_i \cdot T_j)\Bigg[\left(\frac{\theta(\omega)}{\omega}\right)_+ \left(\left(\frac{\pi^2}{3} - \frac{56}{27}\right) - \frac{20}{9}(ij) - \frac{4}{3}(ij)^2\right) + \delta(\omega)\left(\left(\frac{82}{81} - \frac{5\pi^2}{36} - \frac{31\zeta_3}{9}\right) + (ij)\left(\frac{28}{27} - \frac{\pi^2}{6}\right) + \frac{5}{9}(ij)^2 + \frac{2}{9}(ij)^3\right) + \left(\frac{\ln(\frac{\omega}{\mu})}{\omega}\right)_+ \left(\frac{40}{9} + \frac{16}{3}(ij)\right) - \frac{16}{3}\left(\frac{\ln^2(\frac{\omega}{\mu})}{\omega}\right)_+\Bigg], \tag{62}$$

$$S_{ij}^{ren}|_{\text{Divergent}} = \frac{\alpha_S^2}{4\pi^2}\cdot\frac{11C_A - 2N_F}{3} \sum_{i\neq j;i,j\neq 4}^{4} (T_i \cdot T_j)\Bigg[\frac{\delta(\omega)}{2\epsilon^3} + \frac{1}{\epsilon^2}\left(\frac{(ij)\delta(\omega)}{2} - \left(\frac{\theta(\omega)}{\omega}\right)_+\right) + \frac{1}{\epsilon}\left(\left(\frac{(ij)^2}{4} - \frac{\pi^2}{24}\right)\delta(\omega) + 2\left(\frac{\ln(\frac{\omega}{\mu})}{\omega}\right)_+ - (ij)\left(\frac{\theta(\omega)}{\omega}\right)_+\right)\Bigg], \tag{63}$$

$$S_{ij}^{ren}|_{\text{Finite}} = \frac{\alpha_S^2}{4\pi^2}\cdot\frac{11C_A - 2N_F}{3} \sum_{i\neq j;i,j\neq 4}^{4} (T_i \cdot T_j)\cdot\Bigg[\left(\frac{\pi^2}{12} - \frac{(ij)^2}{2}\right)\left(\frac{\theta(\omega)}{\omega}\right)_+ + \delta(\omega)\left(-\frac{7\zeta_3}{6} - \frac{\pi^2(ij)}{24} + \frac{(ij)^3}{12}\right) + 2(ij)\left(\frac{\ln(\frac{\omega}{\mu})}{\omega}\right)_+ - 2\left(\frac{\ln^2(\frac{\omega}{\mu})}{\omega}\right)_+\Bigg]. \tag{64}$$

III. RENORMALIZATION AND RESUMMATION

In this section we perform the renormalization and resummation which can be conveniently done in Laplace space. Therefore, we perform the Laplace transformation first:

$$\tilde{S}(L,\mu) = \int_0^\infty \mathrm{d}\omega\, e^{-\rho\omega} S(\omega,\mu), \quad \rho = \frac{1}{\xi e^{\gamma_E}}. \tag{65}$$

The Laplace-transformed soft function is expressed in terms of the logarithm of the Laplace-space variable $L = \ln(\frac{\mu}{\xi})$. The Laplace transformation is performed using an identity,

$$\int_0^\infty \mathrm{d}\omega\, e^{-\rho\omega}\omega^{-1-n\epsilon} = e^{-n\epsilon\gamma_E}\Gamma(-n\epsilon)\xi^{-n\epsilon}, \tag{66}$$

with $\gamma_E \approx 0.577$ being the Euler constant. For further convenience, we split the scale L and kinematics parts l_{ij} of the logarithms:

$$L_{ij} = \ln\left(\frac{\mu}{\xi}\sqrt{\frac{n_{i4}n_{j4}}{2n_{ij}}}\right) = \underbrace{\ln\left(\frac{\mu}{\xi}\right)}_{=L} + \underbrace{\ln\left(\sqrt{\frac{n_{i4}n_{j4}}{2n_{ij}}}\right)}_{=l_{ij}} = L + l_{ij}, \tag{67}$$

$$L_{ij,kl} = \ln\left(\frac{\mu}{\xi}\sqrt[4]{\frac{n_{i4}n_{j4}n_{k4}n_{l4}}{4n_{ij}n_{kl}}}\right) = \underbrace{\ln\left(\frac{\mu}{\xi}\right)}_{=L} + \underbrace{\frac{1}{2}\ln\left(\sqrt{\frac{n_{i4}n_{j4}}{2n_{ij}}}\right)}_{=\frac{1}{2}l_{ij}} + \underbrace{\frac{1}{2}\ln\left(\sqrt{\frac{n_{k4}n_{l4}}{2n_{kl}}}\right)}_{=\frac{1}{2}l_{kl}} = L + \frac{1}{2}l_{ij} + \frac{1}{2}l_{kl}. \tag{68}$$

NLO contribution takes the form

$$\tilde{S}_G^{(1)}(\xi,\mu) = \frac{\alpha_S}{2\pi}\sum_{i\neq j;i,j\neq 4}^{4}(T_i\cdot T_j)\tilde{S}_{G,ij}^{(1)}(\xi,\mu), \tag{69}$$

$$\begin{aligned}\tilde{S}_{G,ij}^{(1)}(\xi,\mu) = &-\frac{1}{\epsilon^2} - \frac{2L_{ij}}{\epsilon} - 2L_{ij}^2 - \frac{\pi^2}{4} + \epsilon\left(\frac{4L_{ij}^2}{3} - \frac{\pi^2 L_{ij}}{2} - \frac{\zeta_3}{3}\right)\\ &- \epsilon^2\left(\frac{2L_{ij}^4}{3} + \frac{\pi^2}{2}L_{ij}^2 + \frac{2\zeta_3}{3}L_{ij} + \frac{19\pi^4}{480}\right) + O(\epsilon^3).\end{aligned} \tag{70}$$

NNLO contribution consists of four parts:

$$\tilde{S}_G^{(2)}(\xi,\mu) = \tilde{S}^{ab}(\xi,\mu) + \tilde{S}^{C_A}(\xi,\mu) + \tilde{S}^{N_f}(\xi,\mu) + \tilde{S}^{Coup\,Ren}(\xi,\mu). \tag{71}$$

The bare NNLO soft function has the form:

$$\tilde{S}_G(\xi)^{\mathrm{Bare}} = \hat{1} + \tilde{S}_G^{(1)}(\xi,\mu) + \tilde{S}_G^{(2)}(\xi,\mu) + O(\alpha_S^3), \tag{72}$$

where $\hat{1}$ is a unit matrix in a color space. The complete NNLO soft function is obtained by multiplying Eq. (72) by the tree-level color matrix. These matrices for all channels are provided in Appendix B. The Abelian part of NNLO has the form:

$$\tilde{S}^{ab}(\xi,\mu) = \tilde{S}^{ab}(\xi,\mu)_{\mathrm{Divergent}} + \tilde{S}^{ab}(\xi,\mu)_{\mathrm{Finite}}, \tag{73}$$

$$\tilde{S}^{ab}(\xi,\mu)_{\mathrm{Divergent}} = \frac{\alpha_S^2}{4\pi^2}\sum_{i\neq j,i,j\neq 4}^{4}\sum_{k\neq l,k,l\neq 4}^{4}\{T_i\cdot T_j, T_k\cdot T_l\}\left[\frac{1}{4\epsilon^4} + \frac{L_{ij,kl}}{\epsilon^3} + \frac{2L_{ij,kl}^2}{\epsilon^2} + \frac{\pi^2}{8\epsilon^2} + \frac{8L_{ij,kl}^3}{3\epsilon} + \frac{\pi^2 L_{ij,kl}}{2\epsilon} + \frac{\zeta_3}{6\epsilon}\right], \tag{74}$$

$$\tilde{S}^{ab}(\xi,\mu)_{\mathrm{Finite}} = \frac{\alpha_S^2}{4\pi^2}\sum_{i\neq j,i,j\neq 4}^{4}\sum_{k\neq l,k,l\neq 4}^{4}\{T_i\cdot T_j, T_k\cdot T_l\}\left[\frac{8L_{ij,kl}^4}{3} + \pi^2 L_{ij,kl}^2 + \frac{2\zeta_3 L_{ij,kl}}{3} + \frac{17\pi^4}{480}\right]. \tag{75}$$

The double-gluon emission and real-virtual contribution are given by:

$$\tilde{S}^{C_A}(\xi,\mu) = \tilde{S}^{C_A}(\xi,\mu)_{\mathrm{Divergent}} + \tilde{S}^{C_A}(\xi,\mu)_{\mathrm{Finite}}, \tag{76}$$

$$\tilde{S}^{C_A}(\xi,\mu)_{\text{Divergent}} = \frac{\alpha_S^2 C_A}{4\pi^2}\sum_{i\neq j;i,j\neq 4}^{4}(T_i\cdot T_j)\left[-\frac{11}{24\epsilon^3}-\frac{11L_{ij}}{6\epsilon^2}-\frac{67}{72\epsilon^2}+\frac{\pi^2}{24\epsilon^2}-\frac{11L_{ij}^2}{3\epsilon}-\frac{67L_{ij}}{18\epsilon}+\frac{\pi^2 L_{ij}}{6\epsilon}+\frac{7\zeta_3}{4\epsilon}-\frac{101}{54\epsilon}-\frac{55\pi^2}{144\epsilon}\right], \tag{77}$$

$$\tilde{S}^{C_A}(\xi,\mu)_{\text{Finite}} = \frac{\alpha_S^2 C_A}{4\pi^2}\sum_{i\neq j;i,j\neq 4}^{4}(T_i\cdot T_j)\left[-\frac{44L_{ij}^3}{9}+\frac{\pi^2 L_{ij}^2}{3}-\frac{67L_{ij}^2}{9}+7\zeta_3 L_{ij}-\frac{55\pi^2}{36}L_{ij}-\frac{202}{27}L_{ij}-\frac{11\zeta_3}{36}+\frac{7\pi^4}{60}-\frac{335\pi^2}{432}-\frac{607}{162}\right]. \tag{78}$$

The $q\bar{q}$ emission contribution has the form:

$$\tilde{S}^{N_F}(\xi,\mu) = \tilde{S}^{N_F}(\xi,\mu)_{\text{Divergent}} + \tilde{S}^{N_F}(\xi,\mu)_{\text{Finite}}, \tag{79}$$

$$\tilde{S}^{N_F}(\xi,\mu)_{\text{Divergent}} = \frac{\alpha_S^2 N_F T_R}{4\pi^2}\sum_{i\neq j;i,j\neq 4}(T_i\cdot T_j)\left[\frac{1}{6\epsilon^3}+\frac{2L_{ij}}{3\epsilon^2}+\frac{5}{18\epsilon^2}+\frac{4L_{ij}^2}{3\epsilon}+\frac{10L_{ij}}{9\epsilon}+\frac{14}{27\epsilon}+\frac{5\pi^2}{36\epsilon}\right], \tag{80}$$

$$\tilde{S}^{N_F}(\xi,\mu)_{\text{Finite}} = \frac{\alpha_S^2 N_F T_R}{4\pi^2}\sum_{i\neq j;i,j\neq 4}(T_i\cdot T_j)\left[\frac{16L_{ij}^3}{9}+\frac{20L_{ij}^2}{9}+\frac{5\pi^2 L_{ij}}{9}+\frac{56L_{ij}}{27}+\frac{\zeta_3}{9}+\frac{25\pi^2}{108}+\frac{82}{81}\right]. \tag{81}$$

Finally, the renormalization of the coupling constant is:

$$\tilde{S}^{Coup\,Ren}(\xi,\mu) = \tilde{S}^{Coup\,Ren}(\xi,\mu)_{\text{Divergent}} + \tilde{S}^{Coup\,Ren}(\xi,\mu)_{\text{Finite}}, \tag{82}$$

$$\tilde{S}^{Coup\,Ren}(\xi,\mu)_{\text{Divergent}} = \frac{\alpha_S^2\beta_0}{4\pi^2}\sum_{i\neq j;i,j\neq 4}(T_i\cdot T_j)\left[\frac{1}{2\epsilon^3}+\frac{L_{ij}}{\epsilon^2}+\frac{L_{ij}^2}{\epsilon}+\frac{\pi^2}{8\epsilon}\right], \tag{83}$$

$$\tilde{S}^{Coup\,Ren}(\xi,\mu)_{\text{Finite}} = \frac{\alpha_S^2\beta_0}{4\pi^2}\sum_{i\neq j;i,j\neq 4}(T_i\cdot T_j)\left[\frac{2L_{ij}^3}{3}+\frac{\pi^2 L_{ij}}{4}+\frac{\zeta_3}{6}\right]. \tag{84}$$

The soft function is multiplicatively renormalized as

$$\tilde{S}_G(\xi,\mu)^{\text{Renormalized}} = Z(\xi,\mu)\tilde{S}_G(\xi)^{\text{Bare}}(Z(\xi,\mu))^\dagger, \tag{85}$$

$$\begin{aligned} Z(\xi,\mu) = 1 &- \frac{\alpha_S}{2\pi}\sum_{i\neq j;i,j\neq 4}(T_i\cdot T_j)\left(\frac{\Gamma_0}{2\epsilon^2}+\frac{(2L_{ij}-i\pi\lambda_{ij})\Gamma_0+2\gamma_0}{2\epsilon}\right)\\ &+\left(\frac{\alpha_S}{2\pi}\right)^2\sum_{i\neq j;i,j\neq 4}(T_i\cdot T_j)\left(\frac{3\beta_0\Gamma_0}{16\epsilon^3}-\frac{\Gamma_1-\beta_0((2L_{ij}-i\pi\lambda_{ij})\Gamma_0+2\gamma_0)}{8\epsilon^2}-\frac{(2L_{ij}-i\pi\lambda_{ij})\Gamma_1+2\gamma_1}{4\epsilon}\right)\\ &+\frac{1}{4}\left(\frac{\alpha_S}{2\pi}\right)^2\sum_{i\neq j;i,j\neq 4}\sum_{k\neq l;k,l\neq 4}\{T_i\cdot T_j,T_k\cdot T_l\}\left(\frac{\Gamma_0}{2\epsilon^2}+\frac{(2L_{ij}-i\pi\lambda_{ij})\Gamma_0+2\gamma_0}{2\epsilon}\right)\left(\frac{\Gamma_0}{2\epsilon^2}+\frac{(2L_{kl}-i\pi\lambda_{kl})\Gamma_0+2\gamma_0}{2\epsilon}\right). \end{aligned} \tag{86}$$

The anomalous dimension is given by

$$\Gamma(\xi,\mu) = \sum_{i,j=1;i\neq j}^{4} T_i\cdot T_j(2L_{ij}-i\lambda_{ij})\Gamma+2\gamma, \tag{87}$$

$$\Gamma = \sum_{n=1}^{\infty} \Gamma_n \left(\frac{\alpha_S}{2\pi}\right)^n, \qquad \gamma = \sum_{n=1}^{\infty} \gamma_n \left(\frac{\alpha_S}{2\pi}\right)^n, \tag{88}$$

with the first two coefficients given by

$$\Gamma_0 = -1, \qquad \Gamma_1 = \left(\frac{\pi^2}{6} - \frac{67}{18}\right) C_A + \frac{10}{9} n_F T_R, \tag{89}$$

$$\gamma_0 = 0, \qquad \gamma_1 = C_A \left(\frac{11\pi^2}{144} + \frac{7\zeta_3}{4} - \frac{101}{54}\right) + n_F T_R \left(\frac{14}{27} - \frac{\pi^2}{36}\right). \tag{90}$$

Both the finite parts and an interplay of the NLO and the Z_{NLO} (the term in Z proportional to $\sim\alpha_S$) contribute to the renormalized soft function,

$$\tilde{S}_G(\xi,\mu)^{\mathrm{Renormalized}} = 1 + (\tilde{S}_G^{(1)}(\xi,\mu))_{\mathrm{Finite}} + (\tilde{S}_G^{(2)}(\xi,\mu))_{\mathrm{Finite}} + (\tilde{S}_G^{(1)}(\xi,\mu)\cdot Z_{\mathrm{NLO}}^{\dagger})_{\mathrm{Finite}} + (Z_{\mathrm{NLO}}\cdot \tilde{S}_G^{(1)}(\xi,\mu))_{\mathrm{Finite}}, \tag{91}$$

where

$$\begin{aligned}&(\tilde{S}_G^{(1)}(\xi,\mu)\cdot Z_{\mathrm{NLO}}^{\dagger})_{\mathrm{Finite}} + (Z_{\mathrm{NLO}}\cdot \tilde{S}_G^{(1)}(\xi,\mu))_{\mathrm{Finite}} \\ &\quad = \frac{\alpha_S^2}{4\pi^2} \sum_{i\neq j, i,j\neq 4} \sum_{k\neq l, k,l\neq 4} \{T_i\cdot T_j, T_k\cdot T_l\}\left[-\frac{19\pi^4}{960} - \frac{2\zeta_3 L_{ij}}{3} - \frac{\pi^2 L_{ij}^2}{4} - \frac{\pi^2 L_{ij} L_{kl}}{2} + \frac{4 L_{ij} L_{kl}^2}{3} - \frac{L_{ij}^4}{3}\right].\end{aligned} \tag{92}$$

The RG equation has the form:

$$\frac{\mathrm{d}}{\mathrm{d}\ \ln(\mu)} \tilde{S}_G(\xi,\mu) = \Gamma(\xi,\mu)\tilde{S}_G(\xi,\mu) + \tilde{S}_G(\xi,\mu)\Gamma(\xi,\mu)^{\dagger}, \tag{93}$$

and its solution is given by:

$$\begin{aligned}\tilde{S}_G(\xi,\mu)^{\mathrm{Resummed}} &= 1 + \frac{\alpha_S}{2\pi}\sum_{i,j=1,i\neq j}^{4} T_i\cdot T_j[2\Gamma_0 L^2 + 4L(\Gamma_0 l_{ij} + \gamma_0)] + \tilde{S}_G^{(1)}(L=0) \\ &+ \left(\frac{\alpha_S}{2\pi}\right)^2 \sum_{i,j=1,i\neq j}^{4} T_i\cdot T_j\left[\frac{2\beta_0\Gamma_0 L^3}{3} + 2L^2\beta_0(\Gamma_0 l_{ij} + \gamma_0)\right] + L\beta_0 \tilde{S}_G^{(1)}(L=0) \\ &+ \left(\frac{\alpha_S}{2\pi}\right)^2 \sum_{i,j=1,i\neq j}^{4} T_i\cdot T_j(2\Gamma_1 L^2 + 4L(l_{ij} + \gamma_1)) + \frac{1}{4}\left(\frac{\alpha_S}{2\pi}\right)^2 \sum_{i,j=1,i\neq j\ k,}^{4} \sum_{l=1,k\neq l}^{4} \{T_i\cdot T_j, T_k\cdot T_l\} \\ &\times (2\Gamma_0 L^2 + 4L(\Gamma_0 l_{ij} + \gamma_0) + \tilde{S}_{G,ij}^{(1)}(L=0))(2\Gamma_0 L^2 + 4L(\Gamma_0 l_{kl} + \gamma_0) + \tilde{S}_{G,kl}^{(1)}(L=0)) \\ &- 4\pi\left(\frac{\alpha_S}{2\pi}\right)^2 \sum_{i\neq j\neq k\neq 4} f^{abc} T_i^a T_j^b T_l^c(\Gamma_0\lambda_{ij}(2L^2\Gamma_0 l_{jl} + L\tilde{S}_{G,jl}^{(1)}(L=0)) + \tilde{S}_{G,ijl}^{\mathrm{NNLO}}(L=0)) \\ &+ \tilde{S}_G^{(2)}(L=0).\end{aligned} \tag{94}$$

Note that $\tilde{S}_{G,ijl}^{\mathrm{NNLO}}(\xi,\mu) = 0$, as we discussed in Sec. II. In the next section we provide the numerical results and perform the threshold resummation.

IV. NUMERICAL RESULTS

In this section we perform a numerical integration of Eq. (93). To facilitate the usage of our results by other research groups, we make our numerical code publicly available on GitHub[1] repository. The solution is given by:

[1] https://github.com/BalytskyiJaroslaw/NNLLs

$$S_G^{\text{Resummed}} = \exp\left(U\left(\mu; \ln\left(\frac{\mu_s}{\xi}\right)\right)\right) S_G^{\text{Renormalized}}\left(\alpha(\mu_s), \ln\left(\frac{\mu_s}{\xi}\right)\right) \exp\left(U\left(\mu; \ln\left(\frac{\mu_s}{\xi}\right)\right)\right)^\dagger, \tag{95}$$

with the evolution operator being:

$$\begin{aligned} U\left(\mu; \ln\left(\frac{\mu_s}{\xi}\right)\right) &= \sum_{i,j=1,i\neq j}^{4} 2T_i \cdot T_j \int_{\alpha_S(\mu_s)}^{\alpha_S(\mu)} \frac{\mathrm{d}\alpha_S}{\beta(\alpha_S)} \Gamma(\alpha_S) \int_{\alpha_S(\mu_s)}^{\alpha_S} \frac{\mathrm{d}\alpha'_S}{\beta(\alpha'_S)} \\ &+ \int_{\alpha_S(\mu_s)}^{\alpha_S(\mu)} \frac{\mathrm{d}\alpha_S}{\beta(\alpha_S)} \left(\sum_{i,j=1,i\neq j}^{4} T_i \cdot T_j (2l_{ij} - i\lambda_{ij})\Gamma(\alpha_S) + 2\gamma(\alpha_S)\right) \\ &+ \sum_{i,j=1,i\neq j}^{4} 2T_i \cdot T_j \ln\left(\frac{\mu_s}{\xi}\right) \int_{\alpha_S(\mu_s)}^{\alpha_S(\mu)} \frac{\mathrm{d}\alpha_S}{\beta(\alpha_S)} \Gamma(\alpha_S). \end{aligned} \tag{96}$$

In this work, instead of using μ_s and μ, we show equivalently the numerical results for the soft function in terms of the logarithm L and different values of $\alpha_S(\mu_s)$. In the range of ~0.1–1 TeV the value of the strong coupling constant is of order $\alpha_S \sim 0.1$, and the theory is strongly interacting on the scales below 1 GeV [77]. We provide numerical results for several representative values of α_S. We introduce the following reference vectors:

$$\begin{cases} n'_1 = (1, 0, 0, 1), \\ n'_2 = (1, 0, 0, -1), \\ n'_3 = (1, 0, 1, 0), \\ n'_4 = (1, 0, -1, 0). \end{cases}$$

As a reference for other research groups that may want to perform this calculation, we calculate numerically the renormalized soft function for $\alpha_S = 0.118$ and represent our results for each color channel as a function of $L = \ln(\frac{\mu}{\xi})$. To demonstrate the size of the NNLO contribution in comparison with the NLO in Eq. (91), in Figs. 1–3 we plot the ratio of the color traces of the NNLO to the NLO soft function,

$$\frac{\mathrm{Tr}[S_{\text{NNLO}}]}{\mathrm{Tr}[S_{\text{NLO}}]} = \frac{\mathrm{Tr}[[\theta^T\theta] \cdot \tilde{S}_G(\xi,\mu)^{\text{Renormalized}}]}{\mathrm{Tr}[[\theta^T\theta](1 + (\tilde{S}_G^{(1)}(\xi,\mu))_{\text{Finite}})]}, \tag{97}$$

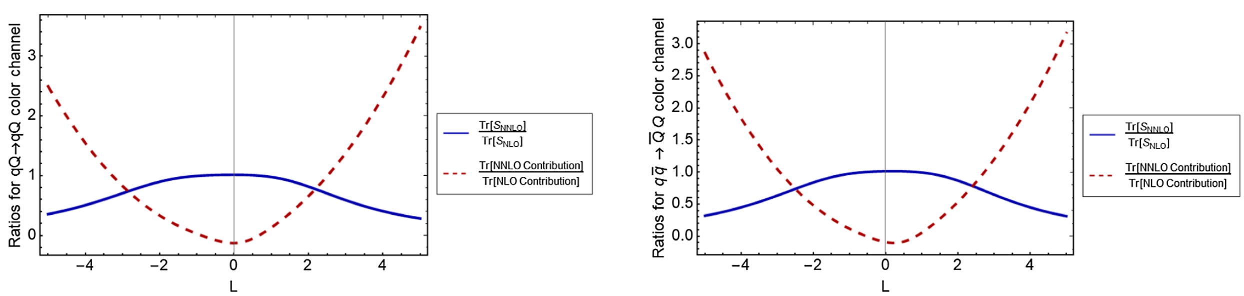

FIG. 1. The ratios of NNLO to NLO soft function and the NNLO to NLO contribution for the $qQ \to qQ$ and $q\bar{q} \to \bar{Q}Q$ color channels.

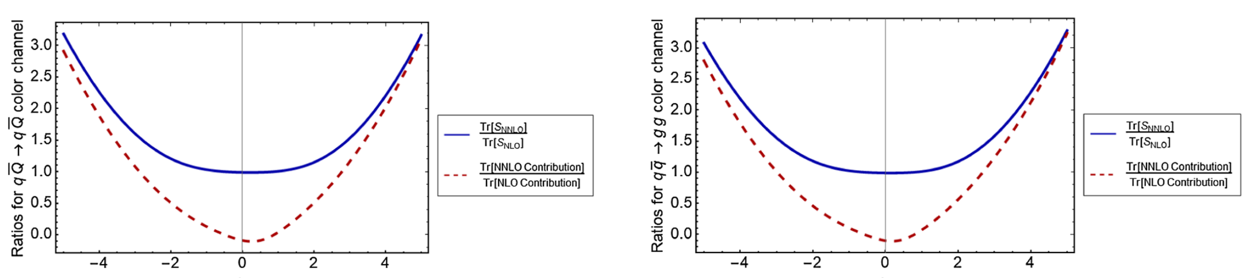

FIG. 2. The ratios of NNLO to NLO soft function and the NNLO to NLO contribution for the $q\bar{Q} \to q\bar{Q}$ and $q\bar{q} \to gg$ color channels.

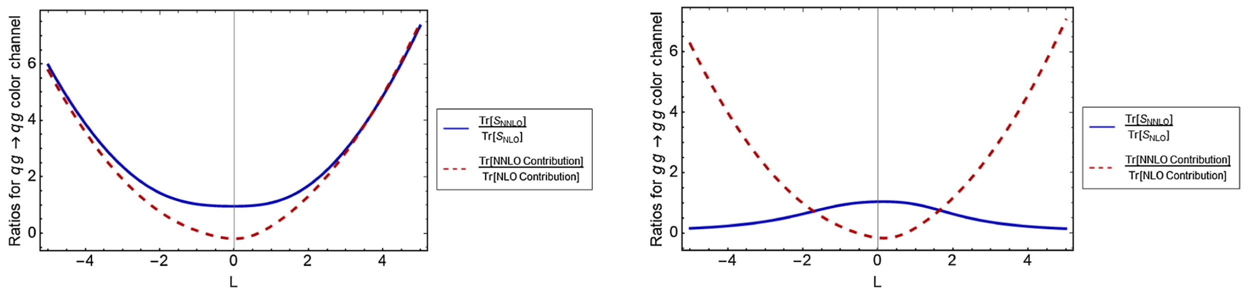

FIG. 3. The ratios of NNLO to NLO soft function and the NNLO to NLO contribution for the $qg \to qg$ and $gg \to gg$ color channels.

and the ratio of the NNLO contribution to the NLO contribution,

$$\frac{\mathrm{Tr}[NNLO\,\text{contribution}]}{\mathrm{Tr}[NLO\,\text{contribution}]} = \frac{\mathrm{Tr}[[\theta^T\theta]\{\tilde{S}_G(\xi,\mu)^{\text{Renormalized}} - \hat{1} - (\tilde{S}_G^{(1)}(\xi,\mu))_{\text{Finite}}\}]}{\mathrm{Tr}[[\theta^T\theta](\tilde{S}_G^{(1)}(\xi,\mu))_{\text{Finite}}]}. \tag{98}$$

Finally, we provide the numerical results for the next-to-next-to-leading logarithmic (NNLL) and next-to-leading logarithmic (NLL) soft functions and compare them. To perform matrix exponentiation and integration numerically, we utilized the *Mathematica* functions MatrixExp and NIntegrate, respectively. To provide a reference for other researchers who would like to perform this calculation, we carry out the resummation for the values of the strong coupling constant at the soft scale $\alpha_S(\mu_s) = [0.12, 0.10, 0.20, 0.40]$ and perform the evolution to the final value at the hard scale $\alpha_S(\mu_h) = 0.118$. To demonstrate the impact of resummation, we provide the ratio of absolute values of the color traces of NNLL vs NLL soft function in Figs. 4–6 as a function of $L = \ln(\frac{\xi}{\mu_s})$. To perform resummation for other input values, one would need to call the function *EvolvedNNLO* $[\alpha Initial, \alpha Final, L]$ in our numerical code for each color channel with the values of $\alpha_S(\mu_s)$ and $\alpha_S(\mu_h)$ corresponding to the $\alpha Initial$ and $\alpha Final$ variables in our code, respectively.

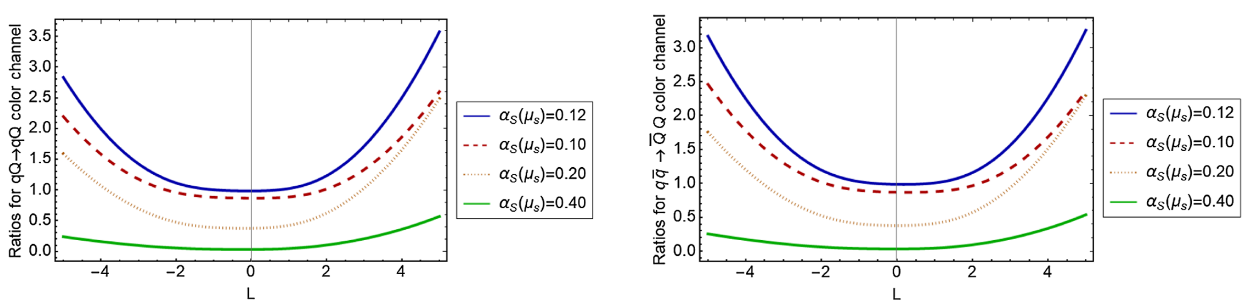

FIG. 4. The ratios of the resummed NNLO and NLO soft function for different values of $\alpha(\mu_s)$ for $qQ \to qQ$ and $q\bar{q} \to \bar{Q}Q$ color channels.

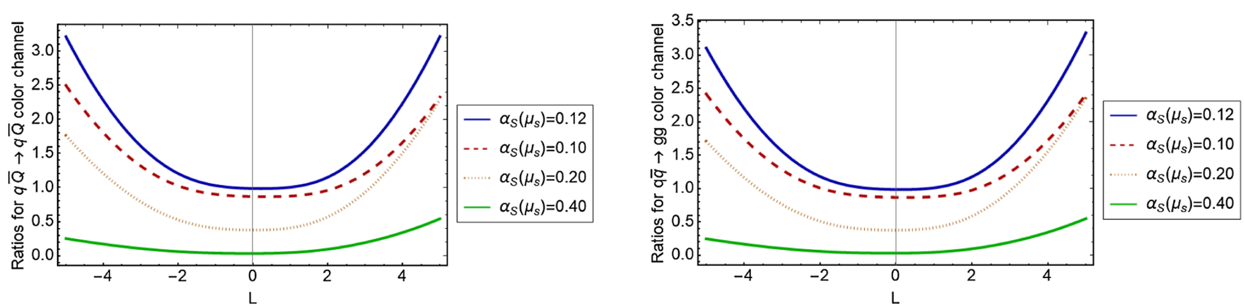

FIG. 5. The ratios of the resummed NNLO and NLO soft function for different values of $\alpha(\mu_s)$ for $q\bar{Q} \to q\bar{Q}$ and $q\bar{q} \to gg$ color channels.

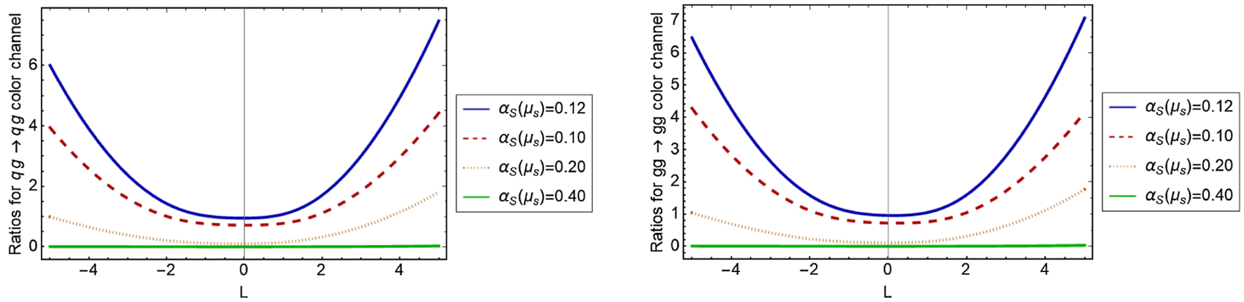

FIG. 6. The ratios of the resummed NNLO and NLO soft function for different values of $\alpha(\mu_s)$ for $qg \to qg$ and $gg \to gg$ color channels.

V. CONCLUSIONS

In this manuscript we performed a calculation of the global soft function for single inclusive jet production involving four light-cone directions. In comparison with the soft function involving three light-cone directions, the color structure is not diagonal, which complicates the color structure. We showed by an explicit calculation that the tripole contribution proportional to $T_1 \cdot T_2 \cdot T_3$ vanishes. We performed resummation for each color channel with the use of the color-space formalism and show that the effects of resummation are quite significant. To give definite predictions, one has to combine the soft function, which we calculated in this paper, with the NNLO jet and hard functions, which we will refer to in our future work.

ACKNOWLEDGMENTS

Y. B. appreciates fruitful discussions with Xiaohui Liu, Sonny Mantry, Ye Li, Thomas Becher, Guido Bell, Maximilian Stahlhofen, and Tobias Neumann. Y. B. is supported by a grant from the U.S. Civilian Research & Development Foundation (CRDF Global). J. G. is supported by Guangdong Introducing Innovative and Enterpreneurial Teams (No. 2016ZT06G025).

Y. B. and J. G. both are contributed equally to this work.

APPENDIX A: INTEGRATIONS

1. Strongly ordered soft function

We consider

$$\begin{aligned} I_1^{s.o.} &= \int \mathrm{d}^d q \delta^{(d)}(q - k_1 - k_2) \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{1}{k_1 \cdot k_2} \frac{n_{ij}}{n_i \cdot k_1 n_j \cdot k_2} \delta(\omega - n_4 \cdot k_1 - n_4 \cdot k_2) \\ &= \int \mathrm{d}^d q \frac{4}{q^2} \delta(\omega - n_4 \cdot q) \int [\mathrm{d}k_1][\mathrm{d}k_2] \frac{n_{ij}}{2 n_i \cdot k_1 n_j \cdot k_2} \delta^{(d)}(q - k_1 - k_2) \\ &= 4\pi^{1-\epsilon} \left[\frac{1}{2\pi}\right]^{2d-2} \frac{\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)} \int \mathrm{d}^d q \delta(\omega - n_4 \cdot q) \\ &\quad \times (q^2)^{-1-\epsilon} q_\perp^{-2-2\epsilon} \left(\frac{2 n_i \cdot q n_j \cdot q}{n_{ij}}\right)^\epsilon {}_2F_1\left(-\epsilon, -\epsilon, 1-\epsilon, \frac{n_{ij} q^2}{2 n_i \cdot q n_j \cdot q}.\right) \end{aligned} \tag{A1}$$

Now we let

$$q^\mu = q_+ \frac{n_i^\mu}{\sqrt{2 n_{ij}}} + q_- \frac{n_j^\mu}{\sqrt{2 n_{ij}}} + q_\perp^\mu, \tag{A2}$$

and further,

$$q_+ = \frac{\omega}{n_4^-} xy, \qquad q_- = \frac{\omega}{n_4^+} \bar{x} y, \qquad q_\perp = \frac{\omega}{\sqrt{n_4^+ n_4^-}} \sqrt{x \bar{x}} y u, \tag{A3}$$

to find

$$4\pi^{1-\epsilon}\left[\frac{1}{2\pi}\right]^{2d-2}\frac{\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Omega_{d-3}}{2}\int \mathrm{d}q_+\,\mathrm{d}q_-\,\mathrm{d}q_\perp\,\mathrm{d}c_\theta\, q_\perp^{1-2\epsilon}s_\theta^{-1-2\epsilon}\delta(\omega-n_4\cdot q)$$
$$\times\left(\frac{\omega^2}{n_4^+n_4^-}\right)^{-2-\epsilon}(x\bar{x})^{-2-\epsilon}[y^2]^{-2-\epsilon}(1-u^2)^{-1-\epsilon}(u^2)^{-1-\epsilon}{}_2F_1(-\epsilon,-\epsilon,1-\epsilon,1-u^2), \quad \text{(A4)}$$

which is:

$$4\pi^{1-\epsilon}\left[\frac{1}{2\pi}\right]^{2d-2}\frac{\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Omega_{d-3}}{2}\int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}u\,\mathrm{d}c_\theta\left(\frac{\omega^2}{n_4^+n_4^-}\right)^{3/2}(x\bar{x})^{1/2}y^2$$
$$\times\left(\frac{\omega^2}{n_4^+n_4^-}x\bar{x}y^2u^2\right)^{1/2-\epsilon}s_\theta^{-1-2\epsilon}\frac{1}{\omega}\delta\left(1-\frac{y}{2}+\sqrt{x\bar{x}}yuc_\theta\right)$$
$$\times\left(\frac{\omega^2}{n_4^+n_4^-}\right)^{-2-\epsilon}(x\bar{x})^{-2-\epsilon}[y^2]^{-2-\epsilon}(1-u^2)^{-1-\epsilon}(u^2)^{-1-\epsilon}{}_2F_1(-\epsilon,-\epsilon,1-\epsilon,1-u^2). \quad \text{(A5)}$$

This simplifies to

$$4\pi^{1-\epsilon}\left[\frac{1}{2\pi}\right]^{2d-2}\frac{\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Omega_{d-3}}{2}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+n_4^-}\right)^{-2\epsilon}\int \mathrm{d}x\,\mathrm{d}u\,\mathrm{d}c_\theta$$
$$\times s_\theta^{-1-2\epsilon}(1-2\sqrt{x\bar{x}}u\,c_\theta)^{4\epsilon}(x\bar{x})^{-1-2\epsilon}(1-u^2)^{-1-\epsilon}(u^2)^{-1/2-2\epsilon}{}_2F_1(-\epsilon,-\epsilon,1-\epsilon,1-u^2). \quad \text{(A6)}$$

Integrating over c_θ gives

$$4\pi^{1-\epsilon}\left[\frac{1}{2\pi}\right]^{2d-2}\frac{\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Omega_{d-3}}{4}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+n_4^-}\right)^{-2\epsilon}4^{1+2\epsilon}\sqrt{\pi}\frac{\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(1-\epsilon)}$$
$$\times\int \mathrm{d}x\,\mathrm{d}u^2{}_2F_1\left(\frac{1}{2}-2\epsilon,-2\epsilon,1-\epsilon,4x\bar{x}u^2\right)(4x\bar{x})^{-1-2\epsilon}(1-u^2)^{-1-\epsilon}(u^2)^{-1-2\epsilon}{}_2F_1(-\epsilon,-\epsilon,1-\epsilon,1-u^2). \quad \text{(A7)}$$

Now we let $4x\bar{x}=t$ to find

$$4\pi^{1-\epsilon}\left[\frac{1}{2\pi}\right]^{2d-2}\frac{\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Omega_{d-3}}{4}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+n_4^-}\right)^{-2\epsilon}4^{1+2\epsilon}\sqrt{\pi}\frac{\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(1-\epsilon)}$$
$$\times\frac{1}{2}\int \mathrm{d}t\,\mathrm{d}u^2{}_2F_1\left(\frac{1}{2}-2\epsilon,-2\epsilon,1-\epsilon,tu^2\right)(t)^{-1-2\epsilon}(1-t)^{-1/2}(1-u^2)^{-1-\epsilon}(u^2)^{-1-2\epsilon}{}_2F_1(-\epsilon,-\epsilon,1-\epsilon,1-u^2), \quad \text{(A8)}$$

which gives

$$4\pi^{1-\epsilon}\left[\frac{1}{2\pi}\right]^{2d-2}\frac{\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\frac{\Omega_{d-3}}{4}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+n_4^-}\right)^{-2\epsilon}4^{1+2\epsilon}\sqrt{\pi}\frac{\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(1-\epsilon)}\frac{1}{2}\sqrt{\pi}\frac{\Gamma(-2\epsilon)}{\Gamma(\frac{1}{2}-2\epsilon)}$$
$$\times\int_0^1 \mathrm{d}u^2(1-u^2)^{-1-\epsilon}(u^2)^{-1-2\epsilon}{}_2F_1(-\epsilon,-\epsilon,1-\epsilon,1-u^2){}_2F_1(-2\epsilon,-2\epsilon,1-\epsilon,u^2), \quad \text{(A9)}$$

which is

$$4\pi^{1-\epsilon}\left[\frac{1}{2\pi}\right]^{2d-2}\frac{\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}\frac{1}{4}\frac{2\pi^{1/2-\epsilon}}{\Gamma(1/2-\epsilon)}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+n_4^-}\right)^{-2\epsilon}4^{1+2\epsilon}\sqrt{\pi}\frac{\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(1-\epsilon)}\frac{1}{2}\sqrt{\pi}\frac{\Gamma(-2\epsilon)}{\Gamma(\frac{1}{2}-2\epsilon)}$$
$$\times\int_0^1 \mathrm{d}u^2(1-u^2)^{-1-\epsilon}(u^2)^{-1-2\epsilon}{}_2F_1(-2\epsilon,-2\epsilon,1-\epsilon,u^2)(1+\epsilon^2\mathrm{Li}_2(1-u^2)+\mathcal{O}(\epsilon^3)). \quad \text{(A10)}$$

Here we need to expand ${}_2F_1(-\epsilon,-\epsilon,1-\epsilon,1-u^2)$ to $\mathcal{O}(\epsilon^3)$. The $\mathcal{O}(\epsilon^3)$ term is lengthy and thus we suppressed its expression.

The integration gives:

$$I_1^{s.o.} = \frac{1}{4\pi^2}\frac{c_R}{16\pi^2}\left[\frac{2}{\epsilon^2} - 2\pi^2 - \frac{112}{3}\zeta_3\epsilon - \frac{\pi^4}{3}\epsilon^2\right]\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+ n_4^-}\right)^{-2\epsilon}\left(-\frac{3}{2\epsilon} - \frac{\pi^2}{6}\epsilon + 13\zeta_3\epsilon^2 + \frac{31}{180}\pi^4\epsilon^3 - \frac{\pi^2}{12}\epsilon - \frac{\pi^4}{720}\epsilon^3 - \epsilon^2\zeta_3\right)$$
$$= \frac{1}{4\pi^2}\frac{c_R}{16\pi^2}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+ n_4^-}\right)^{-2\epsilon}\left(-\frac{3}{\epsilon^3} + \frac{5\pi^2}{2\epsilon} + 80\zeta_3 + \frac{161}{120}\pi^4\epsilon\right). \tag{A11}$$

The rest term for $\mathcal{S}^{s.o.}$ is given by $-(2\pi)^{2-2d} \times I_4$ in [72], which is

$$I_2^{s.o.} = -\frac{1}{4\pi^2}\frac{c_R}{16\pi^2}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+ n_4^-}\right)^{-2\epsilon}\left(-\frac{4}{\epsilon^3} + \frac{10\pi^2}{3\epsilon} + \frac{248}{3}\zeta_3 + \frac{25}{18}\pi^4\epsilon\right). \tag{A12}$$

Therefore we have for the strongly ordered limit contribution

$$S_{G,ij}^{s.o.}(\omega) = \frac{1}{4\pi^2}\frac{c_R}{16\pi^2}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+ n_4^-}\right)^{-2\epsilon}\left(-\frac{2}{\epsilon^3} + \frac{5\pi^2}{3\epsilon} + \frac{232}{3}\zeta_3 + \frac{233}{180}\pi^4\epsilon\right). \tag{A13}$$

2. Calculation of $I_{7,2}$

$(2\pi)^{2-2d} \times I_{7,2}$ is defined as

$$\int \mathrm{d}^d q\delta(\omega - n_4\cdot q)\int [\mathrm{d}k_1][\mathrm{d}k_2]\frac{n_{ij}}{q^2}\frac{n_j\cdot(k_1+2k_2)}{n_j\cdot q n_i\cdot k_1 n_j\cdot k_2}\delta^{(d)}\delta(q-k_1-k_2)$$
$$= \frac{1}{2}I_1^{s.o.} + \int \mathrm{d}^d q\delta(\omega - n_4\cdot q)\frac{n_{ij}}{q^2}\frac{1}{n_j\cdot q}\int [\mathrm{d}k_1][\mathrm{d}k_2]\frac{1}{n_i\cdot k_1}\delta^{(d)}\delta(q-k_1-k_2)$$
$$= \frac{1}{2}I_1^{s.o.} + \pi^{1-\epsilon}\left(\frac{1}{2\pi}\right)^{2d-2}\frac{\Gamma(-\epsilon)}{2\Gamma(1-2\epsilon)}\int \mathrm{d}^d q\delta(\omega - n_4\cdot q)\frac{n_{ij}}{(q^2)^{1+\epsilon}}\frac{1}{n_i\cdot q n_j\cdot q}$$
$$= \frac{1}{2}I_1^{s.o.} + \pi^{1-\epsilon}\left(\frac{1}{2\pi}\right)^{2d-2}\frac{\Gamma(-\epsilon)}{2\Gamma(1-2\epsilon)}I_2$$
$$= \frac{1}{(2\pi)^2}\frac{c_R}{16\pi^2}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+ n_4^-}\right)^{-2\epsilon}\left(-\frac{2}{\epsilon^3} + \frac{3\pi^2}{2\epsilon} + \frac{151}{3}\zeta_3 + \frac{11}{12}\pi^4\epsilon\right). \tag{A14}$$

3. Calculation of I_6

$(2\pi)^{2-2d} \times I_6$ is defined as

$$\int \mathrm{d}^d q\delta(\omega - n_4\cdot q)\frac{n_{ij}}{q^2}\int [\mathrm{d}k_1][\mathrm{d}k_2]\frac{n_j\cdot(k_1-k_2)}{n_i\cdot q n_j\cdot q n_j\cdot k_2}\delta^{(d)}\delta(q-k_1-k_2)$$
$$= \int \mathrm{d}^d q\delta(\omega - n_4\cdot q)\int [\mathrm{d}k_1][\mathrm{d}k_2]\left(\frac{n_{ij}}{q^2}\frac{1}{n_i\cdot q n_j\cdot k_2} - \frac{n_{ij}}{q^2}\frac{2}{n_i\cdot q n_j\cdot q}\right)\delta^{(d)}\delta(q-k_1-k_2)$$
$$= \int \mathrm{d}^d q\delta(\omega - n_4\cdot q)\frac{n_{ij}}{q^2}\frac{1}{n_i\cdot q}\int [\mathrm{d}k_1][\mathrm{d}k_2]\frac{1}{n_j\cdot k_2}\delta^{(d)}\delta(q-k_1-k_2)$$
$$- \int \mathrm{d}^d q\delta(\omega - n_4\cdot q)\frac{n_{ij}}{q^2}\frac{2}{n_i\cdot q n_j\cdot q}\int [\mathrm{d}k_1][\mathrm{d}k_2]\delta^{(d)}\delta(q-k_1-k_2)$$
$$= \left(\frac{1}{2\pi}\right)^{2d-2}\frac{\pi^{1-\epsilon}}{2}\left[\frac{\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)} - \frac{2\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\right]\int \mathrm{d}^d q\delta(\omega - n_4\cdot q)\frac{n_{ij}}{(q^2)^{1+\epsilon}}\frac{1}{n_i\cdot q n_j\cdot q}$$
$$= \left(\frac{1}{2\pi}\right)^{2d-2}\frac{\pi^{1-\epsilon}}{2}\left[\frac{\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)} - \frac{2\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}\right]I_2$$
$$= \left(\frac{1}{2\pi}\right)^{2d-2}\frac{-\pi\csc(2\epsilon\pi)\Gamma^2(-\epsilon)}{(1-2\epsilon)\Gamma(1-4\epsilon)}\frac{1}{\omega}\left(\frac{4\omega^2}{n_4^+ n_4^-}\right)^{-2\epsilon}. \tag{A15}$$

APPENDIX B: COLOR STRUCTURE

For the case of four or more partons involved, the color structure is not diagonal anymore, like in [72], since the color conservation relation alone is not sufficient to express all products $\vec{T}_i \cdot \vec{T}_j$ in terms of $\vec{T}_i^2$. Therefore, we calculate the action of $\vec{T}_i \cdot \vec{T}_j$ on the color basis, and the matrices $(\omega_{ij})_{IJ} = \langle\theta_I|\vec{T}_i \cdot \vec{T}_j|\theta_J\rangle$ factorizing them as $\omega = \langle\theta|A|\theta\rangle = \underbrace{[\theta^T\theta]}_{\text{Tree level}}[A^T]$, where $\underbrace{[\theta^T\theta]}_{\text{Tree level}}$ - tree level color matrix. It is convenient to calculate it in the form: $\omega = \langle\theta|A|\theta\rangle = \underbrace{[\theta^T\theta]}_{\text{Tree level}}[A^T]$, where $\underbrace{[\theta^T\theta]}_{\text{Tree level}}$—tree-level color matrix. We consider each channel separately. Here $N_c = 3$ represents the number of colors, $C_F = \frac{N_c^2-1}{2N_c}$, and $C_A = N_c$.

1. Channel $q_1Q_2 \to q_3Q_4$

Our color basis follows the conventions of [78,79]:

$$\begin{cases} |\theta_1\rangle = (\bar{q}_{i_4}\delta_{i_4i_1}q_{i_1})(\bar{q}_{i_3}\delta_{i_3i_2}q_{i_2}) \\ |\theta_2\rangle = (\bar{q}_{i_4}\delta_{i_4i_2}q_{i_2})(\bar{q}_{i_3}\delta_{i_3i_1}q_{i_1}) \end{cases}; \quad [\theta^T\theta] = \begin{pmatrix} N_c^2 & N_c \\ N_c & N_c^2 \end{pmatrix},$$

$$T_1 \cdot T_2 = \begin{pmatrix} -\frac{1}{2N_c} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2N_c} \end{pmatrix}; \quad T_1 \cdot T_3 = \begin{pmatrix} \frac{1}{2N_c} & 0 \\ -\frac{1}{2} & -C_F \end{pmatrix};$$

$$T_1 \cdot T_4 = \begin{pmatrix} -C_F & -\frac{1}{2} \\ 0 & \frac{1}{2N_c} \end{pmatrix},$$

$$\omega_{12} = \left(\frac{N_c^2-1}{2}\right) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$\omega_{13} = \left(\frac{1-N_c^2}{2}\right) \cdot \begin{pmatrix} 0 & 1 \\ 1 & N_c \end{pmatrix};$$

$$\omega_{14} = \left(\frac{1-N_c^2}{2}\right) \cdot \begin{pmatrix} N_c & 1 \\ 1 & 0 \end{pmatrix}.$$

By color conservation: $\omega_{23} = \omega_{14}$, $\omega_{24} = \omega_{13}$, $\omega_{34} = \omega_{12}$.

2. Channel $q_1\bar{q}_2 \to \bar{Q}_3Q_4$

Our color basis follows the conventions of [78,79]:

$$\begin{cases} |\theta_1\rangle = (\bar{q}_{i_4}\delta_{i_4i_1}q_{i_1})(\bar{q}_{i_2}\delta_{i_2i_3}q_{i_3}) \\ |\theta_2\rangle = (\bar{q}_{i_4}\delta_{i_4i_3}q_{i_3})(\bar{q}_{i_2}\delta_{i_2i_1}q_{i_1}) \end{cases}; \quad [\theta^T\theta] = \begin{pmatrix} N_c^2 & N_c \\ N_c & N_c^2 \end{pmatrix},$$

$$T_1 \cdot T_2 = \begin{pmatrix} \frac{1}{2N_c} & 0 \\ -\frac{1}{2} & -C_F \end{pmatrix}; \quad T_1 \cdot T_3 = \begin{pmatrix} -\frac{1}{2N_c} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2N_c} \end{pmatrix};$$

$$T_1 \cdot T_4 = \begin{pmatrix} -C_F & -\frac{1}{2} \\ 0 & \frac{1}{2N_c} \end{pmatrix},$$

$$\omega_{12} = \left(\frac{1-N_c^2}{2}\right) \cdot \begin{pmatrix} 0 & 1 \\ 1 & N_c \end{pmatrix};$$

$$\omega_{13} = \left(\frac{N_c^2-1}{2}\right) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$\omega_{14} = \left(\frac{1-N_c^2}{2}\right) \cdot \begin{pmatrix} N_c & 1 \\ 1 & 0 \end{pmatrix}.$$

By color conservation: $\omega_{23} = \omega_{14}$, $\omega_{24} = \omega_{13}$, $\omega_{34} = \omega_{12}$.

3. Channel $q_1\bar{Q}_2 \to q_3\bar{Q}_4$

Our color basis follows the conventions of [78,79]:

$$\begin{cases} |\theta_1\rangle = (\bar{q}_{i_2}\delta_{i_2i_1}q_{i_1})(\bar{q}_{i_3}\delta_{i_3i_4}q_{i_4}) \\ |\theta_2\rangle = (\bar{q}_{i_2}\delta_{i_2i_4}q_{i_4})(\bar{q}_{i_3}\delta_{i_3i_1}q_{i_1}) \end{cases}; \quad [\theta^T\theta] = \begin{pmatrix} N_c^2 & N_c \\ N_c & N_c^2 \end{pmatrix},$$

$$T_1 \cdot T_2 = \begin{pmatrix} -C_F & -\frac{1}{2} \\ 0 & \frac{1}{2N_c} \end{pmatrix}; \quad T_1 \cdot T_3 = \begin{pmatrix} \frac{1}{2N_c} & 0 \\ -\frac{1}{2} & -C_F \end{pmatrix};$$

$$T_1 \cdot T_4 = \begin{pmatrix} -\frac{1}{2N_c} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2N_c} \end{pmatrix}.$$

$$\omega_{12} = \left(\frac{1-N_c^2}{2}\right) \cdot \begin{pmatrix} N_c & 1 \\ 1 & 0 \end{pmatrix}; \quad \omega_{13} = \left(\frac{1-N_c^2}{2}\right) \cdot \begin{pmatrix} 0 & 1 \\ 1 & N_c \end{pmatrix};$$

$$\omega_{14} = \left(\frac{N_c^2-1}{2}\right) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By color conservation: $\omega_{23} = \omega_{14}$, $\omega_{24} = \omega_{13}$, $\omega_{34} = \omega_{12}$.

4. Channel $q_1\bar{q}_2 \to g_3g_4$

For convenience, for the channels with two (anti)quarks and two gluons, we rescale the Gell-Mann matrices, leaving the color operators the same.

Our conventions follow [80]:

$$\mathrm{tr}(t^at^b) = \delta^{ab}, \qquad \mathrm{tr}(T^aT^b) = \frac{\delta^{ab}}{2} \Rightarrow t^a = \sqrt{2}T^a.$$

The color basis and tree-level color matrix are:

$$\begin{cases} |\theta_1\rangle = (\bar{\xi}_{i_2}(t^{a_3}t^{a_4})_{i_2i_1}\xi_{i_1})A_{a_3}A_{a_4} \\ |\theta_2\rangle = (\bar{\xi}_{i_2}(t^{a_4}t^{a_3})_{i_2i_1}\xi_{i_1})A_{a_3}A_{a_4}; \\ |\theta_3\rangle = (\bar{\xi}_{i_2}\delta^{a_3a_4}\delta_{i_2i_1}\xi_{i_1})A_{a_3}A_{a_4} \end{cases}$$

$$[\theta^T\theta] = \left(\frac{N_c^2-1}{N_c}\right) \cdot \begin{pmatrix} N_c^2-1 & -1 & N_c \\ -1 & N_c^2-1 & N_c \\ N_c & N_c & N_c^2 \end{pmatrix},$$

$$T_1\cdot T_2=\begin{pmatrix}\frac{1}{2N_c} & 0 & 0\\ 0 & \frac{1}{2N_c} & 0\\ -\frac{1}{2} & -\frac{1}{2} & -C_F\end{pmatrix};\quad T_1\cdot T_3=\begin{pmatrix}0 & 0 & \frac{1}{2}\\ 0 & -\frac{N_c}{2} & -\frac{1}{2}\\ \frac{1}{2} & 0 & 0\end{pmatrix};$$

$$T_1\cdot T_4=\begin{pmatrix}-\frac{N_c}{2} & 0 & -\frac{1}{2}\\ 0 & 0 & \frac{1}{2}\\ 0 & \frac{1}{2} & 0\end{pmatrix},$$

$$T_3\cdot T_4=\begin{pmatrix}-\frac{N_c}{2} & 0 & 0\\ 0 & -\frac{N_c}{2} & 0\\ -\frac{1}{2} & -\frac{1}{2} & -N_c\end{pmatrix},$$

$$\omega_{12}=\left(\frac{N_c^2-1}{N_c}\right)\cdot\begin{pmatrix}-\frac{1}{2N_c} & -\frac{N_c^2+1}{2N_c} & \frac{1}{2}(1-N_c^2)\\ -\frac{N_c^2+1}{2N_c} & -\frac{1}{2N_c} & \frac{1}{2}(1-N_c^2)\\ \frac{1}{2}(1-N_c^2) & \frac{1}{2}(1-N_c^2) & \frac{N_c}{2}(1-N_c^2)\end{pmatrix};$$

$$\omega_{13}=\left(\frac{N_c^2-1}{2}\right)\cdot\begin{pmatrix}1 & 1 & N_c\\ 1 & (1-N_c^2) & -N_c\\ N_c & -N_c & 0\end{pmatrix};$$

$$\omega_{14}=\left(\frac{N_c^2-1}{2}\right)\cdot\begin{pmatrix}1-N_c^2 & 1 & -N_c\\ 1 & 1 & N_c\\ -N_c & N_c & 0\end{pmatrix};$$

$$\omega_{34}=N_c(1-N_c^2)\cdot\begin{pmatrix}\frac{N_c}{2} & 0 & 1\\ 0 & \frac{N_c}{2} & 1\\ 1 & 1 & N_c\end{pmatrix}.$$

By color conservation, $\omega_{24}=\omega_{13}$, $\omega_{23}=\omega_{14}$.

5. Channel $q_1g_2\to g_3q_4$

For convenience, for the channels with two (anti)quarks and two gluons, we rescale the Gell-Mann matrices, leaving the color operators the same.

Our conventions follow [80]:

$$\begin{cases}|\theta_1\rangle=(\bar{\xi}_{i_4}(t^{a_3}t^{a_2})_{i_4i_1}\xi_{i_1})A_{a_3}A_{a_2}\\ |\theta_2\rangle=(\bar{\xi}_{i_4}(t^{a_2}t^{a_3})_{i_4i_1}\xi_{i_1})A_{a_3}A_{a_2};\\ |\theta_3\rangle=(\bar{\xi}_{i_4}\delta^{a_2a_3}\delta_{i_4i_1}\xi_{i_1})A_{a_3}A_{a_4}\end{cases}$$

$$[\theta^T\theta]=\left(\frac{N_c^2-1}{N_c}\right)\cdot\begin{pmatrix}N_c^2-1 & -1 & N_c\\ -1 & N_c^2-1 & N_c\\ N_c & N_c & N_c^2\end{pmatrix},$$

$$T_1\cdot T_2=\begin{pmatrix}-\frac{N_c}{2} & 0 & -\frac{1}{2}\\ 0 & 0 & \frac{1}{2}\\ 0 & \frac{1}{2} & 0\end{pmatrix};\quad T_1\cdot T_3=\begin{pmatrix}0 & 0 & \frac{1}{2}\\ 0 & -\frac{N_c}{2} & -\frac{1}{2}\\ \frac{1}{2} & 0 & 0\end{pmatrix};$$

$$T_1\cdot T_4=\begin{pmatrix}\frac{1}{2N_c} & 0 & 0\\ 0 & \frac{1}{2N_c} & 0\\ -\frac{1}{2} & -\frac{1}{2} & -C_F\end{pmatrix},$$

$$T_2\cdot T_3=\begin{pmatrix}-\frac{N_c}{2} & 0 & 0\\ 0 & -\frac{N_c}{2} & 0\\ -\frac{1}{2} & -\frac{1}{2} & -N_c\end{pmatrix},$$

$$\omega_{12}=\left(\frac{N_c^2-1}{2}\right)\cdot\begin{pmatrix}1-N_c^2 & 1 & -N_c\\ 1 & 1 & N_c\\ -N_c & N_c & 0\end{pmatrix};$$

$$\omega_{13}=\left(\frac{N_c^2-1}{2}\right)\cdot\begin{pmatrix}1 & 1 & N_c\\ 1 & 1-N_c^2 & -N_c\\ N_c & -N_c & 0\end{pmatrix}$$

$$\omega_{14}=\left(\frac{1-N_c^2}{2N_c}\right)\cdot\begin{pmatrix}\frac{1}{N_c} & \frac{N_c^2+1}{N_c} & N_c^2-1\\ \frac{N_c^2+1}{N_c} & \frac{1}{N_c} & N_c^2-1\\ N_c^2-1 & N_c^2-1 & N_c(N_c^2-1)\end{pmatrix};$$

$$\omega_{23}=N_c(1-N_c^2)\cdot\begin{pmatrix}\frac{N_c}{2} & 0 & 1\\ 0 & \frac{N_c}{2} & 1\\ 1 & 1 & N_c\end{pmatrix}.$$

6. Channel $gg\to gg$

Our conventions follow [81]:

$$\begin{cases}|\theta_1\rangle=\mathrm{tr}(t^{a_1}t^{a_2}t^{a_3}t^{a_4})A^{a_1}A^{a_2}A^{a_3}A^{a_4}\\ |\theta_2\rangle=\mathrm{tr}(t^{a_1}t^{a_2}t^{a_4}t^{a_3})A^{a_1}A^{a_2}A^{a_3}A^{a_4}\\ |\theta_3\rangle=\mathrm{tr}(t^{a_1}t^{a_4}t^{a_2}t^{a_3})A^{a_1}A^{a_2}A^{a_3}A^{a_4}\\ |\theta_4\rangle=\mathrm{tr}(t^{a_1}t^{a_3}t^{a_2}t^{a_4})A^{a_1}A^{a_2}A^{a_3}A^{a_4}\\ |\theta_5\rangle=\mathrm{tr}(t^{a_1}t^{a_3}t^{a_4}t^{a_2})A^{a_1}A^{a_2}A^{a_3}A^{a_4}\\ |\theta_6\rangle=\mathrm{tr}(t^{a_1}t^{a_4}t^{a_3}t^{a_2})A^{a_1}A^{a_2}A^{a_3}A^{a_4}\\ |\theta_7\rangle=\mathrm{tr}(t^{a_1}t^{a_2})\mathrm{tr}(t^{a_3}t^{a_4})A^{a_1}A^{a_2}A^{a_3}A^{a_4}\\ |\theta_8\rangle=\mathrm{tr}(t^{a_1}t^{a_3})\mathrm{tr}(t^{a_2}t^{a_4})A^{a_1}A^{a_2}A^{a_3}A^{a_4}\\ |\theta_9\rangle=\mathrm{tr}(t^{a_1}t^{a_4})\mathrm{tr}(t^{a_2}t^{a_3})A^{a_1}A^{a_2}A^{a_3}A^{a_4}.\end{cases};$$

The tree-level matrix in our basis is:

$$[\theta^T\theta] = \frac{C_F}{8C_A}\begin{pmatrix} a & b & b & b & b & c & d & -e & d \\ b & a & b & b & c & b & d & d & -e \\ b & b & a & c & b & b & -e & d & d \\ b & b & c & a & b & b & -e & d & d \\ b & c & b & b & a & b & d & d & -e \\ c & b & b & b & b & a & d & -e & d \\ d & d & -e & -e & d & d & d\cdot e & e^2 & e^2 \\ -e & d & d & d & d & -e & e^2 & d\cdot e & e^2 \\ d & -e & d & d & -e & d & e^2 & e^2 & d\cdot e \end{pmatrix};$$

with the conventions:

$$\begin{cases} a = C_A^4 - 3C_A^2 + 3 \\ b = 3 - C_A^2 \\ c = 3 + C_A^2 \\ d = 2C_A^2 C_F \\ e = C_A \end{cases};$$

$$(T_1\cdot T_2) = \begin{pmatrix} -\frac{C_A}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{C_A}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{C_A}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{C_A}{2} & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & -C_A & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

$$(T_1\cdot T_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{C_A}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{C_A}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & -\frac{C_A}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{C_A}{2} & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & -C_A & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix};$$

$$(T_1\cdot T_4) = \begin{pmatrix} -\frac{C_A}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{C_A}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{C_A}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{C_A}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & -C_A \end{pmatrix}.$$

Using the following conventions:

$$\begin{cases} t = N_c^2 - 1 \\ u = -\frac{N_c^4}{2} + N_c^2 - 1 \\ v = \frac{N_c^2 - 2}{2} \\ x = -N_c^2 - 1 \\ y = N_c\cdot(1 - N_c^2) \\ z = -\frac{N_c^3}{2} \end{cases};$$

ω-matrices are expressed as:

$$\omega_{12} = \frac{C_F}{8}\cdot\begin{pmatrix} u & -1 & v & v & -1 & x & y & 0 & z \\ -1 & u & v & v & x & -1 & y & z & 0 \\ v & v & t & t & v & v & N_c & -z & -z \\ v & v & t & t & v & v & N_c & -z & -z \\ -1 & x & v & v & u & -1 & y & z & 0 \\ x & -1 & v & v & -1 & u & y & 0 & z \\ y & y & N_c & N_c & y & y & N_c\cdot y & -N_c^2 & -N_c^2 \\ 0 & z & -z & -z & z & 0 & -N_c^2 & 0 & N_c^2 \\ z & 0 & -z & -z & 0 & z & -N_c^2 & N_c^2 & 0 \end{pmatrix};$$

$$\omega_{13} = \frac{C_F}{8}\cdot\begin{pmatrix} t & v & v & v & v & t & -z & N_c & -z \\ v & u & -1 & -1 & x & v & z & y & 0 \\ v & -1 & u & x & -1 & v & 0 & y & z \\ v & -1 & x & u & -1 & v & 0 & y & z \\ v & x & -1 & -1 & u & v & z & y & 0 \\ t & v & v & v & v & t & -z & N_c & -z \\ -z & z & 0 & 0 & z & -z & 0 & -N_c^2 & N_c^2 \\ N_c & y & y & y & y & N_c & -N_c^2 & N_c\cdot y & -N_c^2 \\ -z & 0 & z & z & 0 & -z & N_c^2 & -N_c^2 & 0 \end{pmatrix};$$

$$\omega_{14} = \frac{C_F}{8} \cdot \begin{pmatrix} u & v & -1 & -1 & v & x & z & 0 & y \\ v & t & v & v & t & v & -z & -z & N_c \\ -1 & v & u & x & v & -1 & 0 & z & y \\ -1 & v & x & u & v & -1 & 0 & z & y \\ v & t & v & v & t & v & -z & -z & N_c \\ x & v & -1 & -1 & v & u & z & 0 & y \\ z & -z & 0 & 0 & -z & z & 0 & N_c^2 & -N_c^2 \\ 0 & -z & z & z & -z & 0 & N_c^2 & 0 & -N_c^2 \\ y & N_c & y & y & N_c & y & -N_c^2 & -N_c^2 & N_c \cdot y \end{pmatrix}.$$

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