Surface charge of horizon symmetries of a black hole with supertranslation field

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Near-horizon symmetries are studied for static black hole solutions to Einstein equations containing a supertranslation field. A supertranslation field is defined at the unit sphere. We consider general diffeomorphisms which preserve the gauge and the near-horizon structure of the metric. Diffeomorphisms are generated by the vector fields and form a group of near-horizon symmetries. The densities of variation of the surface charge associated to horizon symmetries of the metric are calculated in different coordinate systems connected by "large" transformations containing a supertranslation field in the metric. Variations of the surface charge corresponding to horizon symmetries are calculated in different coordinate systems. It is shown that the variations of the charge in systems connected by a large transformation have different integrability properties over the space of metrics. In the case of a supertranslation field depending only on the spherical angle θ it is shown that, although the variations of the surface charge in two coordinate systems connected by a large transformation are equal to each other, in one coordinate system the charge variation, having the form of variation of a functional of metrics, is integrable, but in another system it is not.

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I. INTRODUCTION

The final state of gravitational collapse is a stationary metric diffeomorphic to the metric of the Kerr black hole [1–3]. General diffeomorphisms contain pure gauge transformations which are changes of coordinates and "large" transformations which change a supertranslation field in metric. Physically large transformations map a physical state to another physical state with a different cloud of soft particles [4–6].

Supertranslations naturally appear in a study of symmetries of the asymptotically flat gravity at the null infinity initiated by Bondi, van der Burg, Metzner and Sachs [7,8]. The infinite-dimensional group of the asymptotic symmetries (the BMS group) extends the Poincaré group and contains a normal subgroup of supertranslations which are the angle-dependent translations of retarded time at the null infinity [9].

BMS algebra can be further enhanced to contain superrotations [10–13]. Exponentiation of the infinitesimal supertranslation and superrotation generators produces finite transformations, but in distinction to supertranslations, exponentiation of the infinitesimal superrotations when acting on physical states produces the states with the energy unbounded from below [14]. One cannot introduce a physical state with a finite superrotation charge, but there exist conserved charges associated with supertranslations and superrotations [10–13]. Supertranslation charges vanish except for a charge corresponding to the mass of a state, but finite superrotation charges differ for states with different supertranslation fields.

BMS transformations are naturally formulated at the null infinity, but there is a complicated problem of extension of an asymptotically defined metric containing a supertranslation field in the bulk. In paper [14] a family of 4D vacua containing a supertranslation field was constructed in the bulk, and in paper [15] a solution-generation technique was developed and the black hole metrics diffeomorphic to the Schwarzschild metric and containing a supertranslation field were obtained.

In this paper we study the near-horizon symmetries of the black holes containing a supertranslation field. The near-horizon symmetries are the main characteristics of horizon microstates which in turn define thermodynamic properties, the entropy and evaporation of a black hole. Near-horizon symmetries were extensively investigated in a large number of papers (a very incomplete list of references is [16–39]).

The near-horizon region is foliated by a set of surfaces enclosing the horizon surface and located at a distance x from the horizon (x is defined differently in different coordinate systems). Near-horizon symmetries are generated by transformations which preserve the horizon structure of a metric, and do not change the power of the leading in x terms in the components of the metric considered at a near-horizon surface at a distance x in the limit $x \to 0$.

We consider the near-horizon transformations of the static black hole solutions of the Einstein equations

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containing a supertranslation field. Transformations are generated by the vector fields ξ^μ. The metric variations $\delta_\xi g$ are elements of the tangent space to the space of metrics and are solutions to the linearized Einstein equations. On the tangent space is defined a bilinear presymplectic form. The presymplectic Lee-Wald form [20] is

$$w^{\mu\mathrm{LW}}(\delta_1 g, \delta_2 g, g) = \delta_1 \Theta^\mu(\delta_2 g, g) - \delta_2 \Theta^\mu(\delta_1 g, g) - \Theta^\mu(g, \delta_{[1,2]} g),$$

where Θ^μ is the boundary term in variation of the Einstein action

$$\delta(\sqrt{-g}R) = E(g)\delta g + \partial_\mu \Theta^\mu(\delta g, g),$$

and $E(g) = 0$ for a solution of the Einstein equations. Other forms of the presymplectic structures [22,38] differ from the Lee-Wald form by the terms vanishing on solutions of the linearized Einstein equations. A symplectic 2-form is defined as an integral over a codimension-1 spacelike surface

$$\Omega^{\mathrm{LW}}(\delta_1 g, \delta_2 g, g) = \int_\Sigma w^{\mu\mathrm{LW}}(\delta_1 g, \delta_2 g, g) d^3 x_\mu.$$

The Lee-Wald presymplectic form contracted with a metric perturbation generated by a vector field ξ^μ and any metric variation δg from the tangent space of metrics satisfies the on-shell relation

$$w^{\mathrm{LW}}(\delta g, \delta_\xi g, g) \simeq dK_\xi^{\mathrm{IW}}(\delta g, g),$$

where K_ξ^{IW} is the Iyer-Wald surface charge form [21,23], and the equality is valid up to terms vanishing on shell. Variation of the surface charge associated with a transformation generated by a vector field ξ^μ is defined as

$$\delta H_\xi = \oint_{\partial\Sigma} K_\xi^{\mathrm{IW}},$$

where integration is over a surface enclosing the horizon [20–23,37–39].

The black hole solutions containing a supertranslation field are obtained from the Schwarzschild solution (in isotropic spherical coordinates) by applying the large transformations containing a supertranslation field. A supertranslation field is a real function on the unit sphere. The event horizon of a metric containing a supertranslation field "constructed in [15] in the ρ-system" is located at a surface which depends on a supertranslation field. It is possible to construct another coordinate system (r-system), connected to the ρ-system by a large transformation, in which the horizon is located at the surface $r = 2M$, where M is the mass of the black hole.

We calculate the surface charge forms $K_\xi^{\mu\nu}$ in different coordinate systems connected by large transformations and also within ρ and r systems in coordinate systems corresponding to different parametrizations of the unit sphere on which a supertranslation field is defined.

To obtain the surface charge H_ξ, variation δH_ξ should be integrated over the space of metrics. The unique surface charge is obtained, if the integral over the space of metrics is independent of a path of integration. We find that in the general case variation of the surface charge δH_ξ cannot be written as a variation of a certain functional over the space of metrics, and integration over the space of metrics does not yield a path-independent charge. We discuss a special case in which the surface charge of horizon symmetries is obtained in the closed form.

The paper is organized as follows.

In Sec. II we review the form of the static vacuum metric containing a supertranslation field in the ρ-system obtained in [15]. Next, by a large transformation we transform the metric to the r-system. In both ρ- and r-systems we obtain the metrics in different parametrizations of the unit sphere on which is defined supertranslation field.

In Sec. III we study diffeomorphisms preserving the near-horizon form of the metric in ρ- and r-systems. We find constraints on the generators of transformations preserving the gauge and the near-horizon form of the metric.

In Sec. IV we consider supertranslations preserving the gauge and the near-horizon structure of the metric which are extendable in the bulk. Supertranslations form a group under the modified bracket [15]. A case of a supertranslation field depending only on an angle θ is considered in detail. It is shown that in the case of a supertranslation field depending only on θ the requirement that supertranslation preserves the gauge and the form of the metric fixes the parameter of the transformation through the supertranslation field $C(\theta)$.

In Sec. V we calculate variations of the surface charge corresponding to horizon symmetries in the ρ- and r-systems. Variation of the charge is obtained by integration of the surface charge forms over the surfaces enclosing the horizon and located at a distance x from the horizon.

In the ρ-system, the variation of the charge δH_ξ receives contributions from the integrals of the charge densities $K_\xi^{\mu\nu}$ with the components $(\mu, \nu) = (t, \rho), (t, \theta), (t, \varphi)$. In the r-system the only contribution is from integration of the component with $(\mu\nu) = (r, t)$ over the horizon sphere.

The surface charge is obtained in the limit $x \to 0$. In the charge densities we separate the leading in x terms in accordance with the power of x coming from the determinant of metric so that the resulting expression for the variation of the charge is independent on x.

In Sec. VI we discuss integrability of the variation of the surface charge in the case of a supertranslation field depending only on a spherical angle θ. In the ρ-system, the variation of the charge cannot be presented as a variation of a functional over the space of metrics. In the r-system we show that the variation of the charge is integrable. Although the variations of the charge have different forms in the ρ- and r-systems, performing the change of coordinates, we show that the expressions are equal.

The last section contains a brief summary of results.

II. STATIC VACUUM SOLUTION OF THE EINSTEIN EQUATIONS WITH A SUPERTRANSLATION FIELD

We begin this section with a short review of a black hole solution with a supertranslation field constructed in [15]. Next, we transform the metric to a form in which the horizon is located at the surface $r = 2M$.

The vacuum solution of the Einstein equations containing a supertranslation field $C(z^a)$ is

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + g_{\rho\rho}d\rho^2 + g_{ab}dz^a dz^b \\ &= -\frac{(1-M/2\rho_s)^2}{(1+M/2\rho_s)^2}dt^2 \\ &\quad + (1+M/2\rho_s)^4[d\rho^2 + (((\rho-E)^2+U)\gamma_{ab} \\ &\quad + (\rho-E)C_{ab})dz^a dz^b], \end{aligned} \tag{2.1}$$

where z^a are coordinates on the unit sphere. A supertranslation field $C(z^a)$ is a real regular function on the unit sphere. Coordinates on the sphere z^a can be realized as spherical coordinates θ, φ with the metric $ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$, or as projective coordinates $z^1 = z = \cot\frac{\theta}{2}e^{i\varphi}$, $z^2 = \bar{z} = \cot\frac{\theta}{2}e^{-i\varphi}$ with the metric $ds^2 = 2\gamma_{z\bar{z}}dzd\bar{z}$, $\gamma_{z\bar{z}} = 2e^{-2\psi}$, $\psi = \ln(1+|z|^2)$. Covariant derivatives D_a are defined with respect to the corresponding metric on the sphere. Here

$$\rho_s(\rho, C) = \sqrt{(\rho - C - C_{00})^2 + D_a C D^a C}. \tag{2.2}$$

C_{00} is the lowest spherical harmonic mode of $C(z^a)$. In the following we do not write C_{00} explicitly understanding $C \to C - C_{00}$. The horizon of metric (2.1) is located at the surface $\rho_s = M/2$. Here $\rho \subset (0, +\infty)$. The tensor C_{ab} and the functions U and E are defined as

$$\begin{aligned} C_{ab} &= -(2D_a D_b - \gamma_{ab}D^2)C, \\ U &= \frac{1}{8}C_{ab}C^{ba}, \\ E &= \frac{1}{2}D^2 C + C. \end{aligned} \tag{2.3}$$

The metric (2.1) in coordinates (ρ, θ, φ) with supertranslation field $C(\theta, \varphi)$ was obtained from the Schwarzschild metric

$$\begin{aligned} ds^2 &= -\left(\frac{1-M/2\rho_s}{1+M/2\rho_s}\right)^2 dt^2 \\ &\quad + (1+M/2\rho_s)^4(dx_s^2 + dy_s^2 + dz_s^2) \\ \rho_s^2 &= x_s^2 + y_s^2 + z_s^2 \end{aligned}$$

by the diffeomorphism [15]

$$\begin{aligned} x_s &= (\rho - C)\sin\theta\cos\varphi + \partial_\varphi C\sin\varphi/\sin\theta - \partial_\theta C\cos\theta\cos\varphi, \\ y_s &= (\rho - C)\sin\theta\sin\varphi - \partial_\varphi C\cos\varphi/\sin\theta - \partial_\theta C\cos\theta\sin\varphi, \\ z_s &= (\rho - C)\cos\theta - \partial_\theta C\sin\theta. \end{aligned} \tag{2.4}$$

In coordinates (ρ, θ, φ) the transformed metric is

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + g_{\rho\rho}[d\rho^2 + 2\tilde{g}_{\theta\varphi}d\theta d\varphi + \tilde{g}_{\theta\theta}d\theta^2 + \tilde{g}_{\varphi\varphi}d\varphi^2] \\ &= -\frac{(\rho_s - M/2)^2}{(\rho_s + M/2)^2}dt^2 \\ &\quad + (1+M/2\rho_s)^4\Big[d\rho^2 + 2(\rho - E)C_{\theta\varphi}d\theta d\varphi \\ &\quad + \left(\rho - E + \frac{1}{2}C_{\theta\theta}\right)^2 d\theta^2 + \sin^2\theta\left(\rho - E - \frac{1}{2}C_{\theta\theta}\right)^2 d\varphi^2\Big], \end{aligned} \tag{2.5}$$

where

$$C_{\theta\theta} = -C'' + C'\cot\theta + \frac{\ddot{C}}{\sin^2\theta} \qquad C_{\varphi\varphi} = -C_{\theta\theta}\sin^2\theta,$$
$$C_{\theta\varphi} = -2(\dot{C}' - \dot{C}\cot\theta).$$

Here dot and prime are derivatives over φ and θ.

In variables $(\rho, z, \bar{z})$ the metric with supertranslation field $C(z, \bar{z})$ has a form

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + g_{\rho\rho}[d\rho^2 + \tilde{g}_{ab}dz^a dz^b] \\ &= -\frac{(1-M/2\rho_s)^2}{(1+M/2\rho_s)^2}dt^2 + (1+M/2\rho_s)^4[d\rho^2 \\ &\quad + 2((\rho-E)^2 + U)\gamma_{z\bar{z}}dzd\bar{z} \\ &\quad + (\rho - E)(C_{zz}dzdz + C_{\bar{z}\bar{z}}d\bar{z}d\bar{z})], \end{aligned} \tag{2.6}$$

where

$$C_{zz} = -2D_z\partial_z C, \qquad C_{\bar{z}\bar{z}} = -2D_{\bar{z}}\partial_{\bar{z}}C, \qquad C_{z\bar{z}} = 0.$$

Transformation from (θ, φ, ρ) to $(z, \bar{z}, \rho)$ is a pure gauge transformation.

If the supertranslation field depends only on $|z|$, or in coordinates (r, θ, φ) only on θ, the metric simplifies with $g_{\theta\varphi} = 0$ in (2.5). On the other hand, if the supertranslation field depends only on $z/\bar{z}$, or on φ, the metric retains its general form.

Next, we transform the metric (2.6) to new variables (r, z^a), where $z^a = (z, \bar{z})$ or (θ, φ). The new variables are chosen so that in new variables the tt component of metric is equal to V [40]

$$g_{tt} = 1 - 2M/r \equiv V.$$

Variable $r \geq 2M$ is defined through the variables ρ, θ, φ by the relation

$$r = \rho_s(\rho, C)\left(1 + \frac{M}{2\rho_s(\rho, C)}\right)^2. \tag{2.7}$$

Inversely, ρ is expressed through r as

$$\rho = C + \sqrt{\frac{K^2}{4} - D_a C D^a C} = C + \frac{K}{2}\sqrt{1 - b_a b^a}, \tag{2.8}$$

where we introduced the functions

$$K = r - M + rV^{1/2}, \qquad \rho_s = \frac{K}{2},$$
$$b_a = \frac{2D_a C}{K}, \qquad b^2 = b_a b^a. \tag{2.9}$$

Differential $d\rho(r, z^a)$ is

$$d\rho = \rho_{,a} dz^a + \rho_{,r} dr$$
$$\rho_{,a} = \frac{K}{2}\left(b_a - \frac{\partial_a b^2}{2\sqrt{1-b^2}}\right), \quad \rho_{,r} = \frac{K}{2\sqrt{1-b^2} r V^{1/2}}. \tag{2.10}$$

Using the relations

$$g_{tt} = \frac{(1 - M/2\rho_s)^2}{(1 + M/2\rho_s)^2} = 1 - \frac{2M}{r} = V,$$
$$g_{\rho\rho} = (1 + M/2\rho_s)^4 = \frac{4r^2}{K^2}, \tag{2.11}$$

we introduce the transformed metric components (in variables $(r, z, \bar{z})$ or (r, θ, φ) the metric components are written with hats, in variables $(\rho, z, \bar{z})$ or (ρ, θ, φ) with tildas)

$$\hat{g}_{rr} = g_{\rho\rho}\rho_{,r}^2 = \frac{1}{Vr^2(1 - b^2)}, \qquad \hat{g}_{ra} = g_{\rho\rho}\rho_{,r}\rho_{,a},$$
$$\hat{g}_{ab} = g_{\rho\rho}(\tilde{g}_{ab} + \rho_a \rho_b). \tag{2.12}$$

We obtain the metric in variables (r, z^a) in a form

$$ds^2 = \hat{g}_{tt}dt^2 + \hat{g}_{rr}dr^2 + 2\hat{g}_{ra}dr dz^a + \hat{g}_{ab}dz^a dz^b$$
$$= -V dt^2 + \frac{4}{K^2}[\rho_r^2 dr^2 + 2\rho_r \rho_a dr dz^a$$
$$+ (\tilde{g}_{ab} + \rho_a \rho_b) dz^a dz^b]. \tag{2.13}$$

For the above expressions to be well defined, we require that $1 - b^2 > 0$. Because K is an increasing function of r which has its minimum at $r = 2M$, the sufficient condition is $1 - (2DC/M)^2 > \text{const} > 0$. In the following we work in the units $M = 1$.

III. DIFFEOMORPHISMS PRESERVING THE NEAR-HORIZON FORM OF THE METRIC

A. The metric in variables ρ, z^a

In this section we study diffeomorphisms which preserve the near-horizon form and the gauge of the metric (2.6) in the ρ-system. Near-horizon foliation of the space-time is done as follows. Let x be a parameter specifying the distance from a near-horizon to horizon surface (the choice of x depends on the choice of coordinates and is specified below). Horizon is located at the surface $\rho_s = 1/2$, where ρ_s is defined in (2.2). The near-horizon surface is defined as a surface $\rho_s = 1/2 + x$. Assuming that at the horizon the equation $\sqrt{(\rho - C)^2 + D_a C D^a C} = 1/2$ has a unique solution,

$$\rho_H(z^a, x) = C + \sqrt{1/4 - D_a C D^a C}, \tag{3.1}$$

by continuity the equation

$$\rho_s = \sqrt{(\rho - C)^2 + D_a C D^a C} = 1/2 + x \tag{3.2}$$

in some vicinity of $\rho_s = 1/2$ also has the unique solution:

$$\rho(x, z^a) = C + \sqrt{(1/2 + x)^2 - D_a C D^a C}.$$

For a small $|x| \ll 1$ we obtain

$$\rho(x, z^a) \simeq \rho_H(z^a) + \tilde{x}, \tag{3.3}$$

where

$$\tilde{x} = \frac{x}{2\sqrt{1/4 - (DC)^2}}. \tag{3.4}$$

There are two branches of $\rho : \rho = C \pm \sqrt{\rho_s^2 - (DC)^2}$. To have a smooth limit to the Schwarzschild metric, $C \to 0$, we choose the plus sign.

In the near-horizon region the metric has a form

$$ds^2 = (\bar{g}_{tt}\tilde{x}^2 + O(\tilde{x}^3))dt^2 + (\bar{g}_{\rho\rho} + O(\tilde{x}))d\rho^2 + (\bar{g}_{ab} + O(\tilde{x}))dz^a dz^b. \quad (3.5)$$

Here $\bar{g}_{\mu\nu}$ are the $O(\tilde{x}^0)$ parts of $g_{\mu\nu}$.

We consider transformations generated by the vector field

$$\xi^i\partial_i = \xi^t\partial_t + \xi^\rho\partial_\rho + \xi^a\partial_a. \quad (3.6)$$

ξ^i are assumed to be independent of t. General near-horizon transformations are required to preserve the gauge of the metric and the power $\tilde{x}^n$ of the leading in $\tilde{x}$ terms in the difference $g_{\mu\nu}(\tilde{x}) - g_{\mu\nu}(0)$ up to a numerical coefficient at the leading term in $\tilde{x}$.

The metric is written in the gauge $g_{\rho a} = g_{\rho t} = g_{at} = 0$. Transformations which preserve the gauge satisfy the relations

$$L_\xi g_{\rho a} = \partial_\rho \xi^b g_{ba} + \partial_a \xi^\rho g_{\rho\rho} = 0, \quad (3.7)$$

$$L_\xi g_{\rho t} = \partial_\rho \xi^t g_{tt} + \partial_t \xi^\rho g_{\rho\rho} = 0, \quad (3.8)$$

$$L_\xi g_{at} = \partial_a \xi^t g_{tt} + \partial_t \xi^b g_{ba} = 0. \quad (3.9)$$

Conditions (3.8) and (3.9) give $\xi^t = \text{const}$.

At the near-horizon surface $\rho_s = 1/2 + x$, or $\rho = \rho_H + \tilde{x}$, the component g_{tt} is

$$g_{tt} \simeq 4(\rho_H - C)^2\tilde{x}^2 + O(\tilde{x}^3). \quad (3.10)$$

Under the action of transformation generated by vector field ξ^k the component g_{tt} is transformed as

$$L_\xi g_{tt} = 4\frac{\rho_s - 1/2}{(\rho_s + 1/2)^3} L_\xi \rho_s, \quad (3.11)$$

where

$$L_\xi \rho_s = (\xi^\rho\partial_\rho + \xi^a D_a)\rho_s = \frac{\xi^\rho 2(\rho - C) + \xi^a(-2(\rho - C)D_a C + D_a(D_b C D^b C))}{2\rho_s}. \quad (3.12)$$

To preserve the near-horizon behavior of g_{tt} (3.10), it is necessary that

$$\xi^\rho 2(\rho - C) + \xi^a(-2(\rho - C)D_a C + D_a(D_b C D^b C) = O(\tilde{x}). \quad (3.13)$$

At the horizon, this condition gives the relation connecting ξ^ρ and ξ^a:

$$\xi^\rho(\rho_H - C) + \xi^a((-\rho_H + C)D_a C + (D_a D_b C)D^b C) = 0. \quad (3.14)$$

Because $g_{\rho\rho}$ is a function of ρ_s, using condition (3.13), we have

$$L_\xi g_{\rho\rho} = \frac{\partial g_{\rho\rho}}{\partial \rho_s} L_\xi \rho_s + 2\partial_\rho \xi^\rho g_{\rho\rho} = 2^5 \partial_\rho \xi^\rho + O(\tilde{x}). \quad (3.15)$$

B. The metric in variables r, z^a

In variables (r, z^a) the horizon of the metric (2.13) is located at the surface $r = 2$. In the foliation of the near-horizon region defined through $\rho_s = 1/2 + x$, in variables (r, z^a) the near-horizon surfaces are at $r = 2 + \hat{x}$, where $\hat{x} = 2x^2 + O(x^3)$. In the near-horizon region the metric (2.13) has a form

$$\begin{aligned} ds^2 &= \hat{g}_{tt}dt^2 + \hat{g}_{rr}dr^2 + 2\hat{g}_{ra}drdz^a + \hat{g}_{ab}dz^a dz^b \\ &= (-\bar{g}_{tt}\hat{x} + O(\hat{x}^2))dt^2 + \left(\frac{\bar{g}_{rr}}{\hat{x}} + O(\hat{x}^{-1/2})\right)d\hat{x}^2 \\ &\quad + 2\left(\frac{\bar{g}_{ra}}{\hat{x}^{1/2}} + O(\hat{x}^0)\right)d\hat{x}dz^a \\ &\quad + (\bar{g}_{ab,0}\hat{x}^0 + O(\hat{x}^{1/2}))dz^a dz^b. \end{aligned} \quad (3.16)$$

$\bar{g}_{\mu\nu} = O(\hat{x}^0)$ are the coefficients at the leading in $\hat{x}$ terms in the metric components. The metric (3.16) is written in the gauge

$$\hat{g}_{rt} = \hat{g}_{ta} = 0.$$

The near-horizon transformations are generated by the vector fields

$$\chi^k\partial_k = \chi^t\partial_t + \chi^r\partial_r + \chi^a\partial_a. \quad (3.17)$$

We assume that χ^k are independent of t. Transformations preserving the gauge conditions are

$$\begin{aligned} L_\chi \hat{g}_{rt} &= \partial_r \chi^t \hat{g}_{tt} + \partial_t \chi^r \hat{g}_{rr} + \partial_t \chi^a \hat{g}_{ar} = 0 \\ L_\chi \hat{g}_{at} &= \partial_a \chi^t \hat{g}_{tt} + \partial_t \chi^r \hat{g}_{ra} + \partial_t \chi^b \hat{g}_{ba} = 0. \end{aligned} \quad (3.18)$$

From conditions (3.18) we obtain that $\chi^t = \text{const}$. Transformations preserving the leading in $\hat{x}$ behavior of the metric components are

$$
\begin{aligned}
L_\chi \hat{g}_{tt} &= \chi^r \partial_r \hat{g}_{tt} = O(\hat{x}),\\
L_\chi \hat{g}_{rr} &= \chi^r \partial_r \hat{g}_{rr} + \chi^a \partial_a \hat{g}_{rr} + 2\partial_r \chi^a \hat{g}_{ar} + 2\partial_r \chi^r \hat{g}_{rr}\\
&= O(\hat{x}^{-1}),\\
L_\chi \hat{g}_{ar} &= \chi^r \partial_r \hat{g}_{ar} + \chi^b \partial_b \hat{g}_{ar} + \partial_a \chi^r \hat{g}_{rr} + \partial_a \chi^b \hat{g}_{br}\\
&\quad + \partial_r \chi^r \hat{g}_{ar} + \partial_r \chi^b \hat{g}_{ab} = O(\hat{x}^{-1/2}),\\
L_\chi \hat{g}_{ab} &= \chi^r \partial_r \hat{g}_{ab} + \chi^c \partial_c \hat{g}_{ab} + \partial_a \chi^r \hat{g}_{rb} + \partial_a \chi^c \hat{g}_{cb}\\
&\quad + \partial_b \chi^r \hat{g}_{ra} + \partial_b \chi^c \hat{g}_{ca} = O(\hat{x}^0).
\end{aligned} \tag{3.19}
$$

We look for the components of the vector field χ^k in a form of expansion in powers in $\hat{x}^{1/2}$:

$$
\chi^k = \chi^k_0 + \hat{x}^{1/2}\chi^k_{1/2} + \hat{x}\chi^k_1 + \cdots .
$$

From the relations (3.19) we obtain the form of generators preserving the near-horizon form of the metric

$$
\chi^r_0 = \chi^r_{1/2} = 0, \quad \chi^r = \chi^r_1 \hat{x}, \quad \chi^a = \chi^a_0 + \hat{x}^{1/2}\chi^a_0. \tag{3.20}
$$

The vector fields generating the near-horizon transformations form the Lie brackets

$$
[\chi_{(1)}, \chi_{(2)}]^k = \chi^k_{(12)}, \tag{3.21}
$$

where

$$
\begin{aligned}
\chi^r_{(12),1} &= \chi^b_{(1),0} \overleftrightarrow{\partial}_b \chi^r_{(2),1},\\
\chi^a_{(12),0} &= \chi^b_{(1),0} \overleftrightarrow{\partial}_b \chi^a_{(2),0},\\
\chi^a_{(12),1/2} &= \chi^b_{(1),0} \overleftrightarrow{\partial}_b \chi^a_{(2),1/2}\\
&\quad + 1/2(\chi^r_{(1),1}\chi^a_{(2),1/2} - (1 \to 2)).
\end{aligned} \tag{3.22}
$$

The vector field (3.17) is connected with the vector field (3.6) by a transformation

$$
\begin{aligned}
\chi^r &= \xi^\rho \frac{\partial r}{\partial \rho} + \xi^a \frac{\partial r}{\partial z^a} = \xi^\rho \frac{\partial r}{\partial \rho_s}\frac{\partial \rho_s}{\partial \rho} + \xi^a \frac{\partial r}{\partial \rho_s}\frac{\partial \rho_s}{\partial z^a},\\
\chi^a &= \xi^\rho \frac{\partial z^a}{\partial \rho} + \xi^b \frac{\partial z^a}{\partial z^b} = \xi^a.
\end{aligned} \tag{3.23}
$$

From (2.7) and (2.8), we have

$$
\begin{aligned}
\partial r/\partial \rho_s &= \frac{K^2-1}{K^2}, \qquad \partial \rho_s/\partial \rho = \sqrt{1-b^2},\\
\partial \rho_s/\partial z^a &= \frac{K}{4}\left[-2b_a\sqrt{1-b^2} + D_a b^2\right].
\end{aligned} \tag{3.24}
$$

Using the relations (3.24), we obtain

$$
\begin{aligned}
\chi^t &= \xi^t,\\
\chi^r &= \frac{K^2-1}{K^2}\left[\xi^\rho\sqrt{1-b^2} + \xi^a \frac{K}{4}(-2b_a\sqrt{1-b^2} + D_a b^2)\right],\\
\chi^a &= \xi^a.
\end{aligned} \tag{3.25}
$$

The expression in the square brackets in χ^r is the same as in (3.14). For $|\hat{x}| \ll 1$ we have

$$
\begin{aligned}
&K \simeq 1 + \sqrt{2\hat{x}}, \qquad b_a = 2\partial_a C(1 - \sqrt{2\hat{x}}),\\
&(K^2-1)/K^2 = O(\hat{x}^{1/2}).
\end{aligned} \tag{3.26}
$$

At the near-horizon surface the metric component $\hat{g}_{tt}$ is

$$
\hat{g}_{tt} = -\frac{\hat{x}}{2} + O(\hat{x}^2). \tag{3.27}
$$

To have the transformed metric component $\hat{g}_{tt}$ of order $O(\hat{x})$, the vector component χ^r should be of order $O(\hat{x})$. It follows that

$$
\begin{aligned}
&\xi^\rho\sqrt{1-b^2} + \xi^a \frac{K}{4}\left(-2b_a\sqrt{1-b^2} + D_a b^2\right)\\
&\quad = O(\hat{x}^{1/2}).
\end{aligned} \tag{3.28}
$$

Noting that $\hat{x} \sim \tilde{x}^2$, we see that condition (3.28) coincides with the condition (3.14).

IV. SUPERTRANSLATIONS EXTENDED IN A BULK: SYMPLECTIC TRANSFORMATIONS

In this section we consider supertranslations preserving the near-horizon form of the metric which are defined not only in a vicinity of the horizon, but extend to the bulk. Supertranslations which preserve the gauge of metric (2.1) were constructed in [15]. Supertranslation field in metric (in coordinates θ, φ) transforms under supertranslations as

$$
\delta_T C(\theta,\varphi) = T(\theta,\varphi),
$$

where $T(\theta,\varphi)$ is an arbitrary smooth function on the unit sphere. Generator of supertranslations preserving the static gauge of the solution (2.1) in coordinates (ρ, z^a) has a form

$$
\xi_T = T_{00}\partial_t - (T - T_{00})\partial_\rho + F^{ab}D_a T D_b, \tag{4.1}
$$

where

$$
F^{ab} = \frac{C^{ab} - 2\gamma^{ab}(\rho - E)}{2((\rho - E)^2 - U)}.
$$

Transformations (4.1) are defined in the bulk and form a commutative algebra under the modified bracket [15]

$$[\xi_1, \xi_2]_{\rm mod} = [\xi_1, \xi_2] - \delta_{T_1}\xi_2 + \delta_{T_2}\xi_1. \tag{4.2}$$

It is explicitly verified that

$$\xi_T^k \frac{\partial \rho_s}{\partial x^k} = \delta_T \rho_s, \tag{4.3}$$

where

$$\delta_T \rho_s(C) = \lim_{\varepsilon\to 0}[\rho_s(C+\varepsilon T) - \rho_s(C)]/\varepsilon, \tag{4.4}$$

and

$$\delta_T g_{\rho t} = \delta_T g_{\rm at} = 0. \tag{4.5}$$

General transformations (4.1) do not respect the near-horizon form of the metric (3.5) changing the component g_{tt}. To preserve the near-horizon form of the metric (3.5), transformation generated by (4.1) must satisfy condition (3.13).

If the supertranslation field depends only on θ, $C = C(\theta)$, the generator of supertranslations simplifies to

$$\xi_T = T_0\partial_t - (T - T_0)\partial_\rho - \frac{T'}{\rho - C - C''}\partial_\theta. \tag{4.6}$$

The near-horizon structure of the metric is preserved provided the parameter of transformation (4.6) $T(\theta)$ satisfies the relation

$$-T(\rho_H - C) + T'C' = O(\tilde{x}). \tag{4.7}$$

At the horizon, condition (4.7) is an ordinary differential equation on $T(\theta)$ with the solution

$$T(\theta) = a \exp \int^\theta d\theta \sqrt{1/4 - C'^2}/C', \tag{4.8}$$

where a is an integration constant. Generators of supertranslations in coordinates (ρ, x^a) and (r, z^a) are connected by the transformation (3.23). In variables (r, z^a) the generator of supertranslations is

$$\begin{aligned} \chi_T &= \chi_T^t\partial_t + \chi_T^r\partial_r + \chi_T^a\partial_a \\ &= T_{00}\partial_t + \frac{K^2-1}{K^2}\left(-T\sqrt{1-b^2} + \frac{K}{4}F^{ab}D_bT\left(-2b_a\sqrt{1-b^2} + D_ab^2\right)\right)\partial_r + F^{ab}D_bT\partial_a, \end{aligned} \tag{4.9}$$

where in F^{ab} it is substituted $\rho - C = K(1-b^2)^{1/2}/2$. Acting by the generator of supertranslations on the component $\hat{g}_{tt}$, we obtain

$$L_{\chi_T}\hat{g}_{tt} = \frac{2}{r^2}\frac{K^2-1}{K^2}\left(-T\sqrt{1-b^2} + \frac{K}{4}F^{ab}D_bT\left(-2b_a\sqrt{1-b^2} + D_ab^2\right)\right). \tag{4.10}$$

In the near-horizon region the relations for K are (3.26). To preserve the form of $\hat{g}_{tt}$, it is necessary that

$$\begin{aligned} &-T\sqrt{1-b^2} + \frac{K}{4}F^{ab}D_bT\left(-2b_a\sqrt{1-b^2} + D_ab^2\right) \\ &\quad = O(\hat{x}^{1/2}). \end{aligned} \tag{4.11}$$

This imposes the condition on $T(z, \bar{z})$:

$$\left[-T\sqrt{1-b^2} + \frac{1}{4}F^{ab}D_bT\left(-2b_a\sqrt{1-b^2} + D_ab^2\right)\right]_{r=0} = 0. \tag{4.12}$$

Equation (4.12) for T is solved in the Appendix. In the case of a supertranslated field in the metric depending only on θ, relation (4.11) turns into (4.7). It is seen that in the near-horizon region in variables r, z^a the generator of supertranslations has the following structure:

$$\chi_T = O(x^0)\partial_t + O(x)\partial_x + O(x^0)\partial_a. \tag{4.13}$$

V. SURFACE CHARGE OF ASYMPTOTIC HORIZON SYMMETRIES

In this section, we calculate the variation of the surface charge corresponding to diffeomorphisms preserving the near-horizon form of the metric. Calculations are performed both in ρ- and r-systems. Variation of the surface charge associated with a symmetry generated by a vector field ξ^μ is

$$\not{\delta} H_\xi(g, h) = \lim_{\partial\Sigma\to\partial\Sigma_H} f14\pi \int (d^2x)_{\mu\nu}\sqrt{-g}K_\xi^{\mu\nu}, \tag{5.1}$$

where $(d^2x)_{\mu\nu} = (1/4)\varepsilon_{\alpha\beta\mu\nu}dx^\alpha dx^\beta$. The charge density is

$$K^{\mu\nu}_\xi = \xi^\mu\nabla^\nu h - \xi^\mu\nabla_\sigma h^{\nu\sigma} + \xi_\sigma\nabla^\mu h^\nu \frac{1}{2} h\nabla^\mu\xi^\nu - h^{\mu\sigma}\nabla_\sigma\chi^\nu + \frac{\alpha}{2} h^{\mu\sigma}(\nabla^\nu\chi_\sigma + \nabla_\sigma\chi^\nu) + (\mu \leftrightarrow \nu), \quad (5.2)$$

where $\alpha = 1$ in the Barnich-Brandt form [22] and $\alpha = 0$ in the Iyer-Wald form [21]. Here $\partial\Sigma$ is a codimension-2 compact spacelike surface enclosing the horizon surface.

A. Variation of the surface charge in the ρ-system

First, let us consider parametrization of the unit sphere in variables $(\rho, z, \bar{z})$. The metric of the black hole is (2.6), and the near-horizon region is foliated by surfaces $\rho_s = \sqrt{(\rho - C)^2 + (DC)^2} = 1/2 + x$, $|x| \ll 1$.

To the variation of the surface charge (5.1) contribute integrations over $(z, \bar{z}), (\rho, z)$ and $(\rho, \bar{z})$:

$$\int \sqrt{-g}\epsilon_{t\rho z\bar{z}} dz \wedge d\bar{z} K^{t\rho}, \qquad \int \sqrt{-g}\epsilon_{t\bar{z}\rho z} d\rho \wedge dz K^{t\bar{z}}, \qquad \int \sqrt{-g}\epsilon_{tz\rho\bar{z}} d\rho \wedge d\bar{z} K^{tz}. \quad (5.3)$$

Taking the differential of the equation of the horizon surface $\sqrt{(\rho - C)^2 + (DC)^2} - 1/2 = 0$, we express $d\rho$ through dz^a:

$$(\rho - C)d\rho = (\rho - C)(C_z dz + C_{\bar{z}} d\bar{z}) - \frac{1}{2}(\partial_z (DC)^2 dz + \partial_{\bar{z}}(DC)^2 d\bar{z}). \quad (5.4)$$

Setting $\epsilon_{t\rho z\bar{z}} = 1$ and introducing evident notations $\rho_{,z}$ and $\rho_{,\bar{z}}$, we obtain

$$\oint H_\xi = \frac{1}{4\pi}\int dz \wedge d\bar{z}\,\sqrt{-g}\,[K^{t\rho} + \rho_{,\bar{z}} K^{t\bar{z}} - \rho_{,z} K^{tz}]. \quad (5.5)$$

The charge density $K^{\rho t}_\xi$ in the Iyer-Wald form is

$$K^{\rho t}_\xi = \xi^\rho\nabla^t h - \xi^\rho\nabla_\sigma h^{t\sigma} + \xi_\sigma\nabla^\rho h^{t\sigma} + \frac{1}{2} h\nabla^\rho\xi^t - h^{\rho\sigma}\nabla_\sigma\xi^t - (\rho \leftrightarrow t). \quad (5.6)$$

From (2.6) we have

$$g(\rho, z, \bar{z}) = g_{tt} g^3_{\rho\rho}\tilde{g}^{(2)},$$

where

$$\tilde{g}^{(2)} = \tilde{g}_{zz}\tilde{g}_{\bar{z}\bar{z}} - \tilde{g}^2_{z\bar{z}} = \gamma^2_{z\bar{z}}[(\rho - E)^2 - U]^2. \quad (5.7)$$

In variables ρ, θ, φ expressions (5.3)–(5.6) have the same functional form as in $\rho, z, \bar{z}$ with the formal change $z, \bar{z} \to \theta, \varphi$:

$$\oint H_\xi = \frac{1}{4\pi}\int d\theta \wedge d\varphi\sqrt{-g}[K^{t\rho} + \rho_{,\varphi} K^{t\varphi} - \rho_{,\theta} K^{t\theta}].$$

At the near-horizon surface $\rho = \rho_H(z^a) + \tilde{x}$ the determinant of the metric is of order $O(\tilde{x}^2)$, and to obtain a nonzero result for $\oint H$, in $K^{\rho t}_\xi$ we collect the terms of order $O(\tilde{x}^{-1})$. The leading in $\tilde{x}$ terms of the metric components are seen from (3.5). Variations of the metric have the same order in $\tilde{x}$ as the metric components.

The five contributions to $K^{\rho t}_\xi$ are

$$\begin{aligned}
&1.\quad \xi^\rho\nabla^t h - \xi^t\nabla^\rho h = \xi^\rho g^{tt}\partial_t h - \xi^t g^{\rho\rho}\partial_\rho h,\\
&2.\quad -\xi^\rho\nabla_s h^{ts} + \xi^t\nabla_s h^{\rho s},\\
&3.\quad \xi_s\nabla^\rho h^{ts} - \xi_s\nabla^t h^{\rho s},\\
&4.\quad \frac{h}{2}(\nabla^\rho\xi^t - \nabla^t\xi^\rho) = \frac{h}{2}(g^{\rho s}\nabla_s\xi^t - g^{ts}\nabla_s\xi^\rho),\\
&5.\quad -h^{\rho s}\nabla_s\xi^t + h^{ts}\nabla_s\xi^\rho.
\end{aligned} \quad (5.8)$$

Because h is independent of t and $g^{\rho\rho} = O(\tilde{x}^0)$, the two terms in the item 1 are of order $\tilde{x}^0$.

The first term in the item 2,

$$\begin{aligned}
-\xi^\rho\nabla_s h^{ts} &= -\xi^\rho(\nabla_t h^{tt} + \nabla_\rho h^{t\rho} + \nabla_a h^{ta})\\
&= -\xi^\rho(2\Gamma^t_{tt}h^{tt} + \Gamma^t_{\rho\rho}h^{\rho\rho} + \Gamma^\rho_{t\rho}h^{tt} + \Gamma^t_{aa}h^{aa} + \Gamma^a_{at}h^{tt})\\
&= 0,
\end{aligned}$$

is zero, because all Γ vanish. The second term in the item 2 is

$$\begin{aligned}
\xi^t\nabla_s h^{\rho s} &= \xi^t(\nabla_t h^{\rho t} + \nabla_\rho h^{\rho\rho} + \nabla_a h^{\rho a})\\
&= \xi^t(\Gamma^\rho_{tt}h^{tt} + \Gamma^t_{t\rho}h^{\rho\rho}) + O(\tilde{x}^0).
\end{aligned}$$

The first term in the item 3 is transformed as

$$\begin{aligned}
\xi_s\nabla^\rho h^{ts} &= g^{\rho\rho}(\xi_t\nabla_\rho h^{tt} + \xi_\rho\nabla_\rho h^{t\rho} + \xi_a\nabla_\rho h^{ta})\\
&= g^{\rho\rho}[\xi_t(\partial_\rho h^{tt} + 2\Gamma^t_{\rho t}h^{tt}) + \xi_\rho(\Gamma^t_{\rho\rho}h^{\rho\rho} + \Gamma^\rho_{\rho t}h^{tt}) + O(x^0)]\\
&= \xi_t g^{\rho\rho}(\partial_\rho h^{tt} + g^{tt}g_{tt,\rho}h^{tt}) + O(\tilde{x}^0).
\end{aligned}$$

The leading in $\tilde{x}$ terms in this expression cancel,

$$\begin{aligned}
\xi_t g^{\rho\rho}(\partial_\rho h^{tt} + g^{tt}g_{tt,\rho}h^{tt}) &= \xi^t g_{tt}g^{\rho\rho}\left(-2\frac{\bar{h}^{tt}}{\tilde{x}^3} + \frac{\bar{g}^{tt}}{\tilde{x}^2}2\tilde{x}\bar{g}_{tt}\frac{\bar{h}^{tt}}{\tilde{x}^2}\right)\\
&= 0,
\end{aligned}$$

and the remaining expression is of order $\tilde{x}^0$. The second term in the item 3,

$$
\begin{aligned}
-\xi_s g^{tt}\nabla_t h^{\rho s} &= -g^{tt}(\xi_t\nabla_t h^{\rho t} + \xi_\rho \nabla_t h^{\rho\rho} + \xi_a \nabla_t h^{\rho a}) \\
&= -\xi^t(\Gamma^\rho_{tt} h^{tt} + \Gamma^t_{t\rho} h^{\rho\rho}) + O(\tilde{x}^0),
\end{aligned}
$$

cancels the corresponding expression in the item 2.

Collecting the items 4 and 5, we obtain

$$
K^{\rho t}_\xi = \nabla_s \xi^t\left(\frac{h}{2}g^{\rho s} - h^{\rho s}\right) - \nabla_s \xi^\rho\left(\frac{h}{2}g^{ts} - h^{ts}\right) + O(\tilde{x}^0). \tag{5.9}
$$

Taking into account the form of the metric in the ρ-system, we write $K^{\rho t}_\xi$ as

$$
K^{\rho t}_\xi = \nabla_\rho \xi^t\left(\frac{h}{2}g^{\rho\rho} - h^{\rho\rho}\right) - \nabla_t \xi^\rho\left(\frac{h}{2}g^{tt} - h^{tt}\right) + O(\tilde{x}^0). \tag{5.10}
$$

The leading in $\tilde{x}$ part of $K^{\rho t}$ of order $\tilde{x}^{-1}$ is

$$
\begin{aligned}
K^{\rho t}_\xi &= \Gamma^t_{\rho t}\xi^t\left(\frac{h}{2}g^{\rho\rho} - h^{\rho\rho}\right) - \Gamma^\rho_{tt}\xi^t\left(\frac{h}{2}g^{tt} - h^{tt}\right) \\
&= \frac{\xi^t}{2} g_{tt,\rho}(h g^{tt} g^{\rho\rho} - g^{tt}h^{\rho\rho} - g^{\rho\rho}h^{tt}) \\
&= \frac{\xi^t}{2} g_{tt,\rho} g^{tt} g^{\rho\rho} h_{ab} g^{ab}.
\end{aligned} \tag{5.11}
$$

The expression $h_{ab}g^{ab}$ can be written in a form

$$
\begin{aligned}
h_{ab}g^{ab} &= (h_{zz}g^{zz} + h_{\bar{z}\bar{z}}g^{\bar{z}\bar{z}} + 2h_{z\bar{z}}g^{z\bar{z}}) \\
&= \frac{1}{g^{(2)}}(h_{zz}g_{\bar{z}\bar{z}} + h_{\bar{z}\bar{z}}g_{zz} - 2h_{z\bar{z}}g_{z\bar{z}}) = \frac{\delta g^{(2)}}{g^{(2)}}.
\end{aligned} \tag{5.12}
$$

Thus, we obtain

$$
K^{\rho t}_\xi = \frac{\xi^t}{2} g_{tt,\rho} g^{tt} g^{\rho\rho} \frac{\delta g^{(2)}}{g^{(2)}}. \tag{5.13}
$$

Calculating the charge density,

$$
\begin{aligned}
K^{zt}_\xi &= \xi^z \nabla^t h - \xi^z \nabla_\sigma h^{t\sigma} + \xi_\sigma \nabla^z h^{t\sigma} + \frac{1}{2} h \nabla^z \xi^t \\
&\quad - h^{z\sigma}\nabla_\sigma \xi^t - (z \leftrightarrow t),
\end{aligned} \tag{5.14}
$$

we note that the contribution from the item 1 is $O(\tilde{x}^0)$, contributions from the items 2 and 3 cancel up to terms $O(\tilde{x}^0)$, and the items 4 and 5 yield

$$
K^{zt}_\xi = \frac{\xi^t}{2} g_{tt,a} g^{tt}\left(\frac{h}{2}g^{za} - h^{za}\right) - \frac{\xi^t}{2} g_{tt,z} g^{zz}\left(\frac{h}{2}g^{tt} - h^{tt}\right). \tag{5.15}
$$

Expression (5.15) is transformed to a form

$$
K^{zt}_\xi = \frac{\xi^t}{2} g^{tt}\left[g_{tt,z}(h_{\rho\rho}g^{\rho\rho}g^{zz} + h_{\bar{z}\bar{z}}g^{(2)-1}) + g_{tt,\bar{z}}\left(\frac{h}{2}g^{z\bar{z}} - h^{z\bar{z}}\right)\right] + O(\tilde{x}^0). \tag{5.16}
$$

In the same way for the charge density $K^{\bar{z}t}_\xi$ we have

$$
K^{\bar{z}t}_\xi = \frac{\xi^t}{2} g^{tt}\left[g_{tt,\bar{z}}(h_{\rho\rho}g^{\rho\rho}g^{zz} + h_{\bar{z}\bar{z}}g^{(2)-1}) + g_{tt,z}\left(\frac{h}{2}g^{z\bar{z}} - h^{z\bar{z}}\right)\right] + O(\tilde{x}^0). \tag{5.17}
$$

In variables t, ρ, θ, φ we obtain the expressions of the form (5.16) and (5.17) with θ, φ substituted for z, $\bar{z}$.

Let us consider the case of a supertranslation field $C(\theta)$ depending on θ. The metric (2.5) takes a form

$$
\begin{aligned}
ds^2 &= g_{tt}dt^2 + g_{\rho\rho}[d\rho^2 + \tilde{g}_{\theta\theta}d\theta^2 + \tilde{g}_{\varphi\varphi}d\varphi^2] \\
&= -\left(\frac{\rho_s - 1/2}{\rho_s + 1/2}\right)^2 dt^2 \\
&\quad + (1 + 1/2\rho_s)^4[d\rho^2 + (\rho - C - C'')^2 d\theta^2 \\
&\quad + \sin^2\theta(\rho - C - C'\cot\theta)^2].
\end{aligned}
$$

Because $\rho_{,\varphi} = 0$, we have

$$
\delta\!\!\!\!\oint H_\xi = \frac{1}{4\pi}\int d\theta \wedge \delta\varphi \sqrt{-g}[K^{\rho t} - \rho_{,\theta}K^{\theta t}]. \tag{5.18}
$$

The charge density $K^{\rho t}$ is

$$
K^{\rho t}_\xi = \frac{\xi^t}{2} g_{tt,\rho} g^{tt} g^{\rho\rho}(\delta g_{\theta\theta}g^{\theta\theta} + \delta g_{\varphi\varphi}g^{\varphi\varphi}). \tag{5.19}
$$

For the density $K^{\theta t}$ we obtain

$$
K^{\theta t} = \frac{\xi^t}{2} g_{tt,\theta} g^{tt} g^{\theta\theta}[\delta g_{\rho\rho}g^{\rho\rho} + \delta g_{\varphi\varphi}g^{\varphi\varphi}]. \tag{5.20}
$$

Variation of the surface charge is

$$
\begin{aligned}
\delta\!\!\!\!\oint H_\xi = \lim_{\rho_s\to 1/2}\frac{1}{4\pi}\int d\theta \wedge d\varphi \sqrt{-g}\frac{\xi^t}{2} g^{tt}\Bigg[& g_{tt,\rho}g^{\rho\rho}\left(\frac{\delta g_{\theta\theta}}{g_{\theta\theta}} + \frac{\delta g_{\varphi\varphi}}{g_{\varphi\varphi}}\right) \\
& - \rho_{,\theta} g_{tt,\theta} g^{\theta\theta}\left(\frac{\delta g_{\rho\rho}}{g_{\rho\rho}} + \frac{\delta g_{\varphi\varphi}}{g_{\varphi\varphi}}\right)\Bigg].
\end{aligned} \tag{5.21}
$$

Here $g_{\theta\theta} = \tilde{g}_{\theta\theta}g_{\rho\rho}$, $g_{\varphi\varphi} = \tilde{g}_{\varphi\varphi}g_{\rho\rho}$. In the near-horizon region we have

$$
\frac{g_{tt,\rho}}{\sqrt{g_{tt}}} \simeq 4(\rho - C), \quad \frac{g_{tt,\theta}}{\sqrt{g_{tt}}} \simeq 4(-C')(\rho - C - C''), \tag{5.22}
$$

and

$$\rho_{,\theta} = -\frac{C'(\rho - C - C'')}{\rho - C}.$$

We obtain variation of the charge as

$$\oint\!\!\!/ H_\xi = \lim_{\rho_s \to 1/2} \frac{1}{4\pi} \int d\theta \wedge d\varphi \frac{\xi^t}{2} \left[(\rho - C)\delta\sqrt{g_{\theta\theta} g_{\varphi\varphi}} + \frac{C'^2(\rho - C - C'')}{4(\rho - C)} \delta\sqrt{g_{\varphi\varphi} g_{\rho\rho}} \right], \tag{5.23}$$

where at the horizon $\rho_H - C = \sqrt{1/4 - C'^2}$.

Integrability of variation of the charge means that the integral of variation over the manifold of metrics is path independent. Integrability is verified explicitly, if variation of the charge is obtained in a form of variation of a functional over the space of metrics. The expression for $\oint\!\!\!/ H_\xi$ (5.23)) is not of the form of variation of a function over the space of metrics.

B. Variables r, z^a

Let us turn to calculation of the variation of the surface charge in variables (r, z^a). First, we perform calculations with the metric (2.13):

$$ds^2 = \hat{g}_{tt} dt^2 + \hat{g}_{rr} dr^2 + 2\hat{g}_{ra} dr dz^a + \hat{g}_{ab} dz^a dz^b.$$

At the surface $r = 2 + \hat{x}$ enclosing the horizon surface $r = 2$, the near-horizon forms of the metric (2.13) and its inverse are

$$\hat{g}_{mn} = \begin{vmatrix} \bar{g}_{tt}\hat{x} & 0 & 0 & 0 \\ 0 & \bar{g}_{rr}/\hat{x} & \bar{g}_{rz}/\sqrt{\hat{x}} & \bar{g}_{r\bar{z}}/\sqrt{\hat{x}} \\ 0 & \bar{g}_{rz}/\sqrt{\hat{x}} & \bar{g}_{zz} & \bar{g}_{z\bar{z}} \\ 0 & \bar{g}_{r\bar{z}}/\sqrt{\hat{x}} & \bar{g}_{z\bar{z}} & \bar{g}_{\bar{z}\bar{z}} \end{vmatrix};$$

$$\hat{g}^{mn} = \begin{vmatrix} \bar{g}^{tt}/\hat{x} & 0 & 0 & 0 \\ 0 & \bar{g}^{rr}\hat{x} & \bar{g}^{rz}\sqrt{\hat{x}} & \bar{g}^{r\bar{z}}\sqrt{\hat{x}} \\ 0 & \bar{g}^{rz}\sqrt{\hat{x}} & \bar{g}^{zz} & \bar{g}^{z\bar{z}} \\ 0 & \bar{g}^{r\bar{z}}\sqrt{\hat{x}} & \bar{g}^{z\bar{z}} & \bar{g}^{\bar{z}\bar{z}} \end{vmatrix}, \tag{5.24}$$

where $\bar{g}_{mn}$ denotes the factor of order $O(\hat{x}^0)$. Variation of the surface charge is

$$\oint\!\!\!/ H_\chi(\hat{g}, h) = \frac{1}{4\pi} \int_{\Sigma_r} (d^2x)_{rt} \sqrt{-\hat{g}} \hat{K}^{rt}_\chi(\delta\hat{g}, \hat{g}), \tag{5.25}$$

where

$$\hat{K}^{\mathrm{IW}rt}_\chi = \chi^r \hat{\nabla}^t \hat{h} - \chi^r \hat{\nabla}_s \hat{h}^{ts} + \chi_s \hat{\nabla}^r \hat{h}^{ts} + \frac{\hat{h}}{2} \hat{\nabla}^r \chi^t - \hat{h}^{rs} \hat{\nabla}_s \chi^t - (r \to t) \tag{5.26}$$

and Σ_r is a surface $r = 2 + \hat{x}$. Here $\hat{g} = g_{tt}\hat{g}^{(3)}$, and $\hat{g}^{(3)}$ is the determinant of the 3D part of the metric:

$$\hat{g}^{(3)}(r, z, \bar{z}) = \hat{g}_{rr}(\hat{g}_{zz}\hat{g}_{\bar{z}\bar{z}} - \hat{g}^2_{z\bar{z}}) - \hat{g}^2_{rz}\hat{g}_{\bar{z}\bar{z}} - \hat{g}^2_{r\bar{z}}\hat{g}_{zz} + 2\hat{g}_{rz}\hat{g}_{r\bar{z}}\hat{g}_{z\bar{z}}. \tag{5.27}$$

Through the variables (ρ, z^a), determinant $\hat{g}^{(3)}$ can be written as $\hat{g}^{(3)} = g_{\rho\rho}\rho^2_r \tilde{g}^{(2)}$. Near the horizon, substituting $\rho_{,r}$ from (2.10) and $g_{\rho\rho}$ (2.11), we have

$$\hat{g}^{(3)} \simeq \frac{\tilde{g}^{(2)}}{V(1 - 4(DC)^2)}, \tag{5.28}$$

where $\tilde{g}^{(2)} = \tilde{g}_{zz}\tilde{g}_{\bar{z}\bar{z}} - \tilde{g}^2_{z\bar{z}}$. From (5.28) it is seen that $\hat{g}^{(3)} = O(\hat{x}^{-1})$, and the determinant of the metric, $\hat{g} = \hat{g}_{tt}\hat{g}^{(3)}$, is of order $O(\hat{x}^0)$. To have an expression nonzero at the horizon, we must select in the surface charge form $\hat{K}^{rt}$ the terms of order $O(\hat{x}^0)$.

The five terms in (5.26) are

1. $\chi^r \hat{\nabla}^t \hat{h} - \chi^t \hat{\nabla}^r \hat{h} = \chi^r \hat{h} g^{tt} \partial_t \hat{h} - \chi^t \hat{g}^{rr} \partial_r \hat{h} - \chi^t \hat{g}^{ra} \partial_a \hat{h} = O(\hat{x}^{1/2})$.
2. $-\chi^r \hat{\nabla}_s \hat{h}^{ts} + \chi^t \hat{\nabla}_s \hat{h}^{rs}$.
3. $\chi_s \hat{\nabla}^r \hat{h}^{ts} - \chi_s \hat{\nabla}^t \hat{h}^{rs} = \chi_t \hat{\nabla}^r \hat{h}^{tt} - \chi_t \hat{\nabla}^t \hat{h}^{rt} - \chi_r \hat{\nabla}^t \hat{h}^{rr} - \chi_a \hat{\nabla}^t \hat{h}^{ra}$.
4. $\frac{\hat{h}}{2}(\hat{\nabla}^r \chi^t - \hat{\nabla}^t \chi^r) = \frac{\hat{h}}{2}[\hat{g}^{rr} \hat{\nabla}_r \chi^t - \hat{g}^{tt} \hat{\nabla}_t \chi^t] + O(\hat{x}^{1/2})$.
5. $-\hat{h}^{rs} \hat{\nabla}_s \chi^t + \hat{h}^{ts} \hat{\nabla}_s \chi^r = -\frac{1}{2}(\hat{h}^{rr} \hat{\nabla}_r \chi^t - \hat{h}^{tt} \hat{\nabla}_t \chi^t) + O(\hat{x}^{1/2})$. (5.29)

Estimating two terms in the item 1, we have

$$\chi^r \hat{g}^{tt} \partial_t \hat{h} - \chi^t (\hat{g}^{rr} \partial_r \hat{h} + \hat{g}^{rr} \partial_a \hat{h}) = O(\hat{x}^{1/2})$$

and the item 1 does not contribute to $\hat{K}^{rt}$.

Because all the terms containing Γ one index t are zero, in the item 2 the first term vanishes:

$$-\chi^r \hat{\nabla}_s \hat{h}^{ts} = -\chi^r (\hat{\nabla}_t \hat{h}^{tt} + \hat{\nabla}_r \hat{h}^{tr} + \hat{\nabla}_a \hat{h}^{ta}) = 0.$$

In the second term

$$\chi^t \hat{\nabla}_r \hat{h}^{rr} = \chi^t (\hat{\nabla}_t \hat{h}^{rt} + \hat{\nabla}_r \hat{h}^{rr} + \hat{\nabla}_a \hat{h}^{ra})$$

the part $\chi^t (\hat{\nabla}_r \hat{h}^{rr} + \hat{\nabla}_a \hat{h}^{ra})$ is estimated as

$$\chi^t\hat{\nabla}_r\hat{h}^{rr} = \chi^t(\partial_r\hat{h}^{rr} + 2\Gamma^r_{rr}\hat{h}^{rr} + 2\Gamma^r_{ra}\hat{h}^{ra}) = \chi^t(\bar{h}^{rr} + 2\Gamma^r_{rr}\hat{h}^{rr} + O(\hat{x}^{1/2}))$$
$$= \chi^t\left[\bar{h}^{rr} + \left(\bar{g}^{rr}\hat{x}\bar{g}_{rr}\left(-\frac{1}{\hat{x}^2}\right) + 2\bar{g}^{ra}\hat{x}^{1/2}\bar{g}_{ra}\left(-\frac{1}{2\hat{x}^{3/2}}\right)\right)\bar{h}^{rr}\hat{x} + O(\hat{x}^{1/2})\right] = O(\hat{x}^{1/2}). \quad (5.30)$$

Because of the identity $\bar{g}^{rr}\bar{g}_{rr} + \bar{g}^{ra}\bar{g}_{ar} = 1$ the sum of the terms in round brackets in (5.30) is equal to $-\bar{h}^{rr}$. The term $\chi^t\hat{\nabla}_a\hat{h}^{ra}$ is of order $O(\hat{x}^{1/2})$. In the item 2 there remains the term $\chi^t\hat{\nabla}_t\hat{h}^{rt}$.

In the item 3 the first term is

$$\chi_s\hat{\nabla}^r\hat{h}^{ts} = \chi_t\hat{\nabla}^r\hat{h}^{tt} + \chi_r\hat{\nabla}^r\hat{h}^{tr} + \chi_a\hat{\nabla}^r\hat{h}^{ta}$$
$$= \chi_t(\hat{g}^{rr}\hat{\nabla}_r + \hat{g}^{ra}\hat{\nabla}_a)\hat{h}^{tt} + \chi_r(\hat{g}^{rr}\hat{\nabla}_r + \hat{g}^{ra}\hat{\nabla}_a)\hat{h}^{tr} + O(\hat{x}^0).$$

The term $\chi^t\hat{g}_{tt}\hat{g}^{ra}\hat{\nabla}_a\hat{h}^{tt}$ is of order $\hat{x}^{1/2}$. In the term

$$\chi^t\hat{g}_{tt}\hat{g}^{rr}\hat{\nabla}_r\hat{h}^{tt} = \chi^t\bar{g}_{tt}\hat{x}\bar{g}^{rr}\hat{x}\left(-\frac{\bar{h}_{tt}}{\hat{x}^2} + \frac{\bar{g}^{tt}}{\hat{x}}\bar{g}_{tt}\frac{\bar{h}^{tt}}{\hat{x}} + O(\hat{x}^0)\right)$$

the leading-order parts cancel, and it is also of order $O(\hat{x}^{1/2})$. The remaining term in the item 3, equal to $-\chi^t\hat{\nabla}_t\hat{h}^{rt}$, cancels the corresponding term in the item 2.

We obtain $\hat{K}^{rt}$ as

$$\hat{K}^{rt}_\chi = \hat{\nabla}_s\chi^t\left(\frac{\hat{h}}{2}\hat{g}^{rs} - \hat{h}^{rs}\right) - \hat{\nabla}_s\chi^r\left(\frac{\hat{h}}{2}\hat{g}^{ts} - \hat{h}^{ts}\right) + O(\hat{x}^{1/2})$$
$$= \frac{\chi^t}{2}\left[\Gamma^t_{rt}\left(\frac{\hat{h}}{2}\hat{g}^{rr} - \hat{h}^{rr}\right) - \Gamma^r_{tt}\left(\frac{\hat{h}}{2}\hat{g}^{tt} - \hat{h}^{tt}\right) + O(\hat{x}^{1/2})\right]. \quad (5.31)$$

Because $g_{tt} = V(r)$, we have $\Gamma^t_{\theta t} = \Gamma^\theta_{tt} = 0$.

In $\hat{K}^{rt}$ the leading terms are

$$\hat{K}^{rt}_\chi = \frac{\chi^t}{2}\hat{g}_{tt,r}[\hat{h}\hat{g}^{rr}\hat{g}^{tt} - \hat{h}^{rr}\hat{g}^{tt} - \hat{g}^{rr}\hat{h}^{tt}]$$
$$= \frac{\chi^t}{2}\hat{g}_{tt,r}\hat{g}^{tt}\hat{h}_{ab}(\hat{g}^{rr}\hat{g}^{ab} - \hat{g}^{ra}\hat{g}^{rb}). \quad (5.32)$$

The combination in the rhs of (5.32) is presented as

$$\hat{h}_{ab}(\hat{g}^{rr}\hat{g}^{ab} - \hat{g}^{ra}\hat{g}^{rb}) = \frac{\hat{h}_{zz}\hat{g}_{\bar{z}\bar{z}} + \hat{h}_{\bar{z}\bar{z}}\hat{g}_{zz} - 2\hat{h}_{z\bar{z}}\hat{g}_{z\bar{z}}}{\hat{g}^{(3)}}$$
$$= \frac{\delta\hat{g}^{(2)}}{\hat{g}^{(3)}}, \quad (5.33)$$

where

$$\hat{g}^{(2)} = \hat{g}_{zz}\hat{g}_{\bar{z}\bar{z}} - \hat{g}^2_{z\bar{z}}.$$

Variation of the surface charge is

$$\delta\!\!\!/ H_\chi(\hat{g},\hat{h}) = \lim_{r\to 2}\frac{1}{4\pi}\int dz\wedge d\bar{z}\sqrt{V\hat{g}^{(3)}}\frac{\chi^t}{2}\hat{g}_{tt,r}\hat{g}^{tt}\frac{\delta\hat{g}^{(2)}}{\hat{g}^{(3)}}. \quad (5.34)$$

Using (5.28) this expression is presented as

$$\delta\!\!\!/ H_\chi(\hat{g},\hat{h}) = \frac{1}{4\pi}\int_{\Sigma_H} dz\wedge d\bar{z}\frac{\chi^t}{2}\frac{\delta\hat{g}^{(2)}}{\sqrt{\hat{g}^{(2)}}}(1/4 - (DC)^2)^{1/2}. \quad (5.35)$$

In the general case the integral (5.35) is not of the form of a variation of a functional over the space of metrics. A special case of the supertranslation field with the integrable variation of the surface charge is discussed in the next section.

VI. INTEGRABLE VARIATION OF SURFACE CHARGE

In this section we consider an example of integrable variation of the charge. We consider the case of a supertranslation field $C(z,\bar{z})$ in coordinate system $(r,z,\bar{z})$ depending only on $|z|$, or in coordinates (r,θ,φ), only on θ.

In coordinates (r,θ,φ) the metric (2.13) with $C = C(\theta)$ takes a form

$$ds^2 = -Vdt^2 + \frac{dr^2}{V(1-b^2)} + 2drd\theta\frac{br(\sqrt{1-b^2} - b')}{(1-b^2)V^{1/2}}$$
$$+ d\theta^2 r^2\frac{(\sqrt{1-b^2} - b')^2}{(1-b^2)}$$
$$+ d\varphi^2 r^2\sin^2\theta\left(b\cot\theta - \sqrt{1-b^2}\right)^2, \quad (6.1)$$

where $b = 2C'(\theta)/K$. The charge density $\hat{K}^{rt}_\chi(\delta\hat{g},\hat{g})$ (5.32) is

$$\hat{K}^{rt}_\chi = \frac{\chi^t}{2}\hat{g}_{tt,r}\hat{g}^{tt}[\hat{h}_{\theta\theta}(\hat{g}^{rr}\hat{g}^{\theta\theta} - (\hat{g}^{r\theta})^2) + \hat{h}_{\varphi\varphi}\hat{g}^{rr}\hat{g}^{\varphi\varphi}]. \quad (6.2)$$

Using the relation

$$\hat{g}^{(2)} = \hat{g}_{rr}\hat{g}_{\theta\theta} - \hat{g}^2_{r\theta} = \frac{\hat{g}_{\theta\theta}}{V} \quad (6.3)$$

and noting that

$$\hat{g}^{rr} = V, \qquad \hat{g}^{rr}\hat{g}^{\theta\theta} - (\hat{g}^{r\theta})^2 = \frac{1}{\hat{g}^{(2)}},$$

we obtain the charge density $\hat{K}^{rt}$ as

$$\hat{K}^{rt}_{\chi} = \frac{\chi^t}{2}\hat{g}_{tt,r}\hat{g}^{tt}\left(V\frac{\hat{h}_{\theta\theta}}{\hat{g}_{\theta\theta}} + V\frac{\hat{h}_{\varphi\varphi}}{\hat{g}_{\varphi\varphi}}\right) = \frac{\chi^t}{2}V_{,r}\left(\frac{\delta\hat{g}_{\theta\theta}}{\hat{g}_{\theta\theta}} + \frac{\delta\hat{g}_{\varphi\varphi}}{\hat{g}_{\varphi\varphi}}\right). \tag{6.4}$$

Variation of the surface charge $\delta\!\!\!/ H_\chi$ is

$$\delta\!\!\!/ H_\chi(\hat{g},\delta\hat{g}) = \lim_{r\to 2}\frac{1}{4\pi}\int_\Sigma d\theta\wedge d\varphi\sqrt{-\hat{g}_{tt}\hat{g}^{(2)}\hat{g}_{\varphi\varphi}}\hat{K}^{rt} = \frac{1}{4\pi}\lim_{r\to 2}\int_\Sigma d\theta\wedge d\varphi\sqrt{\hat{g}_{\theta\theta}\hat{g}_{\varphi\varphi}}\frac{\chi^t}{4}\frac{\delta(\hat{g}_{\theta\theta}\hat{g}_{\varphi\varphi})}{\hat{g}_{\theta\theta}\hat{g}_{\varphi\varphi}}.$$

Finally, we have

$$\delta\!\!\!/ H_\chi(\hat{g},\hat{h}) = \frac{1}{4\pi}\int_{\Sigma_H} d\theta\wedge d\varphi\frac{\chi^t}{2}\delta\sqrt{\hat{g}_{\theta\theta}\hat{g}_{\varphi\varphi}}. \tag{6.5}$$

The expression (6.5) is of the form of the variation of the functional and is integrable.

A. Proof of equality of the variations of the surface charges $\delta\!\!\!/ H_\chi$ and $\delta\!\!\!/ H_\xi$

Let us show that the variation $\delta\!\!\!/ H_\chi$ in variables (r,θ,φ) is equal to the variation $\delta\!\!\!/ H_\xi$ in variables (ρ,θ,φ). Because the charge densities contain derivatives of the metric which transform noncovariantly, we establish the equality by direct calculation.

The $\theta\theta$ and $\phi\phi$ metric components in the r- and ρ-systems are connected as (2.12)

$$\hat{g}_{\theta\theta} = g_{\theta\theta}\frac{K^2}{4(\rho-C)^2}, \qquad \hat{g}_{\varphi\varphi} = g_{\varphi\varphi}. \tag{6.6}$$

Using (6.6), we obtain a connection between the determinants of the metrics

$$\sqrt{-\hat{g}} = \sqrt{-g}\frac{K^2}{(K^2-1)2(\rho-C)}, \qquad \sqrt{-g} = \left(\left(\frac{K-1}{K+1}\right)^2\frac{4r^2}{K^2}g_{\theta\theta}g_{\varphi\varphi}\right)^{1/2}. \tag{6.7}$$

The metric variations are connected as

$$\delta\hat{g}_{\theta\theta} = \delta g_{\theta\theta} + \delta g_{\rho\rho}\left(\frac{\partial\rho}{\partial\theta}\right)^2; \qquad \delta\hat{g}_{\varphi\varphi} = \delta g_{\varphi\varphi}. \tag{6.8}$$

Variation of the charge $\delta\!\!\!/ H_\chi$ is presented as

$$\begin{aligned}\delta\!\!\!/ H_\chi &= \frac{1}{4\pi}\int d\theta d\varphi\sqrt{\hat{g}_{\theta\theta}\hat{g}_{\varphi\varphi}}\frac{\chi^t}{2}V_{,r}\left[\frac{\delta g_{\theta\theta}+\delta g_{\rho\rho}\rho^2_\theta}{g_{\theta\theta}}\frac{4(\rho-C)^2}{K^2} + \frac{\delta g_{\varphi\varphi}}{g_{\varphi\varphi}}\right]\\ &= \frac{1}{4\pi}\int d\theta d\varphi\sqrt{-g}\frac{\chi^t}{2}\frac{K^3}{(K^2-1)(\rho-C)r^2}\left[\frac{\delta g_{\theta\theta}}{g_{\theta\theta}}\frac{4(\rho-C)^2}{K^2} + \frac{\delta g_{\varphi\varphi}}{g_{\varphi\varphi}} + \frac{\delta g_{\rho\rho}}{g_{\theta\theta}}\frac{4(\rho-C)^2}{K^2}\rho^2_{,\theta}\right]\\ &= \frac{1}{4\pi}\int d\theta d\varphi\sqrt{-g}\frac{\chi^t}{2}\frac{4K}{(K^2-1)r^2}\left[\frac{\delta g_{\theta\theta}}{g_{\theta\theta}}(\rho-C) + \frac{\delta g_{\varphi\varphi}}{g_{\varphi\varphi}}\frac{K^2}{4(\rho-C)^2} + \frac{\delta g_{\rho\rho}}{g_{\rho\rho}}\frac{C'^2}{\rho-C}\right].\end{aligned} \tag{6.9}$$

Next, using the relations

$$g_{tt,\rho} = 16\frac{K-1}{(K+1)^3}\frac{\rho-C}{K}, \qquad g_{tt,\theta} = 16\frac{K-1}{(K+1)^3}\frac{(-C')(\rho-C-C'')}{K},$$

we transform $\delta\!\!\!/ H_\xi$ (5.21):

$$\begin{aligned}\delta\!\!\!/ H_\xi &= \frac{1}{4\pi}\int\sqrt{-g}d\theta d\varphi\frac{\xi^t}{2}\left[\frac{g_{tt,\rho}}{g_{tt}g_{\rho\rho}}\left(\frac{\delta g_{\theta\theta}}{g_{\theta\theta}} + \frac{\delta g_{\varphi\varphi}}{g_{\varphi\varphi}}\right) - \rho_{,\theta}\frac{g_{tt,\theta}}{g_{tt}g_{\rho\rho}}\left(\frac{\delta g_{\rho\rho}}{g_{\rho\rho}} + \frac{\delta g_{\varphi\varphi}}{g_{\varphi\varphi}}\right)\right]\\ &= \frac{1}{4\pi}\int\sqrt{-g}d\theta d\varphi\frac{\xi^t}{2}\frac{4K}{(K^2-1)r^2}\left[\frac{\delta g_{\theta\theta}}{g_{\theta\theta}}(\rho-C) + \frac{\delta g_{\varphi\varphi}}{g_{\varphi\varphi}}\frac{K^2}{4(\rho-C)^2} + \frac{\delta g_{\rho\rho}}{g_{\rho\rho}}\frac{C'^2}{\rho-C}\right].\end{aligned} \tag{6.10}$$

Because $\xi^t = \chi^t$, the expressions (6.9) and (6.10) coincide. The variations of the charges (6.9) and (6.10) are written for the surfaces $(\rho-C)^2 + C'^2 = K^2/4$, and for $K=1$, at the horizon, are identical with (5.23).

VII. SUMMARY AND CONCLUSIONS

In this paper we studied the near-horizon symmetries of the metric of a black hole containing a supertranslation field which preserve the gauge and the near-horizon structure of the metric. The aim of the paper was practical calculation of the variation of the surface charge corresponding to asymptotic symmetries preserving the near-horizon form of the metric. The horizon symmetries were considered in different coordinate systems (ρ- and r-systems) connected by a large diffeomorphism containing a supertranslation field and also in coordinate systems obtained by a pure coordinate (gauge) transformations which do not contain the supertranslation field.

In units $M = 1$, where M is the mass of a black hole, in the ρ-system the horizon of a black hole is located at the surface $\rho_s \equiv ((\rho - C)^2 + (DC)^2)^{1/2} = 1/2$, where $C = C(z^a)$ [$z^a = (\theta, \varphi)$ or $(z, \bar{z})$ are different parametrizations of the unit sphere] is a supertranslation field in the metric. In the r-system the horizon is located at the surface $r = 2$. Foliation of the near-horizon region was defined through a smooth deformation of the horizon surface $\rho_s = 1/2$ to $\rho_s = 1/2 + x$.

Horizon transformations preserving the form of the metric were generated by vector fields. Infinitesimal diffeomorphisms preserving the form of the metric were studied in both ρ and r systems. Variation of the surface charge associated with a transformation generated by a vector field ξ^m is the limit $x \to 0$ of the integral of the charge density $K_\xi(\delta_\xi g, g)$ over the near-horizon surface enclosing the horizon surface. Here $g = g_{mn}$ and $\delta_\xi g_{mn}$ are the metric of the black hole and its variation generated by the vector field ξ^m.

In the ρ-system, variation of the surface charge is equal to the sum of three integrals over the surface enclosing the horizon surface with the charge densities $K_\xi^{\rho t}$, $K_\xi^{\theta t}$ and $K_\xi^{\varphi t}$ and corresponding integrations are over the variables $(\theta, \varphi), (\rho, \varphi)$ and (ρ, θ). In the r-system, the surface corresponding to deformation $\rho_s = 1/2 + x$ is the sphere of the radius $r = 2 + 2x^2$.

The square root of the determinant of the metric in the ρ-system is of order x (as for the Schwarzschild metric in the isotropic coordinates), in the r-system it is of order $\hat{x}^0$. Correspondingly, the charge densities were calculated in order x^{-1} in the ρ-system and in order $\hat{x}^0$ in the r-system, so that the resulting integrals for the variation of the surface charge were independent of x.

We discussed symplectic transformations which are extendable from the near-horizon region in the bulk. Under the simplectic transformations the supertranslation field in the metric transforms as $\delta_T C(\theta, \varphi) = T(\theta, \varphi)$. In the case of a supertranslation field depending only on θ a condition that transformation preserves the near-horizon form of metric at the horizon is an ordinary differential equation with a solution for $T(\theta)$ expressed through $C(\theta)$.

In the general case the surface charges obtained by integration of variations of the charges over the space of metrics are path dependent. In a special case of the supertranslation field depending only on a spherical angle θ, variation of the charge in the r-system has a form of variation of a functional over the space of metrics and can be integrated in a path-independent way. By an explicit calculation we show that the variations of the surface charge in both ρ- and r-systems are equal. However, in the ρ-system, in coordinates (ρ, z^a) the expression for the variation of the charge is not integrable.

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APPENDIX: SOLUTION OF EQ. (4.12)

In this Appendix we find a general solution for the function $T(z, \bar{z})$ in the generator of supertranslations Eq. (4.1). We solve Eq. (4.12) which is a condition on the generator T at the horizon (written in notations of Sec. II):

$$\left[-T\sqrt{1-b^2}+\frac{1}{4}F^{ab}D_bT\left(-2b_a\sqrt{1-b^2}+D_ab^2\right)\right]_{r=2}=0. \tag{A1}$$

Equation (A1) can be presented in a form

$$T + F^a D_a T = 0, \tag{A2}$$

where

$$F^a = \frac{1}{2}F^{ac}\left(b_c + \partial_c\sqrt{1-b^2}\right)\Big|_{r=2}.$$

Following the general rules of solving the differential equations with partial derivatives [41], we consider a function $W(T, z, \bar{z})$ satisfying the equation

$$T\frac{\partial W}{\partial T} + F^z\frac{\partial W}{\partial z} + F^{\bar{z}}\frac{\partial W}{\partial \bar{z}} = 0. \tag{A3}$$

Equation (A3) is solved by writing the system of ordinary differential equations

$$\frac{dT}{T} = \frac{dz}{F^z} = \frac{d\bar{z}}{F^{\bar{z}}}. \tag{A4}$$

Let the independent first integrals of Eq. (A4) be

$$\psi_1(T, z, \bar{z}) = C_1, \qquad \psi_2(T, z, \bar{z}) = C_2. \tag{A5}$$

The general solution of Eq. (A3) for $W(T, z, \bar{z})$ is

$$W = f(\psi_1, \psi_2), \tag{A6}$$

where f is an arbitrary smooth function. The function $T(z, \bar{z})$ is implicitly determined from the equation

$$f(\psi_1, \psi_2) = 0. \tag{A7}$$

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