Two-fluid stellar objects in general relativity: The covariant formulation

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We apply the $1 + 1 + 2$ covariant approach to describe a general static and spherically symmetric relativistic stellar object which contains two interacting fluids. We then use the $1 + 1 + 2$ equations to derive the corresponding Tolman-Oppenheimer-Volkoff equations in covariant form in the isotropic noninteracting case. These equations are used to obtain new exact solutions by means of direct resolution and reconstruction techniques. Finally, we show that the generating theorem known for the single-fluid case can also be used to obtain two-fluid solutions from single-fluid ones.

I. INTRODUCTION

The derivation of solutions for relativistic stellar objects is a notoriously complex problem when approached using analytical methods. It is probably for this reason that most of the attempts to find such solutions have, up to now, relied on modeling the matter distribution using a single-fluid description. In realistic situations, however, such an assumption is not physically realistic for several reasons: compact stars can have a structure that can be wildly different, in terms of composition and pressure, and there could be interactions between different components. Hence, in order to build more accurate models, it is necessary to turn to a multifluid description of these objects.

So far, there have been several attempts to model stellar objects containing different fluids. The majority of these rely on using numerical noncovariant approaches (see, e.g., [1]). A different perspective was provided for the first time by Carter and Langlois [2]. They showed that it is possible to formulate a covariant exact model of multifluid (neutron) stars using noninteracting fluids and assuming an equation of state. Although in the end, the analysis of the equations in [2] is still numeric, their attempt shows that a covariant approach to the Tolman-Oppenheimer-Volkof (TOV) equations might be useful to uncover new aspects of these equations.

The present work aims to construct a different approach, which makes full use of covariance, and is oriented towards an analytical investigation of the TOV equations. There are several reasons why it is important to develop analytical studies of the TOV equations in parallel to numerical studies. For example, exact solutions can be used to explore the full parameter space for a given metric, rather than a

single set of values. In addition, exact solutions can be used to test numerical codes, particularly when they entail new languages/approximations schemes.

We will also show that our analytical approach is able to include other equations of state, can be generalized to any number of fluids or fields, and can include fluxes and interactions.

The cornerstone of our formulation will be the TOV equations of hydrostatic equilibrium. These equations were introduced in 1939 [3,4] and provide insights into the pressure profile of a static spherically symmetric object in general relativity (GR). Since their introduction, several authors have tried to solve these equations exactly (see, e.g., [5] for a list of exact solutions). In spite of these efforts, the resolution of the TOV equations still remains a formidable task, particularly if one aims at the deduction of realistic solutions.

In two recent papers, the TOV equations were presented in a fully covariant form and applied to the case of isotropic and anisotropic fluids [6–8]. The generalized equations are written in a covariant dimensionless and autonomous form, thereby providing a combination of the Lane-Emden and homology invariant formulation [9,10]. The covariant (and, therefore, observer independent) form of the equations presents many benefits. For example, one may change coordinate systems with ease, making the description of the system (i.e., its symmetries and properties) easily captured.

The covariant formulation of the TOV equations is based on the so-called *covariant approach*. The original development of the covariant formalisms is due to Ehlers, Ellis, and other authors [11]. Their $1 + 3$ *covariant approach* offers a powerful method for studying the general properties of exact relativistic (and Newtonian) cosmological models [12]. Because all the gravitational and fluid equations can be written down exactly, this approach is well suited to a top-down construction of perturbation theory and has been

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widely used in studies of perturbations of Friedmann-Robertson-Walker models [13] and other backgrounds that admit a high degree of symmetry.

An extension of the $1 + 3$ covariant approach, known as the $1 + 1 + 2$ *covariant approach*, proposed by Greenberg [14] and adapted to locally rotationally symmetric (LRS) spacetimes by Clarkson [15,16] and Betschart [17], allows for the application of the formalism to numerous astrophysical scenarios (for example, lensing in spherically symmetric spacetimes [18] or spherically symmetric spacetimes [19]). The further "split" or "foliation" is in a spatial direction, leaving the other two spatial dimensions unchanged.

The $1 + 1 + 2$ TOV equations have already proven to offer a useful new perspective on the problem of finding exact models for relativistic stars. They are generally easier to solve than the original ones and allow a direct application of reconstruction methods. In addition, in [6,7] it was shown that the generating theorems proposed by Boonserm *et al.* in [20,21] can be easily formulated and extended to more complex cases. The $1 + 1 + 2$ TOV equations have also been used to uncover new features of relativistic stellar objects in Einstein-Cartan gravity in [8].

In the following, we will provide a complete description of the interior of a relativistic star composed of two fluids using the $1 + 1 + 2$ formalism. In particular, we will use the Ricci identities, the Bianchi identities, and the Einstein equations to derive a complete system of covariant TOV equations for two fluids with nonzero fluxes and can be generalized to include anisotropies and interactions. Then we will explore some exact solutions which are physically relevant according to the criteria given in [5]. We will also extend the generating theorems, proving that they can be used to obtain two-fluid solutions from single-fluid ones.

The outline of this paper is as follows: In Sec. II the $1 + 1 + 2$ equations in the case of two fluids are used to construct the covariant TOV equations in the isotropic and noninteracting case. Section III deals with the conditions of physical viability for a given solution of the TOV equations. Section IV, instead, gives a brief sketch of the generalization of Israel's junction conditions to the multifluid case. Section V contains information on some known solutions which will be useful to obtain the main results of the paper. In Sec. VI we obtain the two-fluid generalization of the interior Schwarzschild solution. In Sec. VII we employ a reconstruction algorithm to derive new two-fluid solutions based on the single-fluid ones of Sec. V. The two-fluid extension of the generating theorems of [20] is discussed in Sec. VIII. In Sec. IX we consider some exact two-fluid solutions which include (stationary) fluxes. Finally, a discussion and some concluding remarks can be found in Sec. X. The main equations of the $1 + 1 + 2$ formalism are presented in the Appendix A and the N fluid generalization of the $1 + 1 + 2$ equations, which include anisotropic pressure and interactions, in Appendix B.

II. THE $1+1+2$ EQUATIONS FOR THE TWO-FLUID CASE

Coordinate invariant and tetrad methods are an important way of transforming the equations of general relativity into first-order ordinary differential equations, as opposed to second-order partial differential equations. The approach is most useful in the presence of homogeneity, isotropy, and spacetimes that admit a high degree of symmetry. The $1 + 1 + 2$ formalism that we will employ in the following can be considered a semitetrad approach because it relies on both a timelike and a spacelike threading.[1]

We, therefore, start constructing the $1 + 1 + 2$ formalism from the threading decomposition of the spacetime. In this way, we can construct a set of tensorial objects connected to the properties of the field lines, which make up the set of $1 + 1 + 2$ variables. Following this, we use the Bianchi and Ricci identities, together with the Einstein equations, to derive a closed system of first-order propagation and constraint equations. In this section, we present the equations for the two-fluid case and refer the reader to Appendix A for details of the general equations and formalism.

As it can be seen in Appendix B, the $1 + 1 + 2$ equations can be written down easily for any number of interacting fluids. However, in this work we will limit ourselves to two noninteracting fluids. Considering two fluids is justified, other than by simplicity, by the fact that two-fluids models are already enough to describe systems like neutron stars, which are one of the main applications of the TOV equations [22].

We define a timelike threading vector field u^a associated to the observer's congruence with $u^a u_a = -1$ and a spacelike vector e_a with $e_a e^a = 1$. The u^a and e_a congruences describe a geometry defined by two projection tensors given by

$$h^a{}_b = g^a{}_b + u^a u_b, \qquad h^a{}_a = 3,$$
$$N_a{}^b = h_a{}^b - e_a e^b = g_a{}^b + u_a u^b - e_a e^b, \qquad N^a{}_a = 2,$$

(1)

where $h^a{}_b$ represents the metric of the 3-spaces orthogonal to u_a, and $N_a{}^b$ represents the metric of the 2-spaces orthogonal to u_a and e_a. Any tensorial object may now be split according to the above foliations [6]. The covariant time derivative, orthogonally projected covariant derivative, hat-derivative, and δ-derivative are given by

$$\dot{X}^{a...b}{}_{c...d} \equiv u^e \nabla_e X^{a...b}{}_{c...d},$$
$$D_e X^{a...b}{}_{c...d} \equiv h^a{}_f ... h^b{}_g h^p{}_c ... h^q{}_d h^r{}_e \nabla_r X^{f...g}{}_{p...q},$$
$$\hat{X}_{a...b}{}^{c...d} \equiv e^f D_f X_{a...b}{}^{c...d},$$
$$\delta_e X_{a...b}{}^{c...d} \equiv N_a{}^f ... N_b{}^g N_i{}^c ... N_j{}^d N_e{}^p D_p X_{f...g}{}^{i...j}. \quad (2)$$

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[1]To be precise, we should point out that the $1 + 1 + 2$ formalism is somewhat less general than the tetrad one: the former assumes that the vector fields used for the threading are everywhere regular.

In the following we will consider only LRSII spacetimes. The kinematical variables, which we will employ, are given by (see Appendix A for the complete list of variables for a general LRS spacetime)

$$\mathcal{A} = e_a \dot{u}^a, \tag{3a}$$

$$\phi = \delta_a e^a, \tag{3b}$$

$$\mathcal{E} = C^{ac}_{\ \ bd} u_c u^d \left(e_a e^b - \frac{1}{2} N^b_a \right), \tag{3c}$$

where C_{abcd} is the Weyl tensor.

The energy-momentum tensor is decomposed as

$$T_{ab} = \mu u_a u_b + p(e_a e_b + N_{ab}) + 2Q e_{(a} u_{b)}. \tag{4}$$

The matter variables are given by

$$\mu = T_{ab} u^a u^b,$$

$$p = \frac{1}{3} T_{ab}(e^a e^b + N^{ab}),$$

$$Q = -T_{ab} e^a u^b, \tag{5}$$

where μ is the density, p is the pressure, and Q is the scalar component of the heat flux.

Selecting a frame in which the total flux is zero, the system (A7) for two isotropic fluids is given by[2]

$$\hat{\phi} = -\frac{1}{2}\phi^2 - \frac{2}{3}(\mu_1 + \mu_2) - \mathcal{E}, \tag{6a}$$

$$\hat{\mathcal{E}} = \frac{1}{3}(\hat{\mu}_1 - \hat{\mu}_2) - \frac{3}{2}\phi\mathcal{E}, \tag{6b}$$

$$-\mathcal{A}\phi + \frac{1}{3}(\mu_1 + 3p_1) + \frac{1}{3}(\mu_2 + 3p_2) - \mathcal{E} = 0, \tag{6c}$$

$$\hat{\mathcal{A}} = -\mathcal{A}(\mathcal{A} + \phi) + \frac{1}{2}(\mu_1 + 3p_1) + \frac{1}{2}(\mu_2 + 3p_2), \tag{6d}$$

$$\hat{p}_1 = -\mathcal{A}(\mu_1 + p_1), \tag{6e}$$

$$\hat{p}_2 = -\mathcal{A}(\mu_2 + p_2), \tag{6f}$$

$$K = \frac{1}{3}(\mu_1 + \mu_2) - \mathcal{E} + \frac{1}{4}\phi^2, \tag{6g}$$

$$\hat{Q}_1 = -Q_1(\phi + 2\mathcal{A}), \tag{6h}$$

$$Q_2 = -Q_1, \tag{6i}$$

$$\hat{K} = -\phi K. \tag{6j}$$

Next, we introduce a useful parameter, named ρ, such that $\hat{X} = \phi X_{,\rho}$. In this way the equation for the Gauss's curvature K (6j) can be solved to give

$$K = K_0^{-1} e^{-\rho}. \tag{7}$$

Defining the variables,

$$X = \frac{\phi_{,\rho}}{\phi}, \qquad Y = \frac{\mathcal{A}}{\phi},$$

$$\mathcal{K} = \frac{K}{\phi^2}, \qquad E = \frac{\mathcal{E}}{\phi^2},$$

$$\mathbb{M}_1 = \frac{\mu_1}{\phi^2}, \qquad \mathbb{M}_2 = \frac{\mu_2}{\phi^2},$$

$$P_1 = \frac{p_1}{\phi^2}, \qquad P_2 = \frac{p_2}{\phi^2},$$

$$\mathbb{Q}_1 = \frac{Q_1}{\phi^2}, \qquad \mathbb{Q}_2 = \frac{Q_2}{\phi^2}, \tag{8}$$

and using ρ as parameter, we can recast Eqs. (6) as

$$Y_{,\rho} = -Y(X + Y + 1) + \frac{1}{2}(\mathbb{M}_1 + \mathbb{M}_2) + \frac{3}{2}(P_1 + P_2), \tag{9a}$$

$$\mathcal{K}_{,\rho} = -\mathcal{K}(1 + 2X), \tag{9b}$$

$$P_{1,\rho} = -Y(\mathbb{M}_1 + P_1) - 2XP_1, \tag{9c}$$

$$P_{2,\rho} = -Y(\mathbb{M}_2 + P_2) - 2XP_2, \tag{9d}$$

$$\mathbb{Q}_{1,\rho} = -\mathbb{Q}_1(1 + 2X + 2Y), \tag{9e}$$

with the following constraints:

$$2(\mathbb{M}_1 + \mathbb{M}_2) + 2(P_1 + P_2) + 2X - 2Y + 1 = 0, \tag{10a}$$

$$1 - 4\mathcal{K} - 4(P_1 + P_2) + 4Y = 0, \tag{10b}$$

$$2(\mathbb{M}_1 + \mathbb{M}_2) + 6(P_1 + P_2) - 6Y - 6E = 0, \tag{10c}$$

and

$$E = \frac{1}{3}(\mathbb{M}_1 + \mathbb{M}_2) + P_1 + P_2 - Y, \tag{11}$$

$$X = -\frac{1}{2} - (\mathbb{M}_1 + \mathbb{M}_2) - (P_1 + P_2) + Y, \tag{12}$$

$$\mathbb{Q}_1 = -\mathbb{Q}_2. \tag{13}$$

It is always possible, and sometimes useful, to write the equation for the total pressure, which reads

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[2]The generalization for equations in the case of n fluids and the presence of anisotropic pressure can be found in Appendix B.

$$P_{\text{tot},\rho} + P_{\text{tot}}(Y + 2X) + Y\mathbb{M}_{\text{tot}} = 0, \tag{14}$$

where $P_{\text{tot}} = P_1 + P_2$. The covariant equivalent of the TOV equations can be obtained using the constraints to eliminate all the metric related variables except $\mathcal{K}$. In this way one obtains

$$P_{1,\rho} = -P_1{}^2 + P_1\left[\mathbb{M}_1 - 3\mathcal{K} + \frac{7}{4}\right] + \mathbb{M}_1\left(\frac{1}{4} - \mathcal{K}\right) - P_1(P_2 - 2\mathbb{M}_2) - \mathbb{M}_1 P_2, \tag{15}$$

$$P_{2,\rho} = -P_2{}^2 + P_2\left[\mathbb{M}_2 - 3\mathcal{K} + \frac{7}{4}\right] + \mathbb{M}_2\left(\frac{1}{4} - \mathcal{K}\right) - P_2(P_1 - 2\mathbb{M}_1) - \mathbb{M}_2 P_1, \tag{16}$$

$$\mathcal{K}_{,\rho} = 2\mathcal{K}\left(\frac{1}{4} - \mathcal{K} + \mathbb{M}_1 + \mathbb{M}_2\right), \tag{17}$$

$$\mathbb{Q}_{1,\rho} = \mathbb{Q}_1\left[2\mathcal{K} - 2(\mathbb{M}_1 + \mathbb{M}_2) - \frac{3}{2}\right], \tag{18}$$

$$P_{\text{tot},\rho} = -P_{\text{tot}}{}^2 + P_{\text{tot}}\left[\mathbb{M}_{\text{tot}} - 3\mathcal{K} + \frac{7}{4}\right] + \mathbb{M}_{\text{tot}}\left(\frac{1}{4} - \mathcal{K}\right). \tag{19}$$

In the following we will present some solutions of the above equations for particular cases. It will be useful, then, to give some results which might help the physical interpretation of these solutions. For a generic metric tensor of the form

$$ds^2 = -k_1(x, t)dt^2 + k_2(x, t)dx^2 + k_3(x, t)[dy^2 + k_4(y)dz^2], \tag{20}$$

$$k_4(y) = \begin{cases} \sin y, & \text{closed geometry} \\ y, & \text{flat geometry} \\ \sinh y, & \text{closed geometry}, \end{cases} \tag{21}$$

we can write [17]

$$\phi = \frac{\hat{k}_3}{k_3}, \qquad \mathcal{A} = \frac{\hat{k}_1}{2k_1}. \tag{22}$$

We will use these relations in Sec. IX, where we will deal with models which include fluxes.

In the same way, in order to give a representation of the solutions obtained below in a form more consistent with the current literature, we give some conversion formulae connecting the $1 + 1 + 2$ potentials to the parameter ρ and the area radius r. In terms of ρ, a generic solution of the TOV equations will be written as

$$ds^2 = -k_1(\rho)dt^2 + k_2(\rho)d\rho^2 + k_3(\rho)d\Omega^2, \tag{23}$$

where

$$k_3(\rho) = K_0 e^\rho, \tag{24}$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{25}$$

In these coordinates, we have

$$\phi = \frac{1}{\sqrt{k_2}}, \qquad \mathcal{A} = \frac{k_{1,\rho}}{2k_1\sqrt{k_2}},$$

$$X = -\frac{k_{2,\rho}}{k_2}, \qquad Y = \frac{k_{1,\rho}}{2k_1},$$

$$\mathcal{K} = \frac{k_2}{K_0 e^\rho}, \tag{26}$$

where K_0 is a suitable constant. The solutions we find will, however, be expressed in terms of the area radius r to offer a more familiar representation of our results. The relation between r and ρ is

$$\rho = 2\ln\left(\frac{r}{r_0}\right), \tag{27}$$

$$r = \sqrt{K_0}e^{\rho/2}, \tag{28}$$

where r_0 and K_0 are constants related by $K_0 = r_0^2$.

The conversion from the metric coefficient in ρ and the ones in r can be achieved simply by noting that k_1 and k_3 are scalars, with respect to the change of the radial parameter, and that

$$k_2(\rho) = \frac{r^2}{4}k_2(r). \tag{29}$$

III. CONDITIONS FOR PHYSICAL VIABILITY

Although many solutions can be found to the TOV equations, several of these are not physical. We give here the set of conditions that a solution must satisfy to be physically relevant [5–7,19]. Firstly, we require that each fluid satisfy the weak energy condition:

$$\mu \geq 0, \qquad \mu + p \geq 0. \tag{30}$$

Secondly, they must satisfy the conditions

$$\mu' < 0, \qquad p' < 0, \tag{31}$$

where the prime represents the derivative with respect to the area radius. These conditions are necessary (but not sufficient) for the stability of the solution.

The third requirement is for causality from the speed of sound,

$$0 < \frac{\partial p}{\partial \mu} < 1. \tag{32}$$

We further require that the sources of the Einstein equations are positive definite,

$$p \geq 0. \tag{33}$$

Finally, we require that the matter variables are finite and positive valued at the center of the matter distribution. A possible exception to this rule will be discussed in Sec. IX.

IV. JUNCTION CONDITIONS

Another important aspect of the search for solutions relates to the junction between the interior solution and the exterior vacuum spacetime, characterized by the Schwarzschild metric. The procedure of the determination of the junction conditions is very similar to the case of a single-fluid solution treated in [6,7]. In particular, Israel's junction conditions [23,24] are equivalent to

$$[\mathcal{K}] = 0, \qquad [Y] = 0. \tag{34}$$

Using the constraint in Eq. (11) above, one gets

$$[P_1 + P_2] = 0. \tag{35}$$

This implies that a smooth junction with the Schwarzschild metric requires the total pressure must be zero at the junction. Since pressures must always be positive definite in a realistic solution, the above result implies that the pressure of both fluids must be zero at the junction. As we shall see, however, the most common occurrence in two-fluid solutions is that one of the pressures goes to zero at a specific value of the radial parameter, while the other is not. In this case, one should consider the two-fluid solution up

to that distance from the center and match this solution with a single-fluid one, thereafter. In other words, the inclusion of more than one fluid leads directly to a shelled structure for the matter distribution. As in the case of a single fluid, there is no condition on the energy density and the tangential pressure apart from the ones discussed in the previous section.

In the solution presented below, we will consider vanishing the radial pressure on the vacuum boundary as a desirable feature. Indeed, while there is no need to have a "hard boundary" in a stellar object, the inclusion of a "soft" boundary would require the introduction of types of sources that we have not included in our treatment, i.e., electromagnetic fields, tensions, etc. As we have excluded sources which are not a regular perfect fluid, the request of a hard boundary seems well motivated from a physical point of view.

V. SOME KNOWN SOLUTIONS FOR SINGLE-FLUID RELATIVISTIC STARS

In the following sections, we will consider some indirect resolution methods of the TOV equations (15). These methods can be implemented in an easier way if we rely on the characteristics of a known single-fluid solution. This section aims to introduce three such solutions in a form compatible with the formalism we will employ. We will consider, in particular, the interior Schwarzschild [25], the Tolman IV [3], and Heintzmann IIa [26] solutions.

A. Interior Schwarzschild (constant density) solution

The interior Schwarzschild solution [25] was the very first solution for the interior of a static spherically symmetric relativistic object. It assumes the fluid to be incompressible, a feature introduced in the model by assuming a constant density.

For a metric written in the form of (23), the constant density solution is given by

$$k_1 = a_0(\mathfrak{c}_1 + z)^2,$$
$$k_2 = \frac{3}{z^2},$$
$$k_3 = r^2, \tag{36}$$

where

$$z = \sqrt{3 - r^2 \mu_1}, \tag{37}$$

and a_0, $\mathfrak{c}_1$, and μ_1 are constants. The metric (36) corresponds, via the Einstein equations, to the following expressions for the pressure and energy density, respectively:

$$p_{CD}(r) = -\frac{\mu_1(3z + c_1)}{3(z + c_1)}, \tag{38a}$$

$$\mu_{CD}(r) = \mu_1. \tag{38b}$$

Applying the conditions (30)–(33) with the exception of the first equation in (31), we obtain with $\mu_1 \neq 0$ the maximum radius for the object described by this metric is

$$r < \frac{1}{3}\sqrt{\frac{27 - c_1^2}{\mu_1}}, \tag{39}$$

which corresponds to the well-known Buchdahl limit [27]. In terms of the newly defined variables (8) and the parameter ρ, we have

$$\phi = -\frac{2z_1}{\sqrt{3K_0 e^\rho}}, \tag{40}$$

and (38) correspond to

$$\mathcal{K}(\rho) = \frac{3}{4z_1^2}, \tag{41a}$$

$$P(\rho) = -\frac{\mu_1 K_0 e^\rho (3z_1 + c_1)}{4z_1^2(z_1 + c_1)}, \tag{41b}$$

$$\mathbb{M}(\rho) = \frac{3\mu_1 K_0 e^\rho}{4z_1^2}, \tag{41c}$$

$$Y(\rho) = -\frac{\mu_1 K_0 e^\rho}{2z_1(z_1 + c_1)}, \tag{41d}$$

where

$$z_1 = \sqrt{3 - \mu_1 K_0 e^\rho}. \tag{42}$$

B. Tolman IV solution

The Tolman IV solution was presented in the seminal paper by Tolman [3] and is a well-known solution with no irregularity at $r = 0$. It is characterized by an equation of state that is quadratic in the pressure.

For a metric written in the form of (23), the Tolman IV solution is given by

$$k_1 = B^2\left(1 + \frac{r^2}{A^2}\right),$$
$$k_2 = \frac{R^2(A^2 + 2r^2)}{(R^2 - r^2)(A^2 + r^2)},$$
$$k_3 = r^2, \tag{43}$$

with the following expressions for the pressure and energy density, respectively:

$$p_T(r) = \frac{R^2 - A^2 - 3r^2}{R^2(A^2 + 2r^2)}, \tag{44a}$$

$$\mu_T(r) = \frac{R^2(3A^2 + 2r^2) + 7A^2 r^2 + 3A^4 + 6r^4}{R^2(A^2 + 2r^2)^2}. \tag{44b}$$

Applying the conditions (30)–(33), we find that $A \neq 0$ and that the maximum radius for the object described by this metric is

$$r \leq \sqrt{\frac{R^2 - A^2}{3}}. \tag{45}$$

In terms of the newly defined variables (8) and the parameter ρ, we have

$$\phi_T = -\sqrt{\frac{4(A^2 + K_0 e^\rho)(R^2 - K_0 e^\rho)}{K_0 e^\rho R^2(A^2 + 2K_0 e^\rho)}}, \tag{46}$$

and (44) corresponds to

$$\mathcal{K}_T(\rho) = \frac{R^2(A^2 + 2K_0 e^\rho)}{4(A^2 + K_0 e^\rho)(R^2 - K_0 e^\rho)}, \tag{47a}$$

$$P_T(\rho) = \frac{K_0 e^\rho(A^2 + 3K_0 e^\rho - R^2)}{4(A^2 + K_0 e^\rho)(K_0 e^\rho - R^2)}, \tag{47b}$$

$$\mathbb{M}_T(\rho) = \frac{A^2(2A^2 + R^2)}{4(A^2 + R^2)(A^2 + K_0 e^\rho)} - \frac{A^2}{2(A^2 + 2K_0 e^\rho)}$$
$$+ \frac{R^2(3A^2 + 4R^2)}{4(A^2 + R^2)(R^2 - K_0 e^\rho)} - \frac{3}{4}, \tag{47c}$$

$$Y_T(\rho) = \frac{K_0 e^\rho}{2(A^2 + K_0 e^\rho)}. \tag{47d}$$

C. Heintzmann IIa solution

The Heintzmann IIa solution was presented for the first time in [26]. Its metric can be written as

$$k_1 = b^2(1 + ar^2)^3, \tag{48}$$

$$k_2 = \left(1 - \frac{3ar^2}{2}\frac{1 + c(1 + 4ar^2)^{-\frac{1}{2}}}{1 + ar^2}\right)^{-1}, \tag{49}$$

$$k_3 = r^2, \tag{50}$$

with the pressure and energy density given by

$$p_H(r) = -\frac{3a[7acr^2 + 3(ar^2 - 1)\sqrt{4ar^2 + 1} + c]}{2(ar^2 + 1)^2\sqrt{4ar^2 + 1}}, \tag{51a}$$

$$\mu_H(r) = \frac{3a[c(9ar^2 + 3) + (ar^2 + 3)(4ar^2 + 1)^{3/2}]}{2(ar^2 + 1)^2(4ar^2 + 1)^{3/2}}.$$

(51b)

The Heintzmann IIa metric satisfies conditions (30)–(33) for $a > 0$:

$$r < \frac{1}{\sqrt{a}}.$$

(52)

As before, in order to find some results useful for the next sections, we give some of the variables (8) in terms of the parameter ρ. We have

$$\phi_H^2 = \frac{4e^{-\rho}}{K_0} - \frac{6a}{aK_0e^\rho + 1}$$
$$- \frac{6ac}{(aK_0e^\rho + 1)\sqrt{4aK_0e^\rho + 1}}.$$

(53)

We can write (51) as

$$\mathcal{K}_H(\rho) = \frac{aK_0e^\rho + 1}{4 - 2aK_0e^\rho(1 + \frac{3c}{\sqrt{4aK_0e^\rho+1}})},$$

(54a)

$$P_H(\rho) = \frac{aK_0e^\rho + 1}{2ae^\rho K_0(1 + \frac{3c}{\sqrt{4aK_0e^\rho+1}}) - 4}$$
$$- \frac{3}{2aK_0e^\rho + 2} + \frac{7}{4},$$

(54b)

$$\mathbb{M}_H(\rho) = \frac{3aK_0e^\rho}{4(4aK_0e^\rho + 1)(aK_0e^\rho + 1)}$$
$$\times \frac{3c(3aK_0e^\rho + 1) + (4aK_0e^\rho + 1)^{\frac{3}{2}}(aK_0e^\rho + 3)}{2\sqrt{4aK_0e^\rho + 1} - aK_0e^\rho(\sqrt{4aK_0e^\rho + 1} + 3c)},$$

(54c)

$$Y_H(\rho) = \frac{3aK_0e^\rho}{2(1 + aK_0e^\rho)}.$$

(54d)

VI. TWO-FLUID CONSTANT DENSITY SOLUTION

As a first example of a two-fluid exact solution, we consider the simplified case in which there is no interaction and no fluxes. We assume, in addition, that the fluids have energy densities which are both constant but differ in value. In this case, the Eqs. (15) can be solved directly to obtain a two-fluid generalization of Schwarzschild's interior solution.

Let us name the constant densities for fluids 1 and 2 as μ_1 and μ_2. In this case we have

$$\mathbb{M}_i = \mu_i K_0 e^\rho \mathcal{K}, \quad i = 1, 2.$$

(55)

Substituting this into (17), we obtain the solution

$$\mathcal{K} = \frac{3}{4\mathfrak{z}^2},$$

$$\mathfrak{z} = \sqrt{3 - \mu_{\text{tot}}K_0e^\rho},$$

(56)

which, solving the equation for the total pressure (19) leads to

$$P_{\text{tot}} = -\frac{\mu_{\text{tot}}K_0e^\rho(3\mathfrak{z} + \mathfrak{c}_{\text{tot}})}{4\mathfrak{z}^2(\mathfrak{z} + \mathfrak{c}_{\text{tot}})},$$

(57)

where $\mu_{\text{tot}} = \mu_1 + \mu_2$ and $\mathfrak{c}_{\text{tot}}$ is an integration constant. We now substitute in the first of Eqs. (15) the relation $P_2 = P_{\text{tot}} - P_1$ and the solution above to obtain P_1 and P_2:

$$P_1(\rho) = \frac{e^\rho(4\mathfrak{c}_1 - 3\mu_1 K_0\mathfrak{z})}{4\mathfrak{z}^2(\mathfrak{z} + \mathfrak{c}_{\text{tot}})},$$

(58)

$$P_2(\rho) = -\frac{e^\rho(4\mathfrak{c}_1 - 3K_0\mathfrak{z}\mu_1 + \mu_{\text{tot}}K_0(3\mathfrak{z} + \mathfrak{c}_{\text{tot}}))}{4\mathfrak{z}^2(\mathfrak{z} + \mathfrak{c}_{\text{tot}})},$$

(59)

where $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are constants such that $\mathfrak{c}_{\text{tot}} = \mathfrak{c}_1 + \mathfrak{c}_2$. Mapping ρ to the area radius and utilizing

$$\phi = -\frac{2\mathfrak{z}}{\sqrt{3K_0e^\rho}},$$

(60)

we can calculate the pressures p_1 and p_2 as

$$p_1(r) = \frac{4\mathfrak{c}_1 - 3\mu_1 r_0^2 z}{3r_0^2(z + \mathfrak{c}_{\text{tot}})},$$

(61a)

$$p_2(r) = -\frac{4\mathfrak{c}_1 + r_0^2(3z\mu_2 + \mathfrak{c}_{\text{tot}}\mu_{\text{tot}})}{3r_0^2(z + \mathfrak{c}_{\text{tot}})},$$

(61b)

where $r_0 = K_0^{1/2}$ and

$$z = \sqrt{3 - r^2\mu_{\text{tot}}}.$$

(62)

The variable Y can be found using (10b). Now, breaking covariance and choosing a metric of the form (23), the expressions (26) give

$$k_1 = a_0[\mathfrak{c}_1 + \mathfrak{c}_2 + z]^2,$$

(63a)

$$k_2 = \frac{3}{z^2},$$
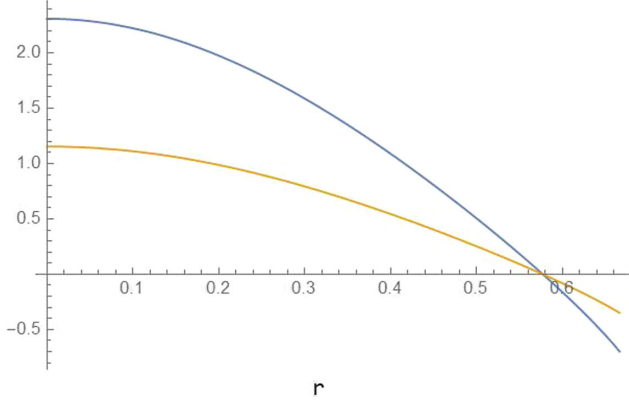
(63b)

$$k_3 = r^2,$$

(63c)

FIG. 1.   Pressure for fluid 1 (blue) and fluid 2 (orange) vs r for the double constant density solution found in Sec. VI with $\mu_1 = 4$ and $\mu_2 = 2$, and parameter values $\mathfrak{c}_1 = 3$ and $\mathfrak{c}_2 = -6$.

where a_0 is a constant, and we have set $r_0 = 1$ without loss of generality. We will use this last convention when writing the area radius solutions in all of the following sections so that the size of the equations is reduced.

Notice that the coefficients of the metric (63), as expected, are the same as the single-fluid solution presented in the previous section. As a consequence, the central pressure will also have the same structure as the single-fluid solution, and the maximum possible mass M_{tot} of this object will be the same, i.e.,

$$M_{\text{tot}} = M_1 + M_2 = \frac{4}{9} R. \tag{64}$$

The difference is that the maximum mass can be achieved with different combinations of the two fluids.

In Fig. 1 we give an example of the behavior of the pressures p_1 and p_2 in a specific case that is compatible with the requirements given in Sec. III.

VII. SOLUTION RECONSTRUCTION

In this section, we will expand the technique proposed in [6,7] to deduce new two-fluid solutions starting from a given metric. We begin with rearranging (9b) as

$$X = -\frac{1}{2} - \frac{\mathcal{K}_{,\rho}}{2\mathcal{K}}. \tag{65}$$

Using (10a) and (10b), we obtain

$$\mathbb{M}_{\text{tot}} = \mathbb{M}_1 + \mathbb{M}_2 = \frac{\mathcal{K}_{,\rho}}{2\mathcal{K}} + \mathcal{K} - \frac{1}{4}. \tag{66}$$

Equation (9a) together with (66) gives

$$P_{\text{tot}} = P_1 + P_2 = \frac{1}{3}[2Y_{,\rho} + 2Y^2 + Y] - \frac{1}{3}\mathcal{K}$$
$$- \left(\frac{2Y+1}{6}\right)\frac{\mathcal{K}_{,\rho}}{\mathcal{K}} + \frac{1}{12}. \tag{67}$$

Given a metric, and therefore the functions Y and $\mathcal{K}$ which satisfy the relation [6,7]

$$(2Y + 1)\mathcal{K}_{,\rho} - 4\mathcal{K}^2 - \mathcal{K}[4Y_{,\rho} + 4(Y - 1)Y - 1] = 0, \tag{68}$$

we can obtain the sum of the pressure and energy density variables of the two fluids.

One could then be tempted to choose the behavior for the energy density and pressure of one of the two fluids and deduce the other. This, however, would be a mistake, as it would ignore the conservation laws associated to the single fluids. Such an additional constraint can be introduced considering also Eq. (9c),

$$P_{1,\rho} = -P_1(2X + Y) - \mathbb{M}_1 Y, \tag{69}$$

which is a first-order differential equation. Solving for $\mathbb{M}_1$, we obtain

$$\mathbb{M}_1 = -\frac{P_1(Y + 2X) + P_{1,\rho}}{Y}, \tag{70}$$

which can be used to derive $\mathbb{M}_1$ once P_1 is given. In fact, remembering the formulae in (26), Eq. (69) can be solved, in general, in terms of the metric coefficients and the energy density to give

$$P_1 = \frac{k_2^2}{\sqrt{k_1}}\left(\mathfrak{c}_1 - \int \frac{\mu_1 k_{1,\rho}}{2\sqrt{k_1}k_2}\, d\rho\right). \tag{71}$$

With this result, one can assign the energy density and derive the pressure. Clearly, because (69) is an ordinary differential equation, the two approaches above are completely equivalent.

Hence, once we choose a geometry, we can then choose the behavior for the energy density or pressure of one of the two fluids and deduce the other quantities via the elementary relations

$$\mathbb{M}_2 = \mathbb{M}_{\text{tot}} - \mathbb{M}_1, \tag{72a}$$

$$P_2 = P_{\text{tot}} - P_1. \tag{72b}$$

At this point, one can obtain the expressions for the energy densities and the pressures, taking into account that the factor ϕ^2 is the one associated with the underlying geometry represented by $\mathcal{K}$ and Y in the formulae above.

In the following, we will construct models of a relativistic star with two fluids using the Tolman IV and
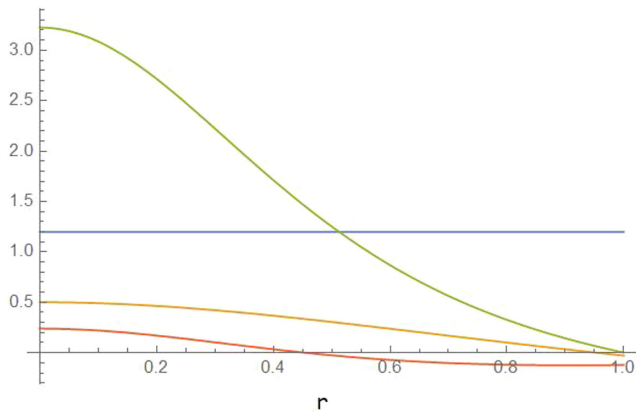
FIG. 2.   The energy density (blue) and pressure (orange) for fluid 1 and the energy density (green) and pressure (red) for fluid 2 vs r for the Tolman IV-constant density object. We used the parameter values $A = 0.95, R = 1.65, \mu_1 = 1.2, \mathfrak{c}_1 = -1.1$, which satisfy the energy and stability conditions (30), (31), and (33), in order to obtain the clearest representation. The pressure for fluid 2 becomes negative close to $r = 0.455$, as indicated by the dashed vertical line.

Heintzmann geometry and employing P and $\mathbb{M}$ of the interior Schwarzschild, Tolman IV, and Heintzmann solutions.[3] These choices are particularly convenient as the constraint (68) is already satisfied. However, the algorithm is general and one can use any other known physical solution to obtain two-fluid solutions. Naturally, these new solutions should be tested against the requirements given in Sec. III to verify their compatibility with physical objects.

A. Tolman IV-constant density relativistic object

We start with the Tolman IV geometry (43) and choose one of the fluids to be the constant density fluid in Sec. VI. In this way the energy density and the pressure of the two fluids will be given by

$$\mu_2 = \frac{3A^4 + 2r^2(3r^2 + R^2) + A^2(7r^2 + 3R^2)}{R^2(A^2 + 2r^2)^2} - \mu_1, \quad (74a)$$

$$p_1 = -\mu_1 - \frac{4\mathfrak{c}_1}{R^2\sqrt{A^2 + r^2}}, \quad (74b)$$

---

[3]It should be noted here that, because of the different forms of ϕ, the relation between P and $\mathbb{M}$ with p and μ is not the usual one. For example, in the Tolman IV-Heintzmann object, P_T and $\mathbb{M}_T$ will not correspond to p_T and μ_T. These last quantities are in fact connected to p_1 and μ_1 by

$$p_1 = p_T \frac{\phi_H^2}{\phi_T^2}, \qquad \mu_1 = \mu_T \frac{\phi_H^2}{\phi_T^2}. \quad (73)$$

In this sense, the name "Tolman IV-Heintzmann object" will indicate the origin of the solution, rather than its actual composition.
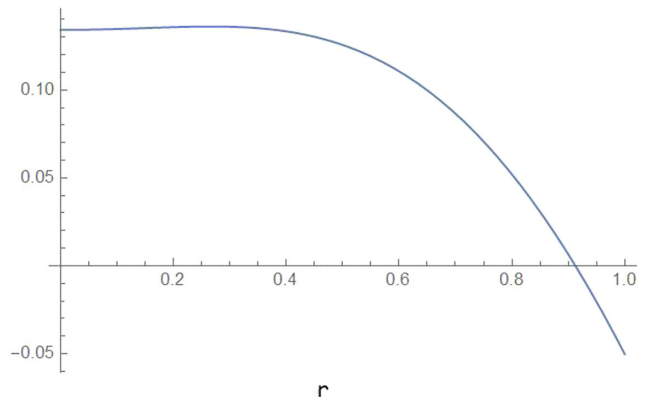

FIG. 3.   Plot of the square of the sound speed $(\frac{\partial p}{\partial \mu})$ for fluid 2 for the Tolman IV-constant density object. We used the parameter values $A = 0.95, R = 1.65, \mu_1 = 1.2, \mathfrak{c}_1 = -1.1$ as in Fig. 2.

$$p_2 = \frac{-A^2 - 3r^2 + R^2}{R^2(A^2 + 2r^2)} + \frac{4\mathfrak{c}_1}{R^2\sqrt{A^2 + r^2}} + \mu_1, \quad (74c)$$

where μ_1 is the constant density of the first fluid, $\mathfrak{c}_1$ is a constant of integration. The properties of the underlying geometry, the energy density, and pressure of the first fluid guarantee that μ_2 and p_2 do not have singularities. Naturally, we should guarantee that the conditions (30), (31), and (33) are all satisfied. Given the number of parameters, it is not practical to give these conditions analytically. Hence, we limit ourselves to show graphically that at least one of these combinations exists. Figure 2 shows the pressure and energy density profiles for both fluids, and Fig. 3 shows the square of the sound speed $\frac{\partial p}{\partial \mu}$.

Notice that unlike in Fig. 1, the pressures do not approach zero at the same value of the radial coordinate. Indeed, we could find no value of the parameters for which p_1 and p_2 are zero at the same r. However, this does not imply that the solution we found should be discarded: for the interval in which p_1 and p_2 are both positive, the combination of (43) with (74) constitutes an acceptable two-fluid solution. It is clear that in order to obtain a complete model of a compact object, such a solution should be joined using, e.g., Israel's prescriptions [23,24], to a shell or another solution which satisfies the conditions (30)–(33), and whose pressure(s) approach zero at some value of the radial coordinate r.

In other words, we obtain naturally a shelled structure for this object: an internal shell in which two fluids are present and an external one which can be matched smoothly to the Schwarzschild solution. To model the external shell, we can use any geometry or number of fluids. An easy setting for the external shell could be a single-fluid Tolman IV solution which has pressure p_1 and energy density $\mu_1 + \mu_2$ at the junction point (where p_2 vanishes). In the following, we will often find ourselves in
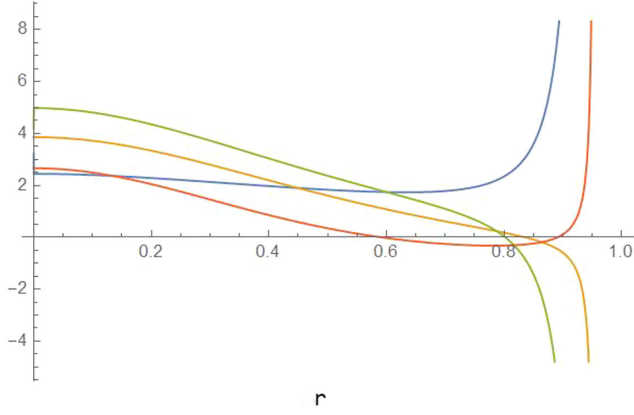
FIG. 4. Energy density of fluid 1 (blue) and fluid 2 (green), as well as the pressure for fluid 1 (orange) and for fluid 2 (red), vs r for the Heintzmann-constant density object. We used the parameter values $a = 1.5, c = 0.1, \mu_1 = 3.3, \mathfrak{c}_1 = -2.5$, which satisfy the energy and stability conditions (30), (31), and (33), in order to obtain the clearest representation. The pressure of fluid 2 becomes negative close to $r = 0.6$.

the same situation. For the sake of brevity, we will not give a description of the single-fluid external shells for each instance.

B. Heintzmann-constant density relativistic object

We start with the Heintzmann IIa geometry (48) and choose one of the fluids to be the constant density fluid in Sec. VI. In this way the energy density and the pressure of the two fluids will be given by

$$\mu_1 = \frac{\mu_1 z_2 z_3}{z_1^4(\mathfrak{c}_1 + z_1)^2} \left(2\mathfrak{c}_1 \mu_1 r^2(\mathfrak{c}_1 + 3z_1)\right.$$
$$\left. + z_1^2(z_4(\mathfrak{c}_1 + z_1)(\mathfrak{c}_1 + 3z_1) + 2\mathfrak{c}_1(\mathfrak{c}_1 + 4z_1) + 18)\right), \tag{75a}$$

$$\mu_2 = -z_3 z_5 + \frac{\mu_1 z_2 z_3}{z_1^4(\mathfrak{c}_1 + z_1)^2} \left(2\mathfrak{c}_1 \mu_1 r^2 z_1(2z_4 + 1)\right.$$
$$\left. - z_1 z_4(\mathfrak{c}_1^2 z_1 + 12\mathfrak{c}_1 + 3z_1^3) - 6(\mathfrak{c}_1 + z_1)(\mathfrak{c}_1 + 3z_1)\right), \tag{75b}$$

$$p_1 = -\frac{\mu_1 z_3(\mathfrak{c}_1 + 3z_1)}{z_1^2(\mathfrak{c}_1 + z_1)}, \tag{75c}$$

$$p_2 = z_3 \left(\frac{\mu_1(\mathfrak{c}_1 + 3z_1)}{z_1^2(\mathfrak{c}_1 + z_1)} + z_6\right), \tag{75d}$$
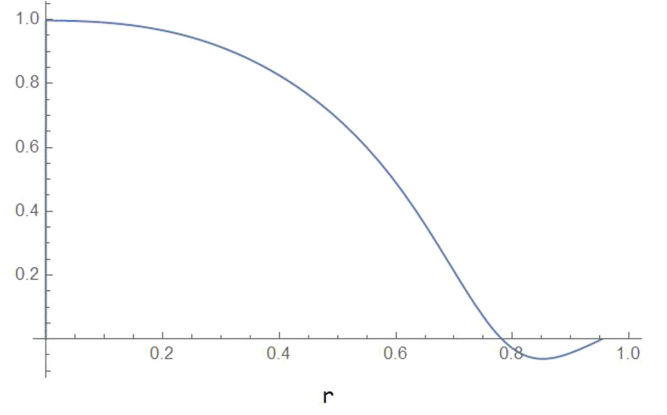
where we have written

FIG. 5. Plot of the square of the sound speed $(\frac{\partial p}{\partial \mu})$ for fluid 2 for the Heintzmann-constant density object. We used the parameter values $a = 1.5, c = 0.1, \mu_1 = 3.3, \mathfrak{c}_1 = -2.5$ as in Fig. 4.

$$z_0 = \sqrt{4ar^2 + 1},$$

$$z_1 = \sqrt{3 - \mu_1 r^2},$$

$$z_2 = \frac{1 + ar^2}{3ar^2},$$

$$z_3 = 1 - \frac{3(z_0 + c)}{2z_2 z_0},$$

$$z_4 = \frac{ar^2\{ar^2[4az_0 r^2 + 9(z_0 - c)] + 18z_0\} + 4z_0}{z_0(ar^2 + 1)[ar^2(3cz_0 + 4ar^2 - 7) - 2]},$$

$$z_5 = \frac{3a[c(9ar^2 + 3) + (ar^2 + 3)z_0^3]}{z_0^2(ar^2 + 1)[ar^2(z_0 + 3c) - 2z_0]},$$

$$z_6 = \frac{3a[7acr^2 + 3(ar^2 - 1)z_0 + c]}{(ar^2 + 1)[ar^2(z_0 + 3c) - 2z_0]}, \tag{76}$$

for compactness, μ_1 is the energy density associated with the constant density solution, $\mathfrak{c}_1$ is a constant of integration, and we have used (48). Figure 4 shows the pressure and energy density profiles of both fluids, and Fig. 5 shows $\frac{\partial p}{\partial \mu}$.

C. Heintzmann-Tolman relativistic object

We start from the Heintzmann IIa geometry (43) and choose one of the fluids to be the Tolman IV fluid (48). In this way the energy density and the pressure of the two fluids will be given by

$$\mu_1 = \frac{z_2 z_3}{z_7^2}[-R^4(A^2(z_4 + 2) + z_4 r^2) - 4A^2 r^4(z_4 + 1)$$
$$+ R^2(A^2 r^2(5z_4 + 12) + A^4(z_4 + 2) + 4r^4(z_4 + 1))$$
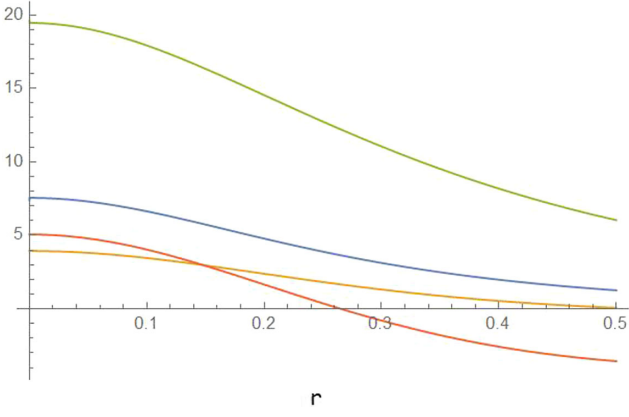$$- A^4 r^2 z_4 - 3r^6 z_4], \tag{77a}$$

FIG. 6. Energy density of fluid 1 (blue) and fluid 2 (green), as well as the pressure for fluid 1 (orange) and for fluid 2 (red), vs r for the Heintzmann-Tolman object. We used the parameter values $A = 0.5, R = 4, a = 3, c = 1$, which satisfy the energy and stability conditions (30), (31), and (33), in order to obtain the clearest representation. The pressure of fluid 2 becomes negative close to $r = 0.25$.

$$\mu_2 = z_3 \left(-z_5 + \frac{z_2}{z_7^2(r+R)^2} \times [3z_7 r^2(z_4+4) \right.$$
$$- R^2 z_7(z_4+2) + A^2(z_7(z_4+2) + 4r^2 R^2 - 10r^4)$$
$$\left. - 2A^4 r^2 - 2r^2(-5r^2 R^2 + 6r^4 + R^4)] \right), \qquad (77b)$$

$$p_1 = \frac{z_8 z_3}{z_7}, \qquad (77c)$$

$$p_2 = \frac{z_3(z_6 z_7 - z_8)}{z_7}, \qquad (77d)$$

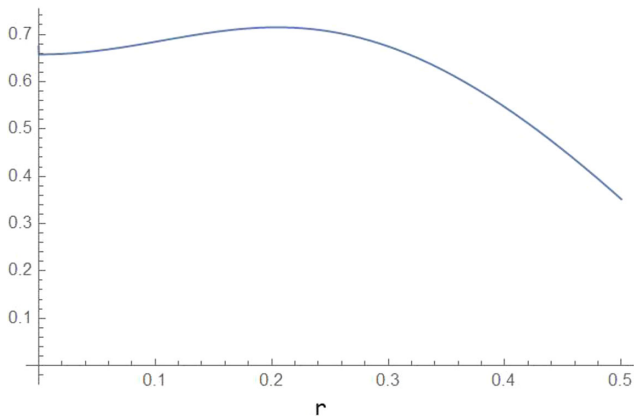where we have used the z_i given in (76) and written

FIG. 7. Plot of the square of the sound speed $(\frac{\partial p}{\partial \mu})$ for fluid 2 for the Heintzmann-Tolman object. We used the parameter values $A = 0.5, R = 4, a = 3, c = 1$ as in Fig. 6.

$$z_7 = (A^2 + r^2)(r^2 - R^2),$$
$$z_8 = A^2 + 3r^2 - R^2, \qquad (78)$$

for compactness, and we have used (44) and (51). Figure 6 shows the pressure and energy density profiles of the two fluids, and Fig. 7 shows the square of the sound speed $(\frac{\partial p}{\partial \mu})$.

VIII. A GENERATING THEOREM

In their 2005 paper, Boonserm *et al.* [20,21] developed transformation theorems able to map one perfect fluid sphere into another called *generating theorems*. The work in [20,21] focuses on the spacetime geometry, starting with a known perfect fluid sphere and applying the theorem in order to obtain a new class of solutions to the Einstein field equations. Certain metrics, known as seed metrics, are transformed to generate new metrics.

In [6,7] it was shown that in the context of the covariant version of the TOV equations, the generating theorems assume the simple forms of linear solution deformation. In this section, we will show that one of such theorems can also be used to obtain two-fluid solutions from single-fluid ones. In the following, we will focus on the simplest cases in which the solution can be obtained analytically. In doing so, we will assume isotropy, no fluxes, and no interaction. As seen in the previous section, and as it happens for the single-fluid generating theorems, it is not obvious that the new solution will be compatible with the conditions (30), (31), and (33). These constraints must be imposed *a posteriori* to evaluate the physical relevance of the solution.[4]

The TOV equations in the case $Q_1 = 0$ reads

$$P_{,\rho} = -P^2 + P\left[\mathbb{M} + 1 - 3\left(\mathcal{K} - \frac{1}{4}\right)\right] - \left(\mathcal{K} - \frac{1}{4}\right)\mathbb{M},$$
$$\mathcal{K}_{,\rho} = -2\mathcal{K}\left(\mathcal{K} - \frac{1}{4} - \mathbb{M}\right). \qquad (79)$$

Given a solution of (79) represented by $P_0, \mathbb{M}_0, \mathcal{K}_0$, and Y_0, we perform the linear deformation

$$P_0 \to P_0 + \tilde{P}, \qquad Y_0 \to Y_0 + \tilde{Y},$$
$$\mathbb{M}_0 \to \mathbb{M}_0, \qquad \mathcal{K}_0 \to \mathcal{K}_0 \qquad (80)$$

corresponding to Theorem 2 of [20]. We can use this theorem to obtain a two-fluid solution from a single-fluid one.

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[4]It is worth adding that relaxing the requirement of deducing an analytical form extends enormously the number of achievable solutions. We will not undertake such analysis here. The reader is reminded, however, that such course of action implies the additional problem to find a justification for the chosen values of the parameters.

In fact, the transformation (80) can be interpreted as a change from a setting in which we have a *single* fluid with pressure $p_0 = P_0\phi^2$ to a new setting in which we have *two fluids* of pressures $p_0 = P_0\phi^2$ and $\tilde{p} = \tilde{P}\phi^2$, where $\tilde{P}$ is given by the equation

$$\tilde{P}_{,\rho} + \tilde{P}^2 + \tilde{P}\left(3\mathcal{K}_0 - \mathbb{M}_0 + 2P_0 - \frac{7}{4}\right) = 0, \quad (81)$$

derived by combining (9a), (9b), and the constraints (10) with the transformation (80).

We then need to determine the energy density of the two fluids. This can be accomplished using the conservation law,

$$\tilde{P}_{,\rho} = -\tilde{P}(2X + Y) - \tilde{\mathbb{M}}Y, \quad (82)$$

that has to hold for the fluid with pressure $\tilde{p}$. Combining (9a), (9b), and the constraints (10) with the transformation (80) and (82), we obtain $\tilde{Y} = \tilde{P}$ and

$$\tilde{\mathbb{M}} = \frac{\tilde{P}(1 - 4\mathcal{K}_0 + 4\mathbb{M}_0 + 4Y_0)}{4(\tilde{P} + Y_0)}. \quad (83)$$

This last equation allows to obtain the energy density corresponding to $\tilde{p}$ as $\tilde{\mu} = \tilde{\mathbb{M}}\phi^2$. Instead, the energy density of the remaining fluid can be obtained, keeping in mind that the total energy density should be μ_0. From (80) we can then write

$$\mathbb{M}_0 = \bar{\mathbb{M}}_0 + \tilde{\mathbb{M}}, \quad (84)$$

where $\bar{\mathbb{M}}_0$ is the energy density variable associated to the fluid with pressure p_0 in the two-fluid picture. The energy density for this last fluid will be, then,

$$\bar{\mu}_0 = \mu_0 - \tilde{\mu}, \quad (85)$$

where $\bar{\mu}_0 = \bar{\mathbb{M}}_0\phi^2$. Hence, starting with a known single-fluid solution characterized by (p_0, μ_0), we obtain a two-fluids solution characterized by a fluid with $(\tilde{p}, \tilde{\mu})$ and a second fluid with $(p_0, \bar{\mu}_0)$.

The transformation (80) induces a transformation of the metric coefficients given by

$$k_1 \rightarrow k_1 \exp\left(\int \tilde{Y} d\rho\right), \quad (86a)$$

$$k_2 \rightarrow k_2, \quad (86b)$$

$$k_3 \rightarrow k_3, \quad (86c)$$

where we have used the form (23). These relations allow to complete the description of the new two-fluid solution.

Let us now suppose that the starting solution is the Tolman IV metric given in (47). The formulas above lead to a new solution, which is sourced by a first fluid, whose pressure is given by

$$\tilde{p}(r) = \frac{4z_1\sqrt{A^2 + r^2}(A^2 + R^2)}{z_4 R^2}$$
$$\times \{[2z_2 z_3 + \mathfrak{c}_1(A^2 + R^2)](A^2 + r^2)$$
$$+ 2z_1 z_4\sqrt{A^2 + r^2}\}^{-1}, \quad (87)$$

where

$$z_1 = \sqrt{R^2 - r^2}, \quad (88a)$$

$$z_2 = E\left(\sin^{-1}\left(\sqrt{\frac{R^2 - r^2}{A^2 + R^2}}\right), \frac{A^2}{A^2 + 2R^2} + 1\right),$$
$$- F\left(\sin^{-1}\left(\sqrt{\frac{R^2 - r^2}{A^2 + R^2}}\right), \frac{A^2}{A^2 + 2R^2} + 1\right), \quad (88b)$$

$$z_3 = \sqrt{A^2 + 2R^2}, \quad (88c)$$

$$z_4 = \sqrt{A^2 + 2r^2}, \quad (88d)$$

and $\mathfrak{c}_1$ is a constant, E is the complete elliptical integral, and F is the elliptical integral of the first kind. The corresponding energy density is given by Eq. (83) and reads,

$$\tilde{\mu}(r) = \frac{4z_3^2\sqrt{A^2 + r^2}(A^2 + R^2)}{R^2 z_4^3}$$
$$\times \{(2z_4\sqrt{A^2 + r^2} + \mathfrak{c}_1 z_1(A^2 + R^2)$$
$$- 2z_1 z_2 z_3)\}^{-1}, \quad (89)$$

where the quantities z_i are the same as (88). Instead, the second fluid will have pressure p_0 given by (44a), i.e.,

$$p_0 = p_T = \frac{R^2 - A^2 - 3r^2}{R^2(A^2 + 2r^2)} \quad (90)$$

and energy density given by (85), i.e.,

$$\bar{\mu}_0 = \frac{R^2(3A^2 + 2r^2) + 7A^2 r^2 + 3A^4 + 6r^4}{R^2(A^2 + 2r^2)^2} \quad (91)$$

$$- \frac{4z_3^2\sqrt{A^2 + r^2}(A^2 + R^2)}{R^2 z_4^3}$$
$$\times \{(2z_4\sqrt{A^2 + r^2} + \mathfrak{c}_1 z_1(A^2 + R^2)$$
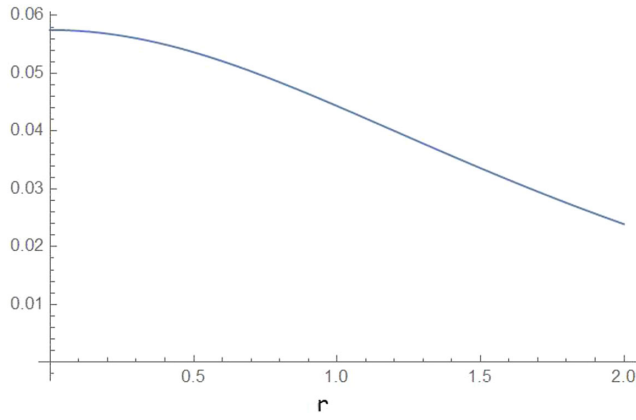$$- 2z_1 z_2 z_3)\}^{-1}. \quad (92)$$

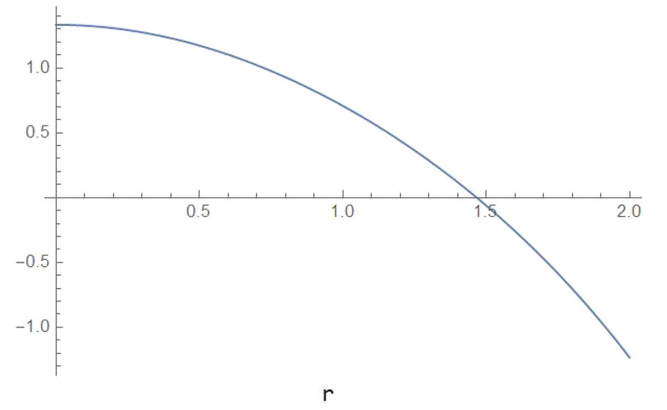FIG. 8. Pressure $\tilde{p}$ vs r for the solution generated in Sec. VIII with constant values $A = -2.55, R = -3.6, \mathfrak{c}_1 = -3.6$.
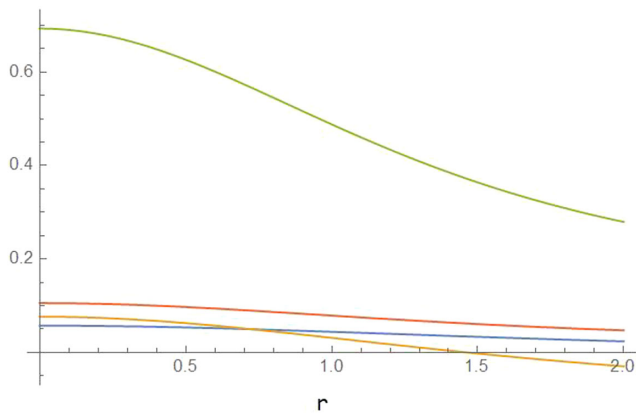

FIG. 9. Pressure p_0 (orange), deformation pressure $\tilde{p}$ (blue), energy density $\bar{\mu}_0$ (green), and deformation energy density $\tilde{\mu}$ (red) vs r for the solution generated in Sec. VIII with constant values $A = -2.55, R = -3.6, \mathfrak{c}_1 = -3.6$.
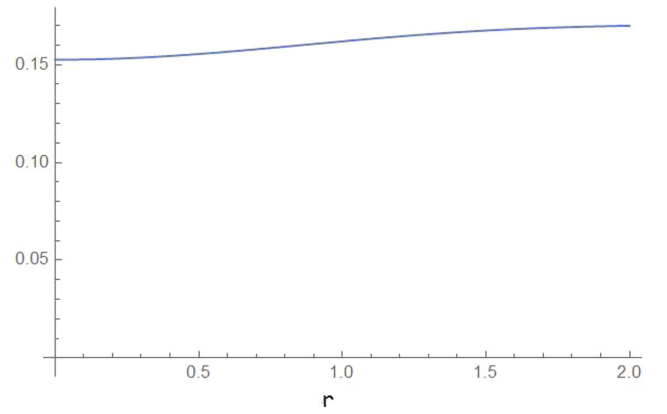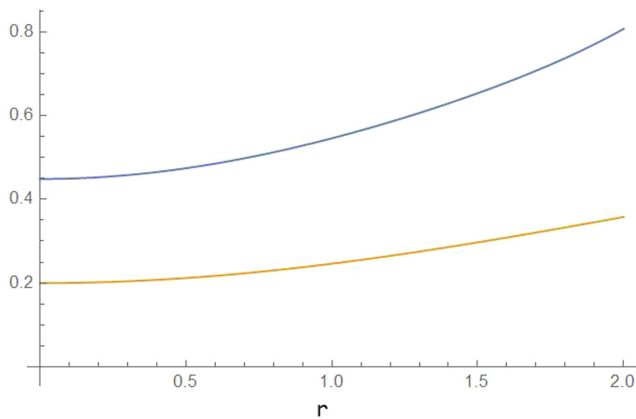

FIG. 10. Square of the sound speed $\left(\frac{\partial p}{\partial \mu}\right)$ for the fluid with pressure p_0 (orange) and with $\tilde{p}$ (blue) vs r for the solution generated in Sec. VIII with constant values $A = -2.55, R = -3.6, \mathfrak{c}_1 = -3.6$.

FIG. 11. $p_0/\tilde{p}$ vs r for the solution generated in Sec. VIII with constant values $A = -2.55, R = -3.6, \mathfrak{c}_1 = -3.6$.

FIG. 12. $\tilde{\mu}/\bar{\mu}_0$ vs r for the solution generated in Sec. VIII with constant values $A = -2.55, R = -3.6, \mathfrak{c}_1 = -3.6$.

Figure 8 shows a plot of the pressure $\tilde{p}$. The four quantities μ_0, $\tilde{\mu}$, p_0, and $\tilde{p}$ are graphically represented in Fig. 9, and the equations of state are shown graphically in Fig. 10, for specific values of the parameters. We also give a plot of the ratio of the two pressures $p_0/\tilde{p}$, shown in Fig. 11, and the two energy densities in Fig. 12.

Finally, using (86), the full expression of the metric coefficients is found to be

$$
k_1 = \left[\frac{2\mathfrak{c}_1}{A^2 + R^2} \left(\frac{z_1 z_4}{\sqrt{A^2 + r^2}} - z_2 z_3 \right) \right. \\
\left. + \sqrt{A^2 + r^2} + \mathfrak{c}_2 \right]^2,
\tag{93}
$$

$$
k_2 = \frac{R^2 z_4{}^2}{z_1{}^2 (A^2 + r^2)},
\tag{94}
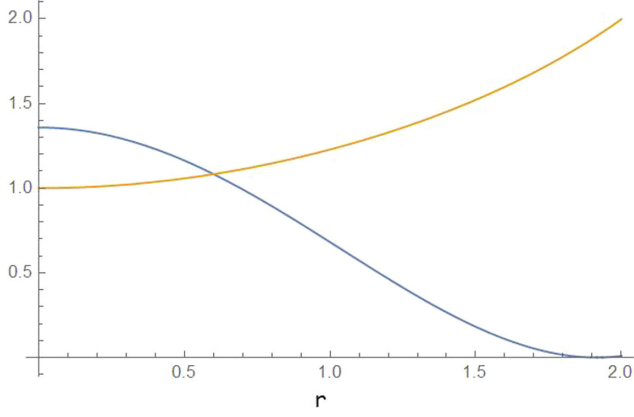$$

$$
k_3 = r^2,
\tag{95}
$$

FIG. 13. Metric coefficients k_1 (blue) and k_2 (orange) vs r for the solution generated in Sec. VIII. We use the constant values $A = -2.55, R = -3.6, \mathfrak{c}_1 = -3.6$ and $\mathfrak{c}_2 = -1.45$.

where the z_i are the same as (88). We show a plot of these metric coefficients in Fig. 13.

IX. INTRODUCING THE FLUXES

Looking at Eq. (6i) it is clear that a two-fluid static solution can be of two types. The first, in which the fluxes are identically zero, and a second, in which the flux of one fluid counteracts exactly the flux of the other. In this section we will consider this last class of solutions. The analysis of the behavior of the fluxes in these solutions is relatively straightforward. Equations (22) allow us to write Eq. (6h) as

$$\hat{Q}_1 = -Q_1 \left(\frac{\hat{k}_3}{k_3} + \frac{\hat{k}_1}{k_1} \right), \tag{96}$$

where we have set the interaction terms to zero. The above equation can be integrated by separation of variables to give

$$Q_1 = \frac{\bar{Q}_1}{k_1 k_3} = -Q_2, \tag{97}$$

where $\bar{Q}_1$ is an integration constant. The same expression holds in terms of the parameter ρ and the area radius r as k_1 and k_3 are scalars for changes in these parameters. Notice that the behavior of the flux depends only on the spacetime geometry and, therefore, it is independent of the properties of matter.

An important aspect of Eq. (97) is that as the flux is inversely proportional to the Gaussian curvature, we expect that in the center, where $k_3 = 0$ (the metric (23) is singular), the flux will diverge. Such divergence is present also without spherical symmetry and even if we relax the assumption of staticity. In fact, in the case of a completely general LRS class II spacetime of the type (20), the $1 + 1 + 2$ equations give [17]
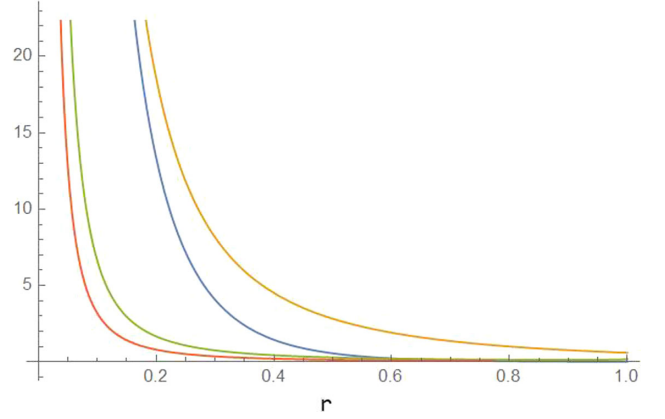

FIG. 14. Flux for interior Schwarzschild (red), Tolman IV (orange), Heintzmann IIa (blue), and the solution found with the generating theorem (green) vs r. We use the constant values $\bar{Q}_1 = 3, a = 3, b = 2, A = 2, B = 2, R = 1, \mu_1 = 2, \mu_2 = 3, \mathfrak{c}_1 = 3, \mathfrak{c}_2 = 4, \mathfrak{c}_3 = 5, \mathfrak{c}_4 = 3$.

$$Q = \frac{\hat{\dot{k}}_3}{k_3} - \frac{\dot{k}_3}{k_3} \frac{\hat{k}_2}{k_2} - \frac{\dot{k}_3}{k_3} \frac{\hat{k}_3}{k_3}, \tag{98}$$

which might also diverge when $k_3 = 0$. The origin of this divergence is fundamentally related to the pathological behavior of the angular coordinates in $r = 0$ and, therefore, we might expect that using a different chart the divergence of Q would disappear, not unlike the case of the divergence of the Schwarzschild horizon.

As an application, we will now give the form of the fluxes for the solutions that we have found in the previous sections. For the two-fluid constant density interior Schwarzschild solution (63) of Sec. VI, we have

$$Q_1 = \frac{\bar{Q}_1}{r^2 [\mathfrak{c}_1 + \mathfrak{c}_2 + z]^2}. \tag{99}$$

For the solution based on the Tolman geometry in Sec. VII, we have

$$Q_1 = \frac{A^2 \bar{Q}_1}{r^2 B^2 (A^2 + r^2)}, \tag{100}$$

whereas, for the solution based on the Heintzmann geometry in Sec. VII, we have

$$Q_1 = \frac{\bar{Q}_1}{r^2 b^2 (1 + ar^2)^3}. \tag{101}$$

Finally, for the solution obtained by the generating theorem in Sec. VIII, we have

$$Q_1 = \frac{\bar{Q}_1 (A^2 + R^2)^2}{r^2} [-2\mathfrak{c}_1 z_2 z_3 \sqrt{A^2 + r^2} + \mathfrak{c}_2 (A^2 + R^2) + 2\mathfrak{c}_1 z_1 z_4]^{-2}, \tag{102}$$

where the z_i are given by (88). The fluxes are graphically represented in Fig. 14.

X. DISCUSSION AND CONCLUSION

In this paper, we have presented a complete set of equations able to describe the interior solution of relativistic objects with more than one fluid source, including possible anisotropic stress, energy flux, and interaction terms. These equations have been written by means of the $1 + 1 + 2$ covariant formalism, which allows a relatively straightforward treatment of the many features of these kinds of systems. As it is done in GR, the $1 + 1 + 2$ equations can be combined to obtain the covariant equivalent of the TOV equations.

The properties of the TOV equations, however, can be more clearly appreciated when they are recast in a form that either contains dimensionless variables or it is written in terms of quantities which are invariant under homological transformation. Using the $1 + 1 + 2$ potential, we have defined variables with similar properties, which allow one to write the TOV equations as a closed system of the Riccati and Bernoulli equations, when the equation of state of matter is included. As in the case of the single-fluid solutions, these equations can be solved exactly with several techniques, other than direct resolution.

We have presented here some of these solutions. In particular, we have obtained the generalization of the interior Schwarzschild solution to the case of two fluids. We have also formulated a reconstruction algorithm for the two-fluid TOV equations. The structure of the reconstruction equations seems to show that there is a degeneracy as a given spacetime metric might correspond to many different combinations of fluids. The origin of such degeneracy is ultimately correlated with the equivalence principle: as all matter gravitates and does it in the same way, several multifluid configurations can produce the same spacetime metric. However, in the two-fluid case, the conservation laws for the individual fluids are independent equations and, therefore, one must include an additional equation. Consequently, an additional constraint must be added to the reconstruction equations, and this additional equation immediately resolves the degeneracy. We found that the additional equation can be seen as a differential constraint for the pressure of one of the fluids or an algebraic one for the energy density. Using the new reconstruction equations and the single-fluid solutions presented in the previous sections, we have been able to obtain two-fluid solutions that are physically viable in the sense of the constraints given in [5] and in Sec. III.

A somewhat surprising, but nonetheless interesting, result concerns the generating theorems that have been shown to hold in the case of single-fluid solutions. We discovered that one of these theorems (Theorem 2 of [20]) can also be employed to construct two-fluid solutions. We used this theorem to obtain a solution representing a two-fluid relativistic star, comprising a perfect fluid and a fluid with a nontrivial equation of state.

As mentioned above, the equations we have constructed contain a complete description of the properties of the fluid, including fluxes. Since one can integrate, in general, the equation for the heat flux for every two-fluid solution known, there is an additional one in which the fluxes are nonzero. We have given the expressions of the flux for all the solutions obtained in the text. We found that in Schwarzschild coordinates these fluxes are always divergent in the center of the matter distribution. As such divergence could be an artifact of the coordinates used for describing the metric, we do not consider the divergence of the flux as a reason to discard solutions, as we have done with other matter potentials. Further investigation on this matter might shed light on the real nature of this feature.

In general, the majority of the solutions we have found present a shell structure. In particular, we found that these solutions have to be completed with an additional shell, which can be joined smoothly to any external spacetime we choose to consider (e.g., the Schwarzschild [25], Vaidya [28], etc.). Such a shell can contain any number of fluids/fields consistent with the junction conditions. This should not be surprising: such configurations are expected to arise in multifluid systems, and the constraints arising from the junction conditions make such occurrences even more likely, as it is more complex to find solutions in which the pressures of two different fluids vanish at the same point.

The set of results presented above indicate clearly that the covariant formulation of the TOV equations is a powerful tool to investigate analytically the physics of interior solutions–even in the multifluid case. The possibility of obtaining physically reasonable and exact toy models allows one to explore more deeply the features of the interior of relativistic objects and their perturbations and can be used as a complement to the large number of numerical models in literature as a testing tool.

We conclude by remarking that the value of our preliminary results goes beyond the scope of standard multifluid relativistic objects. The two-fluid approach described in this paper is also well suited to the study of relativistic objects in theories beyond that of general relativity, where the extra gravitational degree of freedom can be treated as an additional (curvature) fluid coupled to standard matter. This exciting application of our approach will be important in the context of gravitational wave astronomy, where for the first time the validity of general relativity can be probed in the strong-field regime in regions of high density and scalar curvature. These issues will be explored in future papers.

APPENDIX A: MAIN ASPECTS OF THE 1+1+2 COVARIANT APPROACH FOR A SINGLE FLUID

We will present the main aspects of the theory in this section. For more details, the reader is referred to [15,16,17].

We define a timelike threading vector field u^a associated to the observer's congruence with $u^a u_a = -1$ and a space-like vector e_a with $e_a e^a = 1$. The u^a and e_a congruences describe a geometry defined by two projection tensors given by

$$h^a{}_b = g^a{}_b + u^a u_b, \qquad h^a{}_a = 3,$$
$$N_a{}^b = h_a{}^b - e_a e^b = g_a{}^b + u_a u^b - e_a e^b, \qquad N^a{}_a = 2,$$
$$\text{(A1)}$$

where $h^a{}_b$ represents the metric of the 3-spaces orthogonal to u_a, and $N_a{}^b$ represents the metric of the 2-spaces orthogonal to u_a and e_a. Any tensorial object may now be split according to the above foliations [6]. The covariant time derivative, orthogonally projected covariant derivative, hat-derivative, and δ-derivative are given by

$$\dot{X}^{a...b}{}_{c...d} \equiv u^e \nabla_e X^{a...b}{}_{c...d},$$
$$D_e X^{a...b}{}_{c...d} \equiv h^a{}_f...h^b{}_g h^p{}_c...h^q{}_d h^r{}_e \nabla_r X^{f...g}{}_{p...q},$$
$$\hat{X}_{a...b}{}^{c...d} \equiv e^f D_f X_{a...b}{}^{c...d},$$
$$\delta_e X_{a...b}{}^{c...d} \equiv N_a{}^f...N_b{}^g N_i{}^c...N_j{}^d N_e{}^p D_p X_{f...g}{}^{i...j}.$$
$$\text{(A2)}$$

The kinematical variables are given by

$$\mathcal{A} = e_a \dot{u}^a, \qquad \mathcal{A}_a = N_{ab} \dot{u}^b, \qquad \text{(A3a)}$$

$$\Theta = D_a u^a, \qquad \xi = \frac{1}{2} \varepsilon^{ab} \delta_a e_b, \qquad \text{(A3b)}$$

$$\Omega = \frac{1}{2} \varepsilon^{abc} D_{[a} u_{b]} e_a, \qquad \Omega^a = \frac{1}{2} \varepsilon^{cbd} D_{[c} u_{b]} N_d{}^a, \qquad \text{(A3c)}$$

$$\sigma_{ab} = \left(h^c{}_{(a} h_{b)}{}^d - \frac{1}{3} h_{ab} h^{cd} \right) D_c u_d, \qquad \text{(A3d)}$$

$$\Sigma = \sigma^{ab} e_a e_b, \qquad \Sigma_a = \sigma_{cd} e^d N^c{}_a, \qquad \text{(A3e)}$$

$$\Sigma_{ab} = \left(N^c{}_{(a} N_{b)}{}^d - \frac{1}{2} N_{ab} N^{cd} \right) \sigma_{cd}, \qquad \text{(A3f)}$$

$$a_b = e^c D_c e_b = \hat{e}_b, \qquad \phi = \delta_a e^a, \qquad \text{(A3g)}$$

$$\zeta_{ab} = \left(N^c{}_{(a} N_{b)}{}^d - \frac{1}{2} N_{ab} N^{cd} \right) \delta_c e_d, \qquad \text{(A3h)}$$

$$\mathcal{E} = C_{acbd} u^c u^d e^a e^b, \qquad \text{(A3i)}$$

$$\mathcal{E}_a = C_{cedf} u^e u^f e^c N^d{}_a, \qquad \mathcal{E}_{ab} = C_{\{acb\}d} u^c u^d, \qquad \text{(A3j)}$$

$$\mathcal{H} = \frac{1}{2} \varepsilon_{ade} C^{de}{}_{bc} u^c e^a e^b, \qquad \text{(A3k)}$$

$$\mathcal{H}_a = \frac{1}{2} \varepsilon_{cbe} C^{be}{}_{df} u^f e^c N^d{}_a, \qquad \text{(A3l)}$$

$$\mathcal{H}_{ab} = \frac{1}{2} \varepsilon_{\{ade} C^{de}{}_{b\}f} u^f, \qquad \text{(A3m)}$$

where $\varepsilon_{ab} \equiv \varepsilon_{abc} e^c$ and $\varepsilon_{abc} = \eta_{dabc} e^c u^d$ are the volumes of the two hypersurfaces, and C_{abcd} is the Weyl tensor. We represent the symmetrization over the indices of a tensor as $T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba})$ and the antisymmetrization as $T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba})$. We use curly brackets {} to denote the projected symmetric trace free, with respect to the n^a part of a tensor:

$$X_{\{ab\}} \equiv \left(N^c{}_{(a} N_{b)}{}^d - \frac{1}{2} N_{ab} N^{cd} \right) \mathbb{X}_{cd}. \qquad \text{(A4)}$$

The energy-momentum tensor is decomposed as

$$T_{ab} = \mu u_a u_b + (p + \Pi) e_a e_b + \left(p - \frac{1}{2} \Pi \right) N_{ab}$$
$$+ 2Q e_{(a} u_{b)} + 2Q_{(a} u_{b)} + 2\Pi_{(a} e_{b)} + \Pi_{ab}. \qquad \text{(A5)}$$

The matter variables are given by

$$\mu = T_{ab} u^a u^b,$$
$$p = \frac{1}{3} T_{ab} (e^a e^b + N^{ab}),$$
$$\Pi = \frac{1}{3} T_{ab} (2e^a e^b - N^{ab}),$$
$$Q = -T_{ab} e^a u^b,$$
$$Q_a = -T_{cd} N^c{}_a u^d,$$
$$\Pi_a = T_{cd} N^c{}_a e^d,$$
$$\Pi_{ab} = T_{\{ab\}}, \qquad \text{(A6)}$$

where μ is the density, p is the pressure, Q and Q_a represent the scalar and vector parts of the heat flux, and Π and Π^a represent the scalar and vector components of the aniso-tropic pressure. The $1 + 1 + 2$ formalism is most advanta-geous when applied to spacetimes which have a unique preferred spatial direction at each point, exhibiting local rotational symmetry. This direction constitutes a local axis of symmetry. In LRSII spacetimes, we have that the

vorticity terms and the magnetic part of the Weyl tensor are zero, i.e., $\Omega, \xi, \mathcal{H} = 0$. For static and spherically symmetric LRSII spacetimes, we also have $\Theta, \Sigma = 0$, and all the dot derivatives vanish. The remaining $1 + 1 + 2$ scalars, which fully describe the spacetime, can be divided into three categories [17]:

Propagation:

$$\hat{\phi} = -\frac{1}{2}\phi^2 - \frac{2}{3}\mu - \mathcal{E} - \frac{1}{2}\Pi, \tag{A7a}$$

$$Q = 0, \tag{A7b}$$

$$\hat{\mathcal{E}} - \frac{1}{3}\hat{\mu} + \frac{1}{2}\hat{\Pi} = -\frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right). \tag{A7c}$$

Evolution:

$$0 = -\mathcal{A}\phi + \frac{1}{3}(\mu + 3p) - \mathcal{E} + \frac{1}{2}\Pi, \tag{A7d}$$

$$0 = \frac{1}{2}\phi Q. \tag{A7e}$$

Propagation/evolution:

$$\hat{\mathcal{A}} = -\mathcal{A}(\mathcal{A} + \phi) + \frac{1}{2}(\mu + 3p), \tag{A7f}$$

$$\hat{Q} = -Q(\phi + 2\mathcal{A}) + j_u, \tag{A7g}$$

$$\hat{p} + \hat{\Pi} = -\Pi\left(\frac{3}{2}\phi + \mathcal{A}\right) - \mathcal{A}(\mu + p) + j_e, \tag{A7h}$$

$$K = \frac{1}{3}\mu - \mathcal{E} - \frac{1}{2}\Pi + \frac{1}{4}\phi^2, \tag{A7i}$$

where μ, p, Q, and π represent, in general, the total energy density, pressure, total heat flux, and total anisotropic pressure of the fluid. In addition, we have formally included the total particle interaction currents as j_u and j_e according to the definition

$$j_a = j_u u_a + j_e e_a, \tag{A8}$$

with j_u and j_e representing the u_a and e_a components, respectively.

Finally, combining equations (A7), it is not too difficult to prove that the Gauss curvature K satisfies the propagation equation

$$\hat{K} = -\phi K. \tag{A9}$$

APPENDIX B: MULTIFLUID EQUATIONS

The $1 + 1 + 2$ equations in the case of a static spherically symmetric spacetime and N different interacting fluids can be written as

$$\hat{\phi} = -\frac{1}{2}\phi^2 - \frac{2}{3}\sum_{i=1}^{N}\mu_i - \mathcal{E} - \frac{1}{2}\sum_{i=1}^{N}\Pi_i, \tag{B1a}$$

$$\sum_{i=1}^{N} Q_i = 0, \tag{B1b}$$

$$\hat{\mathcal{E}} + \frac{3}{2}\phi\mathcal{E} = \frac{1}{3}\sum_{i=1}^{N}\hat{\mu}_i - \frac{1}{2}\sum_{i=1}^{N}\hat{\Pi}_i - \frac{3}{2}\phi\frac{1}{2}\sum_{i=1}^{N}\Pi_i, \tag{B1c}$$

$$\mathcal{E} + \mathcal{A}\phi = \frac{1}{3}\sum_{i=1}^{N}(\mu_i + 3p_i) + \frac{1}{2}\sum_{i=1}^{N}\Pi_i, \tag{B1d}$$

$$K = \frac{1}{4}\phi^2 - \mathcal{E} + \frac{1}{3}\sum_{i=1}^{N}\mu_i - \frac{1}{2}\sum_{i=1}^{N}\Pi_i, \tag{B1e}$$

$$\hat{\mathcal{A}} = -\mathcal{A}(\mathcal{A} + \phi) + \frac{1}{2}\sum_{i=1}^{N}(\mu_i + 3p_i), \tag{B1f}$$

$$\hat{Q}_i = -Q_i(\phi + 2\mathcal{A}) + \sum_{k \neq i}^{N} j_u^{(i,k)}, \tag{B1g}$$

$$\hat{p}_i + \hat{\Pi}_i = \sum_{k \neq i}^{N} j_e^{(i,k)} - \Pi_i\left(\frac{3}{2}\phi + \mathcal{A}\right) - \mathcal{A}(\mu_i + p_i), \tag{B1h}$$

where the index i represents the ith component, and $j_e^{(i,k)}$ is the interaction term between the component i and k. Notice that $j_u^{(i,j)} = -j_u^{(j,i)}$ and $j_e^{(i,j)} = -j_e^{(j,i)}$.

By summing the (B1h) over i we have the equation for the total energy pressure:

$$\hat{p}_{\text{tot}} + \hat{\Pi}_{\text{tot}} = -\sum_{i=1}^{N}\Pi_1\left(\frac{3}{2}\phi + \mathcal{A}\right) - \mathcal{A}\sum_{i=1}^{N}(\mu_i + p_i). \tag{B2}$$

In terms of the variable ρ defined by the relation $K = K_0^{-1}e^{-\rho}$, the $1 + 1 + 2$ equations can be written as

$$\phi\phi_{,\rho} = -\frac{1}{2}\phi^2 - \frac{2}{3}\sum_{i=1}^{N}\mu_i - \mathcal{E} - \frac{1}{2}\sum_{i=1}^{N}\Pi_i, \tag{B3a}$$

$$\sum_{i=1}^{N} Q_i = 0, \tag{B3b}$$

$$\mathcal{E}_{,\rho} + \frac{3}{2}\mathcal{E} = \frac{1}{3}\sum_{i=1}^{N}\mu_{i,\rho} - \frac{1}{2}\sum_{i=1}^{N}\Pi_{i,\rho} - \frac{3}{2}\phi\frac{1}{2}\sum_{i=1}^{N}\Pi_i, \tag{B3c}$$

$$\mathcal{E} + \mathcal{A}\phi = \frac{1}{3}\sum_{i=1}^{N}(\mu_i + 3p_i) + \frac{1}{2}\sum_{i=1}^{N}\Pi_i, \tag{B3d}$$

$$K = \frac{1}{4}\phi^2 - \mathcal{E} + \frac{1}{3}\sum_{i=1}^{N}\mu_i - \frac{1}{2}\sum_{i=1}^{N}\Pi_i, \tag{B3e}$$

$$\phi\mathcal{A}_{,\rho} = -\mathcal{A}(\mathcal{A} + \phi) + \frac{1}{2}\sum_{i=1}^{N}(\mu_i + 3p_i), \tag{B3f}$$

$$\phi\hat{Q}_{i,\rho} = -Q_i(\phi + 2\mathcal{A}) + \sum_{k\neq i}^{N} j_u^{(i,k)}, \tag{B3g}$$

$$\hat{p}_i + \hat{\Pi}_i = \sum_{k\neq i}^{N} j_e^{(i,k)} - \Pi_i\left(\frac{3}{2}\phi + \mathcal{A}\right) - \mathcal{A}(\mu_i + p_i). \tag{B3h}$$

Using the variables,

$$X = \frac{\phi_{,\rho}}{\phi}, \qquad Y = \frac{\mathcal{A}}{\phi}, \qquad \mathcal{K} = \frac{K}{\phi^2},$$

$$E = \frac{\mathcal{E}}{\phi^2}, \qquad \mathbb{M}_1 = \frac{\mu_1}{\phi^2}, \qquad \mathbb{M}_2 = \frac{\mu_2}{\phi^2},$$

$$P_1 = \frac{p_1}{\phi^2}, \qquad P_2 = \frac{p_2}{\phi^2}, \qquad \mathbb{P}_1 = \frac{\Pi_1}{\phi^2},$$

$$\mathbb{P}_2 = \frac{\Pi_2}{\phi^2}, \qquad \mathbb{Q}_1 = \frac{Q_1}{\phi^2}, \qquad \mathbb{Q}_2 = \frac{Q_2}{\phi^2}, \tag{B4}$$

we obtain the covariant TOV equations as

$$P_{i,\rho} + \mathbb{P}_{i,\rho} = \sum_{i\neq k}^{N} \mathbb{J}_e^{(i,k)} - P_i{}^2 - \mathbb{P}_i{}^2$$

$$+ P_i\left[\mathbb{M}_i - 2\mathbb{P}_i - 3\mathcal{K} + \frac{7}{4}\right]$$

$$+ \mathbb{P}_i\left(\mathbb{M}_i - 3\mathcal{K} + \frac{1}{4}\right) + \mathbb{M}_i\left(\frac{1}{4} - \mathcal{K}\right)$$

$$- P_i\sum_{k\neq i}^{N}(P_k + \mathbb{P}_k - 2\mathbb{M}_k)$$

$$- \mathbb{P}_i\sum_{k\neq i}^{N}(P_k + \mathbb{P}_k - 2\mathbb{M}_k)$$

$$- \mathbb{M}_i\sum_{k\neq i}^{N}(P_k + \mathbb{P}_k), \tag{B5}$$

$$\mathcal{K}_{,\rho} = 2\mathcal{K}\left(\frac{1}{4} - \mathcal{K} + \sum_{k=1}^{N}\mathbb{M}_k\right), \tag{B6}$$

$$\mathbb{Q}_{i,\rho} = \mathbb{Q}_i\left[2\mathcal{K} - 2\sum_{k=1}^{N}\mathbb{M}_k - \frac{3}{2}\right] - \sum_{i\neq k}^{N}\mathbb{J}_u^{(i,k)}, \tag{B7}$$

where we have set

$$\mathbb{J}_u^{(i,k)} = \frac{j_u^{(i,k)}}{\phi^3}, \qquad \mathbb{J}_e^{(i,k)} = \frac{j_e^{(i,k)}}{\phi^3}. \tag{B8}$$

In the case of two fluids the equations above reduce to

$$\hat{\phi} = -\frac{1}{2}\phi^2 - \frac{2}{3}(\mu_1 + \mu_2) - \mathcal{E} - \frac{1}{2}(\Pi_1 + \Pi_2), \tag{B9a}$$

$$\hat{\mathcal{E}} = \frac{1}{3}(\hat{\mu}_1 - \hat{\mu}_2) - \frac{1}{2}(\hat{\Pi}_1 + \hat{\Pi}_2)$$
$$- \frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi_1 + \frac{1}{2}\Pi_2\right), \tag{B9b}$$

$$-\mathcal{A}\phi + \frac{1}{3}(\mu_1 + 3p_1) + \frac{1}{3}(\mu_2 + 3p_2)$$
$$-\mathcal{E} + \frac{1}{2}\Pi_1 + \frac{1}{2}\Pi_2 = 0, \tag{B9c}$$

$$\hat{\mathcal{A}} = -\mathcal{A}(\mathcal{A} + \phi) + \frac{1}{2}(\mu_1 + 3p_1) + \frac{1}{2}(\mu_2 + 3p_2), \tag{B9d}$$

$$\hat{p}_1 + \hat{\Pi}_1 = -\Pi_1\left(\frac{3}{2}\phi + \mathcal{A}\right) - \mathcal{A}(\mu_1 + p_1) + j_e^{(1,2)}, \tag{B9e}$$

$$\hat{p}_2 + \hat{\Pi}_2 = -\Pi_2\left(\frac{3}{2}\phi + \mathcal{A}\right) - \mathcal{A}(\mu_2 + p_2) - j_e^{(2,1)}, \tag{B9f}$$

$$K = \frac{1}{3}(\mu_1 + \mu_2) - \mathcal{E} - \frac{1}{2}(\Pi_1 + \Pi_2) + \frac{1}{4}\phi^2, \tag{B9g}$$

$$\hat{Q}_1 = -Q_1(\phi + 2\mathcal{A}) + j_u^{(1,2)}, \tag{B9h}$$

$$Q_2 = -Q_1, \tag{B9i}$$

$$j_u^{(1,2)} = -j_u^{(2,1)} \tag{B9j}$$

$$j_e^{(1,2)} = -j_e^{(2,1)}, \tag{B9k}$$

which are given in (6) with $\Pi = 0$, $j_u^{(1,2)} = 0$, and $j_e^{(1,2)} = 0$. In terms of ρ,

$$\phi\phi_{,\rho} = -\frac{1}{2}\phi^2 - \frac{2}{3}(\mu_1 + \mu_2) - \mathcal{E} - \frac{1}{2}(\Pi_1 + \Pi_2), \tag{B10a}$$

$$\mathcal{E}_{,\rho} = \frac{1}{3}(\mu_1 - \mu_2) - \frac{1}{2}(\Pi_1 + \Pi_2) - \frac{3}{2}\left(\mathcal{E} + \frac{1}{2}\Pi_1 + \frac{1}{2}\Pi_2\right), \tag{B10b}$$

$$-\mathcal{A}\phi + \frac{1}{3}(\mu_1 + 3p_1) + \frac{1}{3}(\mu_2 + 3p_2)$$
$$-\mathcal{E} + \frac{1}{2}\Pi_1 + \frac{1}{2}\Pi_2 = 0, \tag{B10c}$$

$$\phi\mathcal{A}_{,\rho} = -\mathcal{A}(\mathcal{A} + \phi) + \frac{1}{2}(\mu_1 + 3p_1)$$
$$+ \frac{1}{2}(\mu_2 + 3p_2), \tag{B10d}$$

$$\phi(p_{1,\rho} + \Pi_{1,\rho}) = -\Pi_1\left(\frac{3}{2}\phi + \mathcal{A}\right)$$
$$- \mathcal{A}(\mu_1 + p_1) + j_e^{(1,2)}, \tag{B10e}$$

$$\phi(p_{2,\rho} + \Pi_{2,\rho}) = -\Pi_2\left(\frac{3}{2}\phi + \mathcal{A}\right)$$
$$- \mathcal{A}(\mu_2 + p_2) - j_e^{(1,2)}, \tag{B10f}$$

$$K + \mathcal{E} - \frac{1}{4}\phi^2 = \frac{1}{3}(\mu_1 + \mu_2) - \frac{1}{2}(\Pi_1 + \Pi_2), \tag{B10g}$$

$$\phi Q_{1,\rho} = -Q_1(\phi + 2\mathcal{A}) + j_u^{(1,2)}, \tag{B10h}$$

$$Q_2 = -Q_1, \tag{B10i}$$

and the corresponding TOV equations, for $\Pi = 0$, are given in (15).

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