Entanglement production in Einstein-Cartan theory

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We study the entanglement production for Dirac and Klein-Gordon fields in an expanding spacetime characterized by the presence of torsion. Torsion is here considered according to the Einstein-Cartan theory with a conformally flat Friedmann-Robertson-Walker spacetime. In this framework, torsion is seen as an external field, fulfilling precise constraints gotten directly from the cosmological principle. For Dirac field, we find that torsion increases the amount of entanglement. This turns out to be particularly evident for small values of particle momentum. We discuss the role of Pauli exclusion principle in view of our results, and, in particular, we propose an interpretation of the two maxima that occur for the entanglement entropy in the presence of torsion. For Klein-Gordon field, and differently from the Dirac case, the model can be exactly solved in some cases. We discuss, in particular, conformal coupling to the scalar curvature and the special case of antisymmetric torsion. Again, we show how torsion affects the amount of entanglement, providing a robust physical motivation behind the increase or decrease of entanglement entropy. A direct comparison of our findings is also discussed in view of previous results derived in absence of torsion. To this end, we give prominence on how our expectations would change in terms of the coupling between torsion and the scale factor for both Dirac and Klein-Gordon fields.

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I. INTRODUCTION

Our current understanding of the Universe is undergoing a new revolutionary phase in which observations provide precise measurements that fix bounds on the cosmological parameters characterizing the standard cosmological model [1,2]. In this respect, the interplay between quantum world and gravitation is an ambitious challenge for theoretical physics as it sheds light on how early phases evolve when general relativity breaks down [3]. Applications to quantum gravity could open new windows on the properties of the initial singularity, inflation [4], and likely on the existence of both dark energy [5] and matter [6]. It is therefore of interest to explore different scenarios, choosing them through helpful guiding principles that make use of a minimal number of assumptions and ingredients. Typically, these scenarios lie on postulating the cosmological principle, *i.e.,* assuming the Universe to be homogeneous and isotropic [7]. In this framework, it is interesting to consider the Einstein-Cartan (EC) theory [8], in which the presence of torsion represents the simplest modification of Einstein's gravity [9,10]. More precisely, the torsion tensor is assumed not to vanish as in general relativity, enabling one to match its existence to particle spin. Accordingly, spin plays a dynamical role [11–13], since it couples to the torsion field. This gives rise to interacting terms that act on the overall dynamics of quantum fields. According to these considerations, it is natural to work on particle production and on its applications to quantum cosmology when EC theory is accounted.

Indeed, an intriguing topic that is currently an object of speculation in cosmology is represented by entanglement production in asymptotic phases [14]. Entanglement is a fundamental property of quantum systems implying the existence of global states of composite systems that cannot be written as a product of the states of individual subsystems [15]. It recently started to be a resource in quantum information theory, with several applications that span from quantum communication [16], quantum cryptography [17], quantum teleportation [18] up to quantum computation [19], and, more recently, to its characterization in relativistic frameworks [20,21], such as in curved spacetime [22–25].

Spacetime curvature has nontrivial effects on quantum fields living on the spacetime when compared with their flat-spacetime counterparts [26]. This is especially interesting in the case of dynamical spacetime backgrounds because the gravitational interaction may induce quantum

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correlations in the field state in scenarios such as expanding universes [22–25,27,28]. This is related to the long-known phenomenon of particle-antiparticle production from vacuum [29]. It was shown that Dirac and Klein-Gordon (KG) field have different momentum distribution of entanglement [27]. Within the Dirac field, no qualitative difference appears in the dependence of entanglement from the number of created particles at fixed momentum in going from $1+1$ spacetime to $3+1$ spacetime, hence, including spin [30,31]. However, all these studies were confined to torsionless spacetimes. The inclusion of torsion can shed further light on the differences between the entanglement of bosonic and fermionic fields, and, in particular, concerning the role that spin plays in its generation.

In this paper, we face the problem of investigating entanglement production for bosonic and fermionic particles in an expanding spacetime with the presence of nonzero torsion. In particular, we discuss entanglement for Dirac and KG fields within the EC theory. Thus, assuming the cosmological principle to hold, we adopt the Friedmann-Friedmann-Robertson-Walker (FRW) spacetime, fulfilling constraints for the torsion provided by recent observations [32], and we notice the effect of the torsion appears in the dependence of the particle density from momentum. Thus, invoking a generic torsion source, reinterpreted as an external geometrical source, we describe how Dirac fields are minimally coupled to torsion and how a nonminimal coupling of torsion to KG field is plausible, both introducing significant curvature effects. Afterward, we show how the presence of torsion affects entanglement, in both the cases. In particular, we show how to get from the Dirac equation physical solutions in the presence of torsion in particular spacetime regions. These solutions are not analytical as well as the corresponding entanglement entropy. However, by assuming small corrections due to torsion, we get approximate classes of solutions that resemble previous results developed in the literature, where torsion was not taken into account. As a consequence, we underline how torsion deviates the standard expectations and under which conditions torsion can increase or decrease particle and entanglement productions. In particular, we notice an increase of particles as the momentum p goes to zero in the Dirac case so that torsion could be interpreted as source for dark matter production. Thus, we propose that dark matter particles are under the form of *torsion particles*. The opposite happens for KG field. There, although the torsion effect is modeled in a more complicated way, *i.e.*, adopting two sources instead than one as for Dirac, exact solutions can be argued. We also analyze under which conditions torsion can be described using only one external function, thus providing a similar approach with respect to the Dirac case. According to our findings, we show under which properties torsion can increase the amount of entanglement and how much it is mode dependent. Consequences in cosmology and imprints on observations are discussed. In particular, we interpret our findings in view of the Pauli exclusion principle, explaining the presence of a relative maximum for the Dirac field.

The paper is structured as follows. In Sec. II, basic notions of EC theory are reported, giving emphasis on how to fuel torsion by means of the most generic approach. Thus, in Sec. III, we discuss how to relate EC gravity to Dirac and how torsion modifies the entanglement production. The same is faced for KG field in Sec. IV. A comparison of both the frameworks is extensively discussed throughout Sec. V. In the same section, we also give a physical interpretation of our results, and we stress how to relate our torsion fields to Pauli exclusion principle. Finally, in Sec. VI, we discuss conclusions and perspectives of our work.

II. THE EC THEORY

The EC theory can be introduced starting from the action,

$$\mathcal{L}_{\mathrm{EC}} = -\frac{1}{2\kappa c}\int R(\Gamma)\sqrt{-g}d^4x + \int \mathcal{L}_m\sqrt{-g}d^4x, \quad (1)$$

where $\kappa \equiv 8\pi G$, and g is the determinant of the spacetime metric tensor $g_{\mu\nu}$. The Lagrangian $\mathcal{L}_m$ represents a generic matter contribution. This action is defined in a spacetime with curvature and torsion, usually called Riemann-Cartan (RC) spacetime. The curvature scalar $R(\Gamma) \coloneqq g^{\mu\nu}R_{\mu\nu}$ is constructed out of the Ricci-Cartan tensor $R_{\mu\nu}(\Gamma) \equiv R^{\alpha}_{\mu\alpha\nu}(\Gamma)$, while the torsion tensor $T^{\alpha}{}_{\mu\nu}$ is defined as the antisymmetric part of the affine connection,

$$T^{\alpha}{}_{\mu\nu} \coloneqq \Gamma^{\alpha}{}_{[\mu\nu]} = \frac{1}{2}(\Gamma^{\alpha}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\mu}). \quad (2)$$

Accordingly, the affine connection can be written as the sum of two contributions [33]:

$$\Gamma^{\alpha}{}_{\mu\nu} = \tilde{\Gamma}^{\alpha}{}_{\mu\nu} + K^{\alpha}{}_{\mu\nu}, \quad (3)$$

where $\tilde{\Gamma}^{\alpha}{}_{\mu\nu}$ is the usual Levi-Civita spin connection of general relativity, and $K^{\alpha}{}_{\mu\nu}$ is the contorsion tensor, related to torsion via the formula,

$$K_{\alpha\mu\nu} \coloneqq T_{\alpha\mu\nu} + 2T_{(\mu\nu)\alpha}. \quad (4)$$

In the EC theory, we deal with a set of two field equations: The *first Einstein-Cartan equation* relates the curvature of spacetime to the energy-momentum density of matter, described by the tensor $T_{\mu\nu}$. This equation maintains the same form of standard general relativity, *i.e.*, $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu}$, but without having the *a priori* symmetry of both the Ricci-Cartan and energy-momentum tensors.

The *second Einstein-Cartan equation* couples the spacetime torsion to the matter spin. It can be written as

$$T^{\alpha}{}_{\mu\nu} - T_{\mu}\delta^{\alpha}_{\nu} + T_{\nu}\delta^{\alpha}_{\mu} = -\frac{\kappa}{2}s_{\mu\nu}{}^{\alpha}, \tag{5}$$

where

$$s_{\alpha}{}^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}_m}{\delta K^{\alpha}{}_{\mu\nu}} \tag{6}$$

is the spin tensor of matter.

Now, we want to specify to the case of a spatially homogeneous and isotropic spacetime, described by the conformal FRW line element,

$$ds^2 = a^2(\tau)(-d\tau^2 + dx^2 + dy^2 + dz^2). \tag{7}$$

Here, $a(\tau)$ is the scale factor, determining the spacetime expansion rate, while τ is the conformal time, related to the cosmological time t by $\tau = \int a^{-1}(t)dt$. Given the high symmetry of such a spacetime, the torsion tensor has to satisfy certain constraints. We follow the ansatz of [32] and assume that the only nonzero components of the torsion tensor are[1]

$$T_{\alpha\mu\nu} = f(\tau)\epsilon_{\alpha\mu\nu}, \qquad T^{\alpha}{}_{\mu 0} = h(\tau)\delta^{\alpha}_{\mu}. \tag{8}$$

In this definition, as in the following one, the Greek indices must be all different from zero, *i.e.* $\alpha, \mu, \nu = 1, 2, 3$. Here, $f(\tau)$ and $h(\tau)$ are arbitrary functions of the conformal time, while $\epsilon_{\alpha\mu\nu}$ and δ^{α}_{μ} are the three-dimensional Levi-Civita and Kronecker symbols, respectively. Using the definition (4), from (8), we obtain

$$K_{\alpha\mu\nu} = f(\tau)\epsilon_{\alpha\mu\nu}, \qquad K_{0\mu\nu} = -K_{\mu 0\nu} = 2h(\tau)g_{\mu\nu}. \tag{9}$$

This ansatz is valid for any gravity theory in a RC spacetime if one applies the cosmological principle to the torsion tensor. In doing this, we drop any assumptions about the source of torsion.

In the next sections, we describe the coupling of Dirac and KG field to torsion and discuss entanglement in both cases.

III. DIRAC EQUATION IN PRESENCE OF TORSION

The Dirac Lagrangian in a RC spacetime can be written as

$$\mathcal{L}_D = -\frac{1}{2}[\bar{\psi}\tilde{\gamma}^{\mu}D_{\mu}\psi - (D_{\mu}\bar{\psi})\tilde{\gamma}^{\mu}\psi] - m\bar{\psi}\psi, \tag{10}$$

where the covariant derivatives of spinors ψ and their complex conjugates $\bar{\psi}$ are defined as [32,33]

$$D_{\mu}\psi = \tilde{D}_{\mu}\psi - \frac{1}{4}K_{\alpha\beta\mu}\tilde{\gamma}^{\alpha}\tilde{\gamma}^{\beta}\psi, \tag{11}$$

$$D_{\mu}\bar{\psi} = \tilde{D}_{\mu}\bar{\psi} + \frac{1}{4}K_{\alpha\beta\mu}\bar{\psi}\tilde{\gamma}^{\alpha}\tilde{\gamma}^{\beta}. \tag{12}$$

Here, $K_{\alpha\beta\mu}$ is the contorsion tensor in the fully covariant form, and $\tilde{D}_{\mu}$ is the covariant derivative of a spinor in a torsionless spacetime. Choosing the FRW metric from (7) and introducing the tetrad field,[2]

$$e^{\mu}_{i} = \frac{1}{a(\tau)}\delta^{\mu}_{i}, \tag{13}$$

we have that [36]

$$\tilde{D}_{\mu}\psi = \left(\partial_{\mu} + \frac{1}{4}\frac{\dot{a}}{a}[\gamma_{\mu}, \gamma^{0}]\right)\psi, \tag{14}$$

where the dependence of the scale factor a on τ is understood from now on. Here, γ^{μ} are the flat gamma matrices, chosen according to the notation of [37]. The curved gamma matrices are defined as $\tilde{\gamma}^{\mu} \coloneqq e^{\mu}_{i}\gamma^{i} = a^{-1}\gamma^{i}$. As we have said, in a FRW spacetime, the torsion tensor assumes the general form (8). Now, recalling that for the Dirac field, torsion is equivalent to an axial vector, (see, e.g., Ref. [38] for additional details), we immediately argue that only $f(\tau)$ could describe the torsion itself in such a case. In fact, the torsion tensor can be split, in general, into four terms [34], and in the Dirac scenario, the unique remaining term is represented by the antisymmetric contribution. This part depends only on $f(\tau)$ in our case, as we discuss in detail in Sec. IV. From now on, we thus consider $f(\tau)$ only. Analogous results will be prompted for the KG field.

Doing explicitly the calculations, from the Lagrangian (10), we obtain the Dirac equation in a FRW spacetime with torsion [32],

$$\left[\frac{\gamma^{\mu}}{a}\left(\partial_{\mu} + \frac{1}{4}\frac{\dot{a}}{a}[\gamma_{\mu}, \gamma^{0}]\right) + m\right]\psi = -\frac{3i}{2}f(\tau)a\gamma^{0}\gamma^{5}\psi. \tag{15}$$

Using the ansatz [31],

$$\psi = a^{-3/2}(\gamma^{\nu}\partial_{\nu} - M)\varphi, \tag{16}$$

with $M = ma$, we obtain,

$$(\eta^{\mu\nu}\partial_{\mu}\partial_{\nu} - \gamma^{0}\dot{M} - M^2)\varphi = -\frac{3i}{2}F(\tau)\gamma^{0}\gamma^{5}(\gamma^{\nu}\partial_{\nu} - M)\varphi, \tag{17}$$

[1] We are assuming that torsion is invariant under conformal transformations. For an introduction to conformal properties of torsion, see, for example, [34].

[2] A tetrad is needed when dealing with spinors in a curved spacetime. See, for example, [35].

where $F(\tau) = f(\tau)a^2$, and $\eta^{\mu\nu}$ is the Minkowski metric tensor. Equation (17) is a differential equation for φ. Further, it depends on the functions f and a, respectively, the torsion and scale factor upon specification. In other words, we first have to fix them and then compute the solutions for φ.

We assume an asympotically flat spacetime, with a scale factor of the form,

$$a(\tau) = A + B \tanh(\rho\tau), \tag{18}$$

widely used for the properties of controlling both the volume and the expansion of the Universe [37]. In fact, here, the parameters A and B are related to the volume of the Universe, and ρ describes the rapidity of expansion.

We focus now on the *in* and *out* regions, *i.e.,* the asymptotic regions where we need to compute particle production and, consequently, its modification as due to the torsion presence. In Fig. 1, we portray the torsion field that here we consider with different values of the involved parameters.

In the asymptotic regions, Eq. (17) can be solved with the ansatz,

$$\varphi_{\mathrm{in/out}} = N_{\mathrm{in/out}} e^{-iE_{\mathrm{in/out}}\tau} e^{i\mathbf{p}\cdot\mathbf{x}} \begin{pmatrix} u_d \\ v_d \end{pmatrix}, \tag{19}$$

where $N_{\mathrm{in/out}}$ is a normalization factor, and u_d, v_d $(d = \uparrow, \downarrow)$ are two-component spinors, so that

$$\gamma^0 u_d = -iu_d, \qquad \gamma^0 v_d = iv_d. \tag{20}$$

Inserting (19) into (17), we obtain the equation,

$$\begin{aligned}(E^2_{\mathrm{in/out}} - |\mathbf{p}|^2 - M^2_{\mathrm{in/out}}) \begin{pmatrix} u_d \\ v_d \end{pmatrix} \\ = \frac{3}{2} F_{\mathrm{in/out}} \begin{pmatrix} \boldsymbol{\sigma}\cdot\mathbf{p} & -E_{\mathrm{in/out}} + M_{\mathrm{in/out}} \\ -E_{\mathrm{in/out}} - M_{\mathrm{in/out}} & \boldsymbol{\sigma}\cdot\mathbf{p} \end{pmatrix} \\ \times \begin{pmatrix} u_d \\ v_d \end{pmatrix},\end{aligned} \tag{21}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the set of Pauli matrices, and

$$M_{\mathrm{in/out}} = ma(\tau \to -/+\infty), \tag{22}$$

$$F_{\mathrm{in/out}} = f(\tau \to -/+\infty) a^2(\tau \to -/+\infty). \tag{23}$$

From Eq. (21), we can derive the spinor solution with positive energy, which is found to be

$$w_d = \begin{pmatrix} u_d \\ \frac{[(E^2_{\mathrm{in/out}} - |\mathbf{p}|^2 - M^2_{\mathrm{in/out}}) - \frac{3}{2}F_{\mathrm{in/out}}(\boldsymbol{\sigma}\cdot\mathbf{p})]}{\frac{3}{2}F_{\mathrm{in/out}}(-E_{\mathrm{in/out}} + M_{\mathrm{in/out}})} u_d \end{pmatrix}, \tag{24}$$

and similarly, for the solution with negative energy, with the substitution $p^\mu \to -p^\mu$. Accordingly, the complete positive-energy solution of the Dirac equation (17) in the asymptotic regions can be written as

$$U_{\mathrm{in/out}}(\mathbf{x}, \mathbf{p}, d, \tau) = N_{\mathrm{in/out}} (\gamma^\nu \partial_\nu - M) e^{-iE_{\mathrm{in/out}}\tau} e^{i\mathbf{p}\cdot\mathbf{x}} w_d. \tag{25}$$

Imposing the normalization as in [30], namely $\bar{U}U = iU^\dagger\gamma^0 U = \delta_{d,d'}$, one finds

$$N = \frac{\frac{3}{2}F(E-M)}{(E^2 - M - |\mathbf{p}|^2)\sqrt{(\frac{3}{2}F)^2 + 3F|\mathbf{p}| + |\mathbf{p}|^2 - (E-M)^2}}, \tag{26}$$

where the subscripts in/out have been omitted for brevity.

The only missing element is now the energy correction due to the presence of torsion. We write the total energy as

$$E_{\mathrm{in/out}} = E_0 + x = \sqrt{|\mathbf{p}|^2 + M_{\mathrm{in/out}}} + x, \tag{27}$$

where E_0 is the energy when torsion is not present, and x is the correction due to the torsion contribution. Inserting (27) into (21) and computing the determinant of the corresponding matrix, one finds

$$E^\pm = \sqrt{|\mathbf{p}|^2 + M^2 + \frac{(\frac{3}{2}F)^2 \pm \sqrt{(\frac{3}{2}F)^4 + 8|\mathbf{p}^2|(\frac{3}{2}F)^2}}{2}}, \tag{28}$$

where, again, we have omitted the subscript in/out to simplify the notation. We assume the corrections due to torsion to be small so that the assumption of asymptotic flatness can be preserved. Accordingly, the expression (28) can be simplified to

$$E^\pm = E_0 + \frac{(\frac{3}{2}F)^2 \pm \sqrt{(\frac{3}{2}F)^4 + 8|\mathbf{p}^2|(\frac{3}{2}F)^2}}{4E_0}. \tag{29}$$

Clearly, for antiparticles, the ansatz would be $E_{\mathrm{in/out}} = -E_0 + x$, and so, one finds the opposite of Eq. (29). We remark that if we assume $F_{\mathrm{in/out}} > 0$, the solution E^+ should be excluded, in order to assure that (24) is a positive-energy spinor. Analogously, if we assume $F_{\mathbf{in/out}} < 0$, we should exclude E^- for the same reason.

A. Particle creation and entanglement

To study entanglement for Dirac field in a FRW spacetime with torsion, we should be able to compute the Bogolyubov coefficients that relates the in and out regions [30,31,37]. However, this cannot be done analytically, since the Dirac equation (15) can be solved only in the two asymptotic regions separately.

In order to also simplify the numerical approach, we can imagine that the torsion field becomes negligible during the expansion of the Universe. This assumption can be justified if we recall that torsion shows up at extremely high mass densities, and so, it can play a crucial role only at Planck time scale [39,40]. Accordingly, a suitable form for the torsion function in the asymptotic regions might be

$$F(\tau) = F_0 a(\tau)^{-n}, \qquad n \in \mathbb{N}, \tag{30}$$

and so,

$$f(\tau) = f_0 a(\tau)^{-k}, \qquad k \in \mathbb{N}, \quad k \geq 3. \tag{31}$$

The constant f_0 should assume values much smaller than the mass m (natural units), in order not to deviate significantly from the hypothesis of flatness. Moreover, k should be large enough to assure that torsion quickly falls to zero when the Universe starts its expansion. Of course, the faster the decay is, the better our approximation of zero torsion during expansion works. We also assume charge and angular momentum conservation, as in [31].

With these assumptions, the Bogolyubov transformations that relate the *in* and *out* creation and destruction operators can be written as [31]

$$\begin{aligned} a_{\text{in}}(\mathbf{p},d) &= \mathcal{A}^*(p) a_{\text{out}}(\mathbf{p},d) + \beta^*_{d,-d}(\mathbf{p}) b^\dagger_{\text{out}}(\mathbf{p},-d) \\ b^\dagger_{\text{in}}(-\mathbf{p},d) &= -\beta_{-d,d}(\mathbf{p}) a_{\text{out}}(\mathbf{p},-d) + \mathcal{A}(p) b^\dagger_{\text{out}}(-\mathbf{p},d), \end{aligned} \tag{32}$$

where $p \equiv |p|$. Here, a_{in}, b_{in} and a_{out}, b_{out} are the annihilation operators of particles and antiparticles in the *in* and *out* regions, respectively. The coefficient $\mathcal{A}(p)$ becomes [37]

$$\mathcal{A}(p) = \sqrt{\frac{M_{\text{in}}}{M_{\text{out}}}\frac{E_{\text{in}}}{E_{\text{out}}}\frac{N_{\text{in}}}{N_{\text{out}}}} A(p), \tag{33}$$

and it can be considered real, without loss of generality [31]. Moreover, from the algebra of fermionic operators, it turns out that $|\mathcal{A}(p)|^2 + |\beta_{d,-d}(\mathbf{p})|^2 = 1$.

If the torsion term is negligible during the expansion, the coefficient $A(p)$ can be determined resorting to Hypergeometric functions [37,41]. One thus gets

$$A(p) = \frac{\Gamma(1 - (i/\rho)E_{\text{in}})\Gamma(-(i/\rho)E_{\text{out}})}{\Gamma(1 - (i/\rho)E_+ - imB/\rho)\Gamma(-(i/\rho)E_+ + imB/\rho)}, \tag{34}$$

where $\Gamma(x)$ is the usual gamma function, and

$$E_\pm \equiv \frac{1}{2}(E_{\text{out}} \pm E_{\text{in}}). \tag{35}$$

Inverting Eq. (32), we can compute the number n of particles per mode, created due to the Universe expansion [31],

$$n^p(p,\uparrow) = \langle 0_{\text{in}} | a^\dagger_{\text{out}}(\mathbf{p},\uparrow) a_{\text{out}}(\mathbf{p},\uparrow) | 0_{\text{in}} \rangle = |\beta_{\downarrow\uparrow}|^2, \tag{36}$$

$$n^p(p,\downarrow) = \langle 0_{\text{in}} | a^\dagger_{\text{out}}(\mathbf{p},\downarrow) a_{\text{out}}(\mathbf{p},\downarrow) | 0_{\text{in}} \rangle = |\beta_{\uparrow\downarrow}|^2, \tag{37}$$

and analogously for antiparticles. The unitary operator acting on the Fock space and representing the transformation (32) has been derived in [31] and so, applying it to the *out* vacuum state, we get

$$\begin{aligned} |0_p; 0_{-p}\rangle_{\text{in}} = \mathcal{A}^2 \bigg(& |0_p; 0_{-p}\rangle_{\text{out}} - \frac{\beta^*_{\uparrow\downarrow}}{\mathcal{A}} |\uparrow_p; \downarrow_{-p}\rangle_{\text{out}} \\ & - \frac{\beta^*_{\downarrow\uparrow}}{\mathcal{A}} |\downarrow_p; \uparrow_{-p}\rangle_{\text{out}} + \frac{\beta^*_{\uparrow\downarrow}\beta^*_{\downarrow\uparrow}}{\mathcal{A}^2} |\uparrow\downarrow_p; \uparrow\downarrow_{-p}\rangle_{\text{out}} \bigg). \end{aligned} \tag{38}$$

The particle-antiparticle density operator corresponding to Eq. (38) in the *out* region will be

$$\rho^{(\text{out})}_{p,-p} = |0_p; 0_{-p}\rangle_{\text{in}} \langle 0_p; 0_{-p}|, \tag{39}$$

and taking the partial trace over antiparticles, we obtain the reduced density operator,

$$\begin{aligned} \rho^{(\text{out})}_p = \text{Tr}_{-p}(\rho^{(\text{out})}_{p,-p}) &= \mathcal{A}^4 |0_p\rangle\langle 0_p| + \mathcal{A}^2 |\beta_{\uparrow\downarrow}|^2 |\uparrow_p\rangle\langle\uparrow_p| \\ &+ \mathcal{A}^2 |\beta_{\downarrow\uparrow}|^2 |\downarrow_p\rangle\langle\downarrow_p| + |\beta_{\uparrow\downarrow}|^2 |\beta_{\downarrow\uparrow}|^2 |\uparrow\downarrow_p\rangle\langle\uparrow\downarrow_p|. \end{aligned} \tag{40}$$

If we assume now that

$$n^p(p,\uparrow) = n^p(p,\downarrow) = n^a(p,\uparrow) = n^a(p,\downarrow) = \frac{n(p)}{4}, \tag{41}$$

we obtain that the coefficients in Eq. (40) solely depend on n, which is

$$\mathcal{A}^2 = \frac{4 - n(p)}{4}, \qquad |\beta_{\uparrow\downarrow}|^2 = |\beta_{\downarrow\uparrow}|^2 = \frac{n(p)}{4}. \tag{42}$$

To evaluate the amount of particle-antiparticle entanglement of Eq. (40), we can use the subsystem entropy [31], since the state (39) is pure. Accordingly, we can write

$$S(\rho^{(\text{out})}_p) = -2\left(\frac{4-n}{4}\right)\log_2\left(\frac{4-n}{4}\right) - 2\left(\frac{n}{4}\right)\log_2\left(\frac{n}{4}\right), \tag{43}$$

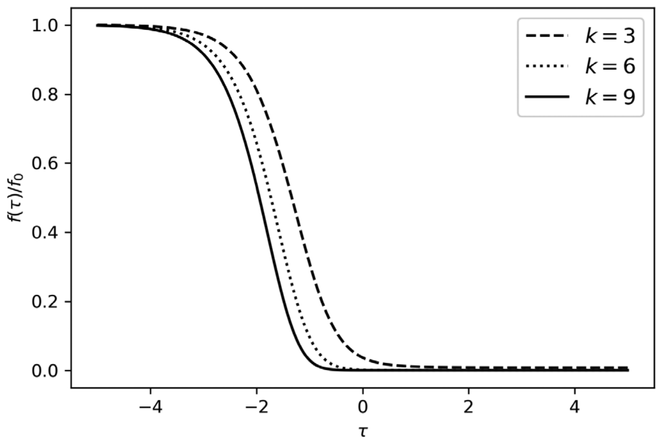

FIG. 1. Plot of the torsion function $f(\tau)/f_0$ for different values of the parameter k. We have assumed $A = 3$, $B = 2$, and $\rho = 1$ for the scale factor (18), in agreement with the subsequent simulations.

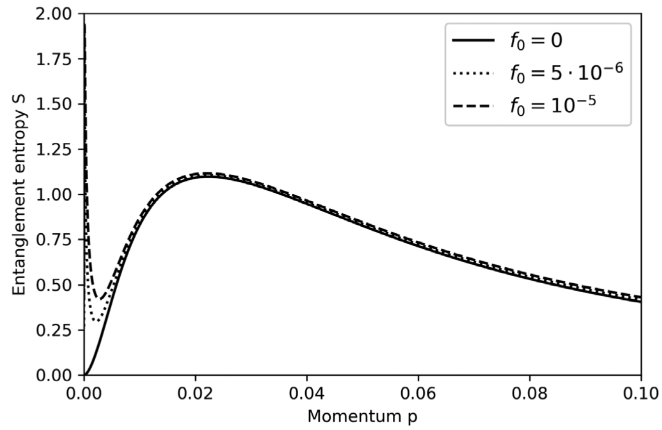

FIG. 2. Entanglement entropy for the Dirac field in the presence of torsion. The values of the parameters are $m = 0.01$, $A = 3$, $B = 2$, $\rho = 1$, and $k = 6$.

where the dependence of n on the momentum p is understood.

Plotting the entropy S as function of p, we see where torsion induces an increase of the entanglement amount for the Dirac field. This mostly occurs for small p, since at $p \to 0$, the particle contribution becomes negligible due to the Pauli exclusion principle, while torsion contribution does not. More precisely, adopting a homogeneous torsion as in Eq. (8), we do not involve the fermionic nature of torsion, albeit $f(\tau)$ guarantees the cosmological principle to hold.

There is no reason to heal the aforementioned effect *a priori*. Thus, one would need to interpret this anomalous excess of entanglement and to suggest a possible physical explanation. We conjecture the existence of exotic *torsion particles* that dominate over fermions at small momenta. We remark that, according to our assumptions, these torsion particles are mainly produced in the early stages of the Universe, since we have assumed that torsion does not play a role in the expansion of the Universe itself, and it is almost negligible in the *out* region. Our model is clearly an approximation, because torsion does not simply "switch off" when the Universe starts its expansion. A better approximation may be to assume that torsion exponentially decays when $\tau \neq \pm\infty$. However, the interaction of torsion with the Dirac field during expansion should be investigated with more refined numerical tools.

Coming back to our results, one can imagine this excess of entanglement, and thus of particles produced, to be associated with *dark matter*. The possibility that dark matter is composed by torsion particles can be checked *a posteriori*, assuming to quantify the torsion amount around $p \to 0$ and comparing this amount with cosmological expectations. See Appendix A for further details.

Alternatively, we could reconcile the behavior of entanglement entropy at $p \to 0$ with the Pauli exclusion principle by a suitable choice of the torsion function, which, however, implies to release the cosmological principle. The discussion of a fermionic torsion, however, is left for future works.

When the mass m increases, the corrections to S due to torsion are almost indistinguishable, as can be seen from Fig. 3. This happens because in this case, the energy corrections due to torsion becomes even smaller with respect to the energy without torsion E_0.

IV. KG EQUATION IN PRESENCE OF TORSION

In a curved spacetime, described by the metric $g_{\mu\nu}$, the minimal interaction of a scalar field with gravity is absent. The nonminimal interaction for a field of mass m has the general form [42,43],

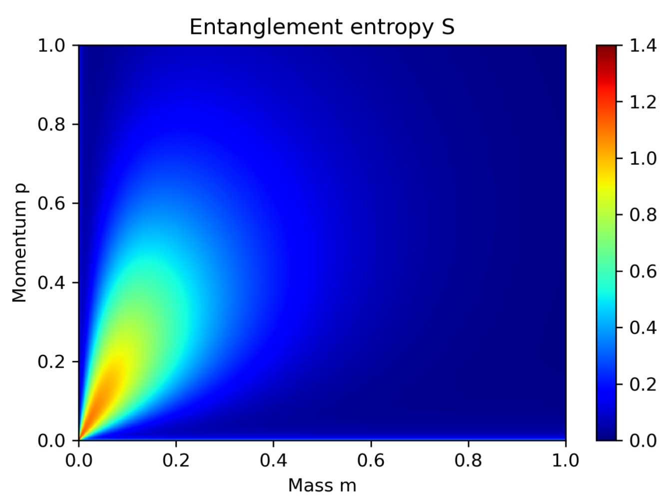

FIG. 3. Entanglement entropy when both the field parameters m and $|\mathbf{p}| = p$ are varied. The other parameters are $f_0 = 10^{-5}$, $A = 2$, $B = 3$, $\rho = 1$, and $k = 6$.

$$\mathcal{L}_{\rm KG} = -\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\left(m^2 + \sum_{i=1}^{5}\xi_i P_i\right)\phi^2. \quad (44)$$

Here, $P_1 = \tilde{R}$ (Riemannian scalar curvature), $P_2 = \nabla_\alpha T^\alpha$, $P_3 = T_\alpha T^\alpha$, $P_4 = S_\alpha S^\alpha$, where S_α is the axial vector $S^\alpha = \epsilon^{\beta\mu\nu\alpha} T_{\beta\mu\nu}$. Finally, $P_5 = q_{\alpha\beta\gamma} q^{\alpha\beta\gamma}$, where $q^\alpha{}_{\beta\gamma}$ is a tensor that satisfies the conditions,

$$q^\alpha{}_{\beta\alpha} = 0 \quad \text{and} \quad \epsilon^{\alpha\beta\mu\nu} q_{\alpha\beta\mu} = 0. \quad (45)$$

From Eq. (44), we see that there can be up to five nonminimal parameters $\xi_{1\ldots5}$. These are, in principle, arbitrary.

The Lagrangian (44) leads to the KG equation,

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g} g^{\mu\nu}\partial_\nu\phi) - \left(m^2 + \sum_{i=1}^{5}\xi_i P_i\right)\phi = 0. \quad (46)$$

To solve this equation, we start by assuming that the KG field is conformally coupled to the Riemannian curvature, *i.e.,* $\xi_1 = 1/6$. As we will see, this assumption allows us to obtain analytical solutions for particle production and entanglement.[3]

For what concerns the parameters $\xi_2, \ldots \xi_5$, we follow two different strategies. Let us first assume[4] that the scalar field couples in the same way to all the components of the Riemann-Cartan curvature scalar R. This means that we need just one parameter[5] ξ to describe the coupling.

Specializing to our conformally flat metric (7), we obtain then

$$\frac{1}{a^2}\Box\phi - \frac{2\dot{a}}{a^3}\dot{\phi} - (m^2 + \xi R)\phi = 0, \quad (47)$$

where $\Box$ is the usual D'Alembertian operator. If torsion is present, in the form described by Eq. (8), we can use Eq. (3) to obtain the scalar curvature in FRW spacetime with torsion,

$$R = 6\left[\frac{\ddot{a}}{a^3} - \frac{f^2(\tau)}{a^6} - \frac{2\dot{h}(\tau)}{a^2} - \frac{4\dot{a}}{a^3}h(\tau) + \frac{4h^2(\tau)}{a^2}\right]. \quad (48)$$

Inserting now the conformal coupling prescription ($\xi = 1/6$), Eq. (47) becomes

$$\frac{1}{a^2}\Box\phi - \frac{2\dot{a}}{a^3}\dot{\phi} - \left[m^2 + \frac{\ddot{a}}{a^3} - \frac{f^2(\tau)}{a^6} - \frac{2\dot{h}(\tau)}{a^2} - \frac{4\dot{a}}{a^3}h(\tau) + \frac{4h^2(\tau)}{a^2}\right]\phi = 0. \quad (49)$$

This equation may be further simplified by making the substitution $\phi \to \chi = a\phi$, to give

$$\Box\chi = \left[a^2 m^2 - \frac{f^2(\tau)}{a^4} - 2\dot{h}(\tau) - \frac{4\dot{a}}{a}h(\tau) + 4h^2(\tau)\right]\chi. \quad (50)$$

The general solution of Eq. (50) can be written in the form [37],

$$\chi_p(\mathbf{x}, \tau) = e^{i\mathbf{p}\cdot\mathbf{x}}\chi_p(\tau), \quad (51)$$

where $\chi_p(\tau)$ satisfies the following differential equation,

$$\ddot{\chi}_p(\tau) + \left[|\mathbf{p}|^2 + m^2 a^2 - \frac{f^2(\tau)}{a^4} - 2\dot{h}(\tau) - \frac{4\dot{a}}{a}h(\tau) + 4h^2(\tau)\right]\chi_p(\tau) = 0. \quad (52)$$

This equation can be solved exactly in some particular cases. If we assume that the scale factor is (18), as in the Dirac case, a solution can be found if we assume

$$f(\tau) = f_0 a^3(\tau) \qquad h(\tau) = h_0 a(\tau), \quad (53)$$

with f_0, h_0 constants. As in the Dirac case, these constants should be small in order to preserve the hypothesis of asymptotic flatness. This also means that the dynamics of the Universe can not be deeply influenced by an ansatz of this form, even if, in principle, one could frame out how the Friedmann equations are modified by (53). This kind of approach can be the subject of future investigations.

We are here interested in particle production and so, in the asymptotic solutions $\chi_p^{\rm in}$, $\chi_p^{\rm out}$, which can be written as

$$\chi_p^{\rm in}(\tau) = \exp\left\{-i\left[\mathcal{E}_+\tau + \frac{1}{\rho}\mathcal{E}_- \ln(2\cosh(\rho\tau))\right]\right\}{}_2F_1 \times \left(1 + \frac{i}{\rho}\mathcal{E}_- - \frac{6h_0 B}{\rho}, \frac{i}{\rho}\mathcal{E}_- + \frac{6h_0 B}{\rho}; 1 - \frac{i}{\rho}\mathcal{E}_{\rm in}; \frac{1+\tanh(\rho\tau)}{2}\right), \quad (54)$$

[3]In principle, one can choose other values for ξ_1, e.g., large values typical of the Higgs inflation. However, using the scale factors (18) or (74), it can be shown that the contribution of this coupling term is negligible in the asymptotic regions, and thus, it does not enhance entanglement. See Appendix B for some details.

[4]The other choice is explored in Sec. IV B.

[5]This does not simply mean that $\xi_2 = \ldots\xi_5 = \xi$. See [34] for the details on how the curvature scalar R can be written in terms of the components P_i.

$$\chi_p^{\rm out}(\tau) = \exp\left\{-i\left[\mathcal{E}_+\tau + \frac{1}{\rho}\mathcal{E}_- \ln(2\cosh(\rho\tau))\right]\right\}{}_2F_1 \times\left(1+\frac{i}{\rho}\mathcal{E}_- - \frac{6h_0B}{\rho}, \frac{i}{\rho}\mathcal{E}_- + \frac{6h_0B}{\rho}; 1+\frac{i}{\rho}\mathcal{E}_{\rm out}; \frac{1-\tanh(\rho\tau)}{2}\right), \tag{55}$$

where ${}_2F_1$ is the Hypergeometric function of second kind, and we have introduced

$$\mathcal{E}_{\rm in} = [|\mathbf{p}|^2 + (m^2 - f_0^2 + 4h_0^2)a^2(\tau\to-\infty)]^{1/2}, \tag{56}$$

$$\mathcal{E}_{\rm out} = [|\mathbf{p}|^2 + (m^2 - f_0^2 + 4h_0^2)a^2(\tau\to+\infty)]^{1/2}. \tag{57}$$

The quantities $\mathcal{E}_\pm$ are defined as in Eq. (35).

A. Particle creation and entanglement

It has already been shown that a dynamical spacetime generates entanglement between particle (p) and antiparticle ($-p$) modes of a KG field [44]. Here, we revisit the mechanism that leads to entanglement, assuming the presence of torsion.

Following the standard quantization procedure, we associate to each mode $\chi_p^{\rm in/out}(\mathbf{x},\tau)$ and to its complex conjugate $\chi_p^{\rm in/out*}(\mathbf{x},\tau)$ annihilation and creation operators $a_{\rm in/out}(\mathbf{p})$, $a^\dagger_{\rm in/out}(\mathbf{p})$. These operators satisfies equal-time commutation relations [44,45], and the two sets of modes define two representations of the scalar field [37],

$$\chi(\mathbf{x},\tau) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{[2\mathcal{E}_{\rm in}]^{1/2}} [\chi_p^{\rm in}(\mathbf{x},\tau)a_{\rm in}(\mathbf{p}) + \chi_p^{\rm in*}(\mathbf{x},\tau)a^\dagger_{\rm in}(\mathbf{p})] = \int\frac{d^3p}{(2\pi)^3}\frac{1}{[2\mathcal{E}_{\rm out}]^{1/2}} [\chi_p^{\rm out}(\mathbf{x},\tau)a_{\rm out}(\mathbf{p}) + \chi_p^{\rm out*}(\mathbf{x},\tau)a^\dagger_{\rm out}(\mathbf{p})]. \tag{58}$$

Expanding now one mode in terms of the other,

$$\chi_k^{\rm in}(\mathbf{x},\tau) = \alpha(p)\chi_p^{\rm out}(\mathbf{x},\tau) + \beta(p)\chi_{-p}^{\rm out*}(\mathbf{x},\tau), \tag{59}$$

and inserting this expression into Eq. (58), we obtain a map between in and out operators:

$$a_{\rm out}(\mathbf{p}) = \left(\frac{\mathcal{E}_{\rm out}}{\mathcal{E}_{\rm in}}\right)^{1/2}[\alpha(p)a_{\rm in}(\mathbf{p}) + \beta^*(p)a^\dagger_{\rm in}(-\mathbf{p})]. \tag{60}$$

The coefficients $\alpha(p)$, $\beta(p)$ are the Bogolyubov coefficients for this transformation. From the commutation relations for bosonic operators, we have [37]

$$|\alpha(p)|^2 - |\beta(p)|^2 = \frac{\mathcal{E}_{\rm in}}{\mathcal{E}_{\rm out}}. \tag{61}$$

Recalling the asymptotic solutions (54) and (55), the Bogolyubov coefficients $\alpha(p)$ and $\beta(p)$ follow from the linear transformation properties of Hypergeometric functions [37]. We have

$$\alpha(p) = \frac{\Gamma(1-(i/\rho)\mathcal{E}_{\rm in})\Gamma(-(i/\rho)\mathcal{E}_{\rm out})}{\Gamma(1-(i/\rho)\mathcal{E}_+ - 6hB/\rho)\Gamma(-(i/\rho)\mathcal{E}_+ + 6hB/\rho)},$$
$$\beta(p) = \frac{\Gamma(1-(i/\rho)\mathcal{E}_{\rm in})\Gamma((i/\rho)\mathcal{E}_{\rm out})}{\Gamma(1+(i/\rho)\mathcal{E}_- + 6hB/\rho)\Gamma((i/\rho)\mathcal{E}_- - 6hB/\rho)}. \tag{62}$$

Now, let us suppose that the KG field is in the vacuum state of the *in* modes, $|0\rangle_{\rm in}$, and we want to evaluate the expectation value of the particle number operator for the *out* modes. We simply have to insert Eq. (60) and its complex conjugate into the expression ${}_{\rm in}\langle 0|a^\dagger_{\rm out}(\mathbf{p}) a_{\rm out}(\mathbf{p})|0\rangle_{\rm in}$, finding

$${}_{\rm in}\langle 0|a^\dagger_{\rm out}(\mathbf{p})a_{\rm out}(\mathbf{p})|0\rangle_{\rm in} = |\beta(p)|^2. \tag{63}$$

Thus, the vacuum *in* state is not empty in the *out* region, and $|\beta(p)|^2$ is interpreted as the number of detected quanta in the mode p.

To discuss entanglement, we write the *in* vacuum as a Schmidt decomposition of *out* states,

$$|0_p;0_{-p}\rangle_{\rm in} = \sum_{n=0}^{\infty} c_n|n_p;n_{-p}\rangle_{\rm out}, \tag{64}$$

where the Schmidt coefficients are [44]

$$c_n = \left(\frac{\beta^*(p)}{\alpha^*(p)}\right)^n c_0, \tag{65}$$

with

$$c_0 = \sqrt{1-\left|\frac{\beta^*(p)}{\alpha^*(p)}\right|^2}. \tag{66}$$

From the state (64), we can write the bipartite density matrix,

$$\rho^{\rm (out)}_{p,-p} = |0_p;0_{-p}\rangle_{\rm in}\langle 0_p;0_{-p}|. \tag{67}$$

Since the Schmidt coefficients (65) are nonzero, it follows that the *in* vacuum is entangled from the point of view of an *out* observer. As for the Dirac case, the amount of particle-antiparticle entanglement is quantified considering the reduced density matrix,

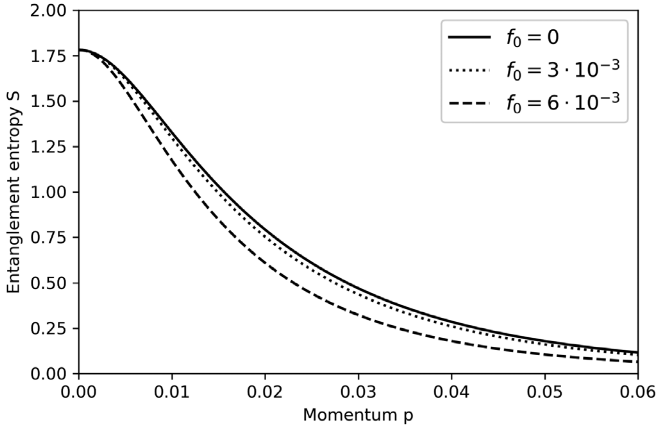

FIG. 4. Entanglement entropy for KG particles for different values of the torsion parameter f_0. The other parameters are $m = 0.01$, $h_0 = 10^{-5}$, $A = 3$, $B = 2$, and $\rho = 1$.

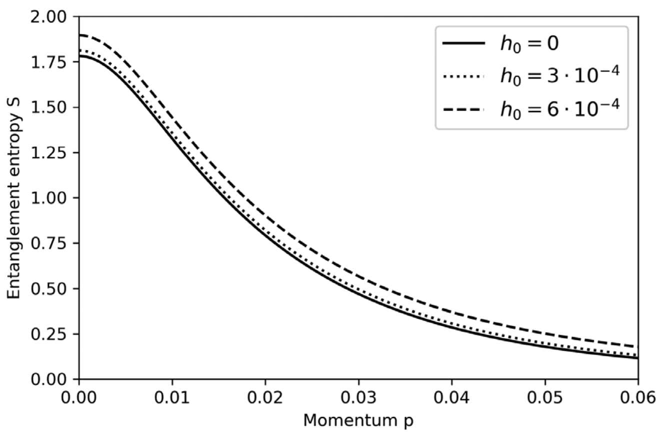

FIG. 5. Entanglement entropy for KG particles for different values of the torsion parameter h_0. The other parameters are $m = 0.01$, $f_0 = 10^{-5}$, $A = 3$, $B = 2$, and $\rho = 1$.

$$\rho_p^{(\mathrm{out})} = \mathrm{Tr}_{-p}(\rho_{p,-p}^{(\mathrm{out})}) = \sum_{m=0}^{\infty} {}_{\mathrm{out}}\langle m_{-p}|\rho_{p,-p}^{(\mathrm{out})}|m_{-p}\rangle_{\mathrm{out}}. \tag{68}$$

Accordingly, the Von Neumann entropy of this state takes the form,

$$\begin{aligned} S(\rho_p^{(\mathrm{out})}) &= -\mathrm{Tr}(\rho_p^{(\mathrm{out})}\log_2\rho_p^{(\mathrm{out})}) \\ &= \log_2 \frac{\gamma^{\gamma/(\gamma-1)}}{1-\gamma}, \end{aligned} \tag{69}$$

where [44]

$$\gamma = \left|\frac{\beta(p)}{\alpha(p)}\right|^2 = \frac{\sinh^2(\pi\mathcal{E}_-/\rho)}{\sinh^2(\pi\mathcal{E}_+/\rho)}. \tag{70}$$

In Figs. 4 and 5, we show how KG entanglement is affected by the presence of the parameters f_0 and h_0. In particular, if h_0 is nonzero, the amount of entanglement is increased, while a nonzero f_0 modifies the mode dependence of S.

B. Nonminimal coupling to completely antisymmetric torsion

Let us restart now from Eq. (46). Following some literature [34,46], we could make the assumption of completely antisymmetric torsion. In this case, the only relevant parameters are ξ_1 and ξ_4. Recalling the explicit form for P_4 and using again the ansatz (8) for torsion, it can be easily shown that

$$P_4 = \frac{36 f^2(\tau)}{a^6}. \tag{71}$$

Assuming again conformal coupling to the Riemannian curvature, the KG equation that corresponds to (50) in case of antisymmetric torsion is found to be

$$\Box\chi = \left[a^2m^2 + 36\xi_4\frac{f^2(\tau)}{a^4}\right]\chi. \tag{72}$$

This equation admits (51) as a general solution, where now, $\chi_p(\tau)$ satisfies the differential equation,

$$\ddot{\chi}_p(\tau) + \left[|\mathbf{p}|^2 + a^2m^2 + 36\xi_4\frac{f^2(\tau)}{a^4}\right]\chi_p(\tau) = 0. \tag{73}$$

An exact solution to (73) can be found if we slightly modify the ansatz (18) for the scale factor to [37],

$$a(\tau) = \sqrt{A + B\tanh(\rho\tau)}, \tag{74}$$

and we assume again $f(\tau) = f_0 a^3(\tau)$ as in (53).

The solutions can be written again in terms of Hypergeometric functions as

$$\begin{aligned} \chi_p^{\mathbf{in}}(\tau) &= \exp\left\{-i\left[\mathcal{E}_+\tau + \frac{1}{\rho}\mathcal{E}_-\ln(2\cosh(\rho\tau))\right]\right\}{}_2F_1 \\ &\quad\times\left(1+\frac{i}{\rho}\mathcal{E}_-, \frac{i}{\rho}\mathcal{E}_-; 1-\frac{i}{\rho}\mathcal{E}_{\mathbf{in}}; \frac{1+\tanh(\rho\tau)}{2}\right), \end{aligned} \tag{75}$$

$$\begin{aligned} \chi_p^{\mathbf{out}}(\tau) &= \exp\left\{-i\left[\mathcal{E}_+\tau + \frac{1}{\rho}\mathcal{E}_-\ln(2\cosh(\rho\tau))\right]\right\}{}_2F_1 \\ &\quad\times\left(1+\frac{i}{\rho}\mathcal{E}_-, \frac{i}{\rho}\mathcal{E}_-; 1+\frac{i}{\rho}\mathcal{E}_{\mathbf{out}}; \frac{1-\tanh(\rho\tau)}{2}\right), \end{aligned} \tag{76}$$

where, now,

$$\mathcal{E}_{\mathbf{in}} = [|p|^2 + (m^2 + 36\xi_4 f_0^2)a^2(\tau\to-\infty)] \tag{77}$$

$$\mathcal{E}_{\mathbf{out}} = [|p|^2 + (m^2 + 36\xi_4 f_0^2)a^2(\tau\to+\infty)]. \tag{78}$$

Following now the usual treatment of Sec. IVA, we can write the analogue of the Eq. (59) for $\chi_p(\mathbf{x},\tau)$ and obtain the corresponding Bogoliubov coefficients [41],

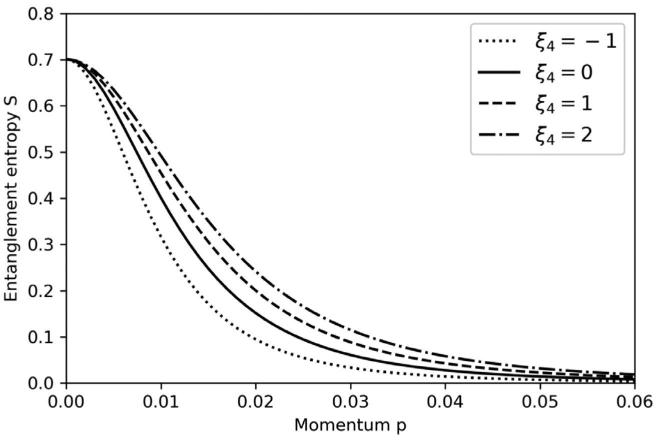

FIG. 6. Entanglement entropy for KG particles for different values of the coupling parameter ξ_4. The other parameters are $m = 0.01$, $f_0 = 10^{-3}$, $A = 3$, $B = 2$, and $\rho = 1$.

$$\alpha(p) = \frac{\Gamma(1-(i/\rho)\mathcal{E}_{\mathbf{in}})\Gamma(-(i/\rho)\mathcal{E}_{\mathbf{out}})}{\Gamma(1-(i/\rho)\mathcal{E}_{+})\Gamma(-(i/\rho)\mathcal{E}_{+})},$$
$$\beta(p) = \frac{\Gamma(1-(i/\rho)\mathcal{E}_{\mathbf{in}})\Gamma((i/\rho)\mathcal{E}_{\mathbf{out}})}{\Gamma(1+(i/\rho)\mathcal{E}_{-})\Gamma((i/\rho)\mathcal{E}_{-})}. \tag{79}$$

from the properties of Hypergeometric functions.

Accordingly, entanglement can be studied following the same steps of the previous section, and the entanglement entropy has the same form of (69). In Fig. 6, we show how entanglement is affected by antisymmetric torsion, for different values of the coupling parameter ξ_4. We notice that negative values for ξ_4 would decrease the total amount of entanglement, except for the limiting cases $p \to 0$ and $p \to \infty$. On the contrary, a positive ξ_4 enhances entanglement. This becomes more evident as ξ_4 increases.

V. DISCUSSION ON PHYSICAL CONSEQUENCES RELATED TO TORSION

In the case of Dirac field, the entanglement entropy S is upper bounded, and no divergences occur. More precisely, S is bounded by $S < \log_2 N$, where N is the Hilbert space dimension of the partial state (reduced density operator). In our picture, the hypothesis of charge and angular momentum conservation [44,45] still holds, so, from Eq. (40), we see that $N = 4$. Accordingly, the maximum value for the entropy is $S = 2$. In the presence of torsion, we see that the entropy provides two maxima: The first is absolute and corresponds to $p = 0$, whereas the second is relative and lies around $p \simeq 2m$. The two maxima are portrayed in Figs. 2 and 3. The relative maximum can be interpreted in view of the Pauli exclusion principle. Indeed, it is not possible to *condensate* fermions as $p \to 0$, and consequently, there would exist a $p \neq 0$ at which the maximum occurs. The absolute maximum could be interpreted in view of the torsion field, adopted in Eq. (31). The function $f(\tau)$ is constructed to guarantee the cosmological principle to hold. Even though appealing, this choice disagrees with the case of particles with half-integer spin that, by virtue of the Pauli exclusion principle, cannot occupy the same quantum state within a quantum system simultaneously. Thus, at $p \to 0$, the particle contribution becomes negligibly small, albeit the torsion source due to f_0 does not, by construction. This implies that at small p the main contribution to particle creation and entanglement is due to torsion, *i.e.,* due to the underlying torsion field. The case without torsion stresses our interpretation since here, at $p \to 0$, the fermionic entropy S goes to zero as expected by construction. If the torsion field is chosen to fulfill simultaneously the Pauli exclusion principle and the cosmological principle, we believe this apparent issue can be healed. This would extend our treatment by means of a refined one. However, the study of a fermionic torsion would require the introduction of a composite (by fermions) torsion field, and for this reason, it is left for future investigations.

Coming back to our bosonic ansatz, the simplest interpretation, as we showed in Sec. III A, leads to the introduction of torsion particles, which can be proposed as dark matter candidates.

In our scenario, since $m \neq 0$, the fact that the relative maximum is around $p \simeq 2m$ is in agreement with our previous discussion because the relative maximum occurs in a region that is far from $p \to 0$. In the region $p \to 0$, the Dirac field seems to resemble the KG framework. However, for the KG field, we have an absolute maximum that always occurs at $p = 0$, and the similarity between the two cases, Dirac and KG curves, is only apparent. Indeed, we believe this apparent similitude is a consequence of the employed torsion field whose functional form is simplified to guarantee the cosmological principle holds. Moreover, again, this can be interpreted by the fact that for bosons, we do not have any Pauli exclusion principle, and at $p = 0$, bosons can *condensate* in the fundamental state to provide the maximum plotted in Figs. 4 and 5.

In the KG field, we first remark that the entanglement entropy is not necessarily bounded, due to the infinite dimension corresponding to the density operator (68). At $p = 0$, the effects due to the torsion field are inferred from a nonminimal coupling between the field and torsion itself. Consequently, torsion does not dominate in any regions of p space, differently from the Dirac case.

Comparing the cases with and without torsion suggests that the shapes of each curve continue to be similar, albeit slightly different. This is direct consequence of the nonminimal coupling above discussed. The most important fact is that for small values of f_0, it seems that S weakly decreases. The opposite happens for h_0; *i.e.,* for small values of h_0, the entropy appears larger than the case without torsion. These two evidences can be interpreted in view of Eqs. (53). Indeed, by construction, $f(\tau)$ and $h(\tau)$ scale as the volume and radius of the Universe, respectively.

Consequently, from Eq. (48), the term $\propto \frac{f^2}{a^6}$ is a constant throughout the Universe evolution, indicating that the curvature is weakly influenced by f_0. This implies that the entropy should be smaller for $f_0 \neq 0$ than the case without torsion. On the other hand, the same does not happen for h since it couples to the term $\propto a^{-2}$ but also to $\propto \dot{a}$. Moreover, the kinematic term $\dot{h}$ is also different from zero, involving the fact that as the Universe radius increases, then its contribution increases as well. This acts on the entropy that is larger than the case without torsion, for $h_0 \neq 0$. Finally, the assumption of completely antisymmetric torsion would allow one to describe the torsion field using only the function $f(\tau)$ if we retain again the hypotheses of homogeneity and isotropy of spacetime. In this scenario, the coupling parameter ξ_4 affects entanglement in a nontrivial way, except for the limiting cases $p \to 0$ and $p \to \infty$. Positive values of ξ_4 produce an increase of the entanglement entropy, while the opposite is true for negative values. In particular, larger values of ξ_4 are responsible for a larger amount of entanglement entropy due to torsion.

In all the aforementioned cases, we underline our findings are in line with previous results found in the literature, certifying that the role of torsion modifies the entanglement measured depending on how it couples with the Universe expansion history.

VI. FINAL REMARKS

In this paper, we investigated particle production and entanglement in the framework of EC theory with fermionic and bosonic fields. Thus, we considered the Dirac and KG equations, solving them when particle spin is not negligible. In the framework of the FRW universe, we took the most general form for the torsion source, whose constraints are imposed in agreement with the cosmological principle.

We showed how torsion affects entanglement in the cases that enabled us to get analytical solutions in the KG field. Even though we demonstrated no analytical entanglement entropy could be obtained for fermions, assuming torsion to be small enough, we got approximate solutions, extending the results when torsion is zero. According to our findings, we showed which properties should be fulfilled by torsion field to get entanglement increase throughout universe's expansion history. In particular, positive values for the function describing torsion are required to increase the amount of entanglement for fermions. In this case, we also noticed that the mode dependence of the entanglement entropy is drastically modified for small values of the particle momentum. We naturally interpreted that the excess of entropy is induced by torsion particles. We conjectured these torsion particles to be related to dark matter. We quantified how much entanglement entropy is requested by these dark matter particles.

For the KG field, the amount and mode dependence of entanglement is slightly modified by the two external functions describing torsion. We interpreted the maxima of entanglement for both Dirac and KG fields. In particular, we showed that the Pauli principle is responsible for the relative maximum in the Dirac case, while the absolute maxima for Dirac and KG are direct consequences of the torsion field, involved in our treatment.

We remark that both for Dirac and KG field, we focused on asymptotic regions and the role of torsion in the Universe expansion is not investigated. This is done mainly to simplify the analysis, which turns out to be highly complicated. For instance, this means that we do not work out how torsion can accelerate or decelerate the Universe. A more detailed analysis should involve how torsion may modify the Universe epochs, and it will be object of incoming works, where we can better specify more complicated paradigms for the torsion source and investigate them in phases far from asymptotic regions. This would relate *de facto* the here-developed quantum approach to a more classical perspective, thus intertwining the dynamical role of torsion with its quantum effects, beside asymptotic regions.

This way to proceed should also allow to better understand how one can relate torsion to dark constituents. Moreover, it would be intriguing to compare the various approaches to torsion sources found in literature and to apply them to our quantum scenario, checking whether entanglement is modified accordingly.

Another interesting avenue of research is related to entanglement extraction from the field modes, using local detector couplings. In particular, it has been shown that modulating a detector's resonance frequency and interaction strength can be useful to optimize the extracted entanglement [47]. In this direction, it would be important to understand what time dependence of interaction would optimize the extraction of information about cosmological parameters and spacetime structure, using this method. This may help to further elucidate the relevance of torsion in the Universe history. Finally, future developments are expected from quantum emulation of the Universe expansion by means of analogue experiments. In particular, using ion traps, it has been shown that ions manifest actual phonon production, if the trap is expanded over a finite time [48]. Moreover, Bose-Einstein condensate models have been proposed [49] to simulate complex inflationary scenarios. There, torsion is expected to play a relevant role, and so, developing a reliable, robust, and highly tunable laboratory test bed for analogue inflation would be of great experimental value to discuss the role of torsion in cosmology.

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APPENDIX A: THE ROLE OF TORSION AT SMALL MOMENTA FOR DIRAC FIELD

We here show how much entanglement due to torsion particles is accounted at small momenta. To do so, let us take into account an exponential decay for the function $f(\tau)$ in terms of p. This procedure would cancel out the degrees of freedom induced by the torsion particle magnitude.

In particular, as $p \to 0$ from the relative maximum (see Fig. 2), namely $p_{\max}$, we propose to cut off the torsion degrees of freedom by

$$f(\tau,p) = \begin{cases} f_0 a^{-k}(e^{cp}+d) & \text{for } p \le p_{\max} \\ f_0 a^{-k} & \text{for } p > p_{\max} \end{cases}. \tag{A1}$$

Immediately, we get $p_{\max} \simeq 0.0223$, $c \simeq 31.08$ and $d = -1$, fulfilling the conditions $f(\tau,0) = 0$ and $f(\tau,p_{\max}) = f_0\, a^{-k}$.

In Fig. 6, we show how the entropy would be affected by this modification in the torsion function, while in Fig. 7, we portray the entanglement entropy for Dirac field in the presence of torsion, assuming an exponential decay for the torsion function. Moreover, if we take S_{tp} as entanglement entropy associated with small momenta, before the maximum momentum $p_{\max}$, and $S^\star$ as total entropy without torsion, then we quantify the amount of entanglement entropy at small momenta by $S_{tp} \simeq 0.167 S^\star$. Even though this treatment is useful to quantify the amount of entanglement that is associated to torsion at small momenta, it suffers from the thorny issue of modifying the style of f, inducing a direct dependence on momenta. Thus, this approach does not guarantee the cosmological principle to hold since the torsion function is no longer a time-dependent function only. Hence, the use of Eq. (A1) can be motivated with the attempt to reconcile torsion with Pauli exclusion principle at the level of entanglement only; albeit, this would imply releasing the cosmological principle. Consequently, if future efforts will show that this amount of entanglement cannot be removed without violating the cosmological principle, then the conjecture of possible *torsion particles* as dark matter candidates can be investigated in more detail.

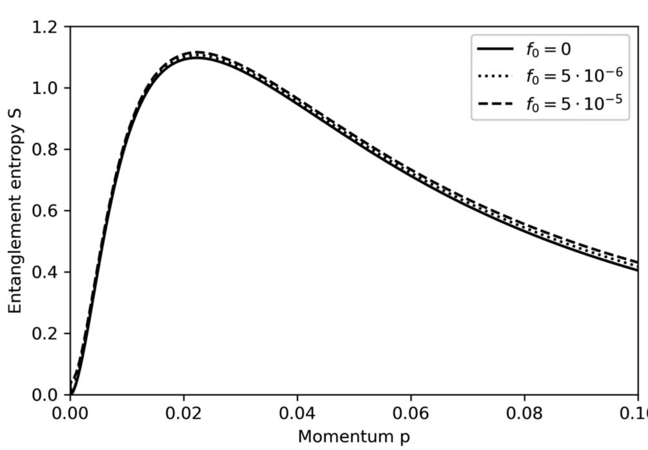

FIG. 7. Entanglement entropy for Dirac field in the presence of torsion, assuming an exponential decay for the torsion function as $p \to 0$. The values of the parameters are the same of Fig. 2: $m = 0.01$, $A = 3$, $B = 2$, $\rho = 1$, and $k = 6$.

APPENDIX B: ENTANGLEMENT IN NONCONFORMAL COUPLING SCENARIOS

We show here that assuming a nonconformal coupling of the KG field to the Riemannian curvature (*i.e.,* $\xi_1 \neq 1/6$) does not affect the entanglement entropy in our toy model.

Let us neglect here the nonminimal coupling of the field to torsion; *i.e.,* we set $\xi_2 = \ldots = \xi_5 = 0$. For a generic ξ_1, the KG equation has the form,

$$\Box\chi + \frac{\ddot{a}}{a}(1-6\xi_1)\chi - a^2 m^2 \chi = 0, \tag{B1}$$

which with the usual ansatz (51), gives the differential equation,

$$\ddot{\chi}_p(\tau) + \left(|\mathbf{p}|^2 + a^2 m^2 + \frac{\ddot{a}}{a}(6\xi_1 - 1)\right)\chi_p(\tau) = 0. \tag{B2}$$

The last term is, of course, zero if we assume conformal coupling as we did in Sec. IV. However, the coupling constant ξ_1 may also take very large values ($\xi_1 \simeq 40k$) in some scenarios, e.g., the Higgs inflation [50].

Equation (B2) should be solved numerically for $\chi_p(\tau)$, but choosing the form (74) for the scale factor, we notice that

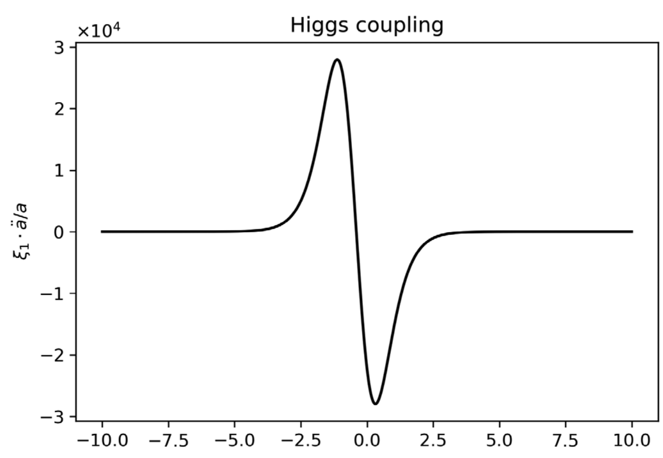

FIG. 8. Nonminimal coupling of the scalar field to the curvature $\tilde{R}$, as function of conformal time. The value $\xi_1 = 50k$ is assumed, typical of the Higgs inflation.

$$\frac{\ddot{a}}{a} = \frac{-2\rho^2 B\mathrm{sech}(\rho\tau)\cdot\mathrm{sech}(\rho\tau)\tanh(\rho\tau)\cdot\sqrt{A+B\tanh(\rho\tau)} - B^2\rho^2\mathrm{sech}(\rho\tau)^4/\sqrt{A+B\tanh(\rho\tau)}}{(A+B\tanh(\rho\tau))^{3/2}}, \tag{B3}$$

and, in Fig. 8, we show the dependence of this coupling term from the conformal time τ. We see that this contribution becomes negligible in the asymptotic regions $\tau \to \pm\infty$. The conclusion would be the same if we take (18) instead of (74) for the scale factor. Accordingly, entanglement would not be affected by large couplings of the scalar field to the Riemannian curvature in our toy model.

[1] A. Joyce, B. Jain, J. Khoury, and M. Trodden, Phys. Rep. **568**, 1 (2015); N. A. Bahcall, J. P. Ostriker, S. Perlmutter, and P. J. Steinhardt, Science **284**, 1481 (1999); A. Aviles, C. Gruber, O. Luongo, and H. Quevedo, Phys. Rev. D **86**, 123516 (2012).

[2] E. J. Copeland, M. Sami, and S. Tsujikawa, Int. J. Mod. Phys. D **15**, 1753 (2006); G. Ellis, Found. Phys. **48**, 1226 (2018); J. Noller, Phys. Rev. D **101**, 063524 (2020); M. Kowalski *et al.*, Astrophys. J. **686**, 749 (2008); K. Bamba, S. Capozziello, S. Nojiri, and S. D. Odintsov, Astrophys. Space Sci. **342**, 155 (2012).

[3] M. Bojowald, Rep. Prog. Phys. **78**, 023901 (2015); A. Vilenkin, Phys. Rev. Lett. **74**, 846 (1995); T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010); A. De Felice and S. Tsujikawa, Living Rev. Relativity **13**, 3 (2010).

[4] A. Borde and A. Vilenkin, Int. J. Mod. Phys. D **05**, 813 (1996).

[5] P. Peebles and B. Ratra, Rev. Mod. Phys. **75**, 559 (2003); T. Padmanabhan, Phys. Rep. **380**, 235 (2003); D. Huterer and M. S. Turner, Phys. Rev. D **64**, 123527 (2001); A. Melchiorri, L. Mersini-Houghton, C. J. Odman, and M. Trodden, Phys. Rev. D **68**, 043509 (2003); J. Frieman, M. Turner, and D. Huterer, Annu. Rev. Astron. Astrophys. **46**, 385 (2008); M. Li, X.-D. Li, S. Wang, and Y. Wang, Commun. Theor. Phys. **56**, 525 (2011).

[6] G. Bertone and T. Tait, Nature (London) **562**, 51 (2018); S. M. Carroll, I. Sawicki, A. Silvestri, and M. Trodden, New J. Phys. **8**, 323 (2006); R. H. Sanders and S. S. McGaugh, Annu. Rev. Astron. Astrophys. **40**, 263 (2002); G. Bertone and D. Hooper, Rev. Mod. Phys. **90**, 45002 (2018); J. G. de Swart, G. Bertone, and G. J. van Dongen, Nat. Astron. **1**, 0059 (2017).

[7] T. Troster *et al.*, Astron. Astrophys. **633**, L10 (2020); P. K. S. Dunsby and O. Luongo, Int. J. Geom. Methods Mod. Phys. **13**, 1630002 (2016); T. M. C. Abbott *et al.*, Phys. Rev. Lett. **122**, 171301 (2019).

[8] E. Huguet, M. Le Delliou, and M. Fontanini, arXiv:2101.07064; R. Aldrovandi and J. G. Pereira, *Teleparallel Gravity: An Introduction* (Springer, New York, 2013), Vol. 173; M. Hohmann, Eur. Phys. J. Plus **136**, 65 (2021).

[9] J. W. Maluf and F. F. Faria, Phys. Rev. D **85**, 027502 (2012); M. Hohmann, L. Jarv, M. Krssak, and C. Pfeifer, Phys. Rev. D **97**, 104042 (2018); M. Hohmann, J. Math. Phys. (N.Y.) **57**, 082502 (2016).

[10] P. Gonzalez, S. Reyes, and Y. Vasquez, J. Cosmol. Astropart. Phys. 07 (2019) 040; J. Formiga, Phys. Rev. D **99**, 064047 (2019); K. Bamba, S. D. Odintsov, and D. Saez Gomez, Phys. Rev. D **88**, 084042 (2013); S. Capozziello, R. DAgostino, and O. Luongo, Int. J. Mod. Phys. D **28**, 1930016 (2019).

[11] T. W. B. Kibble, J. Math. Phys. (N.Y.) **2**, 212 (1961).

[12] D. W. Sciama, Rev. Mod. Phys. **36**, 463 (1964).

[13] M. Blagojevic and F. W. Hehl, *Gauge Theories and Gravitation* (Imperial College Press, London, 2013).

[14] O. Luongo and S. Mancini, Int. J. Geom. Methods Mod. Phys. **16**, 1950114 (2019).

[15] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[16] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal, Phys. Rev. Lett. **83**, 3081 (1999).

[17] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).

[18] C. H. Bennett, G. Brassard, C. Crpeau, R. Josza, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[19] A. M. Steane, Rep. Prog. Phys. **61**, 117 (1998).

[20] T. F. Jordan, A. Shaji, and E. C. G. Sudarshan, Phys. Rev. A **75**, 022101 (2007).

[21] C. Cafaro, S. Capozziello, and S. Mancini, Int. J. Theor. Phys. **51**, 2313 (2012).

[22] I. Fuentes-Schuller and R. B. Mann, Phys. Rev. Lett. **95**, 120404 (2005).

[23] P. M. Alsing, I. Fuentes-Schuller, R. B. Mann, and T. E. Tessier, Phys. Rev. A **74**, 032326 (2006).

[24] R. B. Mann and V. M. Villalba, Phys. Rev. A **80**, 022305 (2009).

[25] J. Leon and E. M. Martinez, Phys. Rev. A **80**, 012314 (2009).

[26] E. Martin-Martinez and N. C. Menicucci, Classical Quant. Grav. **29**, 224011 (2012).

[27] I. Fuentes, R. B. Mann, E. Martin-Martinez, and S. Moradi, Phys. Rev. D **82**, 045030 (2010).

[28] E. Martin-Martinez and N. C. Menicucci, Classical Quant. Grav. **29**, 224003 (2012).

[29] L. Parker, Phys. Rev. Lett. **21**, 562 (1968); Phys. Rev. **183**, 1057 (1969); Phys. Rev. D **3**, 346 (1971).

[30] S. Moradi, R. Pierini, and S. Mancini, Phys. Rev. D **89**, 024022 (2014).

[31] R. Pierini, S. Moradi, and S. Mancini, Int. J. Theor. Phys. **55**, 3059 (2016).

[32] F. Cabral, F. S. N. Lobo, and D. Rubiera-Garcia, J. Cosmol. Astropart. Phys. 10 (2020) 057.

[33] N. J. Poplawski, Phys. Lett. B **694**, 181 (2010).

[34] I. Shapiro, Phys. Rep. **357**, 113 (2002).

[35] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).

[36] D. Benisty, Phys. Rev. D **100**, 043523 (2019).

[37] A. Duncan, Phys. Rev. D **17**, 964 (1978).

[38] F. W. Hehl and B. K. Datta, J. Math. Phys. (N.Y.) **12**, 1334 (1971).

[39] F. W. Hehl, P. von der Heyde, and G. D. Kerlick, Rev. Mod. Phys. **48**, 393 (1976).

[40] D. Puetzfeld, New Astron. Rev. **49**, 59 (2005).

[41] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).

[42] I. L. Buchbinder and I. L. Shapiro, Phys. Lett. B **151**, 263 (1985).

[43] I. L. Buchbinder and I. L. Shapiro, Classical Quant. Grav. **7**, 1197 (1990).

[44] J. L. Ball, I. Fuentes-Schuller, and F. P. Schuller, Phys. Lett. A **359**, 550 (2006).

[45] R. Pierini, S. Moradi, and S. Mancini, Eur. Phys. J. D **73**, 33 (2019).

[46] A. S. Belyaev, M. C. Thomas, and I. L. Shapiro, Phys. Rev. D **95**, 095033 (2017).

[47] S. J. Olson and T. Ralph, Phys. Rev. A **85**, 012306 (2012).

[48] R. Schützhold, M. Uhlmann, L. Petersen, H. Schmitz, A. Friedenauer, and T. Schätz, Phys. Rev. Lett. **99**, 201301 (2007).

[49] U. R. Fischer and R. Schützhold, Phys. Rev. A **70**, 063615 (2004).

[50] G. Degrassi, S. Di Vita, J. Elias-Miró, J. R. Espinosa, G. F. Giudice, G. Isidori, and A. Strumia, J. High Energy Phys. 08 (2012) 098.