Effective field theory for binary cosmic strings

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We extend the effective field theory (EFT) formalism for gravitational radiation from a binary system of compact objects to the case of extended objects. In particular, we study the EFT for a binary system consisting of two infinitely-long cosmic strings with small velocity and small spatial substructure, or "wiggles." The complexity of the system requires the introduction of two perturbative expansion parameters, constructed from the velocity and size of the wiggles, in contrast with the point particle case, for which a single parameter is sufficient. This further requires us to assign new power counting rules in the system. We integrate out the modes corresponding to potential gravitons, yielding an effective action for the radiation gravitons and obtaining the effective potential between the cosmic strings. We show that this action describes a changing quadrupole, sourced by the bending modes of the string, which in turn generates gravitational waves. We study the ultraviolet divergences in this description, and use them to obtain the classical renormalization group flow of the string tension in such a setting.

I. INTRODUCTION

The remarkable observations of gravitational waves by the LIGO and Virgo collaborations [1,2] have opened new possibilities for exploring the universe. Understanding these observations requires the use of different theoretical and numerical tools to reproduce the gravitational waveforms. For example, the merging event of two compact objects consists of three phases: inspiral, merger, and ringdown; where each phase can be described through different techniques. The highly nonlinear merging phase, in particular, requires the use of numerical relativity [3–5]. Fortunately, these time-costly techniques can be supplemented away from the merging phase by the use of theoretical techniques such as the effective one-body (EOB) formalism [6,7] and the self-force formalism [8,9]. During the inspiral phase, perturbative methods such as post-Newtonian(PN) expansions, which consists of a small velocity expansion for virialized systems, and post-Minkowskian expansions (PM), which consists of taking the special relativity limit by taking Newton's constant to be

small, can be used. Note that PN corrections can be computed with various methods [10–13], one of the most successful ones being the nonrelativistic general relativity (NRGR) effective field theory method [14] (see [15,16] for a comprehensive review.) The NRGR formalism consists of considering the effective action of point-particles with gravitational interactions, with higher dimensional operators encoding the finite size effects of the compact object being described. This method has led to great progress in computing PN corrections, not only in the spinless case, but also for spinning compact objects [17]. Recently, amplitudes techniques combined with EFT methods have been used to streamline the calculations in the PM approach [18–24] and for some PN computations [25]. Other EFT formulations for gravitating rotating objects which tackle dissipative effects with a different approach have been proposed in [26–28]. For recent reviews on these different methods see [10,11,13,15,16].

In this paper, we explore the challenges of generalizing the NRGR EFT technique beyond point masses by considering the simple formal extension to one-dimensional sources. To be precise, we provide an EFT formalism to calculate the gravitational radiation from an artificial system of binary infinitely-long cosmic strings living in a $3 + 1$ dimensional spacetime. While we do not expect to find such a simple realization of cosmic strings in nature, the goal is to illuminate and solve some of the issues that arise when one goes beyond the simplest systems in full gravity setup. Notice that other codimension 2 objects have been previously analyzed in different contexts, for example

in [29,30], and the observed behavior is similar to our case, as one might expect for localized objects of the same codimension.

A key feature that allows us to construct an EFT is the large separation of scales involved in the phenomenon being described. At low energies this allows us, by carefully choosing the degrees of freedom, to ignore short-range details of the theory, and to focus on the relevant physical effects at long length scales. For non-relativistic sources, the velocity always serves as a natural expansion parameter. In the case of point particles without gravity, it is easy to expand the worldline of the source to whatever order of velocity is required. However, for systems including gravity, gravitons that only mediate short-range forces, which we refer to as potential gravitons, may contribute to the background potential, making perturbation theory considerably more complicated. One can describe this system by integrating out the potential gravitons and obtaining an effective field theory for the long-distance degrees of freedom, which we refer to as radiation gravitons. Note that the word graviton here, and throughout this paper, refers only to the small metric perturbation around the flat Minkowski background. Although this is a classical concept, it is convenient for us, and makes contact with the particle physics literature, to refer to it in these terms. A successful EFT assigns an order in the expansion parameter (the velocity in the point-particle case) to each term contributing to the effective action, and one may then compute observables up to a desired precision by truncating the EFT at the appropriate order.

The story for the cosmic string system is more involved than the point-particle case. To describe the string action, we must introduce an additional variable, σ, together with the proper time τ, to describe string world sheet. For a non-relativistic infinitely-long cosmic string, the velocity is not the only natural small parameter in the theory—we must also consider the spatial bending of the string. It is therefore natural to expect the effective field theory to contain two expansion parameters. This is made more difficult by the fact that the string equation of motion implies that the velocity and bending should be at the same order. We will show that, from virial theorem, the tiny difference between the velocity and the bending should serve as the second expansion parameter. This novel power counting rule may ultimately be relevant for other studies involving multiple expansion parameters.

The gravitational potential between two binary infinitely-long cosmic strings has been rarely discussed. It is known that the solution of Einstein equations with one infinitely-long straight cosmic string at the lowest order is locally-flat with a deficit angle. For a wiggly string, the stress-energy tensor obtains a small correction leading to a gravitational field that a point-like test particle would feel [31]. However, it is unclear what a test string feels in such a background. Here, we systematically calculate the gravitational potential between two cosmic strings and determine the order at which it becomes non-zero. We also obtain the gravitational radiation arising from a binary string system by calculating the imaginary part of its self-energy diagram. Compared to the point-particle case, it is possible for the cosmic string to generate a time-varying quadrupole itself, inducing gravitational waves with typical frequencies of the inverse of its wiggle scale. We will show that this contribution is larger than the one arising from gravitational interactions between the pair of strings. However, we do not expect such gravitational waves to be detectable in the near future, since the dominant source of gravitational waves from cosmic strings originates in highly relativistic string loops, which our EFT fails to describe.

The paper is organized as follows. In Sec. II, we discuss how to build the EFT for the binary strings and radiation gravitons. We then derive the power counting rules of the EFT from the generalized virial theorem for binary strings. At the end of Sec. II, we explain the scaling of the Feynman graphs in the full theory which allow us to construct our EFT. In Sec. III, we discuss the EFT for nonrelativistic strings. We integrate out the potential graviton to obtain the EFT that only contains radiation gravitons, and use this to calculate the gravitational radiation. In Sec. IV, we consider a static, straight, relativistic string and compute the corrections to its stress-energy tensor from gravitational interactions. As expected, this calculation leads to divergences at 1-loop which can be regularized and renormalized. The renormalization of the string's tension leads to a classical RG flow which can be used to match this EFT coefficient to a UV description. In the Appendix, we show further details of integrals used in the calculations in the main body of the paper. It is important to note that we only work in the classical limit of the theory, and hence that our calculations do not involve $\hbar$ unless explicitly stated otherwise.

II. BUILDING THE EFT

A. General strategy to construct an EFT

The basic principle underlying the EFT idea is that an approximate description of physics at long length scales should not require detailed knowledge of the small-scale structure of the theory. Crucial to constructing an EFT for nonrelativistic strings, therefore, is an understanding of the different relevant length scales in the model. These are: the finite size of the string, r_s; the orbital distance between the binary strings, r_o; the size of the spatial substructure (wiggles) r_w; and the distance from the strings to the gravitational wave detector, r_d. It is the large separations between these scales

$$r_s \ll r_w \ll r_o \ll r_d, \tag{1}$$

that allow us to construct a reliable EFT.[1] The EFT procedure sacrifices the granularity afforded by a complete ultraviolet (UV) description in favor of a calculationally simpler theory, in which high-energy physics is integrated out and its broad effects are instead encoded in higher-dimension operators. This allows us to retain in the EFT only those degrees of freedom that are relevant for the scales of interest in the problem at hand.

In this case, we do not expect that the microphysical details of the core of the string will be important to the question of gravitational radiation, and so we hope to avoid the complications of a field-theoretic description of the string itself, at scales of order r_s or smaller. This is already a considerable simplification, since on scales larger than r_s, we have a good description of the system in terms of the string and Einstein-Hilbert (EH) actions. Nevertheless, what we are really interested in are the effects that a single string or binary strings should yield near the detector at which the gravitational radiation is measured. Thus, we seek an EFT in which all the degrees of freedom describing scales smaller than r_o and r_w are integrated out.

The highly nonlinear structure of general relativity (GR) leads to an infinite number of vertices for graviton self-interactions, with a corresponding degree of calculational complexity. These calculations can be simplified if instead of using Newton's constant as an expansion parameter, we are able to use a new expansion parameter tailored to the specific situation at hand. For example, in nonrelativistic (NR) systems such as compact spherical objects (treated as point-particles), the gravitational dynamics can be computed [14] using a perturbative expansion in their velocity v. This NR approximation is extremely effective, but it should be noted that the formalism is only valid during the binary inspiral phase. One important feature that allows for this expansion is the fact that the virial theorem relates the gravitational coupling $1/M_{pl}$ to the velocity v. This provides a suppression, not only of higher-loop graphs, but also of higher n-point vertices. One can then truncate the infinite number of graviton vertices at a given order in the velocity.

In this paper, for the analogous case of a binary system of orbiting infinite strings, we will consider an extended nonrelativistic limit in which the infinitely-long strings have both small velocity and small bending along their lengths, and will make a further assumption that the difference of the squares of these quantities is parametrically smaller than either of them individually. These assumptions will allow us to satisfy an appropriate version of the virial theorem and will again allow us to construct an EFT in which only a finite number of graphs contribute to physical process at certain order in these expansion

parameters. Similar to the point-particle case, we expect that this approximation is only valid during the binary inspiral phase. Additionally, we will consider the case of a single NR string which can radiate gravitons on its own due to its own substructure, or wiggles. We will show that the binary strings can contribute to the generation of gravitational radiation at a lower order in our expansion parameter than that for the single string.

With the NR effective action for the string in hand, we will then use it to derive two different results. We first compute the gravitational potential between the pair of parallel strings and then calculate the radiation power emitted by a single string.

B. Potential versus radiation gravitons

In this section, we identify and combine the ingredients necessary to construct an effective field theory for non-relativistic binary strings. Our starting point is the action for Einstein gravity coupled to an arbitrary number N of strings

$$S = 2M_{pl}^2 \int d^4x \sqrt{-g} R + S_{string}, \qquad (2)$$

where we have made the convenient but somewhat non-standard definition $M_{pl}^2 \equiv (32\pi G)^{-1}$, with G representing Newton's constant. Here, the string action is given at leading order by the Nambu-Goto action[2]

$$S_{string} = -\sum_{i=1}^{N} T_i \int d^4x \int d^2\sigma_i \sqrt{-\det\gamma}\,\delta(x^0 - \tau_i)$$
$$\times \delta(x^1 - \sigma_i)\delta^2(\mathbf{x} - \mathbf{x_i}), \qquad (3)$$

where $\gamma_{\alpha\beta}$ is the string world sheet metric, $\mathbf{x_i}$ is the position of the ith string, T_i is its tension, and $\{\sigma^0, \sigma^1\} = \{\tau, \sigma\}$ the world sheet coordinates. Note that bold quantities represent 2d vectors in the directions perpendicular to the string. It is worth noting at this early stage that, thanks to the reparametrization invariance of the string action, we are free choose to align the world sheet coordinates with the $x_0 - x_1$ plane.

When constructing an EFT, we initially include all the terms that are consistent with the symmetries of our theory in the Lagrangian. The Einstein-Hilbert action itself contains the graviton self-interactions, while Eq. (3) is the leading order contribution to the string-graviton interactions. In principle, we may include higher order operators which encode the effects of the finite size r_s of the string (see [32] and references therein). For example, we might include the next leading order correction to the string action, given by

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[1]We restrict ourselves to wiggles much larger than the size of the string so that we can consistently include their description in our EFT.

[2]We use the $(-+++)$ signature throughout the paper.

$$S \supset r_s^2 \int d^2\sigma \sqrt{-\det\gamma} \, \mathcal{K}_{a\mu\nu}^2, \tag{4}$$

which corresponds to a nonminimal coupling of the world sheet with the spacetime through the extrinsic curvature $\mathcal{K}_{a\mu\nu}$. Here we use Greek letters for the 4d spacetime coordinates, and Latin letters $a, b, \ldots$ for the spatial directions orthogonal to the string, namely, the x^2 and x^3 components. However, we will not take such higher order terms into account in our treatment. Similarly, we will only consider nonrotating strings by ignoring possible spin degrees of freedom.

An ambitious approach might be to seek an EFT for the strings themselves by integrating out all graviton degrees of freedom. However, this proves remarkably complicated and not suited for a nonrelativistic expansion. Instead, we will take advantage of the different scales in our setup to first construct an EFT for the strings coupled only to those gravitons that can be observed near a gravitational wave detector—the so-called *radiation gravitons*. In order to do this, we begin by defining the relativistic graviton $h_{\mu\nu}$ in the usual way, as a perturbation about the flat background geometry, writing the full metric as

$$g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}. \tag{5}$$

We then split the relativistic graviton into a *potential* contribution, $H_{\mu\nu}$, and a *radiation* field, $\bar{h}_{\mu\nu}$. In this language, the potential gravitons are responsible for mediating the interactions between the strings. They are never on-shell, and thus they cannot appear as external lines in Feynman diagrams. The timescale relevant for these gravitons is set by the strings' velocity $v = \partial x/\partial\tau$ and the strings' orbital distance. Similarly, the length scale in the direction parallel to the strings is set by the strings' bending $x' = \partial x/\partial\sigma$, where prime denotes derivative with respect to σ, and the strings' orbital distance. Following this reasoning, we see that the potential gravitons' 4-momenta scale as:

$$k_{\text{pot}}^\mu \sim \left(\frac{v}{r_o}, \frac{x'}{r_o}, \frac{1}{r_o}\hat{k}\right), \tag{6}$$

where we have defined $v \equiv |v|$, $x' \equiv |x'|$ and $\hat{k}$ is the unit 2-vector transverse to the string. Thus, potential gravitons are always spacelike and correspond to short-range gravitational mediators.

Radiation gravitons, on the other hand, are responsible for the gravitational radiation, and thus can reach all the way to gravitational wave detectors. Since radiation gravitons are physical excitations, and hence are on-shell, $k_\mu k^\mu = 0$ and their momentum scales like

$$k_{\text{rad}}^\mu \sim \left(\frac{v}{r}, \frac{x'}{r}, \frac{\sqrt{v^2 - x'^2}}{r}\hat{k}\right), \tag{7}$$

where, in the case of binary strings r is the orbital scale r_o, and for the case of a single string r is the scale of the wiggles r_w. Since these gravitons are on-shell, we require that $(x')^2 \leq v^2$. The on-shell condition also tells us that the length scale in the direction perpendicular to the string is set by $r/\sqrt{v^2 - x'^2}$, rather than by by r as in the case of the off-shell potential gravitons. This implies that the radiation wavelength scales like $r/\sqrt{v^2 - x'^2}$. Comparing this with those gravitons arising from spherical extended sources [14], we see that in our case the modes have longer wavelengths.

Decomposing the graviton into potential and radiation modes allows us to construct an effective action for the strings and the radiation graviton by integrating out the potential graviton modes according to

$$\exp[iS_{\text{eff}}[x^i, \bar{h}]] = \int \mathcal{D}H_{\mu\nu} \exp\{iS_{\text{EH}}[\bar{h} + H] + iS_{\text{string}}[x^i, \bar{h} + H] + iS_{\text{GF}}[H]\}, \tag{8}$$

where S_{GF} is a gauge fixing term. To keep gauge invariance manifest in the effective action, we choose a gauge fixing term invariant under diffeomorphisms of the background metric $\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu}$. We will work in the harmonic gauge defined by the gauge fixing term

$$S_{\text{GF}} = -M_{\text{pl}}^2 \int d^4x \sqrt{-\bar{g}} \, \Gamma_\mu \Gamma^\mu,$$

$$\Gamma_\mu = D_\alpha H^\alpha{}_\mu - \frac{1}{2}D_\mu H^\alpha{}_\alpha, \tag{9}$$

with D_α the covariant derivative with respect to the metric $\bar{g}_{\mu\nu}$. Taking into account the gauge fixing term, the Green's function for potential gravitons is then given by

$$\langle H_{\mu\nu}(x_1)H_{\alpha\beta}(x_2)\rangle = D_F(x_1 - x_2)P_{\mu\nu,\alpha\beta}, \tag{10}$$

where the Green's function with Feynman boundary condition is given by

$$D_F(x_1 - x_2) = -\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - i\epsilon}e^{-ik\cdot(x_1-x_2)}, \tag{11}$$

and the Lorentz structure is encoded in[3]

$$P_{\mu\nu,\alpha\beta} = \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}). \tag{12}$$

Note that the real part of the Feynman Green's function gives the average of the retarded and advanced Green's functions, which allow us to compute the potential. Meanwhile, the imaginary part is related to energy loss. For potential gravitons, $k^2 = -k_0^2 + k_1^2 + k^2$ is dominated by $k^2 \sim 1/r_o^2$. Therefore, we can approximate the integrand in Eq. (11) in this region by taking $1/(k^2 - i\epsilon) \sim 1/k^2$, and

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[3]In D dimensions, $P_{\mu\nu,\alpha\beta} = \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{D-2}\eta_{\mu\nu}\eta_{\alpha\beta})$.

can then perform the k_0 and k_1 momentum integrals, finding that the Green's function for the potential gravitons reads

$$D_F(x_1 - x_2) \simeq \left(-\int \frac{d^2\boldsymbol{k}}{(2\pi)^2} \frac{i}{\boldsymbol{k}^2} e^{i\boldsymbol{k}\cdot(x_1-x_2)} \right) \delta(x^0)\delta(x^1)$$
$$\sim \mathcal{O}\left(\left(\frac{r_o}{\boldsymbol{v}}\right)^{-1} \left(\frac{r_o}{\boldsymbol{x'}}\right)^{-1} \log\left(\frac{r_o}{r_{\mathrm{IR}}}\right) \right). \quad (13)$$

where $x^0 \sim r_o/v$, $x^1 \sim r_o/x'$, and r_{IR} is an IR cutoff. Further details are explained in Appendix A. Here, we have neglected the epsilon prescription in the Feynman propagator since the potential gravitons never go on-shell. This logarithmic behavior is precisely what we expect for a co-dimension-2 object, and is in agreement with the results of [30].

Rather than performing the path integral to obtain the effective action $S_{\mathrm{eff}}[x^i, \bar{h}]$, we will instead follow a much simpler procedure commonly known as matching [33–35]. This consists of computing the Feynman diagrams in the full theory and constructing the effective action that reproduces them. In the following section, we will establish power counting rules for all the elements involved in the string EFT. Armed with these rules, we will be able to understand the scaling of different terms contributing to the effective action as a function of two distinct quantities which will serve as the expansion parameters. This expansion will then allow us to truncate the result, taking into account only the finite number of graphs that contribute at a given order.

C. Power counting rules

An important feature of the string EFT is the existence of a relationship between the gravitational coupling and the dynamical scales of our system, namely the velocity and the wiggles of the strings. This relationship leads to a truncation of the Feynman diagrams contributing at a given order in the expansion parameters. In the following, we will carefully derive this relation and simultaneously find the correct expansion parameters for our EFT. We summarize the power counting rules in Table I. Once the appropriate expansion parameters have been found, we will show how to obtain the scaling of graphs involving potential

gravitons. Similarly, we will show in detail how to treat the radiation gravitons in our EFT.

1. Virial theorem for binary strings

As in the case of a system of binary point particles, the virial theorem plays a crucial role in enabling us to relate the gravitational coupling to the string velocity, although in the present application things are somewhat more involved. For a stable system bound by potential forces, the virial theorem can be generalized as

$$\left\langle \int d\sigma\, \mathbf{p}^a \dot{\mathbf{x}}_a \right\rangle = -\left\langle \int d\sigma\, \mathbf{x}_a \dot{\mathbf{p}}^a \right\rangle, \quad (14)$$

where in our case the brackets denote a time average over the whole inspiral phase. Using the string equation of motion, we have

$$\int d\sigma\, \mathbf{x}_a \dot{\mathbf{p}}^a = \int d\sigma\, \mathbf{x}_a \partial_\tau \frac{\partial L}{\partial \dot{\mathbf{x}}_a} = \int d\sigma\, \mathbf{x}_a \left(\frac{\partial L}{\partial \mathbf{x}_a} - \partial_\sigma \frac{\partial L}{\partial \mathbf{x}'_a} \right)$$
$$= \int d\sigma \left(\mathbf{x}_a \frac{\partial L}{\partial \mathbf{x_a}} + \mathbf{x'_a} \frac{\partial L}{\partial \mathbf{x'_a}} \right), \quad (15)$$
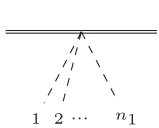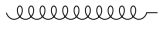
where we have integrated by parts in the last step. Thus, the virial theorem takes the form,

$$\left\langle \int d\sigma \left(\frac{\partial L}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a + \frac{\partial L}{\partial \mathbf{x}'_a} \mathbf{x}'_a \right) \right\rangle = -\left\langle \int d\sigma\, \mathbf{x}_a \frac{\partial L}{\partial \mathbf{x}_a} \right\rangle. \quad (16)$$

This result differs from the point particle case in two ways. First, for a system of point particles, the lhs is proportional to the kinetic energy of the system, while in our string system, an extra term describing wiggles on the string is present. Second, for a power-law potential, the rhs is simply proportional to V, corresponding to the potential energy; but this is not the case here.

We now apply this result to the case of a test string moving in the spacetime generated by a static string with wiggles. After averaging over the wiggles, the stress energy tensor of a wiggly string can be expressed in terms of an effective tension $\tilde{T}$ and an effective energy density $\tilde{u}$ (both of which depend on the original tension T). This effective

TABLE I. Power counting rules for vertices and propagators. The double line represents a cosmic string, the dashed line a potential graviton, and the curly line a radiation graviton. Note that the radiation graviton propagator does not have a logarithmic divergence as explained below.

Power counting rules			
$\frac{\boldsymbol{vx'}}{r^2} \log\left(\frac{r}{r_{\mathrm{IR}}}\right)$	$\frac{r^2}{\boldsymbol{vx'}} \frac{T}{M_{\mathrm{pl}}^{n_1}}$	$\left(\frac{1}{M_{\mathrm{pl}}}\right)^{n_2-2} \sum_i^{n_2} k_i^2\, \delta^4\left(\sum_i^{n_2} k_i^2\right)$	$\frac{\Delta}{r^2}$

stress tensor is valid when we are interested in physics on scales much larger that those of the spatial variations in the string structure. In this limit, the metric around a heavy wiggly static string is given by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with

$$h_{00} = h_{11} = 4G(\tilde{\mu} - \tilde{T}) \log\left(\frac{r}{r_{IR}}\right)$$

$$\sim 4GT x'^2_{\text{heavy}} \log\left(\frac{r}{r_{IR}}\right), \tag{17}$$

$$h_{22} = h_{33} = 4G(\tilde{\mu} + \tilde{T}) \log\left(\frac{r}{r_{IR}}\right)$$

$$\sim 4GT(2 - x'^2_{\text{heavy}}) \log\left(\frac{r}{r_{IR}}\right), \tag{18}$$

where x'_{heavy} denotes the wiggles of the heavy string [36]. In the special case of a straight string, $\tilde{\mu} = \tilde{T} = T$.

Consider a test string living in this spacetime, the world sheet volume element can be expanded as

$$\sqrt{-\det\gamma} = [(1 - v^2 - h_{00} - v^2 h_{22})(1 + x'^2 + h_{11} + x'^2 h_{22}) + (v \cdot x' + v \cdot x' h_{22})^2]^{1/2}$$
$$\simeq [1 - (v^2 - x'^2) - (v^2 - x'^2)h_{22} - (v^2 + x'^2)h_{00} - (v^2 + x'^2)h_{00}h_{22} - h_{00}^2]^{1/2}, \tag{19}$$

where in the second line we have used Eqs. (17) and (18) and have assumed $(v \cdot x')^2 \sim v^2 x'^2$ which holds under time averaging. In order to apply the virial theorem in this setting, it is convenient at this stage to expand the energy order by order in four expansion parameters defined in the following way

$$\epsilon_v \sim v^2 \ll 1, \qquad \epsilon_\Delta \sim \Delta \equiv v^2 - x'^2 \ll 1,$$

$$\epsilon_{GT} \sim \frac{T}{M_{pl}^2} \ll 1, \qquad -\epsilon_h \sim -\frac{T}{M_{pl}^2} \log\left(\frac{r}{r_{IR}}\right) \ll 1. \tag{20}$$

Note that ϵ_h is chosen to be negative since $r/r_{IR} < 1$, so $\log(r/r_{IR}) < 0$. In terms of these expansion parameters we can categorize terms in the action as $v^2 \sim \mathcal{O}(\epsilon_v)$, $x'^2 \sim \mathcal{O}(\epsilon_v - \epsilon_\Delta)$, $h_{00} \sim \mathcal{O}(\epsilon_h(\epsilon_v - \epsilon_\Delta))$, and $h_{22} \sim \mathcal{O}(\epsilon_h(2 - \epsilon_v + \epsilon_\Delta))$, where we have assumed that $\mathcal{O}(x'^2_{\text{heavy}}) = \mathcal{O}(x'^2)$. Note that then

$$\frac{\partial h_{22}}{\partial x^a} x^a = \frac{\partial h_{22}}{\partial r} \frac{\partial r}{\partial x^a} x^a = \frac{\partial h_{22}}{\partial r} r$$
$$\sim \mathcal{O}(\epsilon_{GT}(2 - \epsilon_v + \epsilon_\Delta)) \tag{21}$$

and

$$\frac{\partial h_{00}}{\partial x^a} x^a \sim \mathcal{O}(\epsilon_{GT}(\epsilon_v - \epsilon_\Delta)).$$

Using these expressions, and that for the Lagrangian of the test string, we find that the integrand on the left-hand side of Eq. (16) is of order

$$\mathcal{O}\left(\frac{\partial L}{\partial \dot{x}_a} \dot{x}_a + \frac{\partial L}{\partial x'_a} x'_a\right) \sim 2\epsilon_h \epsilon_v^2 + \epsilon_\Delta, \tag{22}$$

and similarly the integrand on the right-hand side of Eq. (16) is of order

$$\mathcal{O}\left(-x_a \frac{\partial L}{\partial x_a}\right) \sim -\epsilon_{GT} \epsilon_v^2, \tag{23}$$

where we only show the leading order scaling. One can see that the right-hand side is negative, and thus cannot compensate for the ϵ_Δ term on the left-hand side. To satisfy the viral theorem, Eq. (16), one requires that $\mathcal{O}(\epsilon_\Delta) \sim \mathcal{O}(-\epsilon_v^2 \epsilon_h)$, and this leads us to choose the following relationships among our expansion parameters

$$\epsilon_\Delta \sim -\epsilon_v^2 \epsilon_h, \qquad \epsilon_{GT} \lesssim -\epsilon_h. \tag{24}$$

Further details on how this choice allows us to satisfy the virial theorem at higher orders are discussed in Appendix B. Note that, since $-\epsilon_h \ll 1$, Eq. (24) then implies that $\epsilon_\Delta \ll \epsilon_v^2$. Similarly, the second relation in Eq. (24) implies that $|\log(\frac{r}{r_{IR}})| \gtrsim \mathcal{O}(1)$. The above relations are particularly important, since they allow us to relate the gravitational coupling to the velocity v and to the quantity Δ via

$$\left|\frac{T}{M_{pl}^2} \log\left(\frac{r}{r_{IR}}\right)\right| \sim \frac{\Delta}{v^4} \ll 1. \tag{25}$$

This is essential for the success of our EFT formalism. When we integrate out the potential gravitons to obtain an effective action for the radiation gravitons, Eq. (25) allows us to make a consistent truncation of the action. Together, Eqs. (24) and (25) demonstrate that our original four expansion parameters were redundant, and that we only need two parameters, v and Δ.

2. Scaling of graphs involving potential gravitons

Following the techniques of EFT, a convenient way to integrate out the potential gravitons to obtain the effective action for the radiation gravitons is through the matching procedure—using the power of Feynman diagrams to organize the perturbation expansion in terms of the above power counting rules. For example, the two-string interaction term arising from performing the path integral in
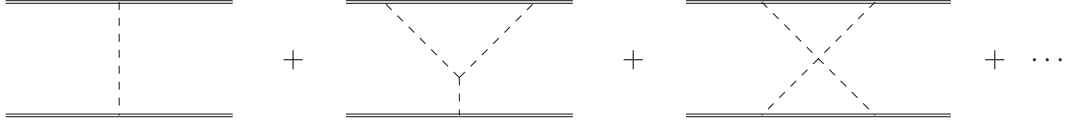
FIG. 1. Representative Feynman diagrams contributing to the expansion of Eq. (8) in the parameters v and Δ. Solid lines represent the source string, dashed lines represent the potential gravitons that are being integrated out.

Eq. (8) can be found by computing and summing a series of diagrams, representative examples of which are shown in Fig. 1. It is necessary to understand how different Feynman diagrams scale with the expansion parameters in order to truncate the infinite series at a desired order in Δ and v. The key elements required to do so are the scaling of the internal lines, of the matter vertex, and of the n-pt graviton vertex. An internal potential graviton line scales as $D_F(x_i - x_j)$ given by Eq. (13). Each matter vertex with n_1 gravitons introduces a factor of $\int d^2\sigma T(1/M_{\text{pl}})^{n_1} \sim \frac{r^2}{vx'}\frac{T}{M_{\text{pl}}^{n_1}}$, and each n_2-point graviton vertex coming from the EH action introduces a factor of $(1/M_{\text{pl}})^{n_2-2}\sum_i^{n_2} k_i^2 \delta^4(\sum_i^{n_2} k_i)$. This condition reduces the number of momentum-space integrations coming from the internal lines by one, and further removes one factor of $\frac{vx'}{r^2}$.

As an example, consider the second graph in Fig. 1. Using the rules described above, this diagram scales as

$$\frac{T^3}{M_{\text{pl}}^4}\int d\sigma_{1,2,3}^2 \int_{\boldsymbol{k}_{1,2,3}} e^{i\sum_{i=1}^3 \boldsymbol{k}_i \cdot \boldsymbol{x}_i}\left(\delta^2(\sigma_1-\sigma_2)\delta^2(\sigma_1-\sigma_3)\delta^2\right.$$
$$\times\left.\left(\sum_{i=1}^3 \boldsymbol{k}_i\right)\prod_{i=1}^3 \frac{1}{k_i^2}\times\left[\sum_{i=1}^3 (k_{i\mu}k_{i\nu})\right]\right), \quad (26)$$

where $\sigma_{1,2,3}$ denote the world sheet coordinates for each of the three string-graviton vertices, and the factor of $[\sum_i^3(k_{i\mu}k_{i\nu})]$ arises from the expansion of the EH term $\partial^2 hhh$. The square brackets indicate that all Lorentz indices are contracted appropriately. In this expression, we have canceled a factor of $\delta^2(\sigma) \sim \frac{r^2}{vx'}$ coming from the internal lines with a factor of $\delta(k^0)\delta(k^1) \sim \frac{vx'}{r^2}$ coming from the vertex. Simplifying the above result we find

$$\frac{T^3}{M_{\text{pl}}^4}\int d\sigma^2 \log\left(\frac{r}{r_{\text{IR}}}\right) \sim \mathcal{O}\left(\frac{L}{v^2}\left(\frac{\Delta}{v^4}\right)^2\right), \quad (27)$$

where we have performed the momentum integral and defined the angular momentum of the string as $L \equiv \int d\sigma T\boldsymbol{r}\cdot\boldsymbol{v} \sim Tr^2 v/x'$. In a similar fashion, we can use the power counting rules to compare the diagrams in Fig. 2. The tree-level graph on the left-hand side is of order $\frac{L}{v^2}\frac{\Delta}{v^4}$, while the right-hand side diagram, which includes a graviton loop, scales as $(\frac{\Delta}{v^4})^2$. This shows that diagrams at

FIG. 2. Examples of diagrams contributing to the effective action with and without a graviton loop. The right diagram is $\frac{\hbar}{L}\frac{\Delta}{v^2}$ suppressed relative to the left one. Notice this graviton loop is a quantum effect.

loop order are $\frac{1}{L}\frac{\Delta}{v^2}$ suppressed[4] and thus we neglect them in our analysis.

3. Scaling of radiation gravitons

Previously, we obtained the scaling of the potential graviton propagator by Fourier transforming to coordinate space. In that case, the 4-momentum was dominated by the directions perpendicular to the string, which led to a logarithmic behavior. As expected, in the case of the radiation graviton propagator the full 4-momentum is relevant. After Fourier transforming, we find that the coordinate-space radiation graviton scaling is

$$\bar{h}_{\mu\nu} \sim \frac{\sqrt{v^2-x'^2}}{r} = \frac{\sqrt{\Delta}}{r}. \quad (28)$$

As discussed in Eq. (7), r is the orbital scale r_o for binary strings and is the wiggle size r_w for a single string. Thus, the scaling of radiation gravitons would be different in these two systems. If we assume $r_w \ll r_o$, then the scaling of the radiation graviton in the binary string system would be parametrically smaller than that in the single string system.

Moreover, we can see that the elements of the EFT that involve radiation gravitons do not have a definite scaling in v and Δ. For example, scattering amplitudes involving internal radiation gravitons behave like

$$\int d^2\sigma_i d^2\sigma_j \int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2-i\epsilon}e^{-ik^0(x_1^0-x_2^0)+ik^1(x_1^1-x_2^1)+i\boldsymbol{k}^a(\boldsymbol{x}_{1a}-\boldsymbol{x}_{2a})}. \quad (29)$$

If we focus on the spatial part $e^{i\boldsymbol{k}^a\boldsymbol{x}_a} \sim 1 + i\boldsymbol{k}^a\boldsymbol{x}_a + \frac{1}{2}(i\boldsymbol{k}^a\boldsymbol{x}_a)^2 + \cdots$, we can see that the exponential contains an infinite series in powers of the expansion parameters.

[4]This requires $L \gtrsim v^3$ which is easily satisfied, since we expect $L \gg 1$.

A similar effect arises for the propagator of potential gravitons coupled to radiation gravitons, where there is an infinite expansion in powers of $\mathbf{k}_{\mathrm{rad}}/\mathbf{k}_{\mathrm{pot}} \sim \sqrt{\Delta}$. To remedy these situations and to obtain an explicit scaling for the EFT action, we will perform a multipole expansion of the radiation graviton

$$
\begin{aligned}
\bar{h}_{\mu\nu}(x^0, x^1, \mathbf{x}^a) &= \bar{h}_{\mu\nu}(x^0, x^1, \mathbf{x}_{\mathrm{CM}}^a) \\
&+ (\mathbf{x}^a - \mathbf{x}_{\mathrm{CM}}^a)\partial_a \bar{h}_{\mu\nu}(x^0, x^1, \mathbf{x}_{\mathrm{CM}}^a) \\
&+ \cdots.
\end{aligned} \tag{30}
$$

In the following expressions, we will suppress the explicit functional dependence of the radiation graviton, but one should remember that it is evaluated at the center of mass $\mathbf{x}_{\mathrm{CM}}$. Since the multipole expansion encodes the effects of the exponential expansion, we will now set $\mathbf{x}_{1a} = \mathbf{x}_{2a}$ at each order. With these power counting rules at hand, we are now ready to construct the EFT of nonrelativistic strings as an example.

III. THE EFT FOR NONRELATIVISTIC STRINGS

To construct the EFT for NR strings, we first need to integrate out the potential gravitons order by order. We begin by expanding the string Lagrangian for small velocity, wiggles, and metric perturbation. We then split the graviton into its potential and radiation parts, and discuss the gravitational potential between two cosmic strings and the gravitational radiation observed at a distant detector.

The world sheet metric of the string is $\gamma_{\alpha\beta} = g_{\mu\nu}\frac{\partial x^\mu}{\partial \sigma^\alpha}\frac{\partial x^\nu}{\partial \sigma^\beta}$, and we define $u^\mu \equiv \frac{\partial x^\mu}{\partial \tau} = (1, 0, \mathbf{v}^a)$ and $w^\mu \equiv \frac{\partial x^\mu}{\partial \sigma} = (0, 1, \mathbf{x}'^a)$, where $\mathbf{v}^a$ and $\mathbf{x}'^a$ stand for the string velocity and bending respectively. From the discussion in Sec. II C 1 we know that the appropriate expansion parameters are v and Δ. However, when expanding the world sheet volume element, we will encounter terms such as $\mathbf{v}^a h_{1a}$, which are not simple to classify according to these expansion parameters. For this reason, it is convenient to expand in factors of v^2 and h, and to perform the split $h = H + \bar{h}$ afterwards. This is useful for performing computations, but note that we will always return to the parameters v and Δ in the effective action.

The world sheet volume element can be expanded as,

$$
\begin{aligned}
\sqrt{-\det \gamma} &= 1 + \frac{1}{2}(-\mathbf{v}^2 + \mathbf{x}'^2 - h_{00} + h_{11} - 2\mathbf{v}^a h_{0a} \\
&\quad + 2\mathbf{x}'^a h_{1a} - \mathbf{v}^a \mathbf{v}^b h_{ab} + \mathbf{x}'^a \mathbf{x}'^b h_{ab} \\
&\quad - v^2 h_{11} - \mathbf{x}'^2 h_{00} + 2\mathbf{v}^a \mathbf{x}'_a h_{01} - h_{00}h_{11} + h_{01}^2) \\
&\quad - \frac{1}{8}[2(-\mathbf{v}^2 + \mathbf{x}'^2)(-h_{00} + h_{11}) \\
&\quad + (-h_{00} + h_{11})^2] + \mathcal{O}(v^2 h^2),
\end{aligned} \tag{31}
$$

and then the action at lowest order is simply given by

$$
S_{L\frac{\Delta}{v^2}}^{\mathrm{eff}} = \frac{1}{2}\sum_i T_i \int d^2\sigma_i(\mathbf{v}_i^2 - \mathbf{x}_i'^2), \tag{32}
$$

where $L\frac{\Delta}{v^2}$ denotes the order of the action and L is the angular momentum defined earlier. The leading order equation of motion (EoM) for $\mathbf{x}$ is then

$$
\ddot{\mathbf{x}} - \mathbf{x}'' = 0. \tag{33}
$$

A simple question is then at which order do gravitational forces start to contribute for a slightly wiggly string with a small velocity? In the following section, we will show that the gravitational potential for binary strings first enters at order $L\Delta^2/v^6$, which is Δ/v^4 smaller than the leading order contribution.

A. The gravitational potential between binary strings

We now focus on the contributions of potential gravitons—$H_{\mu\nu}$—in the expansion of Eq. (31), and compute the gravitational potential between binary strings. The interaction between a string and a potential graviton at $\mathcal{O}(v^0 H)$ reads,

$$
S = \frac{T}{2M_{\mathrm{pl}}} \int d^2\sigma(H_{00} - H_{11}). \tag{34}
$$

The vertex for this interaction is shown in Fig. 3(a) and the corresponding momentum-space Feynman rule is,

$$
\Gamma_{\mu\nu}^{(0)} = \frac{iT}{M_{\mathrm{pl}}} \int d^2\sigma V_{\mu\nu}^{(0)}, \qquad V_{\mu\nu}^{(0)} = \frac{1}{2}(I_{00,\mu\nu} - I_{11,\mu\nu}), \tag{35}
$$

where the tensor $I_{\mu\nu,\alpha\beta} \equiv \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha})$ is symmetric in all of its indices and the superscript (n) denotes the order in v. Using this, we can see that the diagram in Fig. 3(b) is,

$$
\begin{aligned}
\text{Fig. 3(b)} &= \frac{-T_1 T_2}{M_{\mathrm{pl}}^2} \int d^2\sigma_1 d^2\sigma_2 V_{\mu\nu}^{(0)} V_{\alpha\beta}^{(0)} \langle H^{\mu\nu} H^{\alpha\beta} \rangle \\
&\propto V_{\mu\nu}^{(0)} P^{\mu\nu,\alpha\beta} V_{\alpha\beta}^{(0)} = 0,
\end{aligned} \tag{36}
$$

FIG. 3. Left panel: the vertex associated with the Feynman rule in Eq. (35). The solid line represents a NR string source and the dashed line is a potential graviton. Right panel: the leading order contribution to the potential energy between two strings arising from the vertex in Eq. (35); this contribution vanishes, as shown in Eq. (36).
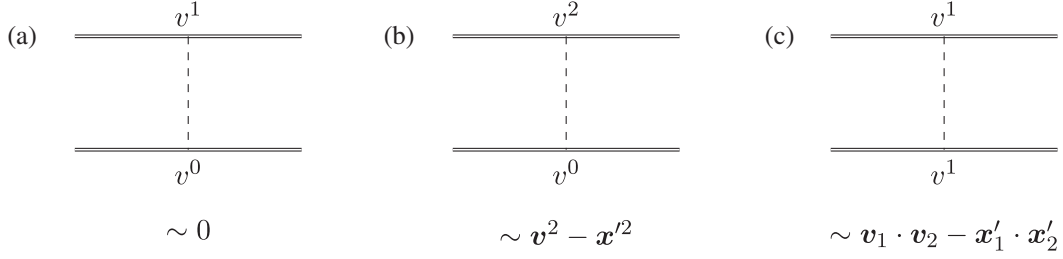
FIG. 4. Diagram (a) contributes at $\mathcal{O}(vH)$, but it vanishes as shown in Eq. (37). Diagrams (b) and (c) contribute at $\mathcal{O}(v^2 H)$.

which vanishes automatically due to the symmetry of the tensor structure.

To understand the leading order gravitational potential between two strings, it is therefore necessary to expand the string action to higher order. As shown in Fig. 4(a), there is only one diagram to consider at $\mathcal{O}(vH)$ and this diagram vanishes since

$$V_{\mu\nu}^{(1)} = -v^a I_{0a,\mu\nu} + x'^a I_{1a,\mu\nu},$$

$$\text{Fig. 4(a)} \propto V_{\mu\nu}^{(1)} P^{\mu\nu,\alpha\beta} V_{\alpha\beta}^{(0)} = 0. \tag{37}$$

At $\mathcal{O}(v^2 H)$, there are two diagrams, shown in Figs. 4(b) and 4(c). To compute the amplitude arising from Fig. 4(b) we need the Feynman rule for the string-potential graviton vertex at order v^2, which is given by $\Gamma_{\mu\nu}^{(2)}$ with

$$V_{\mu\nu}^{(2)} = -\frac{1}{2}(v^a v^b - x'^a x'^b) I_{ab,\mu\nu} - \frac{1}{2}(v^2 I_{11,\mu\nu} + x'^2 I_{00,\mu\nu})$$

$$+ (v \cdot x') I_{01,\mu\nu} - \frac{2}{8}(v^2 - x'^2)(I_{00,\mu\nu} - I_{11,\mu\nu}). \tag{38}$$

The combined contributions of the diagrams give

$$\text{Fig. 4(b)} + \text{(c)} = \frac{-T_1 T_2}{M_{\rm pl}^2} \int d^2\sigma_1 d^2\sigma_2 (V_{\mu\nu}^{(0)} V_{\alpha\beta}^{(2)} + V_{\mu\nu}^{(1)} V_{\alpha\beta}^{(1)}) \langle H^{\mu\nu} H^{\alpha\beta} \rangle$$

$$= \frac{iT_1 T_2}{M_{\rm pl}^2} \int d^2\sigma \int_k e^{-ik(x_1 - x_2)} \frac{1}{k^2} \frac{1}{4} [(v_1 - v_2)^2 - (x'_1 - x'_2)^2]$$

$$= -\frac{iT_1 T_2}{8\pi M_{\rm pl}^2} \int d^2\sigma \log\left(\frac{|x_1 - x_2|}{r_{\rm IR}}\right) [(v_1 - v_2)^2 - (x'_1 - x'_2)^2]. \tag{39}$$

Thus, to this order, the effective action reads

$$S_{L\frac{\Delta^2}{v^6}}^{\rm eff} = -\int d^2\sigma \frac{T_1 T_2}{8\pi M_{\rm pl}^2} \log\left(\frac{|x_1 - x_2|}{r_{\rm IR}}\right)$$

$$\times ((v_1 - v_2)^2 - (x'_1 - x'_2)^2). \tag{40}$$

The negative sign combined with the negative logarithm gives us a negative gravitation potential as in the point particle case. One could use this potential to confirm that the gravitational force between binary strings is attractive, as expected.

B. EFT for a single radiation graviton

In this section, we find the contribution to the effective action which contains only one radiation graviton $\bar{h}$. As indicated below Eq. (7), there are two types of radiation in the binary system. One is purely from the multipole expansion of a single string and has momentum $k \sim \sqrt{\Delta}/r_w$, while the other one involves gravitational interactions between the strings and has typical momentum

scaling as $k \sim \sqrt{\Delta}/r_o$. Since $r_w \ll r_o$, the leading order of the effective action with a single radiation graviton is the one from a single wiggly string. We start by discussing the radiation from a single string, where $\bar{h} \sim \sqrt{\Delta}/r_w$.

The leading order term with a single radiation graviton coming from Eq. (31) is,

$$S_{L\frac{\bar{h}}{v^2 M_{\rm pl}}}^{\rm eff} = \int d^2\sigma \sum_i \frac{T_i}{2M_{\rm pl}} (\bar{h}_{00} - \bar{h}_{11}). \tag{41}$$

Note that, under a gauge transformation $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, the variation of Eq. (41) vanishes after integrating by parts. In fact, our effective action should be gauge invariant order by order, but the proof of this becomes more complex at higher orders.

The next-to-leading order contribution contains terms from two different origins—those arising from the Lagrangian expanded directly to order $\mathcal{O}(v\bar{h})$, and those coming from the multipole expansion around the center mass $x_{\rm CM}$. The effective action obtained from summing these contributions is

$$S^{\text{eff}}_{L\frac{\bar{h}}{v\,M_{\text{pl}}}} = \int d^2\sigma \sum_i \frac{T_i}{2M_{\text{pl}}} (2v_i^a \bar{h}_{0a} - 2x_i'^a \bar{h}_{1a}$$

$$+ (x_i^a - x_{\text{CM}}^a)\partial_a \bar{h}_{00} - (x_i^a - x_{CM}^a)\partial_a \bar{h}_{11})$$

$$= 0, \tag{42}$$

where in the last equality we have integrated by parts and used the fact that $\dot{x}_{\text{CM}} = x'_{\text{CM}} = 0$ in the center of mass frame. Similarly, we find that the next-to-next-to-leading order effective action is

$$S^{\text{eff}}_{L\frac{\bar{h}}{M_{\text{pl}}}} = \int d^2\sigma \sum_i \frac{T_i}{2M_{\text{pl}}} \left(\frac{1}{2} x^a x^b \partial_a \partial_b (\bar{h}_{00} - \bar{h}_{11}) \right.$$

$$+ x^b (2v^a \partial_b \bar{h}_{0a} - 2x'^a \partial_b \bar{h}_{1a})$$

$$+ v^a v^b \bar{h}_{ab} - x'^a x'^b \bar{h}_{ab} + \frac{1}{2}(v^2 + x'^2)(\bar{h}_{00} + \bar{h}_{11})$$

$$\left. - 2(v \cdot x')\bar{h}_{01} \right), \tag{43}$$

where we have now set $x_{CM} = 0$. After integrating by parts, we can express this as

$$S^{\text{eff}}_{L\frac{\bar{h}}{M_{\text{pl}}}} = \int d^2\sigma \sum_i \frac{T_i}{2M_{\text{pl}}} (-R_{0a0b} + R_{1a1b})(x^a x^b)$$

$$+ \frac{1}{2}(v^2 + x'^2)(\bar{h}_{00} + \bar{h}_{11}) - 2(v \cdot x')\bar{h}_{01}. \tag{44}$$

Notice that $\text{Tr}(R_{0a0b} - R_{1a1b}) = 0$, which can be seen by using that $R_{0a0a} - R_{1a1a} = R_{0\mu0\mu} - R_{1\mu1\mu}$, and that $R_{0\mu0\mu}$ and $R_{1\mu1\mu}$ vanish for on-shell gravitons in the harmonic gauge ($\partial_\mu h_{0\mu} = \frac{1}{2}\partial_0 h$). Defining the quadrupole moment,

$$Q_{ab} = \sum_i T_i \left(x^a x^b - \frac{1}{2} x^2 \delta_{ab} \right), \tag{45}$$

we can rewrite Eq. (44) as
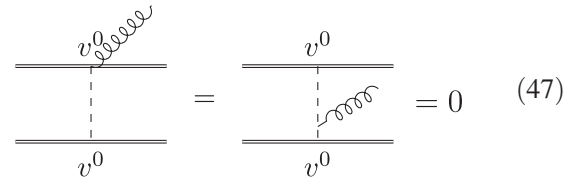
$$S^{\text{eff}}_{L\frac{\bar{h}}{M_{\text{pl}}}} = \int d^2\sigma \left[\frac{Q_{ab}}{2M_{\text{pl}}} (-R_{0a0b} + R_{1a1b}) \right.$$

$$\left. + \sum_i \frac{T_i}{2M_{\text{pl}}} \left(\frac{1}{2}(v^2 + x'^2)(\bar{h}_{00} + \bar{h}_{11}) - 2(v \cdot x')\bar{h}_{01} \right) \right]. \tag{46}$$

The first term in this expression corresponds to a quadrupole coupling, and will give rise to gravitational waves. Meanwhile, the second term is proportional to the total energy of the string at leading order. Since at this order the energy is a conserved quantity, this coupling will not give rise to radiation. The third term also corresponds to a conserved quantity in time—in this case it is proportional to the leading order Noether charge corresponding to invariance under translations in σ. The gauge invariance of this Lagrangian can be checked after integrating by parts and using the conservation of the stress-energy tensor, $\nabla_\mu T^{\mu\nu} = 0$, where $T^{\mu\nu} \propto \int d^2\sigma(\dot{x}^\mu \dot{x}^\nu - x^{\mu'} x^{\nu'}) \sim \int d^2\sigma x^\mu (\ddot{x}^\nu - x''^\nu)$.

It is important to note that, from the results we have just obtained, a single wiggly string will itself give rise to gravitational radiation. Contrary to the analogous calculations in the point-particle case, the contributions we found are gauge invariant on their own. Furthermore, there is no radiation arising from gravitational interactions at this order, which implies that we can already self-consistently obtain the radiation arising from a single string using the effective action found above.

We now discuss the effective action with a single radiation graviton arising from gravitational interactions of the binary string as shown in Fig. 5. In this setting, we will denote the radiation graviton $\bar{h}_b$, since it arises from binary string interactions and thus $\bar{h}_b \sim \sqrt{\Delta}/r_o$. Recall that $r_w \ll r_o$, so the contribution from gravitational interactions in the binary string is not the same order as the contribution arising from a single string, in fact we expect it to be subleading. To confirm this expectation, we analyze the diagrams in Fig. 5. We first consider the case in which the matter vertex is at $\mathcal{O}(v^0)$:

$$\tag{47}$$

The effective action arising from these two diagrams is $\mathcal{O}(\frac{L}{v^2} \frac{\bar{h}_b}{M_{\text{pl}}} (\frac{\Delta}{v^4})^2)$. Due to the symmetry of the vertices' tensor structure, the diagrams shown in Eq. (47) automatically vanish. We then compute the next-leading-order matter
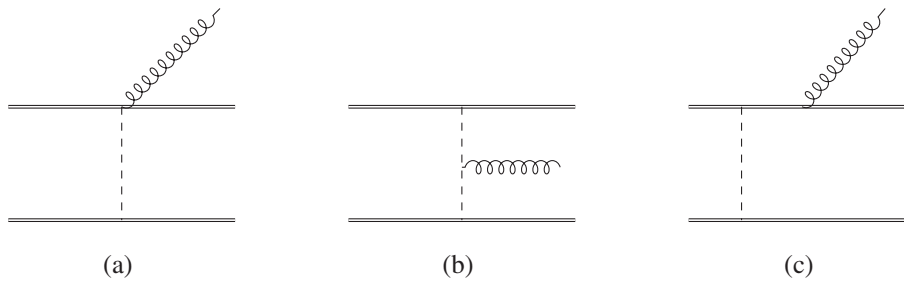

FIG. 5.   The Feynman diagrams contributing to gravitational radiation from binary strings.

vertex $\mathcal{O}(v^1)$ and find that each diagram vanishes on its own.

$$\text{(48)}$$

Another possible diagram is shown in Fig. 5(c), this is the first nonvanishing contribution to the effective action, but it contributes at $\mathcal{O}(L\frac{\Delta^2}{v^6}\frac{\bar{h}_b}{M_{\text{pl}}})$, which is smaller than the single string contribution computed in the previous section.

C. Radiation power analysis

The preceding analysis shows that Eq. (46) is the lowest order contribution to the effective action with a single radiation graviton. To complete the derivation of the full EFT, we would now like to integrate out $\bar{h}$,

$$\exp[iS^{\text{eff}}[x^i]] = \int \mathcal{D}\bar{h}_{\mu\nu} \exp iS^{\text{eff}}[x^i, \bar{h}]. \quad (49)$$

As in the previous case, we are not going to perform the path integral, and instead will again simply use the matching procedure.
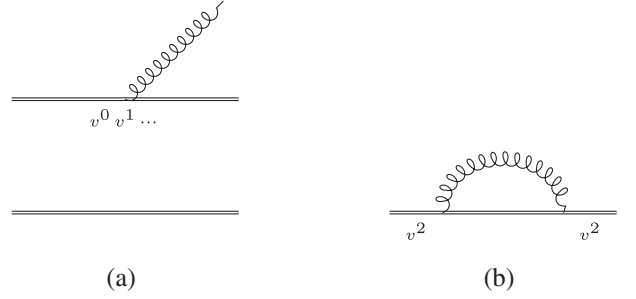
(a)

(b)

FIG. 6. Diagram (a) corresponds to the a graviton radiated from a single string, notice that there are no gravitational interactions between strings. Diagram (b) is the self-energy diagram, whose imaginary part is related to the power radiated.

Since we are interested in what gravitational wave detectors may observe, the relevant quantity we would like to compute is the power radiated by the system. The power radiated is related to the imaginary part of the effective action by

$$\frac{1}{T}\text{Im}[S^{\text{eff}}[x^i]] = \frac{1}{2}\int dEd\Omega \frac{d^2\Gamma}{dEd\Omega}, \qquad dP = Ed\Gamma, \quad (50)$$

where T is the time over which the detector observes the gravitational radiation. At leading order, we can simply match the scattering amplitude corresponding to the self-energy diagram on the right-hand side of Fig. 6, which involves the string-radiation graviton vertex arising from Eq. (46). This leads to

$$S^{\text{eff}}[x] = -\frac{1}{8M_{\text{pl}}^2}\int d^2\sigma_1 d^2\sigma_2 Q_{ab}Q_{cd}\langle(R_{0a0b} - R_{1a1b})(R_{0c0d} - R_{1c1d})\rangle. \quad (51)$$

Using the graviton correlation function, we find that

$$\langle(R_{0a0b} - R_{1a1b})(R_{0c0d} - R_{1c1d})\rangle = \int_k \frac{-i}{k^2 - i\epsilon} e^{-ik^0(x_1^0 - x_2^0) + ik^1(x_1^1 - x_2^1)}(\eta_{ac}\eta_{bd} + \eta_{ad}\eta_{bc})$$
$$\times \frac{1}{4}\left(\frac{1}{2}k_0^4 - \frac{1}{2}k_0^2\boldsymbol{k}^2 + \frac{1}{2}k_1^4 + \frac{1}{2}k_1^2\boldsymbol{k}^2 + \frac{1}{2}k_0^2k_1^2 + \frac{3}{8}\boldsymbol{k}^4\right), \quad (52)$$

where we have ignored terms proportional to $\eta_{ab}\eta_{cd}$, since Q_{ab} is traceless. In order to obtain the imaginary part of the effective action, we make use of

$$\text{Im}\frac{1}{k^2 - i\epsilon} = i\pi\delta(k^2), \quad (53)$$

and the on-shell condition for the radiation graviton in Eq. (A1). These allow us to write the imaginary part of the string effective action as

$$\text{Im}S^{\text{eff}}[x] = -\frac{3}{512M_{\text{pl}}^2}\int \frac{dk^1 d^2\boldsymbol{k}}{(2\pi)^3}\frac{1}{\sqrt{k_1^2 + \boldsymbol{k}^2}}(\boldsymbol{k}^2 + 2k_1^2)^2$$
$$\times |Q_{ab}(k^0, k^1)|^2|_{k^0 = \sqrt{k_1^2 + \boldsymbol{k}^2}}. \quad (54)$$

where $Q(k^0, k^1)$ is the quadrupole in momentum space,

$$Q(k^0, k^1) = \int d^2\sigma e^{-ik^0x^0 + ik^1x^1}Q(x^0, x^1). \quad (55)$$

Using Eq. (50), we finally obtain the power radiated as,

$$P = \frac{2}{T}\frac{1}{4\pi^2}\int dEE^2\left(-\frac{3}{512M_{\text{pl}}^2}(E^2 + k_1^2)^2|Q_{ab}(E, k^1)|^2\right)$$
$$= -\frac{3}{1024\pi M_{\text{pl}}^2}\langle((\partial_\tau^3 + \partial_\tau\partial_\sigma^2)Q_{ab})^2\rangle, \quad (56)$$

where $E = \sqrt{k_1^2 + \boldsymbol{k}^2}$ and angled brackets represent averaging over the observation time T. Here, k_1^{-1} denotes the
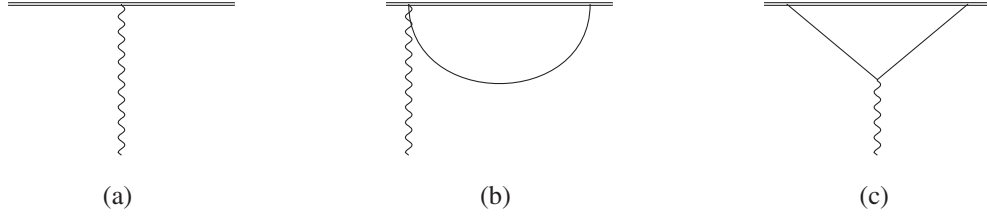
FIG. 7. The divergent piece for $(a): T_{(0)}^{\mu\nu}(k), (b) + (c): T_{(1)}^{\mu\nu}(k)$. The wiggly line denotes the background field $\mathcal{H}$, while the other solid internal lines correspond to the perturbation field $\mathfrak{h}$.

typical wavelength scale of the wiggles. If we were able to detect this gravitational wave signal directly, it would imply a special direction and break cosmological spatial isometry. However, the gravitational waves generated by cosmic strings mainly contribute to the stochastic gravitational wave background thus we do not expect to observe this feature.

IV. THE RENORMALIZED STRING TENSION AND THE BACKGROUND METRIC

An important feature of the relativistic theory described by the actions in Eqs. (2) and (3) is that it gives rise to a classical renormalization group (RG) flow. A nontrivial flow is expected due to the singular nature of the string as described by the Nambu-Goto action. This situation is not exclusive to the string action; other examples of classical RG flows can be found in [14,29,30,37]. In all these cases, the singular nature of the sources leads to UV divergences [38]. This does not imply that our theory is sick; it just signals our ignorance of the structure of the source at small distances. In fact, these UV divergences can be regularized and renormalized by standard field theory methods. In the point-particle case of [14], the coefficients of higher order operators, which encode the effects of the finite size of the sources, are renormalized at 2-loops.[5] By using dimensional analysis, we can see that codimension $n \geq 2$ sources lead to a logarithmic divergence at the $(n-1)$-loop level. Thus, we expect to find this UV divergence at the 1-loop level for the string living in a $(3+1)$d spacetime.

In order to renormalize our relativistic string theory we will consider a single string in a linearized gravitational background and compute the loop corrections to its stress-energy tensor. For simplicity, we will restrict ourselves to the case of a static, straight string. We will also reconstruct the coordinate-space metric, $h_{\mu\nu}(x)$, of the string and verify that at leading order it corresponds to a locally flat spacetime with a global deficit angle [31].

We begin by considering the effective action for the source string in a linearized background, $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ which can be written as[6]

$$\Gamma[g_{\mu\nu}] = -\frac{1}{2M_{\text{pl}}} \int \frac{d^4k}{(2\pi)^4} h_{\mu\nu}(-k)(T^{(0)\mu\nu}(k)$$
$$+ T^{(1)\mu\nu}(k) + T^{(2)\mu\nu}(k) + \cdots). \quad (57)$$

The leading order term corresponds to the original stress-energy tensor of the static, straight string and is given by

$$T^{(0)\mu\nu}(k) = T_{\text{string}}^{\mu\nu}$$
$$= T(2\pi)^2\delta(k^0)\delta(k^1)(u^\mu u^\nu - w^\mu w^\nu), \quad (58)$$

which corresponds to the tree level diagram shown in Fig. 7(a). Here, u^μ and w^μ are unit vectors in the x^0 and x^1 directions respectively.

Higher-order terms in Eq. (57) correspond to corrections to the stress-energy tensor due to gravitational self-interactions. To include self-interactions of the graviton, it is convenient to use the background field method. We first split the metric perturbation $h_{\mu\nu}$ into two terms,

$$h_{\mu\nu} = \mathcal{H}_{\mu\nu} + \mathfrak{h}_{\mu\nu}, \quad (59)$$

where $\mathcal{H}$ is the background and $\mathfrak{h}$ is the fluctuation. We then proceed to integrate out the fluctuation degree of freedom to obtain the effective action for the background field

$$e^{i\Gamma[\mathcal{H}]} = \int \mathcal{D}[\mathfrak{h}_{\mu\nu}]e^{iS_{\text{EH}}[\mathcal{H}+\mathfrak{h}_{\mu\nu}]+iS_{\text{GF}}[\mathfrak{h}_{\mu\nu}]-\frac{i}{2}\int T_{\text{string}}^{\mu\nu}(\mathcal{H}_{\mu\nu}+\mathfrak{h}_{\mu\nu})}$$
$$= e^{-\frac{i}{2}\int T_{\text{string}}^{\mu\nu}\mathcal{H}_{\mu\nu}} \int \mathcal{D}[\mathfrak{h}_{\mu\nu}]e^{iS_{\text{EH}}[\mathcal{H}+\mathfrak{h}_{\mu\nu}]+iS_{\text{GF}}[\mathfrak{h}_{\mu\nu}]-\frac{i}{2}\int T_{\text{string}}^{\mu\nu}\mathfrak{h}_{\mu\nu}}, \quad (60)$$

where the gauge fixing term is given by

$$S_{\text{GF}}[\mathfrak{h}] = -M_{\text{pl}}^2 \int d^4x\sqrt{-\bar{g}}\Gamma_\mu[\mathfrak{h}]\Gamma^\mu[\mathfrak{h}],$$
$$\Gamma_\mu[\mathfrak{h}] = D_\alpha\mathfrak{h}^\alpha_{\ \mu} - \frac{1}{2}D_\mu\mathfrak{h}^\alpha_{\ \alpha}, \quad (61)$$

---

[5]Note that these loops are not real quantum loops as in standard Feynman graphs, since the worldline does not contribute to the Feynman diagram with a propagator. The point-particle is considered to be a classical source. The situation is the same for the string.

[6]The energy-momentum tensor is defined as $T^{\mu\nu} = -\frac{2}{\sqrt{|g|}}\frac{\partial S}{\partial h_{\mu\nu}}$. Here we have an additional $-\frac{1}{2}$ factor in the effective action compared with the convention used in [14].

with D_α the covariant derivative of the background metric $\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \mathcal{H}_{\mu\nu}$. Note that only the fluctuation field $\mathfrak{h}$ is gauge-fixed, since the background field does not propagate. This ensures that the effective action is gauge invariant and simplifies the computation of the corrections to the stress-energy tensor. We can now write the effective action as

$$\Gamma = -\frac{1}{2}\int T^{\mu\nu}_{\text{string}}\mathcal{H}_{\mu\nu}$$
$$- i\log\left(\int \mathcal{D}[\mathfrak{h}_{\mu\nu}]e^{iS_{\text{EH}}+iS_{\text{GF}}-\frac{i}{2}\int T^{\mu\nu}_{\text{string}}\mathfrak{h}_{\mu\nu}}\right), \quad (62)$$

which is the one-particle-irreducible (1PI) effective action. The leading order diagrams contributing to the effective action are shown in Fig. 7. In the following section, we calculate these diagrams, which give the renormalized energy-momentum tensor with two tension insertions.

A. Renormalized energy-momentum tensor with two tension insertions

We proceed to compute the correction to the stress-energy tensor from graphs with two tension insertions. At this order, we have two 1PI diagrams [see Fig. 7, diagrams (b) and (c)]. The two-graviton tension vertex that appears in diagram (b) of Fig. 7 can be obtained from the term of order $v^0\mathfrak{h}\mathcal{H}$ in the expansion of the Lagrangian,

$$\mathcal{V}^{(\mathfrak{h}\mathcal{H})}_{\mu\nu,\alpha\beta} \propto -I_{\mu\nu,01}I_{\alpha\beta,01}$$
$$+\frac{1}{4}(I_{\mu\nu,00}+I_{\mu\nu,11})(I_{\alpha\beta,00}+I_{\alpha\beta,11}). \quad (63)$$

On the other hand, the Feynman rule for the string-$\mathfrak{h}$ vertex can be obtained in a similar way to Eq. (35)

$$\gamma^{(\mathfrak{h})}_{\alpha\beta} = \frac{iT}{M_{\text{pl}}}(2\pi)^2\delta(k^0)\delta(k^1)\mathcal{V}^{(\mathfrak{h})}_{\alpha\beta},$$
$$\mathcal{V}^{(\mathfrak{h})}_{\alpha\beta} = \frac{1}{2}(u^\mu u^\nu - w^\mu w^\nu)I_{\mu\nu,\alpha\beta}. \quad (64)$$

One can see that the contribution from this diagram vanishes due to the symmetries of the vertices involved, as in Eq. (37),

$$\text{Fig. 7(b)} \propto \mathcal{V}^{(\mathfrak{h}\mathcal{H})}_{\mu\nu,\alpha\beta}P^{\mu\nu,\rho\sigma}\mathcal{V}^{(\mathfrak{h})}_{\rho\sigma} = 0. \quad (65)$$

Diagram (c) is more involved since it contains the 3-graviton vertex from the Einstein-Hilbert action. After expanding the EH action around a flat metric, we see that the Lagrangian at order h^3 [39] is

$$\mathcal{L}_{(3)} = -2h^{\alpha,\mu}_\beta h^\nu_\alpha h^\beta_{\mu,\nu} + h^\alpha_\beta h^{\beta,\mu}_\nu h^\nu_{\alpha,\mu} - h^\alpha_\beta h^{\mu,\nu}_\alpha h^\beta_{\nu,\mu} - h^\alpha_\beta h^{\beta,\mu}_\alpha h^\nu_{\nu,\mu}$$
$$-\frac{1}{2}h^\alpha_{\beta,\mu}h^{\beta,\nu}_\alpha h^\mu_\nu - h^{\alpha,\mu}_{\alpha,\nu}h^\beta_\mu h^\nu_\beta + \frac{1}{2}h^\mu_\mu h^\alpha_{\beta,\nu}h^{\nu,\beta}_\alpha$$
$$-\frac{1}{4}h^\alpha_\alpha h^{\mu,\beta}_\nu h^\nu_{\mu,\beta} + \frac{1}{4}h^\alpha_\alpha h^{\beta,\mu}_\beta h^\nu_{\nu,\mu}$$
$$+\frac{1}{2}h^\alpha_\alpha h^{\mu,\nu}_{\mu,\beta}h^\beta_\nu - h^\mu_\nu h^{\nu,\alpha}_{\mu,\beta}h^\beta_\alpha, \quad (66)$$

where the graviton should be split as in Eq. (59). The vertex, $\Gamma_{\mu_1\nu_1,\mu_2\nu_2,\mu\nu}$, arising from the interaction $\mathcal{H}\mathfrak{h}\mathfrak{h}$ can be obtained from the expression above. The gauge fixing term also contributes to the $\mathcal{H}\mathfrak{h}\mathfrak{h}$ vertex, but its contribution to the stress tensor ultimately vanishes, as expected, since a gauge fixing term does not affect the physics.

Having obtained the Feynman rules for all the required vertices, we can now compute diagram (c) in Fig. 7,

$$\text{Fig. 7(c)} = \left(\frac{iT}{M_{\text{pl}}}\right)^2\int_{k_1,k_2}\Gamma_{\mu_1\nu_1,\mu_2\nu_2,\mu\nu}P^{\mu_1\nu_1,\alpha\beta}P^{\mu_2\nu_2,\rho\sigma}\frac{i^2}{k_1^2k_2^2}\mathcal{V}^{(\mathfrak{h})}_{\alpha\beta}\mathcal{V}^{(\mathfrak{h})}_{\rho\sigma}$$
$$= -\frac{T^2}{4M_{\text{pl}}^2}(2\pi)^2\delta(k_3^0)\delta(k_3^1)\left(k_3^2(u_\mu u_\nu - w_\mu w_\nu)\left(1-\frac{5}{2}\epsilon\right) + 2(k_{3\mu}k_{3\nu} - k_3^2\eta_{\mu\nu})\epsilon\right)I_0, \quad (67)$$

where

$$I_0 \equiv \int\frac{d^{d-2}\boldsymbol{k}_1}{(2\pi)^2}\frac{1}{\boldsymbol{k}_1^2(\boldsymbol{k}_1+\boldsymbol{k}_3)^2} = -\frac{1}{2\pi\boldsymbol{k}_3^2}\left(\frac{1}{\epsilon}+\gamma_E+i\pi+\log\left(\frac{\boldsymbol{k}_3^2}{4\pi}\right)\right) \quad (68)$$

is calculated by regularizing the integral via dimensional regularization, as described in Eq. (A2).

From this result we see that the energy-momentum tensor at 1-loop level is,

$$T^{(1)}_{\mu\nu} = \frac{T^2}{8\pi M_{\text{pl}}^2}(2\pi)^2\delta(k_3^0)\delta(k_3^1)\left\{(u_\mu u_\nu - w_\mu w_\nu)\left(\frac{1}{\epsilon}+\log\left(\frac{\boldsymbol{k}_3^2}{4\pi\mu^2}\right)+\gamma_E+i\pi\right)\right.$$
$$\left. +2\left(\frac{k_{3\mu}k_{3\nu}}{\boldsymbol{k}_3^2}-\eta_{\mu\nu}\right)-\frac{5}{2}(u_\mu u_\nu - w_\mu w_\nu)\right\}, \quad (69)$$

As expected, this is a divergent result that can be made finite by adding appropriate counter terms. In fact, since this correction has the same form as Eq. (58), the $1/\epsilon$ divergence can be absorbed by a counterterm

$$T_{\rm CT}^{\mu\nu} = T(2\pi)^2\delta(k^0)\delta(k^1)\delta_T(u^\mu u^\nu - w^\mu w^\nu). \tag{70}$$

This is equivalent to rewriting the bare string tension as $T = T^r(1 + \delta_T)$, where T^r is the renormalized tension, and neglecting higher order δ_T terms. Working in the $\overline{\rm MS}$ scheme, we then find that the renormalized energy-momentum tensor at 1-loop order is,

$$
\begin{aligned}
T_{\mu\nu}^{\rm r}(\boldsymbol{k}) = {} & T^r(2\pi)^2\delta(k^0)\delta(k^1)(u^\mu u^\nu - w^\mu w^\nu) \\
& - \frac{T^{r2}(\mu)}{8\pi M_{\rm pl}{}^2}(2\pi)^2\delta(k^0)\delta(k^1)\left(2\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + (u_\mu u_\nu - w_\mu w_\nu)\left(\log\frac{\boldsymbol{k}^2}{\mu^2} - \frac{5}{2}\right)\right),
\end{aligned}
\tag{71}
$$

where the $\log(\boldsymbol{k}^2/\mu^2)$ term is the expected UV divergence that was found in previous work [40,41] up to the choice of regularization scheme. Although we worked in $d = 4 + 2\epsilon$ dimensions to regulate it, the leftover logarithmic divergence is an ultraviolet one. Notice that the density $\rho = -T^0{}_0$ and pressure $p_1 = T^1{}_1$ have equal magnitude but opposite sign, so they do not contribute to the Newtonian potential $\nabla^2\Phi = 4\pi G(\rho + p_1 + p_2 + p_3)$. However, after diagonalizing the spatial directions, one finds that $p_2 + p_3$ is nonzero. This shows that there is a nonzero Newtonian potential around the static infinitely-long cosmic string at $\mathcal{O}(\frac{T^{r2}(\mu)}{8\pi M_{\rm pl}{}^2})$ and that the locally flat metric result is only correct at leading order. We will confirm this point in the next section by explicitly calculating the induced metric perturbation and the Ricci scalar.

As usual, physics should not depend on the arbitrary renormalization scale, and thus the effective action is independent of μ. This leads to a classical RG flow for the string's tension given by

$$\mu\frac{dT^r}{d\mu} = -\frac{(T^r)^2}{4\pi M_{\rm pl}{}^2}. \tag{72}$$

Once we know the RG flow of the tension we can match this EFT coefficient to the expression arising from a microscopic theory of the string, such as that of [32], at the scale $\mu = 1/r_s$. We refer the reader to [33] for more details on how this matching procedure should be performed.

B. Spacetime metric generated by a static, straight string

From the renormalized stress-energy tensor computed above, we can reconstruct the metric sourced by the string. We calculate $h_{\mu\nu}(k)$ from our effective action (57) as

$$
\begin{aligned}
h_{\mu\nu}(k) &= -\frac{i}{k^2}P_{\mu\nu,\alpha\beta}\frac{i}{M_{\rm pl}}\frac{\delta\Gamma[g]}{\delta h_{\alpha\beta}(-k)}\bigg|_{h=0} \\
&= -\frac{1}{k^2}\frac{1}{2M_{\rm pl}{}^2}\left(T_{\mu\nu}^r - \frac{1}{2}\eta_{\mu\nu}T^{r\alpha}{}_\alpha\right),
\end{aligned}
\tag{73}
$$

where the energy-momentum tensor corresponds to the renormalized one from Eq. (71). Performing an inverse Fourier transform we find at leading order

$$
\begin{aligned}
h_{\mu\nu}^{(0)}(x) &= -\frac{T^r}{2M_{\rm pl}{}^2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{-(k^0)^2 + (k^1)^2 + \boldsymbol{k}^2}(2\pi)^2\delta(k^0)\delta(k^1)e^{-ikx}{\rm diag}\{0,0,1,1\} \\
&= -\frac{T^r}{2M_{\rm pl}{}^2}\int\frac{d^2\boldsymbol{k}}{(2\pi)^2}\frac{1}{\boldsymbol{k}^2}e^{ik\cdot x}{\rm diag}\{0,0,1,1\} \\
&= \frac{T^r}{4\pi M_{\rm pl}{}^2}\log\left(\frac{r}{r_{\rm IR}}\right){\rm diag}\{0,0,1,1\},
\end{aligned}
\tag{74}
$$

where $r^2 = |x_a x^a|$ is the radial distance in the plane orthogonal to the string. This recovers the locally Minkowski metric around a static straight string upon a coordinate transformation [31]. We can now compute the correction arising from the 1-loop calculation. We find that at this order the metric reads

$$h_{00}^{(1)}(x) = -h_{11}^{(1)}(x) = -\frac{(T^r)^2}{32\pi^2 M_{\rm pl}^4} \log\left(\frac{r}{r_{\rm IR}}\right),$$

$$h_{ab}^{(1)}(x) = -\frac{(T^r)^2}{32\pi^2 M_{\rm pl}^4}\left(\eta_{ab} - \frac{x_a x_b}{|\boldsymbol{x}|^2} + \eta_{ab}\left(\log\left(\frac{r r_{\rm IR}}{4e^{-2\gamma_E} r_{\rm UV}^2}\right) + 5\right)\log\left(\frac{r}{r_{\rm IR}}\right)\right), \tag{75}$$

where $r_{\rm UV} = 1/\mu$ is the UV cutoff; the details of the calculation can be found in Appendix A. From this we find that the Ricci scalar at NLO is

$$\begin{aligned}
\mathcal{R}_{\rm NLO} &= \frac{1}{4}h^{(0)}\partial^2 h^{(0)} - \frac{1}{2}h^{(0)\mu\nu}\partial^2 h_{\mu\nu}^{(0)} - \frac{1}{2}\partial^2 h^{(1)} \\
&= \frac{(T^r)^2}{16\pi^2 M_{\rm pl}^4}\frac{1}{r^2},
\end{aligned} \tag{76}$$

where $h^{(0)}$ is short for $h^{(0)\mu}{}_\mu$. This explicitly shows that the metric around the infinitely-long cosmic string is not locally flat at NLO, further confirming the nonvanishing Newtonian potential arising from the spatial part of the stress-energy tensor as discussed in the previous section.

V. CONCLUSION

Effective field theory techniques are playing an increasingly prominent role in our understanding of the gravitational radiation generated by astrophysical sources, such as binary black hole systems. While extensive effort has gone into a detailed understanding of how these methods can be applied to extract large-scale information from point particle sources, less is known about how to extend the techniques to more general settings.

In this paper we have begun such an analysis by focusing on a class of extended objects—cosmic strings—that may well be produced during particle physics phase transitions in the early universe. Cosmic strings can generate gravitational radiation in a variety of ways, with some of the most important and relevant being due to their relativistic motion. Nevertheless, the simplest extension of point-particle techniques is to the more artificial setting in which strings orbit each other in a nonrelativistic way and we have shown that even this simple extension of the basic setup requires careful treatment and contains novel features.

By studying a system of binary infinitely-long cosmic strings, we have seen that the nonrelativistic treatment requires an extra expansion parameter, compared to the point particle case. We have derived the relevant power counting rules and the associated generalized virial theorem, and have established the EFT for a binary cosmic string coupled to gravity. We have calculated the leading-order gravitational potential between a pair of wiggly strings and have obtained the gauge-invariant effective action for radiation gravitons order by order. We have then compared the gravitational radiation from a single NR cosmic string with that from the binary cosmic system, for which both the potential graviton and graviton-sourced radiation are included.

Finally, we have further extended the treatment to the relativistic case of a static straight string, and have analyzed the nontrivial classical RG flow. We obtained the renormalized stress-energy tensor at 1-loop order and showed that the divergences arising in this calculation imply a classical RG flow of the string's tension. Using this result, we have reconstructed the perturbed metric to NLO around a static straight cosmic string, finding that at NLO the metric around the cosmic string is no longer locally flat.

A more realistic setup for the computation of gravitational waves from cosmic strings should involve relativistic cosmic string loops. In this case, we expect that the EFT would also have two expansion parameters, where one could still be the gap between the velocity and bending (Δ), but determining the other one would require a careful analysis of the new setup.

APPENDIX A: USEFUL INTEGRALS

In this Appendix, we discuss some important integrals used throughout the paper. In Sec. III C, we made use of the following integral

$$\int \frac{d^4 k}{(2\pi)^4}\delta((k_0)^2 - k_1^2 - \boldsymbol{k}^2) = \int \frac{dk^0 dk^1 d^2\boldsymbol{k}}{(2\pi)^4}\frac{1}{2\sqrt{\boldsymbol{k}^2 + k_1^2}}\left(\delta\left(k^0 + \sqrt{\boldsymbol{k}^2 + k_1^2}\right) + \delta\left(k^0 - \sqrt{\boldsymbol{k}^2 + k_1^2}\right)\right), \tag{A1}$$

to obtain Eq. (54). Afterwards, the integral I_0 was introduced in Eq. (68). Here, we use dimensional regularization in $d = 4 + 2\epsilon$ to compute it

$$
\begin{aligned}
I_0 &= \int \frac{d^{d-2}\mathbf{k}}{(2\pi)^{d-2}} \frac{1}{\mathbf{k}^2(\mathbf{k}+\mathbf{p})^2} = \int \frac{d^{d-2}\mathbf{k}}{(2\pi)^{d-2}} \int_0^1 dx \frac{1}{((1-x)\mathbf{k}^2 + x(\mathbf{k}+\mathbf{p})^2)^2} \\
&= \int_0^1 dx \int \frac{d^{d-2}\mathbf{k}'}{(2\pi)^{d-2}} \frac{1}{(\mathbf{k}'^2 - \Delta)^2}, \qquad \Delta = (x^2 - x)p^2 \\
&= \int_0^1 dx \frac{1}{(x^2 - x)^{3-d/2}} \frac{1}{(4\pi)^{d/2-1}} \Gamma\left(3 - \frac{d}{2}\right)(p^2)^{d/2-3} \\
&= \frac{1}{(4\pi)^{d/2-1}} \Gamma\left(3 - \frac{d}{2}\right)(p^2)^{d/2-3} \frac{(-1)^{1+\epsilon}\Gamma(\epsilon)^2}{\Gamma(2\epsilon)} \\
&= -\frac{1}{2\pi p^2}\left(\frac{1}{\epsilon} + \gamma_E + i\pi + \log\left(\frac{p^2}{4\pi}\right)\right).
\end{aligned}
\tag{A2}
$$

The other relevant integrals correspond to 2d Fourier transformations arising when we compute the metric at NLO in Eq. (75). Consider the following integral

$$
\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\mathbf{k}^2} e^{i\mathbf{k}\cdot\mathbf{x}} = \int \frac{k\,dk\,d\theta}{(2\pi)^2} \frac{1}{k^2} e^{ikr\cos\theta} = \int \frac{dk}{2\pi} \frac{1}{k} J_0(kr),
\tag{A3}
$$

which diverges logarithmically at small k. Introducing k_ϵ as an IR cutoff, we have

$$
\int_{k_\epsilon}^{\infty} dk \frac{1}{k} J_0(kr) = -\gamma_E + \frac{k_\epsilon^2 r^2}{8} {}_2F_3\left(1, 1; 2, 2, 2; -\frac{k_\epsilon^2 r^2}{4}\right) + \log\left(\frac{2}{k_\epsilon r}\right).
\tag{A4}
$$

We then expand Eq. (A4) around $k_\epsilon r \ll 1$, and rewrite $k_\epsilon = \frac{1}{\tilde{r}_{\text{IR}}}$,

$$
\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\mathbf{k}^2} e^{i\mathbf{k}\cdot\mathbf{x}} = -\frac{1}{2\pi} \log\left(\frac{r}{2e^{-\gamma_E}\tilde{r}_{\text{IR}}}\right) \equiv -\frac{1}{2\pi} \log\left(\frac{r}{r_{\text{IR}}}\right),
\tag{A5}
$$

where we absorbed the constants into r_{IR}, and defined $r \equiv |\mathbf{x}|$. Similarly, we have

$$
\frac{1}{|\mathbf{x}|^2} \int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{(\mathbf{k}\cdot\mathbf{x})^2}{(\mathbf{k}^2)^2} e^{-i\mathbf{k}\cdot\mathbf{x}} = -\frac{1}{4\pi}\left(\log\left(\frac{r}{r_{\text{IR}}}\right) + \frac{1}{2}\right).
\tag{A6}
$$

Thus we have,

$$
\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{k_a k_b}{(\mathbf{k}^2)^2} e^{-i\mathbf{k}\cdot\mathbf{x}} = \left(-\frac{1}{4\pi} \log\left(\frac{r}{r_{\text{IR}}}\right) + \frac{1}{8\pi}\right)\eta_{ab} - \frac{1}{4\pi}\frac{x_a x_b}{|\mathbf{x}|^2}.
\tag{A7}
$$

and

$$
\int \frac{d^2\mathbf{k}}{(2\pi)^2} \frac{1}{\mathbf{k}^2} \log(\mathbf{k}^2) e^{i\mathbf{k}\cdot\mathbf{x}} = \int \frac{dk}{2\pi} \frac{1}{k} \log(k^2) J_0(kr) = \frac{1}{2\pi}\left(\log^2\left(\frac{r}{2e^{-\gamma_E}}\right) - \log^2(\tilde{r}_{\text{IR}})\right).
\tag{A8}
$$

Notice that in Eq. (A8), we cannot have a clean expression in terms of r/r_{IR} as in Eq. (A5) and (A6). With these equations, we are able to obtain the metric perturbation in Eq. (75).

APPENDIX B: POWER COUNTING RULE

In this section, we are going to go show how the power counting rule Eq. (24) holds at higher orders.

We start from the virial theorem for the binary strings which tells us that

$$V = \left\langle \int d\sigma \left(\frac{\partial L}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a + \frac{\partial L}{\partial \mathbf{x}'_a} \mathbf{x}'_a \right) \right\rangle + \left\langle \int d\sigma \mathbf{x}_a \frac{\partial L}{\partial \mathbf{x}_a} \right\rangle = 0.$$

Expanding the 1st and 2nd terms we obtain,

$$\frac{\partial L}{\partial \dot{\mathbf{x}}_a} \dot{\mathbf{x}}_a + \frac{\partial L}{\partial \mathbf{x}'_a} \mathbf{x}'_a = \boldsymbol{v}^2 - \boldsymbol{x}'^2 + (\boldsymbol{v}^2 - \boldsymbol{x}'^2)h_{22} + (\boldsymbol{v}^2 + \boldsymbol{x}'^2)h_{00}$$

$$+ (\boldsymbol{v}^2 + \boldsymbol{x}'^2)h_{00}h_{22} + \cdots$$

$$\sim \epsilon_\Delta + \epsilon_\Delta \epsilon_h + \epsilon_v^2 \epsilon_h - \epsilon_v \epsilon_\Delta \epsilon_h$$

$$+ \epsilon_v^2 \epsilon_h^2 - \epsilon_v \epsilon_\Delta \epsilon_h^2 + \cdots, \qquad (B1)$$

$$-\mathbf{x}_a \frac{\partial L}{\partial \mathbf{x}_a} = -\frac{1}{2}(\boldsymbol{v}^2 - \boldsymbol{x}'^2)\frac{h_{22}}{\partial x^a}x^a - \frac{1}{2}(\boldsymbol{v}^2 + \boldsymbol{x}'^2)\frac{h_{00}}{\partial x^a}x^a$$

$$-\frac{1}{2}(\boldsymbol{v}^2 + \boldsymbol{x}'^2)\left(\frac{h_{00}}{\partial x^a}x^a h_{22} + \frac{h_{22}}{\partial x^a}x^a h_{00} \right) + \cdots$$

$$\sim -\epsilon_\Delta \epsilon_{GT} - \epsilon_v^2 \epsilon_{GT} + \epsilon_v \epsilon_\Delta \epsilon_{GT} - \epsilon_\Delta^2 \epsilon_{GT}$$

$$- \epsilon_v^2 \epsilon_{GT} \epsilon_h + \epsilon_v \epsilon_\Delta \epsilon_h - \epsilon_v^2 \epsilon_{GT} \epsilon_h + \cdots \qquad (B2)$$

The terms appearing in V can be rearrange as a series in $\mathcal{O}(\epsilon_\Delta^i)$ as shown in Table II. As discussed in the main text,

$\mathcal{O}(|\log \frac{r}{r_{\text{IR}}}|) > 1$ so $\mathcal{O}(-\epsilon_h) \gtrsim \mathcal{O}(\epsilon_{GT})$. Thus, from the terms that do not involve ϵ_Δ, $\epsilon_v^2 \epsilon_h$ is the lowest order one that could cancel with ϵ_Δ to satisfy the virial theorem. By choosing $\mathcal{O}(-\epsilon_h \epsilon_v^2) \sim \mathcal{O}(\epsilon_\Delta)$, one can see that each term in the second column of Table II can be matched to one in the first column, i.e., $\mathcal{O}(\epsilon_\Delta \epsilon_h) \sim -\mathcal{O}(\epsilon_v^2 \epsilon_h^2)$, $\mathcal{O}(\epsilon_\Delta \epsilon_{GT}) \sim \mathcal{O}(-\epsilon_v^2 \epsilon_h \epsilon_{GT})$, etc. One may notice the only term left in the first column is $\epsilon_v^2 \epsilon_{GT}$. In fact, it is not necessary to pick a specific order for ϵ_{GT} as this does not affect our result since $\epsilon_v^2 \epsilon_{GT} < \epsilon_\Delta$. Here, we are only analyzing the order of magnitude of the terms in V and the possibility for these to cancel each other and satisfy the Virial theorem. We have verified that $\mathcal{O}(-\epsilon_h \epsilon_v^2) \sim \mathcal{O}(\epsilon_\Delta)$ leads to $V = 0$ up to $\mathcal{O}(\epsilon_\Delta^2)$.

TABLE II. Expansion in orders of magnitude of terms in V as shown in Eqs. (B1) and (B2).

ϵ_Δ^0	ϵ_Δ^1	ϵ_Δ^2
$\epsilon_v^2 \epsilon_h$	ϵ_Δ	$\cdots$
$\epsilon_v^2 \epsilon_{GT}$	$\epsilon_\Delta \epsilon_h$	
$\epsilon_v^2 \epsilon_h^2$	$\epsilon_\Delta \epsilon_{GT}$	
$\epsilon_v^2 \epsilon_h \epsilon_{GT}$	$\cdots$	
$\cdots$		

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