Double field theory and pseudosupersymmetry

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(Received 1 February 2021; accepted 17 June 2021; published 15 July 2021)

Many of the useful features of supergravities, such as admitting supersymmetric bosonic backgrounds governed by first-order Bogomol'nyi–Prasad–Sommerfield equations, can be realized in a much broader setting by relaxing the requirement of closure of the superalgebra beyond the level of quadratic fermion terms. The resulting pseudo-supersymmetric theories can be defined in arbitrary spacetime dimensions. We focus here on the $\mathcal{N}=1$ pseudo-supersymmetric extensions of the arbitrary-dimensional bosonic string action, which were constructed a few years ago. In this paper, we recast these in the language of double field theory. More precisely, we construct the action and the corresponding pseudo-supersymmetry transformation rules in terms of $\mathrm{O}(D)\times\mathrm{O}(D)$ covariant derivatives, and we discuss consistent truncations on manifolds with generalized G structure. We thereby obtain a natural generalization of the previously known results for $\mathcal{N}=1$ supersymmetric double field theory in $D=10$ to arbitrary dimensions. As explicit examples, we discuss Minkowski $\times\, G$ vacuum solutions and their corresponding pseudo-supersymmetry. We also briefly discuss squashed group manifold solutions, including an example with a Lorentzian signature metric on the group manifold G.

DOI: 10.1103/PhysRevD.104.026008

I. INTRODUCTION

Supersymmetry provides a powerful tool for probing aspects of physics that would otherwise be beyond the limits of computability. One important example is that the second-order nonlinear field equations of Einstein gravity or supergravity can be reduced to first-order equations in certain circumstances, namely when there exist supersymmetric bosonic backgrounds that admit one or more Killing spinors. Beyond the classical level, supersymmetry severely restricts quantum corrections and allows some nonperturbative results to be obtained.

A feature of supersymmetry is that it implies restrictions on the dimension of the spacetime. In particular, beyond 11 dimensions, it is not possible to find any supersymmetric extension of gravity without adding higher spin fields, and this rules out having a Lagrangian description. One might, therefore, conclude that supersymmetry would, in general, be of no help in the study of theories in arbitrary higher dimensions. However, these theories can still possess a pseudo-supersymmetry [1–4], which is a weaker notion of supersymmetry that only involves fermionic terms up to second order, in the action and the transformation rules. This is, in fact, sufficient in order to be able to derive many of the useful features of conventional supersymmetric theories, including the existence of pseudo-Killing spinors in certain backgrounds. Thus, pseudo-supersymmetry still allows the second-order field equations for the bosonic fields to be reduced into first-order Bogomol'nyi–Prasad–Sommerfield (BPS) conditions in such backgrounds. Hence, it can provide a powerful tool in the study of solutions of theories of gravity coupled to matter in arbitrary dimensions.

A well-established framework for exploring the landscape of supergravity vacua is provided by (exceptional) generalized geometry [5–10] and the closely related double or exceptional field theory (DFT/ExFT) [11–28].[1] Particularly interesting for our work are the supersymmetric extensions of bosonic DFT [33–40]. All these approaches share one defining property, namely the unification of local diffeomorphisms with form-field gauge transformations

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[1]There has been a considerable amount of original work in this field, and therefore, we only reference a few key contributions here, which is, of course, highly subjective. We refer to the reviews [29–32] for a complete list of references.

into one unified symmetry group. In the most basic setup, the metric and the 2-form B-field potential of a bosonic string action are combined into the generalized metric, giving rise to the $\mathrm{O}(n,n)$ symmetry of DFT. Equivalently, this structure is captured by the generalized tangent bundle $TM \oplus T^*M$ of generalized geometry. Note that, in general, DFT is capable of capturing backgrounds that go beyond supergravity and generalized geometry [41,42]. Here, however, we shall be concerned with the most conservative case, where the section condition is satisfied globally. In this case, DFT is just a rewriting of supergravity, and it is completely equivalent to generalized geometry. Still, there are two important advantages of this rewriting: First, supersymmetry variations have a much simpler form, and second, Abelian T-duality becomes a manifest symmetry of the string's low-energy effective target space action. A natural question in this context is whether pseudo-supersymmetry permits a similar treatment. We answer this question in the affirmative and demonstrate that it is possible to extend the existing results of supersymmetric DFT from 10 dimensions to arbitrary dimensions.

It is worth emphasizing at this point that it is somewhat nontrivial that an extension of the $\mathcal{N}=1$ supersymmetric DFT to arbitrary higher dimensions is possible. First of all, the fact that one can extend 10-dimensional $\mathcal{N}=1$ supersymmetric supergravity to a pseudo-supersymmetric theory in an arbitrary spacetime dimension [4] is itself rather remarkable; the demonstration of the pseudo-supersymmetry of the Lagrangian depends upon the detailed properties of the spinor representations and the Dirac matrices in the different dimensions. Furthermore, not every supergravity theory admits an arbitrary-dimensional pseudo-supergravity extension. For example, although the 10-dimensional $\mathcal{N}=1$ supergravity admits the extension as in [4], it appears not to be possible to obtain a pseudo-supersymmetric extension of 10-dimensional $\mathcal{N}=2$ supergravity in higher dimensions. (One reason for this was discussed in [1].)

Similar remarks apply to the incorporation of $\mathcal{N}=1$ pseudo-supersymmetry within the framework of double field theory. The generalization of the DFT describing the bosonic string to an arbitrary dimension is, of course, well known and completely straightforward. However, the fact that it is possible to generalize the DFT of the 10-dimensional $\mathcal{N}=1$ string to an arbitrary dimension is somewhat nontrivial, and indeed, establishing that this is possible is one of the main purposes of the present paper. We have presented some of the details of the calculations in Appendix B. As with the construction of the pseudo-supersymmetric extension of the arbitrary-dimensional bosonic string Lagrangian in [4], this depends upon the properties of the spinor representations and the Dirac matrices in the different dimensions.

An important application of this work is in the construction of pseudo-supergravity vacua. In particular, we combine the technique of consistent truncations with pseudo-supersymmetry to show how the field equations in arbitrary dimensions can be simplified significantly. More precisely, in the examples we consider, the consistent truncation renders the field equations algebraic but still quadratic. Clearly, this is already a major simplification, but still quadratic equations with multiple variables can be hard to solve. A similar problem arises in the classification of Lie algebras, whose Jacobi identity is a quadratic constraint. In low dimensions, it is possible to solve it, and this gives rise to a complete classifications of real Lie algebra up to six dimensions [43]. Beyond that, solving the quadratic constraint becomes forbiddingly complicated. A similar situation is encountered in the prototypical example of consistent truncations in DFT, namely in generalized Scherk-Schwarz reductions [44–49]. Compared to a standard geometric reduction on a group manifold with isometry group $G_L \times G_R$, which retains the singlets under either G_L or G_R, the consistent reductions in DFT allow one to retain all the gauge bosons of the complete isometry group. This is a much more complicated reduction because of the potentially dangerous trilinear coupling of massive spin-2 modes to bilinears constructed from the $G_L \times G_R$ Yang-Mills bosons [50]. The existence of a consistent reduction of the $(n+D)$ bosonic string to a D-dimensional group manifold keeping all of the $G_L \times G_R$ gauge bosons was conjectured in [51], with further supporting evidence found in [52]. A complete proof of the consistency was obtained in [53], utilizing the $O(D,D)$ formulation of $(n+D)$-dimensional bosonic string [54]. Combining a generalized Scherk-Schwarz reduction with pseudo-supersymmetry, we show how the quadratic field equations for the remaining fields can be reduced, in appropriate backgrounds, to linear equations. Because of the less restrictive nature of pseudo-supersymmetry, in comparison to ordinary supersymmetry, this can be done in arbitrary spacetime dimensions.

The paper is organized as follows. In Sec. II, we give a short review of the $\mathcal{N}=1$ pseudo-supersymmetric theory. In Sec. III, we reformulate it in terms of generalized geometry and then spell out the conditions for the existence of a consistent truncation. In Sec. IV, we explicitly construct solutions of the form $(\mathrm{Minkowski})_{D-\dim G} \times G$, including a description in the framework of generalized geometry. In Sec. V, we discuss their pseudo-supersymmetry, both in standard field theory and in generalized geometry. In four appendices, we record some relevant properties of spinors in general dimensions; we present some details of the calculations showing that the $\mathcal{N}=1$ pseudo-supersymmetric string can be recast in a DFT framework; we give some useful representations for Dirac matrices in dimensionally reduced spacetimes; and we construct an example of a $(\mathrm{Minkowski})_{D-\dim G} \times G$ background, for $G=SO(5)$, where the metric on the group is squashed. It turns out to have Lorentzian signature, and the squashed background breaks all the pseudo-supersymmetry.

II. PSEUDO-SUPERSYMMETRISED BOSONIC STRING

As described in [4], one can construct a pseudo-supersymmetric fermionic extension of the bosonic string Lagrangian in a completely arbitrary dimension D. That is to say, there exist supersymmetry-like transformation rules that leave the Lagrangian invariant, modulo terms beyond the quadratic order in fermions. In practice, many of the desirable features of supersymmetry, such as the existence of Killing spinors in bosonic backgrounds, BPS conditions, and first-order equations, do not directly depend upon the full closure of the transformations. This means that all the useful consequences of having fermionic symmetries in bosonic backgrounds will equally well arise in the much larger arena of pseudo-supersymmetric theories.

A. Lagrangian and pseudo-supersymmetry transformation rules

The Lagrangian for the pseudo-supersymmetric extension of the bosonic string in an arbitrary dimension was constructed in [4], where it was presented both in the Einstein frame and in the string frame. Here, we reproduce the result from [4] in the string frame, with the following notational changes. Firstly, we denote the spacetime dimension by D rather than d, since in this paper, d will be reserved to denote the generalized dilaton of DFT. Secondly, in order to harmonize our notation with some of the DFT literature, we perform the rescalings,

$$\psi_\mu \to \sqrt{2}\psi_\mu, \qquad \lambda \to \sqrt{2}\lambda, \qquad \epsilon \to \sqrt{2}\epsilon, \tag{2.1}$$

on the fermion fields and pseudo-supersymmetry parameter, and finally, we make the replacement $\Gamma_a \to -\Gamma_a$, which, of course, preserves the Clifford algebra. With these replacements, the D-dimensional pseudo-supersymmetric Lagrangian of [4], in the string frame, becomes

$$\begin{aligned} e^{-1}\mathcal{L} = e^{-2\Phi}\Big[& R + 4(\partial\Phi)^2 - \frac{1}{12}H^2 - \bar\psi_\mu\Gamma^{\mu\nu\rho}D_\nu\psi_\rho \\ & + \bar\lambda \not{D}\lambda - 2\mathrm{i}\sqrt{\beta}\bar\lambda\Gamma^{\mu\nu}D_\mu\psi_\nu - 2\bar\psi_\mu\Gamma^\mu\psi_\rho\partial^\rho\Phi \\ & + \frac{2\mathrm{i}}{\sqrt{\beta}}\bar\psi_\mu\Gamma^\nu\Gamma^\mu\lambda\partial_\nu\Phi + H_{\nu\rho\sigma}\Big\{\frac{1}{24}\bar\psi_\mu\Gamma^{\mu\nu\rho\sigma\lambda}\psi_\lambda \\ & + \frac{1}{4}\bar\psi^\nu\Gamma^\rho\psi^\sigma - \frac{1}{24}\bar\lambda\Gamma^{\nu\rho\sigma}\lambda + \frac{\mathrm{i}}{12\sqrt{\beta}}\bar\psi_\mu\Gamma^{\mu\nu\rho\sigma}\lambda\Big\}\Big], \end{aligned} \tag{2.2}$$

and the pseudo-supersymmetry transformation rules are given by

$$\begin{aligned} \delta\psi_\mu &= D_\mu\epsilon - \frac{1}{8}H_{\mu\nu\rho}\Gamma^{\nu\rho}\epsilon, \\ \delta\lambda &= \mathrm{i}\sqrt{\beta}\left(\Gamma^\mu\partial_\mu\Phi - \frac{1}{12}\Gamma^{\mu\nu\rho}H_{\mu\nu\rho}\right)\epsilon, \\ \delta e^a_\mu &= -\frac{1}{2}\bar\psi_\mu\Gamma^a\epsilon, \\ \delta\Phi &= -\frac{\mathrm{i}}{4\sqrt{\beta}}\bar\epsilon\lambda, \\ \delta B_{\mu\nu} &= \bar\epsilon\Gamma_{[\mu}\psi_{\nu]}. \end{aligned} \tag{2.3}$$

Note that $\delta\psi_\mu$ may be reexpressed in terms of a torsionful connection as $\delta\psi_\mu = D_\mu(\omega_-)\epsilon$, where

$$\omega_{\mu\pm}^{ab} \equiv \omega_\mu^{ab} \pm \frac{1}{2}12H_\mu{}^{ab}. \tag{2.4}$$

The constant β, which is either $+1$ or -1 depending on the dimension D and the spinor representation, characterizes the symmetry property of the gamma matrices,

$$\Gamma_\mu^T = \beta C\Gamma_\mu C^{-1}. \tag{2.5}$$

It is listed for each dimension and representation in Table I in Appendix A. Many further properties of spinors in diverse dimensions are summarized in our notation in [4]. All coefficients in (2.2) and (2.3) were determined by the requirement that the Lagrangian be invariant under the pseudo-supersymmetry transformations, provided that one neglects fermionic terms that would arise from higher fermionic powers in the Lagrangian or pseudo-supersymmetry transformations.

TABLE I. Γ-matrix symmetries and spinor representations in diverse dimensions. S denotes symmetric, A denotes antisymmetric, M denotes Majorana, and S-M denotes symplectic Majorana.

D mod 8	$C\Gamma^{(0)}$	$C\Gamma^{(1)}$	$C\Gamma^{(2)}$	$C\Gamma^{(3)}$	$C\Gamma^{(4)}$	$C\Gamma^{(5)}$	Spinor	β
0	S	S	A	A	S	S	M	$+1$
	S	A	A	S	S	A	S-M	-1
1	S	S	A	A	S	S	M	$+1$
2	S	S	A	A	S	S	M	$+1$
	A	S	S	A	A	S	M	-1
3	A	S	S	A	A	S	M	-1
4	A	S	S	A	A	S	M	-1
	A	A	S	S	A	A	S-M	$+1$
5	A	A	S	S	A	A	S-M	$+1$
6	A	A	S	S	A	A	S-M	$+1$
	S	A	A	S	S	A	S-M	-1
7	S	A	A	S	S	A	S-M	-1

It was shown in [4] that, just like in the case of the supersymmetry transformations for 10-dimensional $\mathcal{N}=1$ supergravity, the integrability conditions obtained by taking commutators of the pseudo-supersymmetry transformations on a bosonic background are satisfied if the full set of field equations for the D-dimensional bosonic string are satisfied.

B. Adding a conformal anomaly term

As was shown in [4], one can also add a "conformal anomaly" term to the Lagrangian. In the string frame, after performing the rescalings (2.1) and the replacement $\Gamma_a \to -\Gamma_a$ detailed above, the additional terms in the Lagrangian take the form,

$$e^{-1}\mathcal{L}_c = e^{-2\Phi}\left[-\frac{m^2}{2} - \frac{m}{2\sqrt{2\beta}}(\bar{\psi}_\mu \Gamma^{\mu\nu}\psi_\nu + 2\sqrt{-\beta}\bar{\psi}_\mu\Gamma^\mu\lambda - \bar{\lambda}\lambda)\right]. \tag{2.6}$$

There are associated additional terms in the fermion transformation rules, given by

$$\delta_{\text{extra}}\psi_\mu = 0, \qquad \delta_{\text{extra}}\lambda = \frac{\mathrm{i}}{2\sqrt{2}}m\epsilon. \tag{2.7}$$

Note that the fermionic extension of the conformal anomaly term in (2.6) really requires a doubling of the fermionic degrees of freedom. This is most easily stated in dimensions $D = 2\text{mod}8$, where we can choose $\beta = -1$, and the basic spinors of the pseudo-supersymmetrised bosonic string would be both Majorana and Weyl (with ψ_μ and ϵ being chiral, and λ antichiral). The fermionic terms in (2.6) would vanish under these conditions but will be non-vanishing if the chirality constraints on the fermions are removed. In cases where $\beta = +1$, the first two fermionic terms in (2.6) will vanish identically if the spinors are Majorana or symplectic-Majorana. In these cases, one can still pseudo-supersymmetrize the conformal anomaly term if one doubles the number of fermions by adding an additional doublet index,

$$\psi_\mu \to \psi_\mu^\alpha, \qquad \lambda \to \lambda^\alpha. \tag{2.8}$$

All the previous fermion bilinears in the Lagrangian will now have α and β indices contracted with $\delta_{\alpha\beta}$. The terms in $\mathcal{L}_c$, on the other hand, will have the α and β indices contracted with $\epsilon_{\alpha\beta}$. An $\epsilon_{\alpha\beta}$ should also be inserted in the extra terms (2.7) in transformation rules for ψ_μ and λ.

III. GENERALIZED GEOMETRY AND PSEUDO-SUPERSYMMETRY

It is possible to simplify the Lagrangian (2.2) considerably by introducing the generalized dilaton d and its superpartner ρ:

$$d = \Phi - \frac{1}{2}\log e, \qquad \rho = \Gamma^\mu\psi_\mu + \frac{\mathrm{i}}{\sqrt{\beta}}\lambda. \tag{3.1}$$

Furthermore, we unify the frame field and the B field by introducing the generalized frame field with the components,

$$E_a^{(+)} = \frac{1}{\sqrt{2}}(e_a^\mu\partial_\mu + e_{\mu a}\mathrm{d}x^\mu - \iota_{e_a}B),$$
$$E_a^{(-)} = \frac{1}{\sqrt{2}}(e_a^\mu\partial_\mu - e_{\mu a}\mathrm{d}x^\mu - \iota_{e_a}B), \tag{3.2}$$

and $\iota_{e_a}B = e_a^\mu B_{\mu\nu}\mathrm{d}x^\nu$. Each of these $2D$ components is a generalized vector on the generalized tangent space $TM \oplus T^*M$. After this identification and the redefinitions above, the pseudo-supersymmetry transformation rules (2.3) and (2.7) can be written in the compact form,

$$\delta\psi_\mu = \nabla_\mu^{(-)}\epsilon, \qquad \delta_{\text{extra}}\psi_\mu = 0,$$
$$\delta\rho = \Gamma^\mu\nabla_\mu^{(+)}\epsilon, \qquad \delta_{\text{extra}}\rho = -\frac{1}{2\sqrt{2\beta}}m\epsilon,$$
$$\langle E_b^{(-)}, \delta E_a^{(+)}\rangle = -\frac{1}{2}\bar{\epsilon}\Gamma_b\psi_a,$$
$$\delta d = -\frac{1}{4}\bar{\epsilon}\rho, \tag{3.3}$$

where the $\mathrm{O}(D)\times\mathrm{O}(D)$ covariant derivatives $\nabla_\mu^{(\pm)}$ play a crucial role. They are defined by [8,34]

$$\nabla_\mu^{(-)}\epsilon = \left(D_\mu - \frac{1}{8}H_{\mu\nu\rho}\Gamma^{\mu\nu}\right)\epsilon,$$
$$\Gamma^\mu\nabla_\mu^{(+)}\epsilon = \left(\Gamma^\mu D_\mu - \frac{1}{24}H_{\mu\nu\rho}\Gamma^{\mu\nu\rho} - \Gamma^\mu\partial_\nu\Phi\right)\epsilon, \tag{3.4}$$

and, as we will see in the next subsection, they also have very nice properties when it comes to consistent truncations.

Like the pseudo-supersymmetry transformation rules, also the action (2.2) simplifies considerably once written in string frame and after applying the redefined fields and the adapted covariant derivatives,

$$e^{2d}\mathcal{L}_D = R + 4(\partial\phi)^2 - \frac{1}{12}H_{(3)}^2 - \bar{\psi}^a\Gamma^b\nabla_b^{(+)}\psi_a - \beta\bar{\rho}\Gamma^a\nabla_a^{(+)}\rho + 2\bar{\psi}^a\nabla_a^{(-)}\rho, \tag{3.5}$$

where β is defined by the use of charge conjugation matrix C in (2.5). All details of the computation relating (3.5) and (2.2) for arbitary target space dimensions D is given in Appendix B.

In 10 dimensions, this Lagrangian matches the one of $\mathcal{N}=1$ double field theory [34] after implementing the solution of the section condition, which removes the dependence on the coordinates conjugate to string winding modes and choosing the parameter $\beta = -1$. The interesting observation here is that this result even holds in arbitrary dimensions once we drop the additional constraints imposed by supersymmetry in favor of pseudo-supersymmetry.

The conformal anomaly terms (2.6) can also be formulated in the language of generalized geometry,

$$e^{2d}\mathcal{L}_c = -\frac{m^2}{2} - \frac{m\sqrt{\beta}}{2\sqrt{2}}(\bar{\rho}\rho - \beta\bar{\psi}_\mu\psi^\mu). \qquad (3.6)$$

To satisfy the pseudo-supersymmetry property, it follows from (2.7) that the variation rule of ρ needs to be modified by δ_{extra} in (3.3).

A. Consistent truncations

The crucial observation for constructing consistent truncations in the $\mathcal{N} = 1$ pseudo-supersymmetric theory is that all relevant quantities, like the Lagrangian, the pseudo-supersymmetry transformation rules, and the field equations, can be written in terms of covariant derivatives $\nabla_\mu^{(\pm)}$. Therefore, the insights from constructing consistent truncations in the purely bosonic theory carry over to our pseudo-supersymmetric setup. Here, we review the crucial ingredients of the construction presented in [55] to establish notation that we shall require later. One starts with an $\mathrm{O}(D,D)$ structure, which is defined by the invariant metric,

$$\eta_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & -\eta_{\bar{a}\bar{b}} \end{pmatrix} \quad \text{and} \quad \eta^{AB} = \begin{pmatrix} \eta^{ab} & 0 \\ 0 & -\eta^{\bar{a}\bar{b}} \end{pmatrix}, \qquad (3.7)$$

($\eta_{ab} = \eta_{\bar{a}\bar{b}}$ is the invariant metric of $\mathrm{O}(D)$ or its Lorentzian counterpart), which is also used to raise and lower "doubled" indices $A, B, \ldots$. The group $\mathrm{O}(D,D)$ can further be broken to $\mathrm{O}(D)\times\mathrm{O}(D)$ by requiring a second invariant metric, the generalized metric,

$$\mathcal{H}_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta_{\bar{a}\bar{b}} \end{pmatrix} \quad \text{and} \quad \mathcal{H}^{AB} = \begin{pmatrix} \eta^{ab} & 0 \\ 0 & \eta^{\bar{a}\bar{b}} \end{pmatrix}. \qquad (3.8)$$

It encodes the metric and the B field once it is pulled to the generalized tangent space $TM \oplus T^*M$, where it reads

$$\mathcal{H}^{IJ} = \begin{pmatrix} g_{ij} - B_{ik}g^{kl}B_{lj} & -B_{ik}g^{kj} \\ g^{ik}B_{kj} & g^{ij} \end{pmatrix}. \qquad (3.9)$$

Note that we here have switched from using Greek indices, $\mu, \nu, \ldots$, for spacetime coordinates to Latin indices $i, j, \ldots$. We do this because in DFT, there is a need for capital, doubled indices as well as lowercase, standard indices, and the Greek alphabet does not lend itself to this distinction.

The metrics (3.8) and (3.9) are related by the generalized frame,

$$\mathcal{H}^{IJ} = E_A{}^I E_B{}^J \mathcal{H}^{AB}, \quad \text{with} \quad E_A{}^I \begin{pmatrix} \mathrm{d}x^i \\ \partial_i \end{pmatrix} = \begin{pmatrix} E_a^{(+)} \\ E_{\bar{a}}^{(-)} \end{pmatrix}. \qquad (3.10)$$

In order to construct consistent truncations, one restricts the form of the covariant derivative $\nabla_A = \frac{1}{\sqrt{2}}(\nabla_a^{(+)}, \nabla_{\bar{a}}^{(-)})$ to

$$\nabla_A V^B = \mathcal{D}_A V^B + \omega_{AC}{}^B V^C, \qquad (3.11)$$

where $\mathcal{D}_A$ is a second covariant derivative, which admits some invariant tensors and thus, defines a generalized G structure [55]. G can be any subgroup of $\mathrm{O}(D)\times\mathrm{O}(D)$; we present some examples later, but for the moment, we keep the discussion general. To obtain ∇_A from $\mathcal{D}_A$, the tensor $\omega_{AB}{}^C$ has to be fixed. This is done by imposing four constraints on ∇_A (see for example [56]): First, ∇ is compatible with the η- and the generalized metric, implying that the connection ω_{ABC} is antisymmetric in its last two indices. Moreover, it gives rise to integration by parts,

$$\int e^{-2d}\nabla_A V^A = 0, \quad \text{fixing}$$
$$\omega^B{}_{BA} = 2\mathcal{D}_A d - \partial_I E_A{}^I \coloneqq F_A. \qquad (3.12)$$

Finally, it has vanishing generalized torsion, further constraining the connection by

$$3\omega_{[ABC]} = -\mathcal{T}_{ABC} \coloneqq F_{ABC}, \qquad (3.13)$$

where $\mathcal{T}_{ABC}$ is the generalized torsion of $\mathcal{D}_A$. Still, these constraint do not fix $\omega_{AB}{}^C$ completely. However, all physically relevant quantities, like the action, the field equations, and the pseudo-supersymmetry transformations, use $\nabla_\mu^{(\pm)}$ in such a way that the undefined contributions drop out. To present the partially fixed $\omega_{AB}{}^C$, it is convenient to introduce the projectors,

$$P_{AB} = \frac{1}{2}(\eta_{AB} + \mathcal{H}_{AB}) \quad \text{and} \quad \bar{P}_{AB} = \frac{1}{2}(\eta_{AB} - \mathcal{H}_{AB}), \qquad (3.14)$$

which project onto the two factors of $\mathrm{O}(D)\times\mathrm{O}(D)$, leaving η_{AB} and $\mathcal{H}_{AB}$ invariant,

$$P^A{}_B = \begin{pmatrix} \delta_b^a & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \bar{P}^A{}_B = \begin{pmatrix} 0 & 0 \\ 0 & \delta_{\bar{b}}^{\bar{a}} \end{pmatrix}. \qquad (3.15)$$

The constraints above restrict the form of ω_{ABC} to

$$\omega_{ABC} = \left(\frac{1}{3}P^D{}_A P^E{}_B P^F{}_C + \bar{P}^D{}_A P^E{}_B P^F{}_C + P^D{}_A \bar{P}^E{}_B \bar{P}^F{}_C + \frac{1}{3}\bar{P}^D{}_A \bar{P}^E{}_B \bar{P}^F{}_C\right) X_{DEF}, \tag{3.16}$$

with

$$X_{ABC} = F_{ABC} + \frac{6}{D-1}\eta_{A[B}F_{C]}. \tag{3.17}$$

A consistent truncation arises if the action of ∇_A on a tensor invariant under D_A gives rise to another (or the same) tensor that is invariant under $\mathcal{D}_A$. In this case, the set of all these invariant tensors forms a consistent truncation. By using the definition of ∇_A in (3.11), this requirement translates to the two constraints,

$$\mathcal{D}_A F_{BCD} = 0 \quad \text{and} \quad \mathcal{D}_A F_B = 0. \tag{3.18}$$

IV. MINKOWSKI × G GROUP MANIFOLD COMPACTIFICATIONS

It was observed in [51] that the d-dimensional bosonic string with the added conformal anomaly term admits a vacuum solution of the form $(\text{Minkowski})_{D-\dim G} \times G$, where G is any semisimple compact $\dim G$-dimensional Lie group. Here, we shall study the pseudo-supersymmetry of these vacuum solutions. In order to do this, we first need to establish some basic notation and results for group manifold compactifications.

A. Conventions and geometry for group manifolds

The vacuum solution employs the group manifold G equipped with its bi-invariant metric g_{mn}. This has left-acting and right-acting Killing vectors of the group G, which we denote by K^m_{La} and K^m_{Ra}, respectively. They obey the algebra,

$$[K_{La}, K_{Lb}] = -cf_{ab}{}^c K_{Lc}, \qquad [K_{Ra}, K_{Rb}] = cf_{ab}{}^c K_{Rc}, \qquad [K_{La}, K_{Rb}] = 0, \tag{4.1}$$

where $f_{ab}{}^c$ are the structure constants, and c is a scale-setting constant. The Killing vectors may be normalized so that

$$g_{mn}K^m_{La}K^n_{Lb} = \delta_{ab}, \qquad g_{mn}K^m_{Ra}K^n_{Rb} = \delta_{ab}, \tag{4.2}$$

with δ_{ab} being proportional to the Cartan-Killing metric,

$$-f_{ac}{}^d f_{bd}{}^c = C_A \delta_{ab}, \tag{4.3}$$

where C_A is the quadratic Casimir of the group G. Conversely, one has $g^{mn} = K^m_{La}K^n_{Lb}\delta^{ab} = K^m_{Ra}K^n_{Rb}\delta^{ab}$. It follows that one may view either the K^m_{La} or the K^m_{Ra} Killing vectors as defining a vielbein $e^a = e^a_m dy^m$. We shall consider the left-invariant vielbein,

$$e^a = K^a_R = K^a_{Rm}dy^m. \tag{4.4}$$

Using (4.1), the 1-forms K^a_L and K^a_R obey

$$dK^a_L = \frac{1}{2}cf_{bc}{}^a K^b_L \wedge K^c_L, \qquad dK^a_R = -\frac{1}{2}cf_{bc}{}^a K^b_R \wedge K^c_R, \tag{4.5}$$

The vielbein (4.4), therefore, obeys $de^a = -\frac{1}{2}cf_{bc}{}^a e^b \wedge e^c$, and so the torsion-free spin-connection, defined by $de^a = -\omega^a{}_b \wedge e^b$ and $\omega_{ab} = -\omega_{ba}$, is therefore given by

$$\omega_{ab} = -\frac{1}{2}cf_{abc}e^c. \tag{4.6}$$

Note that since we are taking G to be compact and semisimple, f_{abc} is totally antisymmetric. The curvature 2-forms $\Theta_{ab} = d\omega_{ab} + \omega_{ac} \wedge \omega_{cb}$ and the Riemann tensor (following from $\Theta_{ab} = \frac{1}{2}R_{abcd}e^c \wedge e^d$) are then given by[2]

$$\Theta_{ab} = \frac{1}{8}c^2 f_{abe}f_{cde}e^c \wedge e^d, \qquad R_{abcd} = \frac{1}{4}c^2 f_{abe}f_{cde}. \tag{4.7}$$

Finally, we have the Ricci tensor and Ricci scalar, given by

$$R_{ab} = \frac{1}{4}c^2 C_A \delta_{ab}, \qquad R = \frac{1}{4}c^2 C_A \dim G. \tag{4.8}$$

Note that f_{abc} is covariantly constant. The Lorentz-covariant exterior derivative D acts on Lorentz vector as $DV^a = dV^a + \omega^a{}_b V^b$, so

$$Df^{abc} = df^{abc} + \omega^{[a}{}_d f^{bc]d}, \tag{4.9}$$

and since the f^{abc} are constants, and the spin connection is given by (4.6), we have

$$Df^{abc} = -\frac{1}{2}cf^{[a}{}_{de}f^{bc]d}K^e_R, \tag{4.10}$$

and this vanishes by virtue of the Jacobi identity. Thus, it follows that in coordinate indices, we also have $\nabla_m f_{npq} = 0$.

[2] One needs to use the Jacobi identity to show this.

B. Consistent truncations

Following the discussion in Sec. III A, we now construct a consistent truncation on the group manifolds G. We first define an appropriate covariant derivative $\mathcal{D}_A$ that annihilates the generalized frame field $E_B{}^I$. This renders the corresponding generalized geometry parallelizable or, equally, the generalized structure group trivial. More specifically, we have

$$\mathcal{D}_I E_A{}^J = \partial_I E_A{}^J + \Gamma_{IK}{}^J E_A{}^K = 0, \tag{4.11}$$

thus determining the corresponding connection,

$$\Gamma_{IJK} = \partial_I E^A{}_J E_{AK}. \tag{4.12}$$

Its generalized torsion is given by $\mathcal{T}_{IJK} = 3\Gamma_{[IJK]}$, and therefore, we obtain from (3.13),

$$F_{ABC} = 3E_{[A}{}^I \partial_I E_B{}^J E_{C]J}. \tag{4.13}$$

A consistent truncation requires that (3.18) hold. Thus, we have to find a generalized frame field $E_A{}^I$ on the group manifold G, such that

$$\mathcal{D}_I F_{ABC} = \partial_I F_{ABC} = 0 \tag{4.14}$$

holds. Equivalently stated, the generalized torsion in flat indices must be constant.

This problem does not have a unique solution because there are an infinite number of admissible generalized frame fields that satisfy (4.14) on the Lie group G. For definiteness, we choose here the solution discussed in [53]. It is given by

$$\sqrt{2}E_a^{(+)} = K_{La}^m \partial_m - \eta_{ab}(\iota_{K_L^b} B - K_{Lm}^b \mathrm{d}x^m),$$
$$\sqrt{2}E_a^{(-)} = K_{Ra}^m \partial_m - \eta_{ab}(\iota_{K_R^b} B + K_{Rm}^b \mathrm{d}x^m), \tag{4.15}$$

where K_R and K_L denote the left- and right-invariant vectors fields and their respective duals from Sec. IV A, and $\iota_X B = X^m B_{mn} dx^n$ for any vector X. Additionally, we also have to incorporate a B field whose corresponding H flux yields

$$\mathrm{d}B = -\frac{c}{3!} f_{abc} K_R^a \wedge K_R^b \wedge K_R^c. \tag{4.16}$$

For this generalized frame field, we now compute the generalized torsion (4.13) with the nonvanishing components,

$$F_{abc} = \frac{c}{\sqrt{2}} f_{abc} \quad \text{and} \quad F_{\bar a \bar b \bar c} = \frac{c}{\sqrt{2}} f_{\bar a \bar b \bar c}. \tag{4.17}$$

Note that $f_{abc} = f_{ab}{}^d \delta_{dc} = f_{\bar a \bar b \bar c}$ coincides with the structure coefficients that govern the generators of the Lie group G. They appear here because of the Killing vectors algebra (4.1).

We also need to compute the flux F_A, which captures the dilaton, and check that it is constant, as required by the second equation in (3.18). By combining (3.1) with (3.12), we obtain

$$F_A = 2\mathcal{D}_A \phi - \mathcal{D}_A \log\det e - \partial_I E_A{}^I. \tag{4.18}$$

This equation splits into two contributions, for F_a and $F_{\bar a}$, respectively. Let us take a closer look at

$$F_a = 2K_{La}^m \partial_m \phi - K_{La}^m \partial_m K_{Ln}^b K_{Lb}^n - \partial_m K_{La}^m, \tag{4.19}$$

where we take into account that e_m^a can be identified with K_{Lm}^a. The right-hand side of this relation can be further simplified by using (4.1), yielding

$$F_a = 2K_{La}^m \partial_m \phi - c f_{ab}{}^b. \tag{4.20}$$

The last term vanishes because we take G to be semisimple. An analogous argument applies to $F_{\bar a}$. Hence, we conclude that $F_A = \text{const}$ requires a linear dilaton. Finally, the Bianchi identity for D_I implies that

$$F_{AB}{}^C F_C = 0 \tag{4.21}$$

must hold. Since the generalized torsion F_{ABC} matches the structure coefficients of the isometry group $G_L \times G_R$, F_C is in one-to-one correspondence with an element in the center of this group, but because G is semisimple, so is $G_L \times G_R$. Semisimple Lie groups have a trivial center, and therefore, only $F_A = 0$ is consistent with the Bianchi identity (4.21). Thus, we conclude that the dilaton must be constant in order to give rise to a consistent truncation.

C. The Minkowski × G vacuum

At this point, it is convenient to change the index labeling conventions and notation a little and rewrite the Lagrangian in Sec. II using $\hat\mu, \hat\nu \ldots$ world indices in the full D dimensions, and furthermore to place hats on all D-dimensional fields (and gamma matrices). When needed, D-dimensional tangent-space indices will be written as $\hat a, \hat b, \ldots$. We then use world indices $\mu, \nu, \ldots$ and tangent-space indices $\alpha, \beta, \ldots$ in the $(D - \dim G)$-dimensional spacetime and world indices $m, n, \ldots$ and tangent-space indices $a, b, \ldots$ in the group manifold G. Thus, $\hat\mu = (\mu, m)$ and $\hat a = (\alpha, a)$, etc.

The D-dimensional bosonic field equations for the bosonic string, including conformal anomaly term, are given in the string frame by

$$\hat R - 4(\partial\Phi)^2 + 4\hat\Box\Phi - \frac{1}{12}\hat H^2 - \frac{1}{2}m^2 = 0, \tag{4.22}$$

$$\hat{R}_{\hat{\mu}\hat{\nu}} + 2\hat{\nabla}_{\hat{\mu}}\hat{\nabla}_{\hat{\nu}}\Phi - \frac{1}{4}\hat{H}_{\hat{\mu}\hat{\rho}\hat{\sigma}}\hat{H}_{\hat{\nu}}{}^{\hat{\rho}\hat{\sigma}} = 0, \tag{4.23}$$

$$\hat{\nabla}_{\hat{\mu}}(e^{-2\Phi}\hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}}) = 0. \tag{4.24}$$

We seek a ground-state solution whose metric is a direct sum of a $(D - \dim G)$-dimensional spacetime of maximal symmetry (Minkowski, AdS or dS) times the bi-invariant metric on the group manifold G:

$$d\hat{s}^2 = g_{\mu\nu}dx^\mu dx^\nu + g_{mn}dy^m dy^n. \tag{4.25}$$

The dilaton will be assumed to be constant, and taken, without material loss of generality, to vanish. The components of the 3-form $\hat{M}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ will also be assumed to vanish except those lying entirely in the group-manifold, and for these, we can take

$$\hat{H}_{mnp} = -cf_{mnp}. \tag{4.26}$$

The choice of sign is arbitrary, as far as the bosonic equations of motion are concerned. Our choice of the negative sign is for consistency with the pseudo-supersymmetry; see later. Here, f_{mnp} is constructed from the structure constants f_{abc} using the vielbein K_R^a in the obvious way:

$$f_{mnp} = K^a_{Rm}K^b_{Rn}K^c_{Rp}f_{abc}. \tag{4.27}$$

It follows that we shall have

$$\hat{H}^2_{mn} = c^2 C_A g_{mn}, \qquad \hat{H}^2 = c^2 C_A \dim G. \tag{4.28}$$

Plugging the ansatz into the dilaton field equation (4.22) implies that we can take $\hat{\phi} = 0$ if m is given by

$$m^2 = \frac{1}{3}c^2 C_A \dim G. \tag{4.29}$$

The $\hat{R}_{mn}$ components of the $\hat{R}_{MN}$ equation (4.23) then imply

$$R_{mn} = \frac{1}{4}c^2 C_A g_{mn}, \tag{4.30}$$

which is precisely satisfied if the metric g_{mn} on G is taken to be the one considered in Sec. IV A. The $\hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}}$ equation of motion (4.24) is satisfied identically. Since the equation for the mixed components $\hat{R}_{\mu n}$ of the Einstein equation is satisfied trivially, this leaves only the lower-dimensional spacetime components $\hat{R}_{\mu\nu}$ of the Einstein equation (4.23), and this gives

$$R_{\mu\nu} = 0. \tag{4.31}$$

Thus, we have proved that we indeed have a Minkowski × G vacuum solution with $\hat{\phi} = 0$ and $\hat{H}_{mnp}$ given by (4.26), provided that the coefficient m^2 of the anomaly term is given by (4.29) and that the metric g_{mn} on the group manifold G is chosen as described in Sec. IV A.

An identical conclusion arises from the consistent truncation outlined in the last subsection. It is straightforward to see that a Minkowski space with a constant dilaton is captured by $F_{ABC} = 0$ and $F_A = 0$. Hence, solving the field equations for the product space Minkowski × G boils down to solving the field equations,

$$\mathcal{R}_{AB} = 0 \quad \text{and} \quad \mathcal{R} - \frac{m^2}{2} = 0, \tag{4.32}$$

in the internal space [15]. Here, $\mathcal{R}_{AB}$ denotes the generalized Ricci tensor, and $\mathcal{R}$ is the generalized Ricci scalar. Both admit very simple expressions for the generalized Scherk-Schwarz truncation we are concerned with, namely

$$\mathcal{R}_{AB} = 8P_{(A}{}^C\bar{P}_{B)}{}^D(F_{CEG}F_{DFH}P^{EF}\bar{P}^{GH} + F_{CDE}F_F P^{EF}), \tag{4.33}$$

$$\mathcal{R} = P^{AB}P^{CD}\left(\bar{P}^{EF} + \frac{1}{3}P^{EF}\right)F_{ACE}F_{BDF} - 2P^{AB}F_A F_B. \tag{4.34}$$

According to (4.17), only projections of F_{ABC} using exclusively P or $\bar{P}$ give nonvanishing contributions. For this observation, we immediately see that $\mathcal{R}_{AB}$ vanishes (remember $F_A = 0$) as expected. In the same vein, we obtain

$$\mathcal{R} = \frac{1}{6}c^2 f_{abc}f^{abc} = \frac{1}{6}c^2 C_A \dim G, \tag{4.35}$$

and therefore, recover (4.29) from the second equation in (4.32).

V. PSEUDO-SUPERSYMMETRY OF THE MINKOWSKI × G VACUUM

To check if this background at least partially preserves pseudo-supersymmetry, we need to plug it into the fermionic pseudo-supersymmetry transformation rules to see whether $\delta\hat{\psi}_M$ and $\delta\hat{\lambda}$ vanish for some subset of the parameters $\hat{\epsilon}$. The calculations can be set up along the same lines as those described in [51] for compactifications of $d = 11$ supergravity. In particular, it will involve decomposing the spinors of the D-dimensional spacetime into tensor products of spinors in the $(D - \dim G)$-dimensional spacetime and spinors on the group manifold G. See Appendix C for a summary of how the Dirac matrices may be decomposed in the various cases of even or odd-dimensional spacetime and internal space.

As a preliminary check, consider the dilatino transformation rule in (2.3), together with the conformal anomaly contribution in (2.7). In the background we are considering, with $\Phi = 0$ and the 3-form given by (4.26), we shall have

$$\delta\hat{\lambda} = \frac{1}{12} c f_{abc} \hat{\Gamma}^{abc} \hat{\epsilon} + \frac{\mathrm{i}}{2\sqrt{2}} \hat{\epsilon}. \tag{5.1}$$

We shall consider the case $\beta = -1$, where, as we discussed before, the conformal anomaly extension is simpler. Using

$$\hat{\Gamma}^{abc}\hat{\Gamma}_{def} = \hat{\Gamma}^{abc}{}_{def} + 9\hat{\Gamma}^{[ab}{}_{[de}\delta^{c]}_{f]} - 18\hat{\Gamma}^{[a}{}_{[d}\delta^{bc]}_{ef]} - 6\delta^{abc}_{def}, \tag{5.2}$$

it follows that if we define

$$\hat{Q} \equiv \frac{1}{6} f_{abc} \hat{\Gamma}^{abc}, \tag{5.3}$$

then

$$\hat{Q}^2 = -\frac{1}{6} f_{abc} f^{abc} = -\frac{C_A \dim G}{6} \tag{5.4}$$

times the identity matrix. By the Cayley-Hamilton theorem, and noting that $\mathrm{tr}\hat{Q} = 0$, this means that $\hat{Q}$ has the eigenvalues,

$$\pm \mathrm{i}\sqrt{\frac{C_A \dim G}{6}}, \tag{5.5}$$

with equal numbers of each. Thus, with m given, from (4.29), by

$$m = c\sqrt{\frac{c_A \dim G}{3}}, \tag{5.6}$$

we see that if $\hat{\epsilon}$ is any of the eigenvectors with eigenvalue $-\mathrm{i}\sqrt{\frac{C_A \dim G}{6}}$, we shall get $\delta\hat{\lambda} = 0$. The dilatino transformations suggest, therefore, that the Minkowski $\times\, G$ background preserves one half of the pseudo-supersymmetry.

To confirm this, we now turn to the gravitino transformation rule. Assuming again that $\beta = -1$ we have, from (2.3),

$$\delta\hat{\psi}_{\hat{\mu}} = \hat{D}_{\hat{\mu}}\hat{\epsilon} - \frac{1}{8} \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} \hat{\Gamma}^{\hat{\nu}\hat{\rho}}. \tag{5.7}$$

In the internal group manifold directions, we have

$$\begin{aligned} \delta\hat{\psi}_m &= \partial_m\hat{\epsilon} + \frac{1}{4}(\omega_{ab})_m \hat{\Gamma}^{ab}\hat{\epsilon} + \frac{c}{8} f_{mnp}\hat{\Gamma}^{np}\hat{\epsilon}, \\ &= \partial_m\hat{\epsilon}, \end{aligned} \tag{5.8}$$

after using the expression (4.6) for the spin connection on the group manifold. Finally, in the Minkowski spacetime directions, we have

$$\delta\hat{\psi}_\mu = \partial_\mu\hat{\epsilon}. \tag{5.9}$$

Thus, we see that the pseudo-supersymmetry variations of both the dilatino and the gravitino vanish, provided that $\hat{\epsilon}$ is an eigenstate of $\hat{Q}$ with eigenvalue $-\mathrm{i}\sqrt{\frac{C_A \dim G}{6}}$, and that $\hat{\epsilon}$ is independent of all the coordinates.

Again, we rederive this result using the relation between generalized geometry and pseudo-supersymmetry established in Sec. III where we consider the transformation of the gravitino first. Combining (3.3), (3.11), (3.16), and (4.17) yields

$$\delta\hat{\psi}_{\bar{a}} = \nabla^{(-)}_{\bar{a}}\hat{\epsilon} = \sqrt{2}\mathcal{D}_{\bar{a}}\hat{\epsilon} + \frac{1}{\sqrt{2}}\omega_{\bar{a}bc}\hat{\Gamma}^{bc}\hat{\epsilon} = k^m_{R\bar{a}}\partial_m\hat{\epsilon} = 0, \tag{5.10}$$

which tells us that the spinor $\hat{\epsilon}$ has to be constant. In the same vein, we compute the variation of the generalized dilatino. It consists of two contributions: First, we evaluate

$$\delta\hat{\rho} = \hat{\Gamma}^a\nabla^{(+)}_a\hat{\epsilon} = \frac{1}{\sqrt{2}}\omega_{abc}\hat{\Gamma}^a\hat{\Gamma}^{bc}\hat{\epsilon} = \frac{1}{12} c f_{abc}\hat{\Gamma}^{abc}\hat{\epsilon} = \frac{1}{2} c\hat{Q}\,\hat{\epsilon}, \tag{5.11}$$

where we take into account that partial derivatives on $\hat{\epsilon}$ have to vanish. Second, we include the conformal anomaly term, which alters the transformation of the generalized dilatino according to

$$\delta_{\text{extra}}\rho = \frac{\mathrm{i}}{2\sqrt{2}} m\hat{\epsilon} \tag{5.12}$$

for $\beta = -1$. Together, (5.11) and (5.12) yield

$$\hat{Q}\,\hat{\epsilon} = -\frac{\mathrm{i}}{\sqrt{2}} m\hat{\epsilon}, \tag{5.13}$$

which leads to the same result as already discussed above.

One may also consider more general vacuum solutions of the form $(\text{Minkowski})_{D-\dim G} \times G_s$, where G_s is a group manifold endowed with a "squashed" metric that, while still being invariant under the left action of the group G_L, is no longer invariant under the right action of the full group G_R. We have looked at examples where G is taken to be $SU(3)$ or $SO(5)$, and although these can indeed give rise to squashed solutions, we find that there is no surviving pseudo-supersymmetry in these backgrounds. The details of the $SO(5)$ example are described in Appendix D.

ACKNOWLEDGMENTS

We are grateful to Daniel Butter, Hong Lü, and Aritra Saha for useful discussions. This work is supported in part by DOE Grant No. DE-FG02-13ER42020.

APPENDIX A: SPINORS IN D DIMENSIONS

Here, we reproduce a table from [4], showing the types of spinor and the corresponding values of the constant β that can arise in each dimension.

APPENDIX B: REWRITING OF THE PSEUDO-SUPERGRAVITY LAGRANGIAN

In this Appendix, we discuss the proof that the pseudo-supersymmetric DFT Lagrangian (3.5) reduces to the pseudo-supergravity Lagrangian (2.2) after imposing the section condition. Most importantly, we do this computation for arbitrary dimensions D and take into account the dimension-dependent properties of the spinor representations (as in Appendix A).

Before we begin, let us review the relevant properties of the fermionic fields we encounter. All of them are scalars under $\mathrm{O}(D,D)$ transformations and generalized diffeomorphisms. Furthermore, the gravitino ψ_a transforms as a vector under $\mathrm{O}(D)_L$ and a spinor under $\mathrm{O}(D)_R$, while the dilatino λ, the generalized dilatino ρ, and the pseudo-supersymmetric variation parameter ϵ transform as spinors under $\mathrm{O}(D)_R$. Moreover, we will frequently use the two relations,

$$\bar{\lambda}\Gamma_{\mu_1-\mu_n}\chi = t_n\bar{\chi}\Gamma_{\mu_1-\mu_n}\lambda, \qquad \overline{\Gamma_{\mu_1-\mu_n}\lambda} = t_0 t_n\bar{\lambda}\Gamma_{\mu_1-\mu_n}, \tag{B1}$$

which hold in arbitrary dimensions for both Majorana and symplectic-Majorana spinors (as appropriate), where $t_n = -\beta^{n-1}(-1)^{\frac{n(n-1)}{2}}$. The first relation arises from the observation that $C\Gamma^{(n)}$'s symmetry properties are always opposite for Majorana spinors and symplectic-Majorana spinors when they share the same value for β (see Table I). The second relation originates from the definition of the Majorana conjugate $\bar{\lambda} = \lambda^T C$ and the property,

$$\Gamma_\mu^T = \beta C\Gamma_\mu C^{-1}. \tag{B2}$$

Because the generalized dilatino ρ has the same transformation behavior as the pseudo-supersymmetry variation parameter ϵ under double Lorentz transformations, we immediately obtain from (3.4) that

$$\begin{aligned}\bar{\psi}^a\nabla_a^{(-)}\rho &= \bar{\psi}^\mu D_\mu\rho - \frac{1}{8}\bar{\psi}^\mu H_{\mu\nu\rho}\Gamma^{\nu\rho}\rho,\\ \bar{\rho}\Gamma^a\nabla_a^{(+)}\rho &= \bar{\rho}\Gamma^\mu D_\mu\epsilon - \frac{1}{24}\bar{\rho}H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\rho - \frac{1}{24}\bar{\rho}\Gamma^\mu\partial_\mu\Phi\rho,\end{aligned} \tag{B3}$$

where the last term in the second line vanishes because of (B1):

$$\bar{\rho}\Gamma_\mu\rho = -\bar{\rho}\Gamma_\mu\rho = 0. \tag{B4}$$

Rewriting the last remaining term in the fermionic part of (3.5), *i.e.*, $-\bar{\psi}^a\Gamma^b\nabla_b^{(+)}\psi_a$, is more involved because the gravitino ψ_a is a spinor under $\mathrm{O}(D)_R$ and a vector under $\mathrm{O}(D)_L$. Eventually, we find

$$\begin{aligned}\bar{\psi}^a\Gamma^b\nabla_b^{(+)}\psi_a &= \bar{\psi}^\mu\gamma^\nu D_\nu\psi_\mu - \frac{1}{24}\bar{\psi}^\sigma H_{\mu\nu\rho}\Gamma^{\mu\nu\rho}\psi_\sigma\\ &\quad - \frac{1}{24}\bar{\psi}^\sigma\Gamma^\mu\partial_\mu\Phi\psi_\sigma - \frac{1}{2}H_{\mu\nu\rho}\bar{\psi}^\mu\Gamma^\nu\psi^\rho,\end{aligned} \tag{B5}$$

where the third term on the right-hand side will also vanish for the same reason as in (B4).

After rewriting the fermionic part of the DFT Lagrangian (3.5) as

$$\begin{aligned}e^{2d}\mathcal{L}_{DF} &= -\bar{\Psi}^\mu\gamma^\nu D_\nu\Psi_\mu + \frac{1}{4}\bar{\Psi}^\mu H\!\!\!/\Psi_\mu + \frac{1}{2}H_{\mu\nu\sigma}\bar{\Psi}^\mu\gamma^\nu\Psi^\sigma\\ &\quad - \beta\bar{\rho}\gamma^\mu D_\mu\rho + \frac{\beta}{24}\bar{\rho}H_{\mu\nu\sigma}\gamma^{\mu\nu\sigma}\rho\\ &\quad + 2\bar{\Psi}^\mu D_\mu\rho - \frac{1}{4}\bar{\psi}^\mu H_{\mu\nu\sigma}\Gamma^{\nu\sigma}\rho,\end{aligned} \tag{B6}$$

we start to match it with the fermionic part of pseudo-supersymmetric Lagrangian (2.2) expressed in terms of ρ instead of λ,

$$\begin{aligned}e^{2d}\mathcal{L}_F &= -\bar{\Psi}_\mu\Gamma^{\mu\nu\sigma}D_\nu\Psi_\sigma - \bar{\Psi}_\mu\Gamma^\mu\Gamma^\nu\Gamma^\delta D_\nu\Psi_\delta + 2\bar{\Psi}_\sigma\Gamma^\sigma\Gamma^{\mu\nu}D_\mu\Psi_\nu - D_\rho\bar{\Psi}_\mu\Gamma^\mu\Psi^\rho - \bar{\Psi}_\mu\Gamma^\mu D^\sigma\Psi_\sigma\\ &\quad - D_\nu\bar{\Psi}_\mu\Gamma^\nu\Gamma^\mu\Gamma^\delta\Psi_\delta - \bar{\Psi}_\mu\Gamma^\nu\Gamma^\mu D_\nu\Gamma^\delta\Psi_\delta - \beta\rho D\!\!\!/\rho + \frac{\beta}{24}\bar{\rho}\Gamma^{\nu\gamma\sigma}\rho H_{\nu\gamma\sigma} + \beta\bar{\rho}D_\nu\Gamma^\nu\Gamma^\delta\Psi_\delta\\ &\quad + \frac{1}{24}(\bar{\Psi}_\mu\Gamma^{\mu\nu\gamma\sigma\lambda}\Psi_\lambda + \beta\Gamma^\mu\bar{\Psi}_\mu\Gamma^{\nu\gamma\sigma}\Gamma^\delta\Psi_\delta - 2\bar{\Psi}_\mu\Gamma^{\mu\nu\gamma\sigma}\Gamma^\delta\Psi_\delta)H_{\nu\gamma\sigma} + \bar{\Psi}_\mu\Gamma^\mu\Gamma^\nu D_\nu\rho\\ &\quad - 2\beta\bar{\rho}\Gamma^{\mu\nu}D_\mu\Psi_\nu + D_\nu\bar{\Psi}_\mu\Gamma^\nu\Gamma^\mu\rho + \bar{\Psi}_\mu\Gamma^\nu\Gamma^\mu D_\nu\rho - \frac{\beta}{24}\bar{\rho}\Gamma^{\nu\rho\sigma}\Gamma^\delta\Psi_\delta H_{\nu\rho\sigma}\\ &\quad + \frac{1}{12}\bar{\Psi}_\mu\Gamma^{\mu\nu\gamma\sigma}\rho H_{\nu\gamma\sigma} + \frac{1}{4}H_{\nu\gamma\sigma}\bar{\psi}^\nu\Gamma^\gamma\psi^\sigma - \frac{\beta}{24}\Gamma^\mu\bar{\Psi}_\mu\Gamma^{\nu\gamma\sigma}\rho H_{\nu\gamma\sigma},\end{aligned} \tag{B7}$$

where we have substituted $\lambda = \frac{\sqrt{\beta}}{i}(\rho - \Gamma^\mu \Psi_\mu)$ and performed partial integrations to remove $\partial_\mu \Phi$ terms).

Suppressing Γ matrices, numerical factors, and indices, we may identify the following contributions,

$$\bar{\psi}\psi H, \quad \bar{\rho}\Psi, \quad \bar{\Psi}\Psi, \quad \bar{\rho}\Psi H, \quad \bar{\rho}\rho, \quad \bar{\rho}\rho H, \tag{B8}$$

to (B7). We now analyze each of them individually to prove that (B6) agrees with (B7):

(1) $\bar{\psi}\psi H$

In this category, the terms,

$$\frac{1}{24}(\bar{\Psi}_\mu \Gamma^{\mu\nu\rho\sigma\lambda}\Psi_\lambda + \beta\Gamma^\mu \bar{\Psi}_\mu \Gamma^{\nu\rho\sigma}\Gamma^\delta \Psi_\delta - 2\bar{\Psi}_\mu \Gamma^{\mu\nu\rho\sigma}\Gamma^\delta \Psi_\delta) H_{\nu\rho\sigma} + \frac{1}{4} H_{\nu\rho\sigma}\bar{\psi}^\nu \Gamma^\rho \psi^\sigma, \tag{B9}$$

contribute. Using the relations,

$$\begin{aligned}
\Gamma^\mu \bar{\psi}_\mu &= \beta \bar{\psi}_\mu \Gamma^\mu, \\
\bar{\psi}_\mu \Gamma^\mu \Gamma^{\nu\rho\sigma}\Gamma^\delta \psi_\delta H_{\nu\rho\sigma} &= \bar{\psi}_\mu \Gamma^{\mu\nu\rho\sigma\delta}\psi_\delta H_{\nu\rho\sigma} - \bar{\psi}_\mu \Gamma^{\nu\rho\sigma}\psi^\mu H_{\nu\rho\sigma} + 3\bar{\psi}^\mu \Gamma^{\rho\sigma\delta}\psi_\delta H_{\mu\rho\sigma} + 3\bar{\psi}_\mu \Gamma^{\mu\nu\rho}\psi^\sigma H_{\nu\rho\sigma} + 6\bar{\psi}^\nu \Gamma^\rho \psi^\sigma H_{\nu\rho\sigma}, \\
\bar{\psi}_\mu \Gamma^{\mu\nu\rho\sigma}\Gamma^\delta \psi_\delta H_{\nu\rho\sigma} &= \bar{\psi}_\mu \Gamma^{\mu\nu\rho\sigma\delta}\psi_\delta H_{\nu\rho\sigma} - \bar{\psi}_\mu \Gamma^{\nu\rho\sigma}\psi^\mu H_{\nu\rho\sigma} + 3\bar{\psi}_\mu \Gamma^{\mu\nu\rho}\psi^\sigma H_{\nu\rho\sigma},
\end{aligned} \tag{B10}$$

(B9) can be written as

$$\frac{1}{4}\bar{\psi}^\mu \not{H}\psi_\mu + \frac{1}{2}H_{\mu\nu\sigma}\bar{\psi}^\mu \Gamma^\nu \psi^\sigma. \tag{B11}$$

(2) $\bar{\rho}\Psi$

Here, the terms,

$$\beta\bar{\rho}D_\nu \Gamma^\nu \Gamma^\delta \Psi_\delta + \bar{\Psi}_\mu \Gamma^\mu \Gamma^\nu D_\nu \rho - 2\beta\bar{\rho}\Gamma^{\mu\nu}D_\mu \Psi_\nu + D_\nu \bar{\Psi}_\mu \Gamma^\nu \Gamma^\mu \rho + \bar{\Psi}_\mu \Gamma^\nu \Gamma^\mu D_\nu \rho, \tag{B12}$$

contribute. The second term and the last term can be combined to

$$2\bar{\Psi}^\mu D_\mu \rho, \tag{B13}$$

while the sum of the remaining terms vanishes.

(3) $\bar{\Psi}\Psi$

There are seven terms what contribute to this category:

$$-\bar{\Psi}_\mu \Gamma^{\mu\nu\rho}D_\nu \Psi_\rho - \bar{\Psi}_\mu \Gamma^\mu \Gamma^\nu \Gamma^\delta D_\nu \Psi_\delta + 2\bar{\Psi}_\sigma \Gamma^\sigma \Gamma^{\mu\nu}D_\mu \Psi_\nu - D_\rho \bar{\Psi}_\mu \Gamma^\mu \Psi^\rho - \bar{\Psi}_\mu \Gamma^\mu D^\rho \Psi_\rho - D_\nu \bar{\Psi}_\mu \Gamma^\nu \Gamma^\mu \Gamma^\delta \Psi_\delta - \bar{\Psi}_\mu \Gamma^\nu \Gamma^\mu D_\nu \Gamma^\delta \Psi_\delta. \tag{B14}$$

Using the relations,

$$\begin{aligned}
\Gamma^\mu \Gamma^\nu \Gamma^\delta &= \Gamma^{\mu\nu\delta} + 2\eta^{\mu[\nu}\Gamma^{\delta]} + \Gamma^\mu \eta^{\nu\delta}, \\
\Gamma^\sigma \Gamma^{\mu\nu} &= \Gamma^{\sigma\mu\nu} + 2\eta^{\sigma[\mu}\Gamma^{\nu]},
\end{aligned} \tag{B15}$$

one finds that (B14) simplifies to

$$\begin{aligned}
&-\bar{\Psi}_\mu \Gamma^{\mu\nu\rho}D_\nu \Psi_\rho - \bar{\Psi}_\mu \Gamma^{\mu\nu\delta}D_\nu \Psi_\delta - \bar{\Psi}_\mu \Gamma^\delta D^\mu \Psi_\delta + \bar{\psi}_\mu \Gamma^\nu D_\nu \Psi^\mu - \bar{\psi}_\mu \Gamma^\mu D_\nu \Psi^\nu + 2\bar{\Psi}_\sigma \Gamma^{\sigma\mu\nu}D_\mu \Psi_\nu + 2\bar{\Psi}_\sigma \Gamma^\nu D^\sigma \Psi_\nu \\
&- 2\bar{\Psi}_\sigma \Gamma^\mu D_\mu \Psi^\sigma - D_\rho \bar{\psi}_\mu \Gamma^\mu \psi^\rho - \bar{\Psi}_\mu \Gamma^\mu D^\rho \Psi_\rho - D_\nu \bar{\Psi}_\mu \Gamma^{\nu\mu\delta}\Psi_\delta - D_\nu \bar{\Psi}^\nu \Gamma^\delta \Psi_\delta + D_\nu \bar{\Psi}_\mu \Gamma^\mu \Psi^\nu - D_\nu \bar{\Psi}_\mu \Gamma^\nu \Psi^\mu \\
&- \bar{\Psi}_\mu \Gamma^{\nu\nu\delta}D_\nu \Psi_\delta - \bar{\Psi}_\mu \Gamma^\delta D^\mu \Psi_\delta + \bar{\Psi}_\mu \Gamma^\mu D_\nu \Psi^\nu - \bar{\Psi}_\mu \Gamma^\nu D_\nu \Psi^\mu.
\end{aligned} \tag{B16}$$

All terms with $\Gamma^{(3)}$,

$$-\bar{\Psi}_\mu \Gamma^{\mu\nu\rho}D_\nu \Psi_\rho - \bar{\Psi}_\mu \Gamma^{\mu\nu\delta}D_\nu \Psi_\delta + 2\bar{\Psi}_\sigma \Gamma^{\sigma\mu\nu}D_\mu \Psi_\nu - D_\nu \bar{\Psi}_\mu \Gamma^{\nu\mu\delta}\Psi_\delta - \bar{\Psi}_\mu \Gamma^{\nu\mu\delta}D_\nu \Psi_\delta, \tag{B17}$$

cancel after using (B1). The remaining terms,

$$- \bar{\Psi}_\mu \Gamma^\delta D^\mu \Psi_\delta + \bar{\psi}_\mu \Gamma^\nu D_\nu \Psi^\mu - \bar{\psi}_\mu \Gamma^\mu D_\nu \Psi^\nu + 2\bar{\Psi}_\sigma \Gamma^\nu \nabla^\sigma \Psi_\nu - 2\bar{\Psi}_\sigma \Gamma^\mu D_\mu \Psi^\sigma - D_\rho \bar{\psi}_\mu \Gamma^\mu \psi^\rho - \bar{\Psi}_\mu \Gamma^\mu D^\rho \Psi_\rho$$
$$- D_\nu \bar{\Psi}^\nu \Gamma^\delta \Psi_\delta + D_\nu \bar{\Psi}_\mu \Gamma^\mu \Psi^\nu - D_\nu \bar{\Psi}_\mu \Gamma^\nu \Psi^\mu - \bar{\Psi}_\mu \Gamma^\delta D^\mu \Psi_\delta + \bar{\Psi}_\mu \Gamma^\mu D_\nu \Psi^\nu - \bar{\Psi}_\mu \Gamma^\nu D_\nu \Psi^\mu, \quad \text{(B18)}$$

fall into three different categories:

(a) A Γ matrix contracted with a gravitino,

$$-\bar{\Psi}_\mu \Gamma^\delta D^\mu \Psi_\delta + 2\bar{\Psi}_\sigma \Gamma^\nu D^\sigma \Psi_\nu - \bar{\Psi}_\mu \Gamma^\delta D^\mu \Psi_\delta. \quad \text{(B19)}$$

(b) A Γ matrix contracting with a derivative,

$$\bar{\Psi}_\mu \Gamma^\nu D_\nu \Psi^\mu - 2\bar{\Psi}_\sigma \Gamma^\mu D_\mu \Psi^\sigma - D_\nu \bar{\Psi}_\mu \Gamma^\nu \Psi^\mu - \bar{\Psi}_\mu \Gamma^\nu D_\nu \Psi^\mu. \quad \text{(B20)}$$

Again, (B1) is used to simplify this expression to

$$-\bar{\Psi}^\mu \gamma^\nu D_\nu \Psi_\mu. \quad \text{(B21)}$$

(c) Finally, a derivative is contracted with a gravitino,

$$- \bar{\Psi}_\mu \Gamma^\mu D_\nu \Psi^\nu - D_\rho \bar{\Psi}_\mu \Gamma^\mu \Psi^\rho - \bar{\Psi}_\mu \Gamma^\mu D^\rho \Psi_\rho + \bar{\Psi}_\mu \Gamma^\mu D_\nu \Psi^\nu + D_\nu \bar{\Psi}_\mu \Gamma^\mu \Psi^\nu - D_\nu \bar{\Psi}^\nu \Gamma^\delta \Psi_\delta. \quad \text{(B22)}$$

All these terms cancel.

(4) $\bar{\rho}\Psi H$

The following terms contribute:

$$-\frac{\beta}{24}\bar{\rho}\Gamma^{\nu\gamma\sigma}\Gamma^\delta \Psi_\delta H_{\nu\gamma\sigma} - \frac{\beta}{24}\overline{\Gamma^\mu \Psi_\mu}\Gamma^{\nu\gamma\sigma}\rho H_{\nu\gamma\sigma} + \frac{1}{12}\bar{\Psi}_\mu \Gamma^{\mu\nu\gamma\sigma}\rho H_{\nu\gamma\sigma}. \quad \text{(B23)}$$

Using the relations,

$$\overline{\Gamma_\mu \lambda} = t_0 t_1 \bar{\lambda}\Gamma_\mu = \beta \bar{\lambda}\Gamma_\mu,$$
$$\Gamma^\mu \Gamma^{\nu\rho\sigma} H_{\nu\rho\sigma} = \Gamma^{\mu\nu\rho\sigma} H_{\nu\rho\sigma} + 3\eta^{\mu\nu}\Gamma^{\rho\sigma} H_{\nu\rho\sigma}, \quad \text{(B24)}$$

and exploiting the total antisymmetry of $H_{\mu\nu\rho}$, we find that (B23) is equal to

$$-\frac{1}{4}\bar{\Psi}^\mu \Gamma^{\gamma\sigma}\rho H_{\mu\gamma\sigma}. \quad \text{(B25)}$$

(5) $\bar{\rho}\rho$

Only one term, $-\beta\rho \not{D}\rho$, has the required structure. It does not admit further simplifications.

(6) $\bar{\rho}\rho H$

Again, there is only one term, $\frac{\beta}{24}\bar{\rho}\Gamma^{\nu\rho\sigma}\rho H_{\nu\rho\sigma}$, with no need for further simplification.

We conlcude that the fermionic Lagrangians,

$$\mathcal{L}_F = \mathcal{L}_{DF}, \quad \text{(B26)}$$

match for arbitrary dimensions D. For $D = 10$, this match has already been established in [34]. Matching the bosonic contribution is straightforward if the section condition is imposed and follows the known calculations in the literature [14,15].

APPENDIX C: DECOMPOSITION OF DIRAC MATRICES

In a Kaluza-Klein reduction, we need to write the higher-dimensional Dirac matrices $\hat{\Gamma}_A$ in terms of tensor products of lower-dimensional spacetime Dirac matrices γ_α and internal space Dirac matrices Γ_a. The way this works depends upon whether the various space(time)s are even-dimensional or odd-dimensional. A table of how the decompositions may be made is given in Appendix A of [57]:

$$\begin{aligned}
(\text{Even, odd}):&\ \hat{\Gamma}_\alpha = \gamma_\alpha \otimes \mathbb{1}, && \hat{\Gamma}_a = \gamma_* \otimes \Gamma_a,\\
(\text{Odd, even}):&\ \hat{\Gamma}_\alpha = \gamma_\alpha \otimes \Gamma_*, && \hat{\Gamma}_a = \mathbb{1} \otimes \Gamma_a,\\
(\text{Even, even}):&\ \hat{\Gamma}_\alpha = \gamma_\alpha \otimes \mathbb{1}, && \hat{\Gamma}_a = \gamma_* \otimes \Gamma_a,\\
\text{or}&\ \hat{\Gamma}_\alpha = \gamma_\alpha \otimes \Gamma_*, && \hat{\Gamma}_a = \mathbb{1} \otimes \Gamma_a,\\
(\text{Odd, odd}):&\ \hat{\Gamma}_\alpha = \sigma_1 \otimes \gamma_\alpha \otimes \mathbb{1}, && \hat{\Gamma}_a = \sigma_2 \otimes \mathbb{1} \otimes \Gamma_a,
\end{aligned} \quad \text{(C1)}$$

where the first entry in the pair enclosed in parentheses indicates whether the lower-dimensional spacetime is even or odd-dimensional, and the second entry indicates whether the internal space is even or odd-dimensional. γ_* denotes the chirality operator in even-dimensional lower-dimensional spacetimes, and Γ_* denotes the chirality operator in even-dimensional internal spaces (with $\gamma_*^2 = +1$ and $\Gamma_*^2 = +1$). In the (odd, odd) case, the extra factor involving the Pauli matrices σ_1 and σ_2 ensures that the $\hat{\Gamma}_A$ matrices obey the Clifford algebra. They are needed because the Dirac matrices $\hat{\Gamma}_A$ in this case are twice the size of the tensor products of the lower-dimensional and the internal Dirac matrices.

APPENDIX D: SQUASHED GROUP MANIFOLD SOLUTIONS

It is well known that any compact semisimple group manifold other than $SU(2)$ or $SO(3)$ admits at least one additional, inequivalent Einstein metric, over and above the standard bi-invariant metric. This raises the possibility that there might exist Minkowski $\times$ G vacua in which the metric on the group manifold G is not the bi-invariant one. Such solutions would not necessarily involve a squashed Einstein metric on G, since the form of the 3-form field strength H_{mnp} in the squashed vacuum may also change. One approach to looking for such squashed solutions is to consider families of squashed metrics on G, with an associated deformation of the 3-form field. The families of metrics in question here will be homogeneous, invariant still under the left-acting copy of G, but no longer invariant under the full right action of G. Such metrics can be obtained by rescaling the left-invariant vielbeins by constant factors. A detailed discussion of the construction of squashed Einstein metrics using this procedure can be found, for example, in [58].

One can look for squashed vacuum solutions on a case by case basis. We have checked two examples, one being a family of squashed metrics on the $SU(3)$ group manifold and the other a family of squashed metrics on the $SO(5)$ group manifold. In neither case do we find any squashed vacua in the bosonic string for which the squashed metric on the group manifold is of positive-definite signature. For the case of $SO(5)$, we do find one squashed example for which the metric has Lorentzian signature [that is, (1,9) signature]. Since this may be of some interest, we shall present some details below.

We make define left-invariant 1-forms L_{IJ} for $SO(5)$, with I and J ranging over 1 to 5, and $L_{IJ} = -L_{JI}$, obeying the exterior algebra,

$$dL_{IJ} = L_{IK} \wedge L_{KJ}. \tag{D1}$$

Following [58], we split the indices into $I = (1, 2, i)$, and here, we define the "unsquashed" vielbein,

$$\begin{aligned} &\bar{e}_i = L_{1i}, \qquad \bar{e}_{i+3} = L_{2i}, \qquad \bar{e}_7 = L_{34}, \\ &\bar{e}_8 = L_{35}, \qquad e_9 = L_{45}, \qquad \bar{e}_{10} = L_{12}. \end{aligned} \tag{D2}$$

When we need to assign specific numerical values to tangent-space indices, it is more convenient to put the index downstairs. We consider metrics,

$$\begin{aligned} ds_{10}^2 = {} & x_1(\bar{e}_1^2 + \bar{e}_2^2 + \bar{e}_3^2) + x_2(\bar{e}_4^2 + \bar{e}_5^2 + \bar{e}_6^2) \\ & + x_3(\bar{e}_7^2 + \bar{e}_8^2 + \bar{e}_9^2) + x_4\bar{e}_{10}^2, \end{aligned} \tag{D3}$$

where (x_1, x_2, x_3, x_4) are constants. Correspondingly, we have a vielbein,

$$e_1 = \sqrt{x_1}\bar{e}_1, \ \ e_2 = \sqrt{x_1}\bar{e}_2, \ldots, e_9 = \sqrt{x_3}\bar{e}_9, \ \ e_{10} = \sqrt{x_4}\bar{e}_{10}. \tag{D4}$$

Thus, we have a four-parameter family of homogeneous metrics on $SO(5)$, which are invariant under the left action of the full $SO(5)$ group but invariant only under an $SO(3)$ subgroup of right-acting transformations. As was discussed in [58], there are three inequivalent Einstein metrics in this family, corresponding to (up to overall scale),

$$(x_1, x_2, x_3, x_4) = (1,1,1,1), \quad (14,14,4,19), \quad (1,2,1,2). \tag{D5}$$

The first of these is the standard bi-invariant metric.

In order to obtain a solution of the bosonic string of the form (Minkowski) $\times$ G_{squashed}, we need also to construct a 3-form $G_{(3)}$ that is closed and also co-closed (*i.e.*, the 3-form must be harmonic). In the case of the bi-invariant vacuum, we just used a constant multiple of the structure constants f_{abc}. In fact, we could write

$$G_{(3)} = \frac{1}{3} d\bar{e}^a \wedge \bar{e}^a = \frac{1}{6} f_{abc}\bar{e}^a \wedge \bar{e}^b \wedge \bar{e}^c. \tag{D6}$$

This is manifestly closed, and one can easily verify that it is also co-closed in the bi-invariant metric.

There must always exist an harmonic 3-form regardless of whether the metric is bi-invariant or squashed, since the topological number b_3 (the third Betti number) is equal to 1 regardless of the metric. One way to construct the required harmonic 3-form is by a brute-force Mathematica calculation, starting with a general 3-form,

$$G_{(3)} = \frac{1}{6} G_{abc} e^a \wedge e^b \wedge e^c, \tag{D7}$$

and solving for the (constant) components G_{abc}, such that $dG_{(3)} = 0 = d * G_{(3)}$. In fact, we find that the harmonic 3-form is exactly the same as the one constructed in Eq. (D6) (*i.e.*, still written using the bi-invariant vielbein $\bar{e}^a$). Of course, when one calculates the Hodge dual and $d * G_{(3)}$ (or equivalently, the divergence $\nabla^a G_{abc}$), the fact that the metric is squashed enters in the calculation.

Substituting $\hat{H}_{mnp} = \pm c G_{mnp}$ and the direct sum of the Minkowski metric and the squashed $SO(5)$ metric (D3) into the equations of motion for the bosonic string (with dilaton set to zero), we find two inequivalent solutions. Up to scaling, they are

$$(x_1, x_2, x_3, x_4) = (1,1,1,1), \quad c = \frac{1}{3}, \quad m^2 = 20, \tag{D8}$$

which is the bi-invariant solution of the kind we found earlier for a general group G, and

$$(x_1, x_2, x_3, x_4) = (1,1,3,-3), \quad c = 1, \quad m^2 = \frac{56}{3}. \tag{D9}$$

We see from (D3) that the squashed metric on $SO(5)$ in this solution has Lorentzian (1,9) signature, because x_4 is negative.

It was noted in [58] that although various examples of squashed group manifolds were checked and many squashed Einstein metrics were found, all them had either Euclidean signature or else more than one timelike direction. Our squashed $SO(5)$ bosonic string vacuum (D9) thus provides a first example of a Lorentzian signature group manifold metric arising as a solution in a theory of physical interest. We can, of course, take the flat directions in the vacuum solution to be Euclidean space rather than Minkowski spacetime in this case so that the signature of the entire higher-dimensional bosonic string spacetime will be $(1, D-1)$.

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