Path-integral action in the generalized uncertainty principle framework

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Various gedanken experiments of quantum gravity phenomenology in search of a complete theory of gravity near the Planck scale indicate a modification of the Heisenberg uncertainty principle to the generalized uncertainty principle (GUP). This modification leads to nontrivial contributions on the Hamiltonian of a nonrelativistic particle moving in an arbitrary potential. In this paper we study the path-integral representation of a particle moving in an arbitrary potential using the most general form of the GUP containing both the linear and quadratic contributions in momentum. First we work out the action of the particle in an arbitrary potential and hence find an upper bound to the velocity of a free particle. This upper bound interestingly imposes restrictions on the relation between the GUP parameters α and β. Analysis shows that $\beta > 4\alpha^2$. We then deduce the mathematical expressions of classical action and the quantum fluctuations for both free particle and the harmonic oscillator systems.

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I. INTRODUCTION

General relativity (GR), proposed by Einstein in 1915 [1], can explain and predict a large number of physical phenomena in astrophysics and cosmology. On the other hand, quantum mechanics (QM) is the most successful candidate to describe the dynamics of a particle in the microscopic world. Since GR is a classical theory, it fails to characterize the universe near the Planck scale. Though these two fundamental theories (GR and QM) are very successful in their own domain, a complete quantum theory of gravity to explore the universe close to the Planck epoch is absolutely essential. Interestingly substantial investigation of various theories of quantum gravity such as string theory [2], loop quantum gravity [3,4], noncommutative geometry [5], and some gedanken experiments in quantum gravity phenomenology hint at the existence of a minimal length, namely, the Planck length. This fundamental hypothesis of the observer independent Planck length together with the Heisenberg uncertainty principle (HUP), one of the main pillars of QM, leads to a modification of the HUP [6]. This modification is well known in the literature as the generalized uncertainty principle (GUP). A large area in theoretical physics which includes black hole thermodynamics [7–9], and various quantum systems like particle in a box, Landau levels, and simple harmonic oscillator [10–13], have been extensively studied under the GUP framework. We would also like to mention that the simplest form of the minimal length uncertainty relation is motivated by the scattering of strings in the first quantized formulation of string theory. A cubic form of an uncertainty relation known as the spacetime uncertainty relation also appears in the literature getting its motivation from M theory [14–16]. Interestingly, in the nonperturbative formulation of open string field theory such cubic structures arise [17,18]. This makes it quite natural to look for cubic algebraic structures in string theory. In [19], some speculative comments on the generalization of two-bracket algebraic structures to three-bracket (cubic) algebraic structures have been mentioned and its possible role in string theory has been discussed. The issue of gauge invariance in the presence of a minimal length is another important problem that has been discussed in the literature [20]. This is important as it seems to contradict the robustness of the symplectic form appearing in Gromov's nonsqueezing theorem [21]. Also the breaking of Lorentz covariance due to the minimal length may lead to a violation of the second law of thermodynamics [22]. These are some of the fundamental issues that make the study of GUP even more interesting.

Recently studies have been done to investigate the path-integral formalism of a nonrelativistic particle in the presence of the GUP [23,24]. In [24], the Feynman propagator of a particle under any arbitrary potential has been calculated using the simplest form of the GUP, in which the modification to the HUP involves a term proportional to the quadratic in momentum. This modification of the HUP, proportional to the quadratic in momentum is motivated by

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black hole physics and string theory. However, theories of doubly special relativity suggest that there can be a modification involving a term linear in momentum. Hence, combining both possibilities, the most general form of the GUP has been first introduced in [13]. In this paper we want to explore the path-integral formalism for a nonrelativistic particle moving under any arbitrary potential using this form of the GUP which contains both the linear and quadratic modifications in momentum. We would also like to comment that the path-integral formalism is consistent with two-bracket algebraic structures; however, it is not quite clear at present how to figure out from the path-integral formalism the underlying algebraic structures that would lead to cubic algebraic structures.

The modified uncertainty principle between the position q_i and its conjugate momentum p_j incorporating both the contributions, linear and quadratic in momentum, is given by [10]

$$\Delta q_i \Delta p_i \geq \frac{\hbar}{2}\left[1 - \alpha the\left\langle p + \frac{p_i p_i}{p}\right\rangle - (\alpha^2 - \beta)((\Delta p)^2 + \langle p\rangle^2) - (\alpha^2 - 2\beta)((\Delta p_i)^2 + \langle p_i\rangle^2)\right] \tag{1}$$

where $p^2 \equiv |\vec{p}|^2 = \eta_{ij} p^i p^j$; $i,\ j = 1,\ 2,\ 3$, the parameters α and β bear the signature of the GUP, and are defined as $\alpha = \alpha_0/(M_{\mathrm{Pl}}c)$ and $\beta = \beta_0/(M_{\mathrm{Pl}}c)^2$, with M_{Pl} being the Planck mass and c is the speed of light in free space. The dimensions of α and β are $(\text{momentum})^{-1}$ and $(\text{momentum})^{-2}$, respectively. The above uncertainty principle is consistent with the following modified Heisenberg algebra:

$$[q_i, p_j] = i\hbar\left[\delta_{ij} - \alpha\left(\delta_{ij} p + \frac{p_i p_j}{p}\right) + \beta(\delta_{ij} p^2 + 2 p_i p_j) - \alpha^2(\delta_{ij} p^2 + p_i p_j)\right]. \tag{2}$$

It can be easily shown that the above commutator follows the Jacobi identity, $[q_i, q_j] = [p_i, p_j] = 0$. The modified variables (q_j, p_j) can be expressed in terms of the usual variables (q_{0j}, p_{0j}) in such a way that they obey the commutation relation (2). Hence we have

$$\begin{aligned} q_j &= q_{0j}, \\ p_j &= p_{0j}(1 - \alpha p_0 + \beta {p_0}^2), \end{aligned} \tag{3}$$

where (q_{0j}, p_{0j}) satisfies the usual commutation relation $[q_{0i}, p_{0j}] = i\hbar\delta_{ij}$. With the relation (3) in hand, we now construct the GUP modified Hamiltonian describing a particle moving under any arbitrary potential $V(q)$. In the subsequent discussion we shall work in one spatial dimension. Therefore using Eq. (3), the Hamiltonian of the particle in an arbitrary potential $V(q)$ up to order $\mathcal{O}(\alpha^2, \beta)$ reads

$$\hat{H} = \frac{\hat{p}_0^2}{2m} - \frac{\alpha}{m}\hat{p}_0^3 + \frac{1}{m}\left(\frac{\alpha^2}{2} + \beta\right)\hat{p}_0^4 + V(\hat{q}) + \mathcal{O}(\alpha\beta, \alpha^3, \beta^2). \tag{4}$$

We are now ready to construct the path-integral representation of a particle moving in an arbitrary potential in the GUP modified Hamiltonian (4). Note that we have used the most general form of the GUP incorporating both the contributions, linear and quadratic in momentum. In this paper, first we calculate the classical action and the explicit form of the propagation kernel for a particle moving under any arbitrary potential using the Hamiltonian (4) in Sec. I. In this section, we derive the equation of motion of the particle which gives an interesting relation between the GUP parameters α and β. In Sec. II, we evaluate the explicit form of the propagation kernel both for the free particle and the harmonic oscillator. Then we proceed to Sec. III to calculate the quantum fluctuations of the propagation kernel for the harmonic oscillator. Finally, we summarize our results in the concluding section.

II. PROPAGATION KERNEL FOR AN ARBITRARY POTENTIAL

To investigate the path-integral formalism of a particle described by the GUP modified Hamiltonian (4), we write the general form of the propagation kernel as

$$\langle q_f, t_f | q_0, t_0\rangle = \lim_{n\to\infty}\int_{-\infty}^{+\infty}\prod_{j=1}^{n} dq_j \langle q_f, t_f | q_n, t_n\rangle\langle q_n, t_n|\ldots|q_1, t_1\rangle\langle q_1, t_1|q_0, t_0\rangle. \tag{5}$$

Then we compute the propagator over a small segment in the above path integral. Here we use the following completeness relation:

$$\int_{-\infty}^{+\infty} dp|p\rangle\langle p| = \mathbf{1}. \tag{6}$$

Hence, the Hamiltonian given by Eq. (4) along with the above completeness relation ([3]) gives the infinitesimal propagator, which reads

$$\begin{aligned}\langle q_{j+1}, t_{j+1}|q_j, t_j\rangle &= \langle q_{j+1}|e^{-\frac{i}{\hbar}\hat{H}\tau}|q_j\rangle \\ &= \left\langle q_{j+1}\middle|1 - \frac{i}{\hbar}\hat{H}\tau + \Theta(\tau^2)\middle|q_j\right\rangle \\ &= \int_{-\infty}^{+\infty} \frac{dp_j}{2\pi\hbar} e^{\frac{i}{\hbar}p_j(q_{j+1}-q_j)} e^{-\frac{i}{\hbar}\tau(\frac{p_j^2}{2m}-\frac{\alpha}{m}p_j^3+\frac{1}{m}(\frac{\alpha^2}{2}+\beta)p_j^4+V(q_j))} + \mathcal{O}(\tau^2).\end{aligned} \tag{7}$$

Substituting the above expression in Eq. (5), the propagation kernel takes the form (apart from a constant factor)

$$\langle q_f, t_f|q_0, t_0\rangle = \lim_{n\to\infty}\int_{-\infty}^{+\infty}\prod_{j=1}^{n} dq_j \prod_{j=0}^{n} dp_j \exp\left(\frac{i}{\hbar}\sum_{j=0}^{n}\left[p_j(q_{j+1}-q_j) - \tau\left\{\frac{p_j^2}{2m} - \frac{\alpha}{m}p_j^3 + \frac{1}{m}\left(\frac{\alpha^2}{2}+\beta\right)p_j^4 + V(q_j)\right\}\right]\right). \tag{8}$$

Taking the $\tau \to 0$ limit, the propagation kernel given by Eq. (5) in the path-integral representation reads

$$\langle q_f, t_f|q_0, t_0\rangle = \int \mathcal{D}q\mathcal{D}p \exp\left(\frac{i}{\hbar}\mathcal{A}\right) \tag{9}$$

where the phase-space action $\mathcal{A}$ is given by

$$\mathcal{A} = \int_{t_0}^{t_f} dt\left[p\dot{q} - \left\{\frac{p^2}{2m} - \frac{\alpha}{m}p^3 + \frac{1}{m}\left(\frac{\alpha^2}{2}+\beta\right)p^4 + V(q)\right\}\right], \qquad t_f - t_0 = T. \tag{10}$$

Before doing the p integral, let us look at the jth phase-space integral up to order β. Setting $\alpha = 0$, this takes the form

$$\langle q_{j+1}, t_{j+1}|q_j, t_j\rangle = \int_{-\infty}^{+\infty} dq_j dp_j \exp\left(\frac{i}{\hbar}\left[p_j(q_{j+1}-q_j) - \tau\left\{\frac{p_j^2}{2m} + \frac{\beta}{m}p_j^4\right\}\right]\right). \tag{11}$$

The above expression can be rewritten in the following form:

$$\langle q_{j+1}, t_{j+1}|q_j, t_j\rangle = \int_{-\infty}^{+\infty} \frac{dq_j d\tilde{p}_j}{(1+3\beta\tilde{p}_j{}^2)} \exp\left(\frac{i\tilde{\tau}}{\hbar}\left[\tilde{p}_j\left(\frac{q_{j+1}-q_j}{\tau}\right) - \frac{\tilde{p}_j{}^2}{2m}\left(1+\frac{\beta}{m}\tilde{p}_j{}^4\right)\right]\right) \tag{12}$$

where $\tilde{p}_j = p_j(1+\beta p_j^2)$ and $\tilde{\tau} = \tau(1+\beta p_j^2)^{-1}$. From the above expression one can easily see that the usual phase-space volume $dqdp$ gets corrected by a factor of $(1+3\beta p^2)^{-1}$. Interestingly, this correction factor in the weighted phase-space volume has been obtained in [25] using the canonical approach. Note that the factor 3β in our case appears due to the choice of 2β as the coefficient of $p_i p_j$ in the commutation relation (2).

Now we compute the momentum integral in Eq. (7). Evaluating this keeping terms up to $\mathcal{O}(\alpha^2,\beta)$ yields

$$\begin{aligned}\langle q_{j+1},t_{j+1}|q_j,t_j\rangle &= \sqrt{\frac{m}{2\pi i\hbar\tau}}\left[1+\left\{\frac{i\alpha m^2(q_{j+1}-q_j)^3}{\hbar\tau^2}+\frac{1}{2}\left(\frac{i\alpha m^2(q_{j+1}-q_j)^3}{\hbar\tau^2}\right)^2\right\}+\frac{3\alpha m(q_{j+1}-q_j)}{\tau}\right.\\ &\quad +\frac{3i\hbar m\beta}{\tau}-\frac{6i\hbar m\alpha^2}{\tau}-6m^2\left(\frac{\alpha^2}{2}+\beta\right)\left(\frac{q_{j+1}-q_j}{\tau}\right)^2-\frac{im^3\beta(q_{j+1}-q_j)^4}{\hbar\tau^3}m\\ &\quad\left.+\frac{7i\alpha^2m^3(q_{j+1}-q_j)^4}{\hbar\tau^3}+\frac{45\alpha^2m^2(q_{j+1}-q_j)^2}{2\tau^2}\right]\times\exp\left(\frac{im(q_{j+1}-q_j)^2}{2\hbar\tau}+\frac{i}{\hbar}\tau V(q_j)\right)\\ &= \sqrt{\frac{m}{2\pi i\hbar\tau}}\left[1-\frac{i}{\hbar}\left\{\frac{m^3\beta(q_{j+1}-q_j)^4}{\tau^3}-\frac{4\alpha^2m^3(q_{j+1}-q_j)^4}{\tau^3}\right\}+\frac{3\alpha m(q_{j+1}-q_j)}{\tau}\right.\\ &\quad\left.+\frac{3\beta i\hbar m}{\tau}-\frac{6i\hbar m\alpha^2}{\tau}-6m^2\left(\frac{\alpha^2}{2}+\beta\right)\left(\frac{q_{j+1}-q_j}{\tau}\right)^2+\frac{45\alpha^2m^2}{2}\left(\frac{q_{j+1}-q_j}{\tau}\right)^2\right]\\ &\quad\times\exp\left(\frac{im(q_{j+1}-q_j)^2}{2\hbar\tau}+\frac{i}{\hbar}\tau V(q_j)-\frac{i\alpha m^2(q_{j+1}-q_j)^3}{\hbar\tau^2}\right)\\ &= \sqrt{\frac{m}{2\pi i\hbar\tau}}\left[\frac{3\alpha m(q_{j+1}-q_j)}{\tau}+\frac{3\beta i\hbar m}{\tau}-\frac{6i\hbar m\alpha^2}{\tau}-6m^2\beta\left(\frac{q_{j+1}-q_j}{\tau}\right)^2\right.\\ &\quad\left.+\frac{39\alpha^2m^2}{2}\left(\frac{q_{j+1}-q_j}{\tau}\right)^2\right]\times\exp\left(\frac{im(q_{j+1}-q_j)^2}{2\hbar\tau}+\frac{i}{\hbar}\tau V(q_j)-\frac{i\alpha m^2(q_{j+1}-q_j)^3}{\hbar\tau^2}\right.\\ &\quad\left.-\frac{i\beta m^3(q_{j+1}-q_j)^4}{\hbar\tau^3}+\frac{4i\alpha^2m^3(q_{j+1}-q_j)^4}{\hbar\tau^3}\right)+\mathcal{O}(\beta^2).\end{aligned}\tag{13}$$

Using the above result in Eq. (8), we obtain the propagation kernel up to a constant factor as

$$\begin{aligned}\langle q_f,t_f|q_0,t_0\rangle &= \int_{-\infty}^{+\infty}\prod_{j=1}^{n}dq_j\exp\left(\frac{i}{\hbar}\sum_{j=0}^{n}\tau\left[\frac{m}{2}\left(\frac{q_{j+1}-q_j}{\tau}\right)^2\left\{1+2\alpha m\left(\frac{q_{j+1}-q_j}{\tau}\right)+8\alpha^2m^2\left(\frac{q_{j+1}-q_j}{\tau}\right)^2\right.\right.\right.\\ &\quad\left.\left.\left.-2\beta m^2\left(\frac{q_{j+1}-q_j}{\tau}\right)^2\right\}-V(q_j)\right]\right).\end{aligned}\tag{14}$$

To get the configuration space path-integral representation of a particle moving in an arbitrary potential $V(q)$, we take the limit $\tau\to 0$. This gives

$$\langle q_f,t_f|q_0,t_0\rangle = \tilde{F}(T,\alpha,\beta)\int\mathcal{D}q e^{\frac{i}{\hbar}S}\tag{15}$$

where the action of the particle moving in the presence of an arbitrary potential $V(q)$ in the configuration space is given by

$$S=\int_{t_0}^{t_f}dt\left[\frac{m}{2}\dot{q}^2(1+2\alpha m\dot{q}+8\alpha^2m^2\dot{q}^2-2\beta m^2\dot{q}^2)-V(q)\right].\tag{16}$$

From the above action one can readily write down the Lagrangian to be

$$L=\frac{m}{2}\dot{q}^2(1+2\alpha m\dot{q}+8\alpha^2m^2\dot{q}^2-2\beta m^2\dot{q}^2)-V(q).\tag{17}$$

Equations (16) and (17) are the general forms of the action and the Lagrangian of a nonrelativistic particle moving under an arbitrary potential $V(q)$ in the presence of the GUP. It is to be noted that here we take the generalized structure of the GUP (3) where both the linear and quadratic modifications in momentum p_j are present. We now proceed to investigate the free particle and the harmonic oscillator systems.

III. FREE PARTICLE AND HO SYSTEM

With the above results in hand, we now proceed to investigate the free particle and harmonic oscillator potential in this section.

For the free particle case we have $V(q) = 0$. Hence, from the action in Eq. (16) one can easily find the classical equation of motion. This reads

$$m(1 + 6\alpha m\dot{q} + 48\alpha^2 m^2 \dot{q}^2 - 12\beta m^2 \dot{q}^2)\ddot{q} = 0$$
$$\Rightarrow \ddot{q} = 0 \quad \text{or} \quad (1 + 6\alpha m\dot{q} + 48\alpha^2 m^2 \dot{q}^2 - 12\beta m^2 \dot{q}^2) = 0. \tag{18}$$

Before going further we now analyze the above result. Interestingly, both the possibilities indicate $\ddot{q} = 0$. Moreover, the presence of the GUP puts a bound on the free particle velocity. The upper bound on the velocity of the free particle obtained from the action reads

$$\dot{q}_{\max} = \frac{-\alpha - \sqrt{(2\beta - 7\alpha^2)}}{2m(4\alpha^2 - \beta)}. \tag{19}$$

Note that in the limit $\alpha \to 0$, this maximum particle speed agrees with the result obtained in [24]. Now the free particle velocity given by Eq. (19) cannot be imaginary and must be finite. This restriction gives a relation between the GUP parameters α and β which is

$$\beta > 3.5\alpha^2 \quad \text{and} \quad \beta \neq 4\alpha^2. \tag{20}$$

It should also be noted that we have taken the negative sign before the square root. The reason behind this choice is that there is no value of $\beta > 3.5\alpha^2$ for which $\dot{q}_{\max}$ is positive. With the negative sign before the square root, the positivity of $\dot{q}_{\max}$ implies $\beta > 4\alpha^2$. We now want to point out some relevant and interesting facts about the above relations. In [10], the authors showed that $\beta = 2\alpha^2$ (see the Appendix). But from our analysis we find that the relation $\beta = 2\alpha^2$ is not possible. From the analysis in [10] (see the Appendix), we have

$$\beta = (n + 1)\alpha^2. \tag{21}$$

Therefore, Eq. (20) together with Eq. (21) gives the relation between α and β. This is an important result in our paper.

We now calculate the classical action for the free particle in the presence of the GUP. To do this first we have to solve $\ddot{q} = 0$, imposing the boundary conditions that at $t = t_0$, $q = q_0$; $t = t_f$, $q = q_f$. The classical trajectory of the free particle then comes out to be

$$q_c(t) = q_0 + \frac{q_f - q_0}{T} t, \qquad t_f - t_0 = T. \tag{22}$$

Substituting Eq. (22) in Eq. (16), the classical action for the free particle in the presence of the GUP takes the form

$$S_c = \frac{m}{2T}(q_f - q_0)^2 \left[1 + 2\alpha m \left(\frac{q_f - q_i}{T}\right) + 8\alpha^2 m^2 \left(\frac{q_f - q_i}{T}\right)^2 - 2\beta m^2 \left(\frac{q_f - q_0}{T}\right)^2\right]. \tag{23}$$

Using the above expression for the classical action in Eq. (15), we obtain

$$\langle q_f, t_f | q_0, t_0 \rangle = \tilde{F}(T, \alpha, \beta) e^{\frac{m}{2T}(q_f - q_0)^2 [1 + 2\alpha m (\frac{q_f - q_i}{T}) + 8\alpha^2 m^2 (\frac{q_f - q_i}{T})^2 - 2\beta m^2 (\frac{q_f - q_0}{T})^2]}. \tag{24}$$

The above action reduces to that in [24] in the $\alpha \to 0$ limit. Our next step is to evaluate the constant $\tilde{F}(T, \alpha, \beta)$ which contains the quantum fluctuations. We now use the following identity:

$$\langle q_f, t_f | p \rangle = \int_{-\infty}^{+\infty} dq_0 \langle q_f, t_f | q_0, t_0 \rangle \langle q_0, t_0 | p \rangle \tag{25}$$

and the overlaps

$$\langle q_0, 0 | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{i}{\hbar} p q_0\right); \qquad \langle q_f, T | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left\{\frac{-iT}{\hbar}\left(\frac{p_0^2}{2m} - \frac{\alpha}{m} p_0^3 + \frac{\gamma}{m} p_0^4\right)\right\} \exp\left(\frac{i}{\hbar} p q_f\right) \tag{26}$$

with $t_0 = 0$ and $t_f = T$. Equations (24) and (25), along with Eq. (26), yield the quantum fluctuations to be

$$\tilde{F}(T, \alpha, \beta) = \sqrt{\frac{m}{2\pi i\hbar T}} \left[1 + \frac{3\alpha m(q_f - q_0)}{T} + \frac{3i\beta\hbar m}{T} - \frac{6i\alpha^2\hbar m}{T} - 6\left(\frac{\alpha^2}{2} + \beta\right)\frac{m^2(q_f - q_0)^2}{T^2} + \frac{45\alpha^2 m^2 (q_f - q_0)^2}{2T^2}\right]. \tag{27}$$

Hence, from Eqs. (24) and (27), the propagation kernel for a free particle in the GUP framework, containing both the linear and quadratic corrections in momentum, can be recast as

$$\langle q_f, t_f | q_0, t_0 \rangle = \sqrt{\frac{m}{2\pi i\hbar T}} \left[1 + \frac{3\alpha m(q_f - q_0)}{T} + \frac{3i\beta\hbar m}{T} - \frac{6i\alpha^2\hbar m}{T} - 6\left(\frac{\alpha^2}{2} + \beta\right) \frac{m^2(q_f - q_0)^2}{T^2} + \frac{45\alpha^2 m^2(q_f - q_0)^2}{2T^2} \right] \\ \times e^{\frac{m}{2T}(q_f - q_0)^2[1 + 2\alpha m(\frac{q_f - q_i}{T}) + 8\alpha^2 m^2 (\frac{q_f - q_i}{T})^2 - 2\beta m^2 (\frac{q_f - q_0}{T})^2]}. \tag{28}$$

Now we proceed to investigate the harmonic oscillator potential with $V(q) = \frac{1}{2} m\omega^2 q^2$ in Eq. (16). This yields

$$S = \int_0^T dt \left[\frac{m}{2} \dot{q}^2 (1 + 2\alpha m\dot{q} + 8\alpha^2 m^2 \dot{q}^2 - 2\beta m^2 \dot{q}^2) - \frac{1}{2} m\omega^2 q^2 \right]. \tag{29}$$

From the above action one can easily find out the classical equation of motion which reads

$$\ddot{q}(t) + 6\alpha m\dot{q}(t)\ddot{q}(t) + 48\alpha^2 m^2 \dot{q}^2(t)\ddot{q}(t) - 12\beta m^2 \dot{q}^2(t)\ddot{q}(t) + \omega^2 q(t) = 0. \tag{30}$$

Now we carry out a consistency check of the path-integral formalism by deriving the above equation of motion of the one-dimensional harmonic oscillator up to $\mathcal{O}(\alpha^2, \beta)$ using Hamilton's equations of motion. This gives

$$\dot{q} = \{q, H\} = \frac{p}{m} - \frac{3\alpha}{m} p^2 + \frac{2\alpha^2}{m} p^3 + \frac{4\beta}{m} p^3 \\ \dot{p} = \{p, H\} = -m\omega^2 q. \tag{31}$$

A simple algebra now shows that the above equations agree with the equation of motion (30). This ensures the validity of the path-integral formalism.

We now proceed to solve Eq. (30) to get the classical trajectory of the harmonic oscillator in the presence of the GUP up to order $\mathcal{O}(\alpha^2, \beta)$. The solution can be recast as

$$q(t) = q_{(0)}(t) + \alpha q_{(1)}(t) + \alpha^2 q_{(2)}(t) + \beta q_{(3)}(t) \tag{32}$$

where

$$\begin{aligned} q_{(0)}(t) &= A\cos(\omega t) + B\sin(\omega t), \\ q_{(1)}(t) &= C_1 \cos(\omega t) + C_2 \sin(\omega t) + m\omega(A^2 - B^2)\sin(2\omega t) - 2ABm\omega\cos(2\omega t), \\ q_{(2)}(t) &= C_3 \cos(\omega t) + C_4 \sin(\omega t) + 3m^2\omega^2(A^2 + B^2)(A - Bt\omega)\cos(\omega t) - 2m\omega(BC_1 + AC_2)\cos(2\omega t) \\ &\quad - \frac{3}{4} m^2\omega^2 A(A^2 - 3B^2)\cos(3\omega t) + 3m^2\omega^3 tA(A^2 + B^2)\sin(\omega t) + 2m\omega(AC_1 - BC_2)\sin(2\omega t) \\ &\quad + \frac{3}{4} m^2\omega^2 B(B^2 - 3A^2)\sin(3\omega t), \\ q_{(3)}(t) &= C_5 \cos(\omega t) + C_6 \sin(\omega t) + \frac{1}{8}[6m^2\omega^2(A^2 + B^2)(2Bt\omega - A)\cos(\omega t) - 3m^2\omega^2 A(A^2 - 3B^2)\cos(3\omega t) \\ &\quad - 6m^2\omega^2(A^2 + B^2)(B + 2At\omega)\sin(\omega t) + 3m^2\omega^2 B(B^2 - 3A^2)\sin(3\omega t)]. \end{aligned} \tag{33}$$

The constants A, B, C_1, C_2, C_3, C_4, C_5, and C_6 read

$$
\begin{aligned}
A &= q_0 \\
B &= [q_f - q_0\cos(\omega T)]\csc(\omega T) \\
C_1 &= 2m\omega AB \\
C_2 &= \frac{1}{\sin(\omega T)}[2ABm\omega\cos(2\omega T) - m\omega(A^2 - B^2)\sin(2\omega T) - 2m\omega AB\cos(\omega T)] \\
C_3 &= -\frac{5}{4}m^2\omega^2 AB^2 + 2m\omega AC_2 - \frac{9}{4}m^2\omega^2 A^3 \\
C_4 &= -\frac{1}{\sin(\omega T)}[C_3\cos(\omega T) + 3m^2\omega^2(A^2 + B^2)(A - B\omega T)\cos(\omega T) - 2m\omega(BC_1 + AC_2)\cos(2\omega T) \\
&\quad - \frac{3}{4}m^2\omega^2 A(A^2 - 3B^2)\cos(3\omega T) + 3m^2\omega^3 TA(A^2 + B^2)\sin(\omega T) + 2m\omega(AC_1 - BC_2)\sin(2\omega T) \\
&\quad + \frac{3}{4}m^2\omega^2 B(B^2 - 3A^2)\sin(3\omega T)] \\
C_5 &= \frac{3}{8}m^2\omega^2(3A^3 - AB^2) \\
C_6 &= -\frac{1}{8\sin(\omega T)}[\{6m^2\omega^2(A^2 + B^2)(2B\omega T - A)\cos(\omega T) - 3nm^2\omega^2(A^3 - 3A^2 B)\cos(3\omega T) \\
&\quad - 6m^2\omega^2(A^2 + B^2)(B + 2A\omega t)\sin(\omega T) + 3m^2\omega^2(B^3 - 3A^2 B)\sin(3\omega T)\} - C_5\cos(\omega t)].
\end{aligned} \tag{34}
$$

Therefore, the classical action for the harmonic oscillator in the framework of the GUP algebra (2) can be obtained by using Eqs. (32), (33), and (34) in Eq. (30). This yields

$$
S_c = S_c(0) + S_c(\alpha) + S_c(\alpha^2) + S_c(\beta). \tag{35}
$$

Here, $S_c(0)$ is the classical action for the ordinary harmonic oscillator. $S_c(\alpha)$, $S_c(\alpha^2)$, and $S_c(\beta)$ are the corrections due to the presence of the GUP. The forms of $S_c(0)$, $S_c(\alpha)$, $S_c(\alpha^2)$, and $S_c(\beta)$ are

$$
S_c(0) = \frac{1}{2}mw\csc(Tw)[(q_0^2 + q_f^2)\cos(Tw) - 2q_0 q_f] \tag{36}
$$

$$
S_c(\alpha) = -\frac{\alpha}{6}m^2 w^2(q_0 - q_f)\csc^2(Tw)[(q_0^2 + q_0 q_f + q_f^2)\cos(2Tw) - 12q_0 q_f\cos(Tw) - q_0 q_f + 5(q_0^2 + q_f^2)] \tag{37}
$$

$$
\begin{aligned}
S_c(\alpha^2) = \frac{\alpha^2}{16}m^3 w^3\csc^4(Tw)[&(q_0^4 + q_f^4)\sin(4Tw) - 4q_0 q_f(21q_0^2 - 20q_0 q_f + 21q_f^2)\sin(Tw) \\
&- 4q_0 q_f(5q_0^2 - 4q_0 q_f + 5q_f^2)\sin(3Tw) + 24q_0^2 q_f^2 Tw\cos(2Tw) - 48q_0 q_f Tw(q_0^2 + q_f^2)\cos(Tw) \\
&+ 12Tw(q_0^4 + 4q_0^2 q_f^2 + q_f^4) + 4(6q_0^4 - 8q_0^3 q_f + 23q_0^2 q_f^2 - 8q_0 q_f^3 + 6q_f^4)\sin(2Tw)]
\end{aligned} \tag{38}
$$

$$
\begin{aligned}
S_{(\beta)} = -\frac{\beta}{32}m^3 w^3\csc^4(Tw)[&(q_0^4 + q_f^4)\sin(4Tw) - 44q_0 q_f(q_0^2 + q_f^2)\sin(Tw) - 12q_0 q_f(q_0^2 + q_f^2)\sin(3Tw) \\
&+ 24q_0^2 q_f^2 Tw\cos(2Tw) - 48q_0 q_f Tw(q_0^2 + q_f^2)\cos(Tw) + 12Tw(q_0^4 + 4q_0^2 q_f^2 + q_f^4) \\
&+ 4(2q_0^4 + 15q_0^2 q_f^2 + 2q_f^4)\sin(2Tw)].
\end{aligned} \tag{39}
$$

It is reassuring to note that we recover the free particle classical action (23) in the limit $\omega \to 0$. Therefore, the propagator for the harmonic oscillator reads

$$
\langle q_f, t_f | q_0, t_0 \rangle = \sqrt{\frac{m\omega}{2\pi i\hbar\sin(\omega T)}}\tilde{F}_1(T, \alpha, \beta)e^{\frac{i}{\hbar}S_c}. \tag{40}
$$

Now we will calculate the quantum fluctuations $\tilde{F}_1$ from the Schrödinger equation in the subsequent section.

IV. CALCULATION OF THE QUANTUM FLUCTUATION

In this section we apply a different approach to evaluate the explicit form of the kernel of a particle moving in a harmonic potential in the GUP framework. We calculate the Feynman propagator and the quantum fluctuations $\tilde{F}_1$ from the Schrödinger equation up to order $\mathcal{O}(\alpha, \beta)$. Note that in this section though we give the complete expression of eigenfunctions and energy eigenvalues retaining the terms in α^2 in the final expression of the quantum fluctuation for harmonic oscillator we neglect the terms of the order $\mathcal{O}(\alpha^2, \beta^2)$.

To do this first we write the Schrödinger equation for the harmonic oscillator bearing the GUP effects for both linear and quadratic corrections in momentum. This reads

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2}-\frac{i\alpha\hbar^3}{m}\frac{\partial^3}{\partial q^3}+\frac{\alpha^2\hbar^4}{2m}\frac{\partial^4}{\partial q^4}+\frac{\beta\hbar^4}{m}\frac{\partial^4}{\partial q^4}+\frac{1}{2}m\omega^2q^2+\mathcal{O}(\beta^2)\right]\psi_n(q)=E_n\psi_n(q) \tag{41}$$

where $\psi_n(q)$ and E_n are nth order eigenfunction and eigenvalue of the Schrödinger equation. Hence, the Feynman propagator $\langle q_f, t_f|q_0, t_0\rangle$ can be recast as

$$\langle q_f, t_f|q_0, t_0\rangle = \sum_n \psi_n(q_f)\psi_n^*(q_0)e^{-(i/\hbar)E_n(t_f-t_0)}. \tag{42}$$

We now solve the Schrödinger equation (41) by treating the GUP contributions as time independent perturbations. Then the perturbation piece of the Hamiltonian up to the order $\mathcal{O}(\alpha^2, \beta)$ can be written as

$$H(\alpha, \alpha^2, \beta) = -\frac{\alpha}{m}p_0^3+\frac{1}{m}\left(\frac{\alpha^2}{2}+\beta\right)p_0^4+\frac{1}{2}m\omega^2q^2+\mathcal{O}(\alpha\beta, \alpha^3, \beta^2). \tag{43}$$

We can now obtain the eigenstates and eigenvalues by applying time independent perturbation theory. This yields

$$\begin{aligned}\psi_n(q) = {}& \phi_n(q)-\frac{i\alpha}{m\hbar\omega}\left(\frac{\hbar m\omega}{2}\right)^{\frac{3}{2}}\left[\frac{\sqrt{n(n-1)(n-2)}}{3}\phi_{n-3}(q)-3n\sqrt{n}\phi_{n-1}(q)-3(n+1)\sqrt{n+1}\phi_{n+1}(q)\right.\\ & \left.+\frac{(n+1)(n+2)(n+3)}{3}\phi_{n+3}(q)\right]+\left(\frac{\alpha^2}{2}+\beta\right)(m\hbar\omega)\left[\frac{(2n+3)\sqrt{(n+1)(n+2)}}{4}\phi_{n+2}(q)\right.\\ & \left.-\frac{(2n-1)\sqrt{n(n-1)}}{4}\phi_{n-2}(q)+\frac{\sqrt{n(n-1)(n-2)(n-3)}}{16}\phi_{n-4}(q)-\frac{\sqrt{(n+1)(n+2)(n+3)(n+4)}}{16}\phi_{n+4}(q)\right]\\ & +\mathcal{O}(\alpha^3)+\mathcal{O}(\beta^2)+\mathcal{O}(\alpha\beta)\end{aligned} \tag{44}$$

and

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega\left[1+\frac{3(2n^2+2n+1)}{2(2n+1)}\left(\frac{\alpha^2}{2}+\beta\right)(m\hbar\omega)\right]+\mathcal{O}(\beta^2), \tag{45}$$

where $n = 0, 1, 2, \ldots$, and

$$\phi_n(q) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}H_n\left(\sqrt{\frac{m\omega}{\hbar}}q\right)\exp\left[-\frac{m\omega}{2\hbar}q^2\right]. \tag{46}$$

Note that the energy eigenvalue obtained in Eq. (45) for the $1-d$ harmonic oscillator agrees with the result reported in [26] up to order $\mathcal{O}(\beta)$. This can be seen by setting $\alpha = 0$ in Eq. (45) and the coefficient of $p_i p_j$ in [26] to 2β. Further, since the result for the energy eigenvalue in [26] is given for any arbitrary dimension d, one needs to set $d = 1$ to see the agreement. These are the complete forms of the eigenstates and energy eigenvalues of the harmonic oscillator in the presence of the GUP, with the general GUP structure containing both linear and quadratic contributions in momentum up to order $\mathcal{O}(\alpha, \alpha^2, \beta)$. Using Eqs. (44) and (45) in Eq. (42), the Feynman propagator reads

$$\langle q_f, t_f | q_0, t_0 \rangle = J + \frac{i\alpha}{m\omega\hbar}\left(\frac{m\omega\hbar}{2}\right)^{\frac{3}{2}}[M_1 + M_2] + (\beta m\hbar\omega)[N_1 + N_2] + \mathcal{O}(\beta^2) + \mathcal{O}(\alpha^2, \beta) \tag{47}$$

where

$$\begin{aligned}
J &= \sum_{n=0}^{\infty} \phi_n(q_f)\phi_n(q_0)\exp\left[-\frac{i}{\hbar}\left(n+\frac{1}{2}\right)\hbar\omega T\left\{1 + \frac{3(2n^2+2n+1)}{2(2n+1)}(\beta m\hbar\omega)\right\}\right]\\
M_1 &= \left[\sum_{n=3}^{\infty}\frac{\sqrt{n(n-1)(n-2)}}{3}[\phi_n(q_f)\phi_{n-3}(q_0) - \phi_n(q_0)\phi_{n-3}(q_f)]\right.\\
&\quad \left. + \sum_{n=0}^{\infty}\frac{\sqrt{(n+1)(n+2)(n+3)}}{3}[\phi_{n+3}(q_0)\phi_n(q_f) - \phi_{n+3}(q_f)\phi_n(q_0)]\right]\exp\left[-\frac{i}{\hbar}\left(n+\frac{1}{2}\right)\hbar\omega T\right]\\
M_2 &= -3\left[\sum_{n=1}^{\infty} n\sqrt{n}[\phi_n(q_f)\phi_{n-1}(q_0) - \phi_{n-1}(q_f)\phi_n(q_0)] + \sum_{n=0}^{\infty}(n+1)\sqrt{n+1}[\phi_n(q_f)\phi_{n+1}(q_0) - \phi_{n+1}(q_f)\phi_n(q_0)]\right]\\
&\quad \times \exp\left[-\frac{i}{\hbar}\left(n+\frac{1}{2}\right)\hbar\omega T\right]\\
N_1 &= \left[\sum_{n=0}^{\infty}\frac{(2n+3)\sqrt{(n+1)(n+2)}}{4}[\phi_n(q_f)\phi_{n+2}(q_0) + \phi_n(q_0)\phi_{n+2}(q_f)]\right.\\
&\quad \left. - \sum_{n=2}^{\infty}\frac{(2n-1)\sqrt{n(n-1)}}{4}[\phi_n(q_f)\phi_{n-2}(q_0) + \phi_n(q_0)\phi_{n-2}(q_f)]\right]\exp\left[-\frac{i}{\hbar}\left(n+\frac{1}{2}\right)\hbar\omega T\right]\\
N_2 &= \left[\sum_{n=4}^{\infty}\frac{\sqrt{n(n-1)(n-2)(n-3)}}{16}[\phi_n(q_f)\phi_{n-4}(q_0) + \phi_n(q_0)\phi_{n-4}(q_f)]\right.\\
&\quad \left. - \sum_{n=0}^{\infty}\frac{\sqrt{(n+1)(n+2)(n+3)(n+4)}}{16}[\phi_n(q_f)\phi_{n+4}(q_0) + \phi_n(q_0)\phi_{n+4}(q_f)]\right].\\
&\quad \times \exp\left[-\frac{i}{\hbar}\left(n+\frac{1}{2}\right)\hbar\omega T\right].
\end{aligned} \tag{48}$$

Now using the exact form of $\phi_n(q)$ given by Eq. (46) in Eq. (48), we have

$$J=\sqrt{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{2\hbar}(q_0^2+q_f^2)\right]\exp\left(-\frac{i\omega T}{2}\right)\sum_{n=0}^{\infty}\left(\frac{\exp(-i\omega T)}{2}\right)^n\frac{1}{n!}H_n\left(\sqrt{\frac{m\omega}{\hbar}}q_0\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}q_f\right)\times\left[1-\frac{3i\beta m\omega^2\hbar T}{4}(2n^2+2n+1)\right]$$

$$M_1=-\sqrt{\frac{m\omega}{\pi\hbar}}\frac{i}{3\sqrt{2}}\exp\left[-\frac{m\omega}{2\hbar}(q_0^2+q_f^2)\right]\exp(-2i\omega T)\sin\left(\frac{3\omega T}{2}\right)\sum_{n=0}^{\infty}\left(\frac{\exp(-i\omega T)}{2}\right)^n\frac{1}{n!}\times\left[H_{n+3}\left(\sqrt{\frac{m\omega}{\hbar}}q_f\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}q_0\right)-H_{n+3}\left(\sqrt{\frac{m\omega}{\hbar}}q_0\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}q_f\right)\right]$$

$$M_2=\sqrt{\frac{m\omega}{\pi\hbar}}3i\sqrt{2}\exp\left[-\frac{m\omega}{2\hbar}(q_0^2+q_f^2)\right]\exp(-i\omega T)\sin\left(\frac{\omega T}{2}\right)\sum_{n=0}^{\infty}(n+1)\left(\frac{\exp(-i\omega T)}{2}\right)^n\frac{1}{n!}\times\left[H_{n+1}\left(\sqrt{\frac{m\omega}{\hbar}}q_f\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}q_0\right)-H_{n+1}\left(\sqrt{\frac{m\omega}{\hbar}}q_0\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}q_f\right)\right]$$

$$N_1=\sqrt{\frac{m\omega}{\pi\hbar}}3i\sqrt{2}\exp\left[-\frac{m\omega}{2\hbar}(q_0^2+q_f^2)\right]\exp\left(-\frac{3i\omega T}{2}\right)\sin(\omega T)\sum_{n=0}^{\infty}\left(\frac{\exp(-i\omega T)}{2}\right)^n\frac{1}{n!}\times\left[H_{n+2}\left(\sqrt{\frac{m\omega}{\hbar}}q_f\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}q_0\right)+H_{n+2}\left(\sqrt{\frac{m\omega}{\hbar}}q_0\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}q_f\right)\right]. \tag{49}$$

Similarly, N_2 can be recast in terms of the Hermite polynomials.

Now to evaluate the constants J, M_1, M_2, N_1, and N_2, we use the extended Mehler's formula [27]

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}H_{k+m}(x)H_{k+n}(y)=(1-4t^2)^{-(m+n+1)/2}\exp\left[\frac{4txy-4t^2(x^2+y^2)}{1-4t^2}\right]\times\sum_{k=0}^{\min(m,n)}2^{2k}k!\binom{m}{k}\binom{n}{k}t^kH_{m-k}\left(\frac{x-2ty}{\sqrt{1-4t^2}}\right)H_{n-k}\left(\frac{y-2tx}{\sqrt{1-4t^2}}\right). \tag{50}$$

Using this we get

$$J=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}e^{\frac{i}{\hbar}S_0}\tilde{J},\qquad M_1=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}e^{\frac{i}{\hbar}S_0}\tilde{M}_1,\qquad M_2=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}e^{\frac{i}{\hbar}S_0}\tilde{M}_2,$$
$$N_1=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}e^{\frac{i}{\hbar}S_0}\tilde{N}_1,\qquad N_2=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}e^{\frac{i}{\hbar}S_0}\tilde{N}_2, \tag{51}$$

where

$$\tilde{J}=1-\frac{3i\beta m\omega^2T}{8\hbar\sin^4\omega T}[-3i\hbar m\omega(q_0^2+q_f^2)\sin 2\omega T+m^2\omega^2(q_0^2+q_f^2-2q_0q_f\cos\omega T)^2+4i\hbar m\omega\sin\omega T(2+\cos 2\omega T)q_0q_f-\hbar^2\sin^2\omega T(2+\cos 2\omega T)] \tag{52}$$

$$\tilde{M}_1=\frac{1}{3}\sqrt{\frac{m\omega}{2\hbar}}\frac{\sin\frac{3\omega T}{2}}{\hbar\sin^2\omega T\sin\frac{\omega T}{2}}(q_0-q_f)[-m\omega(q_0^2+4q_0q_f+q_f^2)+2m\omega(q_0^2+q_0q_f+q_f^2)\cos\omega T-3i\hbar\sin\omega T]$$
$$\tilde{M}_2=-\frac{3\sqrt{2}}{8\hbar}\sqrt{\frac{m\omega}{\hbar}}\frac{(q_0-q_f)}{\sin^2\frac{\omega T}{2}\cos^2\frac{\omega T}{2}}[-i\hbar\sin 2\omega T+m\omega(q_0^2-2q_0q_f\cos\omega T+q_f^2)-i\hbar\sin\omega T] \tag{53}$$

$$\begin{aligned}\tilde{N}_1 &= -\frac{i}{8\hbar^2\sin^3\omega T}[-4m^2\omega^2 q_0 q_f(q_0^2+q_f^2)(3+\cos 2\omega T)+3\hbar^2(\cos 3\omega T-\cos\omega T)\\&\quad+4m\omega\cos\omega T\{m\omega(q_0^4+6q_0^2q_f^2+q_f^4)+12i\hbar q_0q_f\sin\omega T\}-3i\hbar m\omega(q_0^2+q_f^2)(5\sin\omega T+\sin 3\omega T)]\\ \tilde{N}_2 &= -\frac{i\cos\omega T}{16\hbar^2\sin^3\omega T}[12m^2\omega^2q_0^2q_f^2-3\hbar^2(1-\cos 2\omega T)+2m\omega\{m\omega\cos 2\omega T(q_0^4+q_f^4)\\&\quad-4m\omega q_0q_f(q_0^2+q_f^2)\cos\omega T-6i\hbar\sin\omega T\{(q_0^2+q_f^2)\cos\omega T-2q_0q_f\}\}].\end{aligned} \tag{54}$$

Therefore, $\langle q_f,t_f|q_0,t_0\rangle$ can be recast as [up to $\mathcal{O}(\alpha,\beta)$]

$$\langle q_f,t_f|q_0,t_0\rangle=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega T}}[1+\alpha f(q_0,q_f;T)+\beta g(q_0,q_f;T)+\mathcal{O}(\alpha,\beta)]e^{\frac{i}{\hbar}(S_c(0)+S_c(\alpha)+S_c(\beta))} \tag{55}$$

where

$$\begin{aligned}S_c(0)&=\frac{m\omega}{2}\csc\omega T[(q_0^2+q_f^2)\cos\omega T-2q_0q_f]\\ S_c(\alpha)&=-\frac{\alpha}{6}m^2\omega^2(q_0-q_f)\csc^2\omega T[(q_0^2+q_0q_f+q_f^2)\cos 2\omega T-12q_0q_f\cos\omega T-q_0q_f+5(q_0^2+q_f^2)]\\ S_c(\beta)&=-\frac{\beta m^3\omega^3}{32}\csc^4\omega T[\{12\omega T+8\sin 2\omega T+\sin 4\omega T\}(q_0^4+q_f^4)\\&\quad-4\{12\omega T\cos\omega T+11\sin\omega T+3\sin 3\omega T\}q_0q_f(q_0^2+q_f^2)\\&\quad+12\{4\omega T+2\omega T\cos 2\omega T+5\sin 2\omega T\}q_0^2q_f^2]\end{aligned} \tag{56}$$

with the functions f and g being given by

$$\begin{aligned}f(q_0,q_f;T)&=-(q_0-q_f)m\omega\csc^2\omega T[\sin\omega T+\sin 2\omega T]\\ g(q_0,q_f:T)&=\frac{3i\hbar m\omega}{8\sin^2\omega T}(2\omega T+5\sin\omega T\cos\omega T+\omega T\cos 2\omega T)\\&\quad-\frac{3m^2\omega^2}{8\sin^3\omega T}[2\omega T\{3\cos\omega T(q_0^2+q_f^2)-2(2+\cos 2\omega T)q_0q_f\}\\&\quad+10\sin\omega T(q_0^2+q_f^2-2q_0q_f\cos\omega T)-6\sin^3\omega T(q_0^2+q_f^2)].\end{aligned} \tag{57}$$

Note that in this method we calculate the exact expression for the quantum fluctuation up to first order in α, β. This calculation can be extended for higher order in α^2.

V. CONCLUSION

We now summarize the results in this paper. In this paper we have constructed the path-integral formalism of the propagation kernel in the presence of the generalized uncertainty principle incorporating both the contributions proportional to linear and quadratic terms in momentum. We obtained the action of a nonrelativistic particle moving in an arbitrary potential in the framework of the generalized uncertainty principle. After getting the general form of the action we have moved on to investigate the free particle and harmonic oscillator systems. From the free particle analysis, we have seen that the action imposes an upper bound on the free particle velocity which depends on the mass of the particle. This feature is consistent with the results obtained earlier [23,24,28]. Moreover, the fact that the particle velocity must be real and finite leads us to a relation between parameters α and β. We show that $\beta > 4\alpha^2$. This is an interesting result in our paper. Then we have calculated the Feynman propagator for a harmonic oscillator. In the limiting case $\omega \to 0$, the classical action for the harmonic oscillator reduces to the free particle result. We have explored another approach to get the propagation kernel. We have constructed the Schrödinger equation for a harmonic oscillator in the framework of the generalized uncertainty principle. Solving the Schrödinger equation we have got expressions for the nth order eigenfunction and energy eigenvalue bearing the effects of the generalized uncertainty principle. Using these results, we derive the expression for the propagation kernel for the harmonic oscillator. We have obtained the explicit form of the quantum fluctuations up to first order in α and β. These results would be important to derive the thermodynamics of the harmonic oscillator system in the general uncertainty principle framework. This we hope to report in future.

APPENDIX: RELATION BETWEEN THE GUP PARAMETERS

The most general algebra [10] for the commutation relation between position q_j and its conjugate momentum p_j with linear and quadratic modifications in momentum reads

$$[q_i, p_j] = i\hbar\left(\delta_{ij} + \delta_{ij}\alpha_1 p + \alpha_2 \frac{p_i p_j}{p} + \beta_1 \delta_{ij} p^2 + \beta_2 p_i p_j\right). \tag{A1}$$

Therefore, the coordinates and its conjugate momentum follow the Jacobi identity

$$-[[q_i, q_j], p_k] = [[q_j, p_k], q_i] + [[p_k, q_i], q_j] = 0. \tag{A2}$$

Now we expand the right-hand side of the Jacobi identity and using Eq. (A1), we get

$$\begin{aligned} &i\hbar\{-\alpha_1\delta_{jk}[q_i, p] - \alpha_2([q_i, p_j]p_k p^{-1} + p_j[q_i, p_k]p^{-1} + p_j p_k[q_i, p^{-1}]) - \beta_1\delta_{jk}([q_i, p_l]p_l + p_l[q_i, p_l]) \\ &\quad - \beta_2([q_i, p_j]p_k + p_j[q_i, p_k])\} - (i \leftrightarrow j) = 0. \end{aligned} \tag{A3}$$

We can easily evaluate the following commutator up to $\mathcal{O}(p)$:

$$[q_i, p] = i\hbar\{p_i p^{-1} + (\alpha_1 + \alpha_2)p_i\} \tag{A4}$$

and

$$[q_i, p^{-1}] = -i\hbar p_i p^{-3}\{1 + (\alpha_1 + \alpha_2)p\}. \tag{A5}$$

Using the above commutation relations in Eq. (A3), we get

$$\{(\alpha_1 - \alpha_2)p^{-1} + (\alpha_1^2 + 2\beta_1 - \beta_2)\}(p_i\delta_{jk} - p_j\delta ik) = 0. \tag{A6}$$

Thus, the above equation is satisfied only when $\alpha_1 = \alpha_2 = \alpha$ ($\alpha > 0$ [29]) and $\beta_2 = 2\beta_1 + \alpha_1^2$. Now from dimensional analysis we have $\beta_1 \sim \alpha^2$. Let $\beta_1 = n\alpha^2$, where n is positive number. Then we have $\beta_2 = (2n+1)\alpha^2$. Note that in [10] $\beta_1 = \alpha^2$ (that is $n = 1$) has been taken into account for mathematical simplicity. Putting the values of β_1 and β_2 in Eq. (A1), the commutation relation takes the form as

$$[q_i, p_j] = i\hbar\left[\delta_{ij} - \alpha\left(p\delta_{ij} + \frac{p_i p_j}{p}\right) + n\alpha^2 p^2 \delta_{ij} + (2n+1)\alpha^2 p_i p_j\right]. \tag{A7}$$

Now the most general form of the momentum p_j in terms of p_{0j} can be written as

$$p_j = p_{0j} + a p_0 p_{0j} + b p_0^2 p_{0j}, \tag{A8}$$

where $a \sim \alpha$ and $b \sim \alpha^2$. Hence the commutation relation can be recast as

$$\begin{aligned} [q_i, p_j] &= [q_i, p_{0j} + a p_0 p_{0j} + b p_0^2 p_{0j}] \\ &= i\hbar\delta_{ij} + i\hbar a(p\delta_{ij} + p_i p_j p^{-1}) + i\hbar(2b - a^2)p_i p_j + (b - a^2)p^2\delta_{ij}. \end{aligned} \tag{A9}$$

Comparing the above relation with (A7), finally we get $a = -\alpha$, $n\alpha^2 = b - a^2$ and $(2n+1)\alpha^2 = 2b - a^2$. Hence,

$$b = (n+1)\alpha^2. \tag{A10}$$

Note that if we take $n = 1$ for mathematical simplicity, then we get $\beta = 2\alpha^2$ [10]. Now using the above relations we define two parameters, bearing the signature of the GUP as $a = -\alpha$ and $(n+1)\alpha^2 = \beta$. Therefore, Eq. (A8) yields

$$p_j = p_{0j} - \alpha p_0 p_{0j} + \beta p_0^2 p_{0j}, \quad (A11)$$

where $\beta = (n+1)\alpha^2$. This is Eq. (3) in this paper. In our analysis, Eq. (20) shows that

$$\beta > 4\alpha^2. \quad (A12)$$

This implies $n > 3$.

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