Generalized Landau-Khalatnikov-Fradkin transformations for arbitrary N-point fermion correlators

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We examine the nonperturbative gauge dependence of arbitrary configuration space fermion correlators in quantum electrodynamics (QED). First, we study the dressed electron propagator (allowing for emission or absorption of any number of photons along a fermion line) using the first quantized approach to quantum field theory and analyze its gauge transformation properties induced by virtual photon exchange. This is then extended to the N-point functions where we derive an exact, generalized version of the fully nonperturbative Landau-Khalatnikov-Fradkin (LKF) transformation for these correlators. We discuss some general aspects of application in perturbation theory and investigate the structure of the LKF factor about $D = 2$ dimensions.

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I. INTRODUCTION

The nonperturbative structure of the N-point functions in QED is an important aspect of quantum field theory, yet analyzing such aspects of the theory remains a difficult problem and still attracts significant attention. In a general theory, such information plays an important role in determining its phase structure, such as for dynamical chiral symmetry breaking or confinement in the well-known example of QCD. It is often desirable to determine the gauge dependence of various quantities or to use results found in a certain gauge to extract information about the same quantity in a different gauge. There exist limited nonperturbative analyses of relatively simple objects, such as the Ball-Chiu decomposition of the QED vertex [1] and its perturbative determination at one-loop order in various gauges [2,3]; results of similar calculations of the three-point vertex in three-dimensional QED_3 have been reported in [4–9] and for scalar QED in [10,11]. Ball and Chiu generalized their work to analyze the one-loop quark gluon vertex in QCD [12], later extended to two-loop order [13,14] in a particular renormalization scheme, and its gauge structure in arbitrary covariant gauge and dimension for an $SU(N)$ symmetry group [15]—this information is crucial for the determination of the transverse part of the vertex. Similarly, it has been possible to calculate the three- and four-gluon vertices, off shell, in covariant gauges for special kinematics up to two-loop order [16–23], which bears strongly on both the Dyson-Schwinger equations and infrared divergences within QCD.

Although physical observables such as cross sections are gauge invariant, the N-point Green functions of a given gauge theory generally have a strong dependence on the gauge choice for internal photons (the gauge transformations of external photons are well understood via the Ward-Takahashi [24] or Slavanov-Taylor [25,26] identities, whereas perturbation theory requires the gauge of internal photons to be fixed in order to define their propagator). Various covariant gauges can offer significant advantages for specific computations: Feynman gauge, besides minimizing the number of terms in loop calculations, leads to simple ghost-free Ward identities [27]. Landau gauge has become the favorite in Yang-Mills theory and QCD since it leads to an UV-finite ghost-gluon vertex [25] and an infrared (IR) fixed point of the renormalization group flow accessible on the lattice [28,29]. The Yennie-Fried gauge ($\xi = 3$) [30] is useful for eliminating spurious IR divergences in $D = 4$ [2], and a similar role is played by the traceless gauge $\xi = -1$ in $D = 2$ [31]. It is thus of great interest to develop efficient techniques for transforming Green's functions from one gauge to another. Such studies

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were initiated for the QED two-point function—or propagator—and the electron-photon vertex for the family of linear covariant gauges by Landau and Khalatnikov [32] and independently by Fradkin [33] and were later revisited using functional techniques in [34,35]; see also [36].

These LKF transformations are framed in coordinate space and are fully nonperturbative; denoting the propagator with covariant gauge parameter ξ by $S(x;\xi)$, a variation in the gauge $\xi \to \xi + \Delta\xi$ leads to the transformation ($\xi = 0$ corresponds to Landau gauge and $\xi = 1$ to Feynman gauge),

$$S(x;\xi+\Delta\xi) = S(x;\xi)\mathrm{e}^{-i\Delta\xi[\Delta_D(x)-\Delta_D(0)]}, \tag{1}$$

where $\Delta_D(x)$ is a function that fixes the gauge [32] [see (9)]. Aside from specifying the way in which the N-point functions vary under a gauge transformation at a given loop order, the linear dependence of Δ_D on α shows that one can construct gauge-dependent parts of higher-loop diagrams from knowledge of lower order terms [37,38]. The utility of these transformations is well illustrated by the restrictions imposed on the nonperturbative three-point vertex for it to be compatible with the LKF transformation of the fermion propagator [39–41]. Unfortunately, despite their importance in restraining the structure of gauge theory interactions, the LKF transformations have been studied far less than the more familiar constraints arising from the Ward-Takahashi and Slavanov-Taylor identities [42,43].

However, the LKF transformations for the particular case of the propagator have been examined to varying degrees of detail. In the massless case, it has been shown for both scalar and spinor QED and, in particular, circumstances for QCD that wavefunction renormalization takes a multiplicative power law form in four dimensions [37,44,45]. Moreover, the LKF transformations link the wave function normalization constants in different gauges [34] and relate strongly to chiral symmetry breaking. It has also been shown that the LKF rules for the fermion propagator lead to an enhancement to the quark anomalous magnetic moment [46] and constrain Anstze for the vertex operator often employed in analyses of the Schwinger-Dyson equations [47]. Some similar results for the propagator in reduced QED are given in [7,48].

Going beyond the propagator, there has been a recent resurgence in studying the generalized LKF transformations for N-point functions, especially in the context of QCD and extensions of the Gribov-Zwanziger (GZ) scenario [49–51] away from Landau gauge [52,53], among other nonperturbative properties of Yang-Mills theories. The use of the LKF transformations in perturbation theory is discussed in detail in [54,55]. More modern treatments have arrived at the transformation rules by including auxiliary "Stueckelberg-type" fields and Becchi-Rouet-Stora-Tyutin (BRST) invariance [56,57] (see also [58,59]), methods that were recently applied to the gluon propagator [60].

The LKF transformations in scalar QED for both the propagator and the generalized case of the N-point functions were also derived in [61,62] using the alternative *worldline approach* to quantum field theory. There, it is shown that the ordered (quenched) ($N = 2n$)-point amplitudes completely fix the LFK transformations. Denoting such an amplitude that corresponds to the contraction of n fields $\phi(x_i)$ with n conjugate fields $\phi^\dagger(x'_{\pi(i)})$ for $\pi \in S_n$ as $\mathcal{A}(x_1,\ldots,x_n;x'_{\pi(1)},\ldots,x'_{\pi(n)}|\xi)$ when covariant gauge parameter ξ is used for internal photons, they find the amplitudes in different gauges are related by

$$\begin{aligned}&\mathcal{A}(x_1,\ldots,x_n;x'_{\pi(1)},\ldots,x'_{\pi(n)}|\xi+\Delta\xi)\\&\quad=\prod_{k,l=1}^{N}\mathrm{e}^{-\Delta_\xi S_{i\pi}^{(k,l)}}\mathcal{A}(x_1,\ldots,x_n;x'_{\pi(1)},\ldots,x'_{\pi(n)}|\xi),\end{aligned} \tag{2}$$

which is the natural generalization of (1), constructed now from functions $\Delta_\xi S_{i\pi}^{(k,l)}$ to be defined below in (22). This same worldline formalism was recently applied to extend this work to the case of spinor QED [63]. In this companion paper, we provide further calculational details on this worldline derivation of the LKF transformation of the fermion propagator and the N-point fermionic Green functions that enter the calculation of scattering amplitudes in perturbation theory and discuss various applications of these recent results.

The worldline formalism is an alternative, first quantized approach to quantum field theory that has its roots in work due to Feynman at the same time that the more familiar second quantized approach was developed [64,65]. Strassler later developed perturbation theory within this framework [66], motivated by the seminal works of Bern and Kosower [67,68]. The essential idea is to reexpress the field theory scattering matrix in terms of path integrals over relativistic point particle trajectories, which reproduce the so-called Master Formulae of Bern and Kosower; a detailed review and a more recent report describing these methods can be found in [69,70]. Despite remaining lesser known than the "standard" perturbation theory based on Feynman diagrams that came to dominate the development of quantum field theory, the first quantized worldline approach has been applied with great success in a wide variety of problems. Of particular importance in the context of LKF transformations, the nontrivial task of extracting the form factor decomposition of the QCD vertex that is usually done by analyzing the Ward identities is quite cumbersome for rich tensorial structures. The worldline calculation has shown its efficiency in the decomposition of the three- and four-gluon vertices and the generalization of the N-point Ward identity (see [18,21,23]), and we expect similar simplifications in analyses of the gauge structure of the propagator.

Early successful applications of the worldline approach include a semiclassical "instanton" based determination of the Schwinger pair production rate [71]. By now, it is recognized that the worldline formalism has several advantages over standard methods, including representations of amplitudes where virtual momenta have already been integrated over that combine multiple Feynman diagrams related by permutation of external legs [31,70,72–74] and that are gauge invariant even at the level of the integrand [66,69]. Initial work using worldline techniques was largely focused on loop amplitudes, and although there are a few preliminary representations of propagators and tree-level processes using worldline techniques [72,75–80], it is only recently that a complete description of the scalar [61,62,81–85], spinor [86–89], and quark [90,91] propagators that are needed to study the LKF transformations has been achieved that retains the familiar benefits of the first quantized approach (see also [92]).

We are motivated by several main goals. First, this article expands upon the brief report of the main results given in [63], where we sought to compare the forms of the generalized transformation of the N-point functions between spinor and scalar QED; second, the theoretical developments presented here for spinor QED are far from trivial and will serve as a stepping stone to the more complicated transformations in QCD or more general gauge theories (worldline techniques have been extended to the non-Abelian case in a series of recent articles [93–98]); finally, a systematic study of the variation of N-point functions under a change of gauge is crucial for understanding how gauge invariant information can be extracted from calculations carried out in particular gauges—we have in mind, for instance, the truncation of Dyson-Schwinger equations to a particular order or numerical evaluation of such quantities on the lattice. Our use of the worldline formalism will be seen to simplify both the derivation of the LFK transformations and their implementation in perturbation theory.

A. Overview

The thrust of our approach and the main results can be summarized as follows. The worldline representation of the spinor propagator was recently developed in [86,87] and uses the second order formalism of the Dirac field [99,100]. The fundamental procedure is to decompose the configuration space representation of a Dirac fermion propagating in an electromagnetic field with gauge potential $\mathrm{A}(x) = A_\mu(x)dx^\mu$ as

$$S^{x'x}[A] = [m + i\not{D}']K^{x'x}[A], \tag{3}$$

where $D'_\mu(x) = \partial'_\mu + ieA_\mu(x')$ is the covariant derivative acting at x', and $K^{x'x}[A]$ is a matrix-valued auxiliary kernel to be discussed below. Here, we shall describe N-point functions by extending this representation to multiple open fermionic lines. To isolate the gauge transformation of internal photons, it is convenient to utilize the background field method [101,102] to split the gauge field $A = \mathrm{A}^\gamma + \bar{\mathrm{A}}$ into a part A^γ representing external photons and a "quantum" piece $\bar{\mathrm{A}}$—the path integral over $\bar{\mathrm{A}}$ will produce the virtual photons joining the spinor lines. These photons' propagators will be evaluated in a particular covariant gauge with parameter ξ, and we examine the dependence of the propagator on this choice. Related to this, we define the "backgroundless" propagator by

$$S_0^{x'x}[A^\gamma + \bar{A};\xi] = \langle [m + i(\not{\partial}' + ie\not{A}^\gamma)]K^{x'x}[A^\gamma + \bar{A}]\rangle_{\bar{A},\xi}, \tag{4}$$

without the insertion of $-e\bar{\not{A}}(x')$ in the prefactor of (3). The expectation value is taken over configurations of the background field that produces loop photons attached to the fermion line with the photon propagator taken in the covariant gauge with parameter ξ.

Our principle results are the following. The backgroundless part of the $(N = 2n)$-point function will be shown to transform according to the following rules for its partial amplitudes (again $\pi \in S_n$):

$$\begin{aligned}
&\mathcal{S}_{0\pi}(x_1,\ldots,x_n;x'_{\pi(1)},\ldots,x'_{\pi(n)}|\xi+\Delta\xi)\\
&= \left\langle \prod_{i=1}^{n}[m+i(\not{\partial}'_i+ie\not{A}^\gamma)]K^{x'_{\pi(i)}x_i}[A^\gamma+\bar{A}]\right\rangle_{\bar{A},\xi+\Delta\xi}\\
&= \left\langle \prod_{i=1}^{n}[m+i(\not{\partial}'_i+ie\not{A}^\gamma)]K^{x'_{\pi(i)}x_i}[A^\gamma+\bar{A}]\right\rangle_{\bar{A},\xi}\prod_{k,l=1}^{N}\mathrm{e}^{-\Delta_\xi S^{(k,l)}_{i\pi}},
\end{aligned} \tag{5}$$

with the *same* scalar factor as in (2). In this case, however, the derivatives $\not{\partial}'_i \coloneqq \gamma^\mu\partial_{x_i'^\mu}$ act through onto the trailing exponential and generate additional terms involving the derivatives of the $\Delta_\xi S^{(k,l)}_{i\pi}$. The additional contractions that appear from the $\bar{\not{A}}$ multiplying the kernels $K^{x_{\pi(i)}x_i}$ in the complete $\mathcal{S}^{x_{\pi(i)}x_i}$ precisely cancel these to allow for the exponential factor to be commuted to the left of the expectation value in (5). In this way, we arrive at the generalized LKF transformations of the partial fermionic N-point functions,

$$\begin{aligned}
\mathcal{S}_\pi(x_1,\ldots,x_n;x'_{\pi(1)},\ldots,x'_{\pi(n)}|\xi+\Delta\xi) &= \left\langle \prod_{i=1}^{n}[m+i(\not{\partial}'_i+ie(\not{A}^\gamma+\bar{\not{A}}))]K^{x'_{\pi(i)}x_i}[A^\gamma+\bar{A}]\right\rangle_{\bar{A},\xi+\Delta\xi} \\
&= \prod_{k,l=1}^{N}\mathrm{e}^{-\Delta_\xi S^{(k,l)}_{i\pi}}\left\langle \prod_{i=1}^{n}[m+i(\not{\partial}'_i+ie(\not{A}^\gamma+\bar{\not{A}}))]K^{x'_{\pi(i)}x_i}[A^\gamma+\bar{A}]\right\rangle_{\bar{A},\xi} \\
&= \prod_{k,l=1}^{N}\mathrm{e}^{-\Delta_\xi S^{(k,l)}_{i\pi}}\mathcal{S}_\pi(x_1,\ldots,x_n;x'_{\pi(1)},\ldots,x'_{\pi(n)}|\xi), \qquad (6)
\end{aligned}$$

which is the direct generalization of (2) to the spinor case reported in [63] (we emphasize that the exponential factor is identical to the scalar result).

Moreover, we shall show that the exponential prefactor, once summed over k and l, is independent of the permutation and thus, factorizes out of the sum over partial amplitudes, giving a simple multiplicative transformation for the N-point correlator itself, denoted S,

$$S(x_1,\ldots,x_n;x'_1,\ldots,x'_n|\xi+\Delta\xi) = T_n S(x_1,\ldots,x_n;x'_1,\ldots,x'_n|\xi), \qquad (7)$$

with T_n as the exponential prefactor for any chosen permutation. Likewise, a similar simplification lifts the result for the partial amplitudes of scalar QED (2) to the complete scalar propagator. Indeed, the congruence of these results mirrors the original outcome of Landau and Khalatnikov's analysis [32], which makes clear that the propagator's transformation is essentially independent of the particular field theory under study; here, we prove this to hold for arbitrary correlators.

This paper has the following structure: In Sec. II, we review the precise form of the LKF transformations and recent work on their application. In Sec. III, we use the recently developed worldline representation of the fermion propagator [86] to study its gauge transformation properties, followed by the generalization to the LKF transformations of the N-point functions in Sec. IV. We then illustrate the application of our results in perturbation theory in Sec. V before a conclusion and discussion of ongoing and future work.

II. GAUGE TRANSFORMATIONS OF GREEN FUNCTIONS

The LKF transformations show how field theory Green functions change between linear covariant gauges and contain information about their gauge-dependent part to all orders in the coupling to internal gauge bosons. We begin by reviewing the original construction of the coordinate space LKF transformations for the propagator (two-point functions) presented in [32,33,35].

The coordinate space photon propagator corresponding to the covariant gauge parameter ξ can be written as

$$\begin{aligned}
G_{\mu\nu}(x-x';\xi) &\coloneqq \langle \bar{A}_\mu(x)\bar{A}_\nu(x')\rangle_\xi \\
&= G_{\mu\nu}(x-x';\hat{\xi}) + \Delta\xi\partial_\mu\partial_\nu\Delta_D(x-x'), \qquad (8)
\end{aligned}$$

where $\hat{\xi}$ refers to a reference covariant gauge (chosen arbitrarily), and $\Delta\xi = \xi-\hat{\xi}$. Here, Δ_D is a function that fixes the gauge (see [32,103]), given by

$$\begin{aligned}
\Delta_D(y) &= -ie^2(\mu)\mu^{4-D}\int d^D\bar{k}\,\frac{\mathrm{e}^{-ik\cdot y}}{k^4} \\
&= -\frac{ie^2(\mu)}{16\pi^{\frac{D}{2}}}\Gamma\left[\frac{D}{2}-2\right](\mu y)^{4-D}, \qquad (9)
\end{aligned}$$

where we used $d^Dk = (2\pi)^D d^D\bar{k}$ for brevity and introduced the mass scale μ by identifying $e^2 \to \mu^{4-D}e^2(\mu)$ to maintain a dimensionless coupling constant $e(\mu)$ (on the far right-hand side, y indicates the magnitude of the vector). The relation (8) can be found by considering a gauge transformation $\bar{A}_\mu \to \bar{A}_\mu - \partial_\mu\phi$ and interpreting the function ϕ as a Stueckelberg-type scalar field. Quantization in momentum space in covariant gauge with parameter ξ yields [57,104] the correlation functions,

$$\langle \bar{A}_\mu(k)\phi(-k)\rangle_\xi = \frac{i\xi}{k^4}k_\mu \qquad (10)$$

$$\langle \phi(k)\phi(-k)\rangle_\xi = \frac{\xi}{k^4}, \qquad (11)$$

so that under $\bar{A}_\mu(k) \to \bar{A}_\mu(k) - ik_\mu\phi(k)$, the photon two-point function changes, according to

$$\langle \bar{A}_\mu(k)\bar{A}_\nu(-k)\rangle_\xi \longrightarrow \langle \bar{A}_\mu(k)\bar{A}_\nu(-k)\rangle_\xi - \xi\frac{k_\mu k_\nu}{k^4}. \qquad (12)$$

This reproduces the Fourier space representation of $\partial_\mu\partial_\nu\Delta_D$ with the familiar result that it is only the longitudinal part[1] of the photon propagator that varies with ξ.

[1]We refer to the usual momentum space decomposition of the photon propagator $G_{\mu\nu}(k) = \Delta(k^2)P_{\mu\nu} + \xi/k^2 L_{\mu\nu}$ into its transverse projector, $P_{\mu\nu} \coloneqq \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$, and longitudinal part, $L_{\mu\nu} \coloneqq \frac{k_\mu k_\nu}{k^2}$, where at tree level, of course, $\Delta(k^2) = \frac{1}{k^2}$.

It is the function Δ_D that appears in (1) that transforms the matter field's propagator between covariant gauges. For $D=3$, $\Delta_3(x) = -\frac{i}{2}\alpha x$ with the familiar fine structure constant, $4\pi\alpha = e^2$, so that (8) gives [105]

$$S(x;\xi) = S(x;\hat{\xi})\mathrm{e}^{-\frac{\Delta\xi}{2}\alpha x}. \tag{13}$$

In four dimensions, however, Δ_D requires regularization; expanding (9) about $D = 4-\epsilon$ leads to the result [38] $\Delta_4(x) = \frac{i\alpha}{4\pi}[\frac{2}{\epsilon} + \gamma_E + \ln(\pi) + 2\ln(\mu x) + \mathcal{O}(\epsilon)]$ so that, upon introducing a cut-off to regularize the x-dependent logarithm, we arrive firstly at

$$\Delta_4(x_0) - \Delta_4(x) = -i\ln\left[\frac{x^2}{x_0^2}\right]^{\frac{\alpha}{4\pi}}, \tag{14}$$

and subsequently, to the LKF transformation,

$$S(x;\xi) = S(x;\hat{\xi})\left[\frac{x^2}{x_0^2}\right]^{-\frac{\Delta\xi\alpha}{4\pi}}. \tag{15}$$

As is clear in the original derivations of the LKF transformations, this transformation is the same for scalar and spinor QED. We shall shortly rederive this result using worldline techniques.

A. Transformation of N-point functions

The LKF transformation of the N-point correlation functions has been studied to differing extents in scalar and spinor QED and QCD. Here, we shall review the calculation in scalar QED presented in [61,62], where a first quantized approach was used to derive the transformation for arbitrary correlators, but we refer to [57,60] and references therein for an examination of these transformations using BRST symmetry in the standard formulation.

The position space worldline representation of the partial $N = 2n$-point function in which the field $\phi(x_i)$ is connected to the conjugate field $\phi^\dagger(x'_{\pi(i)})$ for $i \in \{1,\ldots n\}$ and $\pi \in S_n$ is given in [61,62] as

$$\mathcal{A}(x_1,\ldots,x_n;x'_{\pi(1)},\ldots,x'_{\pi(n)}|\xi) = \prod_{i=1}^{n}\int_0^\infty dT_i \mathrm{e}^{-m^2T_i}\int_{x_i(0)=x'_{\pi(i)}}^{x_i(T_i)=x_i}\mathcal{D}x_i(\tau_i)\mathrm{e}^{-\sum_{l=1}^{n}(S_0^l+S_\gamma^l)-\sum_{k,l=1}^{n}S_{i\pi}^{(k,l)}(\xi)}, \tag{16}$$

where m is the mass of the field, and the path integral is over trajectories that travel between $x'_{\pi(i)}$ and x_i in (Schwinger) proper time T. The worldline action has been split up into the free particle actions,

$$S_0^l[x_l] = \int_0^{T_l} d\tau_l \frac{\dot{x}_l^2}{4}, \tag{17}$$

the interaction of these particles with external photons,

$$S_\gamma^l[x_l, A^\gamma] = ie\int_0^{T_l} d\tau_l \dot{x}_l\cdot A^\gamma(x_l(\tau_l)), \tag{18}$$

and

$$S_{i\pi}^{(k,l)}[x_k, x_l, \bar{A}|\xi] = \frac{e^2}{2}\int_0^{T_k} d\tau_k\int_0^{T_l} d\tau_l \dot{x}_k^\mu G_{\mu\nu}(x_k - x_l;\xi)\dot{x}_l^\nu, \tag{19}$$

which produces the electromagnetic interaction due to the exchange of virtual photons between particle worldlines k and l in the chosen covariant gauge (we shall derive the equivalent action for spinor QED below).

To study the LKF transformations, we recall the explicit form of the configuration space photon propagator in an arbitrary covariant gauge,

$$G_{\mu\nu}(y;\xi) = \frac{1}{(4\pi^{\frac{D}{2}})}\left\{\frac{1+\xi}{2}\Gamma\left[\frac{D}{2}-1\right]\frac{\delta_{\mu\nu}}{y^{2\frac{D}{2}-1}} + (1-\xi)\Gamma\left[\frac{D}{2}\right]\frac{y_\mu y_\nu}{y^{2\frac{D}{2}}}\right\}. \tag{20}$$

Under a change in the gauge parameter, $\xi \to \xi + \Delta\xi$, the integrand of the action S_i changes by a total derivative, $S_{i\pi}^{(k,l)}(\xi) \to S_{i\pi}^{(k,l)}(\xi) + \Delta_\xi S_{i\pi}^{(k,l)}$, where

$$\Delta_\xi S_{i\pi}^{(k,l)} = \frac{\Delta\xi e^2}{32\pi^{\frac{D}{2}}}\Gamma\left[\frac{D}{2}-2\right]\int_0^{T_k} d\tau_k\int_0^{T_l} d\tau_l \partial_{\tau_k}\partial_{\tau_l}[(x_k(\tau_k) - x_l(\tau_l))^2]^{2-\frac{D}{2}} \tag{21}$$

$$= \frac{\Delta\xi e^2}{32\pi^{\frac{D}{2}}}\Gamma\left[\frac{D}{2}-2\right]\{[(x_k - x_l)^2]^{2-\frac{D}{2}} - [(x_k - x'_{\pi(l)})^2]^{2-\frac{D}{2}} - [(x'_{\pi(k)} - x_l)^2]^{2-\frac{D}{2}} + [(x'_{\pi(k)} - x'_{\pi(l)})^2]^{2-\frac{D}{2}}\}. \tag{22}$$

As we shall discuss below for the fermionic amplitudes, the effect of changing the gauge of the external photons also introduces a total derivative term in the action [see Eq. (32)]. Consequently, the contributions from gauge transforming the internal and external photons vanish for photons with at least one leg on a closed scalar loop. This allows us to focus on the quenched amplitudes, which completely fix the form of the LKF transformation, as discussed in [61,62].

Even for quenched amplitudes, there are further simplifications, since the gauge transformation of external photons is well understood through the Ward identity. As shall be made clear below, although such a gauge transformation produces a nonvanishing boundary contribution when the photon is attached to an open line, it does not contribute to the Lehmann-Symanzik-Zimmermann (LSZ) formula for on-shell matrix elements. As such, to study the nontrivial gauge transformation of the propagator, we may restrict our attention to only those virtual photons that mediate interactions between open lines. Then (22), which does not depend upon the path integral variables nor the proper time, T, is the full contribution to the generalized LKF transformation,

$$\begin{aligned}&\mathcal{A}(x_1, \ldots, x_n; x'_{\pi(1)}, \ldots, x'_{\pi(n)} | \xi + \Delta\xi) \\ &\quad = \prod_{k,l=1}^{n} \mathrm{e}^{-\Delta_\xi S_{i\pi}^{(k,l)}} \mathcal{A}(x_1, \ldots, x_n; x'_{\pi(1)}, \ldots, x'_{\pi(n)} | \xi), \end{aligned} \tag{23}$$

as reported in (2); the full amplitude is constructed by simply summing the ordered amplitudes over the permutations $\pi \in S_n$. More information and a discussion about the pole structure in dimensional regularization are given in [61] and are elaborated for spinor QED below, where we will also compute the product over k and l explicitly. The application to perturbation theory in position space is also given in [61], which we shall repeat for the spinor case presently.

III. THE FERMION PROPAGATOR IN FIRST QUANTISATION

The fermion propagator has only recently been given a satisfactory worldline description that maintains the familiar advantages of the first quantized approach (see earlier attempts in [76,77,79]). Contrary to the one- or multiloop case, this involves a path integral over *open* lines joining the end points of the propagation and is also a function of the initial and final spin states. The path integral formulation of this construction is given in [86,87], and the reader is referred to [106] for an alternative approach.

The fermion propagator in a background electromagnetic field $\mathrm{A} = A_\mu dx^\mu$ is defined in position space by the matrix elements,

$$S^{x'x}_{\beta\alpha}[A] \coloneqq \langle x', \beta | [m - i\not{D}]^{-1} | x, \alpha \rangle, \tag{24}$$

where the covariant derivative is given by $D_\mu \coloneqq \partial_\mu + ieA_\mu$, and α and β indicate the spin at the points x and x', respectively. Applying the Gordon identity, we can rewrite this as

$$\begin{aligned}S^{x'x}_{\beta\alpha}[A] &= [m + i\not{D}']_{\beta\sigma} \langle x', \sigma | \left[-D^2 + m^2 + \frac{ie}{2}\gamma^\mu F_{\mu\nu}\gamma^\nu \right]^{-1} | x, \alpha \rangle \\ &\equiv [m + i\not{D}']_{\beta\sigma} K^{x'x}_{\sigma\alpha}[A], \end{aligned} \tag{25}$$

where the covariant derivative acts on x'. The matrix element, which we call the kernel $K^{x'x}_{\sigma\alpha}$, now takes the form of the propagator for a scalar particle in the presence of a matrix valued potential, and it is well known how to give a path integral representation for this object. As discussed in [86], this path integral can be written as

$$K^{x'x}[A] = 2^{-\frac{D}{2}} \mathrm{symb}^{-1} \int_0^\infty dT e^{-m^2 T} \int_{x(0)=x}^{x(T)=x'} \mathcal{D}x(\tau) \int_{\psi(0)+\psi(T)=0} \mathcal{D}\psi(\tau) \mathrm{e}^{-\int_0^T d\tau [\frac{\dot{x}^2}{4} + \frac{1}{2}\psi\cdot\dot{\psi} + ie\dot{x}\cdot A(x) - ie(\psi+\eta)\cdot F(x)\cdot(\psi+\eta)]}. \tag{26}$$

Here, the path integrals are over trajectories from x to x', on which lives a one-dimensional field theory described by the action that couples the bosonic embedding coordinates $x^\mu(\tau)$ and the antiperiodic Grassmann variables $\psi^\mu(\tau)$ to the background field; the former produce the orbital interaction, while the latter generate the spin coupling (the so-called Feynman "spin factor" [65]). The spin structure of the kernel arises from the "symbol map" acting on the constant Grassmann variables η^μ, according to

$$\mathrm{symb}\{\gamma^{[\mu_1} \cdots \gamma^{\mu_n]}\} \equiv (-i\sqrt{2})^n \eta^{\mu_1} \ldots \eta^{\mu_n}, \tag{27}$$

where the square brackets indicate antisymmetrization of the product with the appropriate combinatorial factor. As mentioned in [86], this representation has the advantage of expressing the propagator directly in the Dirac basis of the Clifford algebra.

In the following, we are interested in analyzing scattering amplitudes involving an arbitrary number of external photons attached to the particle worldline and any number of virtual photon exchanges along the line. To achieve this, we employ the background field method for the virtual photons, decomposing $\mathrm{A} = \mathrm{A}^\gamma + \bar{\mathrm{A}}$, and we shall quantize $\bar{\mathrm{A}}$ in the path integral formalism choosing a particular linear covariant gauge for the internal photons. The internal photons are thus produced by Wick contractions between distinct factors of $\bar{A}$, while external photons are represented by A^γ. Hence, we write the full propagator (we subsequently suppress the spinor indices for brevity),

$$\begin{aligned} S(x,x'|\xi) &\coloneqq \langle S^{x'x}[A^\gamma+\bar{A}]\rangle_{\bar{A},\xi} \\ &= \langle [m+i\not{D}^{\gamma\prime}-e\bar{\not{A}}]K^{x'x}[A^\gamma+\bar{A}]\rangle_{\bar{A},\xi}, \end{aligned} \tag{28}$$

where we have extracted the "backgroundless" part of the covariant derivative, $D^{\gamma\prime}_\mu=\partial'_\mu+ieA^\gamma_\mu$. We follow the notation used to define (8) and establish expectation values in the path integral approach in the Appendix.

The final identification that can be made to connect this propagator to photon amplitudes is the specification of the external photon source as a sum of plane waves of fixed momenta, $k_{i\mu}$, with polarizations $\varepsilon_{i\mu}$,

$$A^\gamma_\mu(x)=\sum_{i=1}^N \varepsilon_{i\mu}\mathrm{e}^{ik_i\cdot x}, \tag{29}$$

after which, the amplitude (28) is expanded to multilinear order in the polarizations. Substituted into $K[A^\gamma+\bar{A}]$, this leads to the insertion of photon vertex operators under the path integral (26),

$$\begin{aligned} V^{x'x}_\eta[k,\varepsilon] &= \int_0^T d\tau[\varepsilon\cdot\dot{x}(\tau)-i(\psi(\tau)+\eta)\cdot f\cdot(\psi(\tau)+\eta)]\mathrm{e}^{ik\cdot x(\tau)} \\ &= \int_0^T d\tau \mathrm{e}^{ik\cdot x(\tau)+\varepsilon\cdot\dot{x}(\tau)-i(\psi(\tau)+\eta)\cdot f\cdot(\psi(\tau)+\eta)}\big|_\varepsilon. \end{aligned} \tag{30}$$

Here, we introduced the photon field strength tensor $f_{\mu\nu}\coloneqq 2k_{[\mu}\varepsilon_{\nu]}$ and borrowed the trick often employed in string theory for such vertex operators by exponentiating the prefactor with the instruction that only the part linear in ε should be taken. In this way, (26) becomes

$$\begin{aligned} K^{x'x}[k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N|\bar{A}] &= (-ie)^N\int_0^\infty dT\mathrm{e}^{-m^2T}\int_{x(0)=x}^{x(T)=x'}\mathcal{D}x(\tau)\mathrm{e}^{-\int_0^T d\tau[\frac{\dot{x}^2}{4}+ie\dot{x}\cdot\bar{A}(x)]} \\ &\quad\times 2^{-\frac{D}{2}}\mathrm{symb}^{-1}\int_{\psi(0)+\psi(T)=0}\mathcal{D}\psi(\tau)\mathrm{e}^{-\int_0^T d\tau[\frac{1}{2}\psi\cdot\dot{\psi}-ie(\psi+\eta)\cdot\bar{F}(x)\cdot(\psi+\eta)]}\prod_{i=1}^N V^{x'x}_\eta[k_i,\varepsilon_i]|_{\varepsilon_1\ldots\varepsilon_N}. \end{aligned} \tag{31}$$

We have now separated the contributions to the kernel from external photons and the interaction with the background field, Ā, which will produce the virtual photons running along the line.

A. Gauge transformations

Here, we consider how changing the gauge of the photons attached to the particle line affects the propagator. To begin with, we consider the gauge transformation of the external photons represented by the vertex operators in (30). The (momentum-space) gauge transformation of photon i takes the form $\varepsilon_{i\mu}\to\varepsilon_{i\mu}+\lambda k_{i\mu}$ for an arbitrary constant λ. Under this, the vertex changes as

$$\begin{aligned} V^{x'x}_\eta[k_i,\varepsilon_i] &\to V^{x'x}_\eta[k_i,\varepsilon_i]+i\lambda\int_0^T d\tau_i\partial_{\tau_i}\mathrm{e}^{ik_i\cdot x(\tau_i)} \\ &= V^{x'x}_\eta[k_i,\varepsilon_i]+i\lambda(\mathrm{e}^{ik_i\cdot x'}-\mathrm{e}^{ik_i\cdot x}). \end{aligned} \tag{32}$$

Here, the last term-which depends only upon the end points of the trajectory—does not contribute to the on-shell matrix elements by the Ward identity (once Fourier transformed to momentum space, the exponential factors shift the location of the poles away from the mass shell, so they cannot contribute in the LSZ formula). Moreover, for an external photon attached to a closed fermion loop, the total derivative integrates to zero. This means that the nontrivial gauge transformation properties of the propagator come only from the transformation of the internal photons (of course, the effect of a gauge transformation of the internal photons must also drop out of observable quantities, but this transformation is not well understood as it is for external photons[2]), which we go on to determine in the following section.

B. Variation of the propagator

First, we rederive the original LKF transformation of the two-point function using the worldline techniques presented above. For this analysis, we split (28) into two parts:

$$\begin{aligned} S(x;x'|\xi) &= [m+i\not{D}^{\gamma\prime}]\langle K^{x'x}[A^\gamma+\bar{A}]\rangle_{\bar{A},\xi} \\ &\quad-\langle e\bar{\not{A}}(x')K^{x'x}[A^\gamma+\bar{A}]\rangle_{\bar{A},\xi}. \end{aligned} \tag{33}$$

[2]The plane wave decomposition of the external photons also enters in the prefactor of (25), yet it has been shown in [87] that these terms do not contribute on shell for the same reason that they lack the correct LSZ poles; as such, their gauge transformation need not be considered here.

After the gauge transformation, the first term will produce the multiplicative LKF law seen in the scalar case (23), plus an additional, unwanted derivative of this exponential factor. This extra derivative term will be canceled by the nonmultiplicative part of the gauge transformation of the second term involving $\bar{\not{A}}$ (this is reminiscent of how the covariant derivative of the Wilson line works out to be gauge covariant).

It will be convenient to define the more general ξ-dependent functional,

$$\mathcal{I}[J,M;\xi) \coloneqq \left\langle \mathrm{e}^{ie\int d^D x J[x]\cdot\bar{A}[x]} \prod_{j=1}^{M} K_j^{x'_{\pi(j)}x_j}[A^\gamma + \bar{A}] \right\rangle_{\bar{A},\xi}, \tag{34}$$

which can be used to generate insertions of $\bar{A}$ through functional differentiation. To evaluate this, we require the path integral representation of the kernel (26). It is clear that the functional integral over $\bar{A}$ will then be Gaussian. However, further simplifications can be engendered by taking advantage of a supersymmetry in the worldline action. The action is invariant under the transformations,

$$\delta x^\mu = -2\zeta\psi^\mu, \qquad \delta\psi^\mu = \zeta\dot{x}^\mu, \tag{35}$$

with ζ as a constant Grassmann number, which motivates us to formulate the worldline theory in superspace. So, we extend our parameter domain to 1-1 superspace, $\tau \to \tau|\theta$, by introducing the Grassmann parameter θ. We can then define the superfield and superderivative,

$$\mathbb{X}^\mu(\tau,\theta) = x^\mu(\tau) + \sqrt{2}\theta(\psi^\mu(\tau) + \eta^\mu) \tag{36}$$

$$\mathbb{D} = \partial_\theta - \theta\partial_\tau. \tag{37}$$

Integrals of superfields over the whole of superspace, such as $\int d\tau \int d\theta \mathbb{X}$, are invariant (up to boundary terms) under supersymmetric transformations. In particular, we can express (26) as

$$K^{x'x}[A] = 2^{-\frac{D}{2}}\mathrm{symb}^{-1}\int_0^\infty dT \mathrm{e}^{-m^2 T}\int \mathcal{D}\mathbb{X}\mathrm{e}^{-S_0[\mathbb{X}]-S_{\mathrm{A}}[\mathbb{X}]}, \tag{38}$$

in which appear the free particle action, $S_0[\mathbb{X}]$, and the interaction with the gauge field, $S_{\mathrm{A}}[\mathbb{X}]$, which (up to total derivatives) can be written in superspace as[3]

$$S_0[\mathbb{X}] = \int_0^T d\tau \int d\theta \left[-\frac{1}{4}\mathbb{X}\cdot\mathbb{D}^3\mathbb{X}\right] \tag{39}$$

$$S_{\mathrm{A}}[\mathbb{X}] = \int_0^T d\tau \int d\theta[-ieA[\mathbb{X}]\cdot\mathbb{D}\mathbb{X}]. \tag{40}$$

The boundary conditions on $\mathbb{X}$ are inherited from x and ψ.

To determine $\mathcal{I}$, we decompose the gauge field into external and internal photons. The path integral over the potential of the internal photons, $\bar{A}$, is then determined by using the superspace representation of the kernels in $\mathcal{I}$ and completing the square to arrive at

$$\mathcal{I}[J,M;\xi) = \prod_{j=1}^{M} 2^{-\frac{D}{2}}\mathrm{symb}^{-1}\int_0^\infty dT_j \mathrm{e}^{-m^2 T_j} \times \int \mathcal{D}\mathbb{X}_j \mathrm{e}^{-\sum_{l=1}^{M} S_{0\gamma}^{(l)}[\mathbb{X}_l] - S_i[\mathbb{X},J]}, \tag{41}$$

where $S_{0,\gamma}^{(l)}[\mathbb{X}]$ consists of the free action for trajectory l along with its coupling to the external photons, and we have defined the generalized interaction term,

$$S_i[\mathbb{X},J] = \frac{e^2}{2}\iint d^D y d^D y' \mathcal{J}(y)\cdot G(y-y';\xi)\cdot\mathcal{J}(y'), \tag{42}$$

$$\mathcal{J}^\mu(y) = J^\mu(y) + \sum_{l=1}^{M}\int_0^{T_l} d\tau_l \int d\theta_l \delta^D(y-\mathbb{X}_l)\mathbb{D}_l\mathbb{X}_l^\mu. \tag{43}$$

Note that in the current case, the inverse symbol map must first order the variables η_l in ascending order to reproduce the numeration of variables in the product in (34) *before* converting them into products of γ matrices.

The crucial observation is that, using (20), a change in the gauge parameter $\xi \to \xi + \Delta\xi$ causes a variation in S_i that is again a total derivative:

$$\mathcal{J}(y)\cdot\Delta_\xi G\cdot\mathcal{J}(y') = \frac{\Delta\xi}{16\pi^{\frac{D}{2}}}\Gamma\left[\frac{D}{2}-2\right]\mathcal{J}(y)\cdot\partial_y\mathcal{J}(y')\cdot\partial_{y'}[(y-y')^2]^{2-\frac{D}{2}}. \tag{44}$$

Using this in the variation of (41), we get the gauge variation of the exponent divided into three term. The contribution independent of the external source, J, coincides with (22) from the scalar case, according to

[3]We define Grassmann integration by $\int d\theta = 0$ and $\int d\theta\theta = 1$. Then, $\int d\theta S_0[\mathbb{X}]$ provides a τ integrand that gives the kinetic terms of (26), plus $-\frac{1}{4}\partial_\tau(x\cdot\dot{x} + 2\psi\cdot\eta)$ that is independent of the gauge field A, and $\int d\theta S_{\mathrm{A}}[\mathbb{X}]$ gives exactly the interaction terms of (26).

$$\begin{aligned}\Delta_\xi S_{i\pi}^{(k,l)} &= -\Delta\xi \frac{e^2}{32\pi^{\frac{D}{2}}}\Gamma\left(\frac{D}{2}-2\right)\int_0^{T_k} d\tau_k \int_0^{T_l} d\tau_l \int d\theta_k \int d\theta_l \mathbb{D}_k \mathbb{D}_l [(\mathbb{X}_k - \mathbb{X}_l)^2]^{2-\frac{D}{2}} \\ &= \Delta\xi \frac{e^2}{32\pi^{\frac{D}{2}}}\Gamma\left(\frac{D}{2}-2\right)\int_0^{T_k} d\tau_k \int_0^{T_l} d\tau_l \partial_{\tau_k}\partial_{\tau_l}[(x_k - x_l)^2]^{2-\frac{D}{2}}. \end{aligned} \tag{45}$$

This corresponds to the transformation of the propagator caused by a change of gauge in the internal photon propagators that couple to the worldline trajectories. Aside from this, there are two terms in (44) involving the source that provide the term denoted by $\Delta_\xi I_M$ in [63]. It can be split up into the sum of

$$\begin{aligned}\Delta_\xi I_M^{(1)} &= -e^2 \sum_{i=1}^{M} \int d^D y \int_0^{T_i} d\tau_i \int d\theta_i J[y] \cdot G(y - \mathbb{X}_i; \xi) \cdot \mathbb{D}_i \mathbb{X}_i \\ &= \frac{\Delta\xi e^2}{16\pi^{\frac{D}{2}}}\Gamma\left[\frac{D}{2}-2\right]\sum_{i=1}^{M}\int_0^{T_i} d\tau_i \int d^D x J(x) \cdot \partial_x \partial_{\tau_i}[(x - x_i)^2]^{2-\frac{D}{2}}, \end{aligned} \tag{46}$$

and

$$\begin{aligned}\Delta_\xi I_M^{(2)} &= -\frac{e^2}{2}\int d^D y \int d^D z J[y] \cdot G(y - z; \xi) \cdot J[z] \\ &= -\frac{\Delta\xi e^2}{32\pi^{\frac{D}{2}}}\Gamma\left[\frac{D}{2}-2\right]\iint d^D x d^D x' J(x) \cdot \partial_x J(x') \cdot \partial_{x'}[(x - x')^2]^{2-\frac{D}{2}}. \end{aligned} \tag{47}$$

Put together, these imply that $\mathcal{I}$ transforms with the simple multiplicative law,

$$\mathcal{I}[J, M; \xi + \Delta\xi) = \mathcal{I}[J, M; \xi) \mathrm{e}^{-\sum_{k,l=1}^{M}\Delta_\xi S_{i\pi}^{(k,l)} + \Delta_\xi I_M}, \tag{48}$$

which generalizes slightly the original LFK transformation.

With this, we can analyze how the propagator (33) transforms. We can express it in terms of $\mathcal{I}$ as

$$S(x; x'|\xi) = [m + i\not{D}'^{\gamma}]\mathcal{I}[0, 1; \xi) + i\frac{\delta}{\delta \not{J}'}\mathcal{I}[J, 1; \xi)|_{J=0}, \tag{49}$$

where the functional derivative is taken at the point x' to produce the insertion of $\bar{\not{A}}(x')$. Making a transformation of the gauge parameter, we have

$$\begin{aligned} S(x; x'|\xi + \Delta\xi) &= [m + i\not{D}'^{\gamma}](\mathcal{I}[0, 1; \xi)\mathrm{e}^{-\Delta_\xi S_i}) \\ &\quad + i\frac{\delta}{\delta \not{J}'}(\mathcal{I}[J, 1; \xi)\mathrm{e}^{-\Delta_\xi S_i + \Delta_\xi I_1}])\Big|_{J=0}, \end{aligned} \tag{50}$$

where both partial and functional derivatives act through onto the exponential factors. However, the two terms that arise from applying the derivatives to the exponents cancel due to the general relation,

$$\frac{\delta}{\delta \not{J}(x_i')}\Delta_\xi I_M^{(1)} = \not{\partial}_i' \sum_{k,l=1}^{M} \Delta_\xi S_{i\pi}^{(k,l)}, \tag{51}$$

and the fact that $\frac{\delta}{\delta \not{J}(x_i')}\Delta_\xi I_M^{(2)} = 0$ when $J = 0$. So, we see that the extra factor of $\bar{\not{A}}$ in the second term of (33) produces a nonmultiplicative contribution to the gauge transformed propagator [from the functional derivative of the exponent in (50)] that is precisely what is needed to cancel the unwanted partial derivative of $\Delta_\xi S$ arising from the first term of S. This allows for the exponential factors in both terms of (50) to be commuted to the left, resulting in the following transformation for the propagator:

$$S(x; x'|\xi + \Delta\xi) = \mathrm{e}^{-\Delta_\xi S_i} S(x; x'|\xi), \tag{52}$$

which corresponds to the original LKF transformation. It takes the same form as in the scalar case, as obtained in the derivation of Landau and Khalatnikov that is independent of the details of the matter field.

IV. *N*-POINT FUNCTIONS

The main contribution of this paper is to provide additional details that prove the generalization of the LKF transformation of the propagator to arbitrary correlators, expanding upon the results reported in [63]. To this end, we generalize the propagator, which corresponds to the field theory correlator $\langle \bar{\Psi}(x)\Psi(x')\rangle$, to the correlator of an arbitrary even number, $N = 2n$, of fields, $\langle \bar{\Psi}(x_1)\cdots\bar{\Psi}(x_n)\Psi(x_1')\cdots\Psi(x_n')\rangle$. This N-point function can be decomposed into partial amplitudes,

$$S(x_1\ldots x_n;x'_1\ldots x'_n|\xi) = \sum_{\pi\in S_n}\mathcal{S}_\pi(x_1\ldots x_n;x'_{\pi(1)}\ldots x'_{\pi(n)}|\xi), \tag{53}$$

where the partial amplitude $\mathcal{S}_\pi$ represents the contribution in which the field $\Psi(x'_{\pi(i)})$ is contracted with the conjugate field $\bar\Psi(x_i)$, defined as

$$\mathcal{S}_\pi(x_1\ldots x_n;x'_{\pi(1)}\ldots x'_{\pi(n)}|\xi) = \langle[m+i\not{D}'_1]K_1^{x'_{\pi(1)}x_1}\cdots[m+i\not{D}'_n]K_n^{x'_{\pi(n)}x_n}\rangle_{\bar A,\xi}. \tag{54}$$

The generalized LKF transformation will be determined in terms of the transformation of these partial amplitudes. We shall express the kernels in terms of path integrals over particle trajectories; as discussed above, the gauge transformation due to the external photons is fixed by the Ward identity, so we are again free to focus on the variation induced by changing the gauge parameter of the internal, virtual photons.

Before continuing with the complete derivation of the gauge transformation, it is worth stressing how the final result comes about. For the propagator, the previous section showed that there is a cancellation between undesirable derivatives of the LKF factor $\Delta_\xi S$: in (50), the insertion of $\not{A}$ gave a term that canceled against the partial derivative of this factor. Now, transforming (54), there will be multiple partial derivatives of an LKF factor produced by the $\not{D}'_i$—these will be seen to be in exact correspondence with the nonmultiplicative transformation of the additional insertions of $\not{A}$ from the covariant derivatives and will cancel against with them.

To illustrate this cancellation mechanism, we consider the four-point case ($N=4$) explicitly for the identity permutation, I. We organize the calculation of the gauge transformed function with respect to the $i\not{\partial}_i{}' - \bar{\not{A}}_i$ (for notational brevity, we use $e=1$ throughout this example):

$$\begin{aligned}\mathcal{S}_I(x_1,x_2;x'_1x'_2|\xi+\Delta\xi) = {}&\langle[m-\not{A}_1^\gamma]K_1^{x'_1x_1}[m-\not{A}_2^\gamma]K_2^{x'_2x_2}\rangle_{\bar A,\xi+\Delta\xi} + \langle[i\not{\partial}_1{}'-\bar{\not{A}}_1]K_1^{x'_1x_1}[m-\not{A}_2^\gamma]K_2^{x'_2x_2}\rangle_{\bar A,\xi+\Delta\xi}\\ &+\langle[m-\not{A}_1^\gamma]K_1^{x'_1x_1}[i\not{\partial}_2{}'-\bar{\not{A}}_2]K_2^{x'_2x_2}\rangle_{\bar A,\xi+\Delta\xi} + \langle[i\not{\partial}_1{}'-\bar{\not{A}}_1]K_1^{x'_1x_1}[i\not{\partial}_2{}'-\bar{\not{A}}_2]K_2^{x'_2x_2}\rangle_{\bar A,\xi+\Delta\xi}.\end{aligned} \tag{55}$$

The first three terms of the right-hand side are already known to transform in the desired way; the first (no partial derivatives) is trivial, while the transformation of second and third terms (one partial derivative) can be obtained by the same method applied in the previous section for the two-point function case. It is in the last term where new undesired terms that don't appear in the two-point function case have arisen, as can be seen if we organize the various terms as follows:

$$\begin{aligned}\langle[i\not{\partial}'_1-\bar{\not{A}}_1]K_1^{x'_1x_1}[i\not{\partial}'_2-\bar{\not{A}}_2]K_2^{x'_2x_2}\rangle_{\bar A,\xi+\Delta\xi} = {}& i\partial_1'^\mu i\partial_2'^\nu\langle\gamma_\mu K_1^{x'_1x_1}\gamma_\nu K_2^{x'_2x_2}\rangle_{\bar A,\xi+\Delta\xi} - i\partial_1'^\mu\langle\gamma_\mu K_1^{x'_1x_1}\bar{\not{A}}_2K_2^{x'_2x_2}\rangle_{\bar A,\xi+\Delta\xi}\\ &- i\partial_2'^\nu\langle\bar{\not{A}}_1K_1^{x'_1x_1}\gamma_\nu K_2^{x'_2x_2}\rangle_{\bar A,\xi+\Delta\xi} + \langle\bar{\not{A}}_1\bar{\not{A}}_2K_1^{x'_1x_1}\bar{\not{A}}_2K_2^{x'_2x_2}\rangle_{\bar A,\xi+\Delta\xi}.\end{aligned} \tag{56}$$

We can again apply our knowledge of how the two-point function transforms to obtain the following transformations for the first three contributions (derivatives inside expectation values do not act through while those outside do):

$$\begin{aligned}i\partial_1'^\mu i\partial_2'^\nu\langle\gamma_\mu K_1^{x'_1x_1}\gamma_\nu K_2^{x'_2x_2}\rangle_{\bar A,\xi+\Delta\xi} = {}&\{\langle i\not{\partial}'_1K_1^{x'_1x_1}i\not{\partial}'_2K_2^{x'_2x_2}\rangle_{\bar A,\xi} + \langle\gamma_\mu K_1^{x'_1x_1}i\not{\partial}'_2K_2^{x'_2x_2}\rangle_{\bar A,\xi}i\partial_1'^\mu\\ &+\langle i\not{\partial}'_1K_1^{x'_1x_1}\gamma_\nu K_2^{x'_2x_2}\rangle_{\bar A,\xi}i\partial_2'^\nu + \langle\gamma_\mu K_1^{x'_1x_1}\gamma_\nu K_2^{x'_2x_2}\rangle_{\bar A,\xi}i\partial_1'^\mu i\partial_2'^\nu\}\mathrm{e}^{-\sum_{k,l=1}^2\Delta_\xi S_i^{(k,l)}},\end{aligned} \tag{57}$$

$$\begin{aligned}-i\partial_1'^\mu\langle\gamma_\mu K_1^{x'_1x_1}\bar{\not{A}}_2K_2^{x'_2x_2}\rangle_{\bar A,\xi+\Delta\xi} = {}&\{-\langle i\not{\partial}'_1K_1^{x'_1x_1}\bar{\not{A}}_2K_2^{x'_2x_2}\rangle_{\bar A,\xi} - \langle\gamma_\mu K_1^{x'_1x_1}\not{A}_2K_2^{x'_2x_2}\rangle_{\bar A,\xi}i\partial_1'^\mu\\ &-\langle i\not{\partial}_1{}'K_1^{x'_1x_1}\gamma_\nu K_2^{x'_2x_2}\rangle_{\bar A,\xi}i\partial_2'^\nu - \langle\gamma_\mu K_1^{x'_1x_1}\gamma_\nu K_2^{x'_2x_2}\rangle_{\bar A,\xi}i\partial_1'^\mu i\partial_2'^\nu\}\mathrm{e}^{-\sum_{k,l=1}^2\Delta_\xi S_i^{(k,l)}},\end{aligned} \tag{58}$$

$$\begin{aligned}-i\partial_2'^\nu\langle\bar{\not{A}}_1K_1^{x'_1x_1}\gamma_\nu K_2^{x'_2x_2}\rangle_{\bar A,\xi+\Delta\xi} = {}&\{-\langle\bar{\not{A}}_1K_1^{x'_1x_1}i\not{\partial}_2{}'K_2^{x'_2x_2}\rangle_{\bar A,\xi} - \langle\gamma_\mu K_1^{x'_1x_1}i\not{\partial}_2{}'K_2^{x'_2x_2}\rangle_{\bar A,\xi}i\partial_1'^\mu\\ &-\langle\not{A}_1K_1^{x'_1x_1}\gamma_\nu K_2^{x'_2x_2}\rangle_{\bar A,\xi}i\partial_2'^\nu - \langle\gamma_\mu K_1^{x'_1x_1}\gamma_\nu K_2^{x'_2x_2}\rangle_{\bar A,\xi}i\partial_1'^\mu i\partial_2'^\nu\}\mathrm{e}^{-\sum_{k,l=1}^2\Delta_\xi S_i^{(k,l)}}.\end{aligned} \tag{59}$$

The fourth term, involving $\bar{\not{A}}_1\bar{\not{A}}_2$ in the expectation value, must be calculated explicitly to show that it transforms in such a way as to cancel the partial derivatives of the LKF exponent in the previous expressions. To do this, we start in the gauge ξ and generate the insertions of $A's$ via two functional derivatives of $\mathcal{I}[J,2;\xi]$. Then, applying the transformation $\xi\to\xi+\Delta\xi$, we get

$$\langle \bar{A}_1 K_1^{x_1'x_1} \bar{A}_2 K_2^{x_2'x_2} \rangle_{\bar{A},\xi+\Delta\xi} = \Big\{ \langle \bar{A}_1 K_1^{x_1'x_1} \bar{A}_2 K_2^{x_2'x_2} \rangle_{\bar{A},\xi} - \langle \gamma^\mu K_1^{x_1'x_1} \bar{A}_2 K_2^{x_2'x_2} \rangle_{\bar{A},\xi} i \frac{\delta}{\delta J_1'^\mu} - \langle \bar{A}_1 K_1^{x_1'x_1} \gamma^\nu K_2^{x_2'x_2} \rangle_{\bar{A},\xi} i \frac{\delta}{\delta J_2'^\nu} + \langle \gamma^\mu K_1^{x_1'x_1} \gamma^\nu K_2^{x_2'x_2} \rangle_{\bar{A},\xi} i \frac{\delta}{\delta J_1'^\mu} i \frac{\delta}{\delta J_2'^\nu} \Big\} e^{-\sum_{k,l=1}^2 \Delta_\xi S_i^{(k,l)} + \Delta_\xi I_2} \big|_{J=0}. \tag{60}$$

Using relation (51) along with the simple extension,

$$\frac{\delta}{\delta J_i'^\mu} \frac{\delta}{\delta J_j'^\nu} \Delta_\xi I^{(2)} = -\partial_i'^\mu \partial_j'^\nu \sum_{k,l=1}^n \Delta_\xi S_i^{(k,l)} = -2\partial_i'^\mu \partial_j'^\nu \Delta_\xi S_i^{(i,j)}, \tag{61}$$

we can rewrite (60) as

$$\langle \bar{A}_1 K_1^{x_1'x_1} \bar{A}_2 K_2^{x_2'x_2} \rangle_{\bar{A},\xi+\Delta\xi} = \{ \langle \bar{A}_1 K_1^{x_1'x_1} \bar{A}_2 K_2^{x_2'x_2} \rangle_{\bar{A},\xi} + \langle \gamma^\mu K_1^{x_1'x_1} \bar{A}_2 K_2^{x_2'x_2} \rangle_{\bar{A},\xi} i\partial_1'^\mu + \langle \bar{A}_1 K_1^{x_1'x_1} \gamma^\nu K_2^{x_2'x_2} \rangle_{\bar{A},\xi} i\partial_2'^\nu + \langle \gamma^\mu K_1^{x_1'x_1} \gamma^\nu K_2^{x_2'x_2} \rangle_{\bar{A},\xi} i\partial_1'^\mu i\partial_2'^\nu \} e^{-\sum_{k,l=1}^2 \Delta_\xi S_i^{(k,l)}}. \tag{62}$$

Now, we may compare the above expressions and see that only the first term of each remains uncanceled. The fourth terms in (57) and (62) are canceled by the fourth terms of (58) and (59). The second term of (57) is canceled by the second term of (59), and the second term of (58) is canceled by the second term of (62) etc. Finally, putting everything together, we find that indeed the four-point correlation function as

$$\mathcal{S}_I(x_1, x_2; x_1'x_2' | \xi + \Delta\xi) = e^{-\sum_{k,l=1}^2 \Delta_\xi S_i^{(k,l)}} \mathcal{S}_I(x_1, x_2; x_1'x_2' | \xi + \Delta\xi), \tag{63}$$

under the gauge parameter transformation, as desired.

A. Arbitrary N

In order to lift this result to the general case, in this section, we continue to apply functional methods to evaluate correlators expressed in path integral form; a complementary approach that focuses more on the combinatorial aspects is developed in the Appendix. We begin with an intermediate result that extends $\mathcal{I}$ of (34). We consider the gauge transformation of the following function (for $M \geq 1$):

$$\left\langle [m + i\not{D}_1'] e^{ie\int d^D x J[x]\cdot\bar{A}[x]} \prod_{i=1}^M K^{x_i'x_i} \right\rangle_{\bar{A},\xi}. \tag{64}$$

Applying the change $\xi \to \xi + \Delta\xi$, our analysis is a straightforward generalization of the steps that led to (50), except that we no longer set $J = 0$ at the end of the calculation. Explicitly, we repeat the trick of generating $\bar{A}$ with a functional derivative with respect to $J(x')$ to find

$$\left\langle [m + i\not{D}_1'] e^{ie\int d^D x J[x]\cdot\bar{A}[x]} \prod_{i=1}^M K^{x_i'x_i} \right\rangle_{\bar{A},\xi+\Delta\xi} = \left[m + i\not{D}_1^{\gamma\prime} + i\frac{\delta}{\delta \not{J}_1'} \right] \left(\mathcal{I}[J, M; \xi] e^{-\sum_{k,l=1}^M \Delta_\xi S_{i\pi}^{(k,l)} + \Delta_\xi I_M} \right). \tag{65}$$

The derivatives act through and produce, aside from the original function, various nonmultiplicative terms. However, (51) and the similar relation,

$$\frac{\delta}{\delta \not{J}(x_i')} \Delta_\xi I_M^{(2)} = -\not{\partial}_i' \Delta_\xi I_M^{(1)}, \tag{66}$$

imply that most of these additional contributions cancel. One may then verify that there is really only one new term produced when $J \neq 0$,

$$\left\langle [m + i\not{D}_1'] e^{ie\int d^D x J[x]\cdot\bar{A}[x]} \prod_{i=1}^M K^{x_i'x_i} \right\rangle_{\bar{A},\xi+\Delta\xi} = e^{-\sum_{k,l}^M \Delta_\xi S_i^{(k,l)} + \Delta_\xi I_M} \left\langle [m + i\not{D}_1' + i\not{\partial}_1{}'(\Delta_\xi I_M^{(2)})] e^{ie\int d^D x J[x]\cdot\bar{A}[x]} \prod_{i=1}^M K^{x_i'x_i} \right\rangle_{\bar{A},\xi}. \tag{67}$$

This warm-up calculation illustrates the generalization that will be used below to prove the transformation rule of the N-point correlator: in the calculations that follow, we will work with nonzero $J(x)$, which will eventually be put to zero when we apply the results to the N-point function.

To this end, it is useful to consider a slightly more general functional, defined as (again $\pi \in S_M$)

$$\begin{aligned} &\mathcal{J}[J,K,M;\xi) \\ &:= \left\langle \mathrm{e}^{ie\int d^D x J[x]\cdot\bar{A}[x]} \prod_{i=1}^{K}[m+i\not{D}'_i]\prod_{j=1}^{M}K_j^{x'_{\pi(j)}x_j}[A^\gamma+\bar{A}]\right\rangle_{\bar{A},\xi}, \\ &\quad K\leq M. \end{aligned} \tag{68}$$

When we use the path integral representation of the various kernels, the symbol map continues to order the Grassmann variables according to the two products in $\mathcal{J}$, before converting them back to γ matrices. In fact, we should stress that the ordering of the γ matrices in $\mathcal{J}[J,K,M;\xi)$ does not yet correspond to that in the partial N-point amplitude, but we shall see that this is easily remedied under the symbol map.

The additional insertions of the Dirac operators (product over i) suggest that this functional will transform in a way that generalizes (67), introducing additional derivatives of $\Delta_\xi I_M^{(2)}$. Indeed, we claim that $\mathcal{J}$ transforms in the following way:

$$\mathcal{J}[J,K,M;\xi+\Delta\xi) = \left[\mathcal{J}[J,K,M;\xi) + \sum_{k=0}^{K-1}\prod_{l=1}^{k}[m+i\hat{\not{D}}'_l](i\not{\partial}'_{k+1}\Delta_\xi I_M^{(2)})\mathcal{J}^{(k+2)}[J,K,M;\xi)\right]\mathrm{e}^{-\sum_{k,l=1}^{M}\Delta_\xi S_{i\pi}^{(k,l)}+\Delta_\xi I_M}, \tag{69}$$

with $\hat{D}'_\mu = \partial'_\mu + \frac{\delta}{\delta J^\mu(x')} + ieA^\gamma_\mu(x')$ a generalized differential operator, whose derivatives act through onto everything to their right. The superscript in $\mathcal{J}^{(k+2)}[J,K,M;\xi)$ indicates that the variable i in (68) runs from $k+2$ to K. The proof is most easily done by induction on K with the transformations for (34) and (64) derived above, validating the base cases corresponding to $K=0$ and $K=1$, respectively.

Supposing this transformation, we examine the case $\mathcal{J}[J,K+1,M;\xi+\Delta\xi)$ (maintaining $K+1\leq M$). This introduces an additional factor of the Dirac operator, $[m+i\not{D}']$ in the product over i, to whose argument we assign the subscript 1, and we consequently shift the labels of the other insertions by one. We generate the additional insertion of $\bar{A}$ that appears in this operator by functional differentiation with respect to J so that

$$\begin{aligned} &\mathcal{J}[J,K+1,M;\xi+\Delta\xi) \\ &\quad = [m+i\hat{\not{D}}'_1]\mathcal{J}^{(2)}[J,K+1,M;\xi+\Delta\xi). \end{aligned} \tag{70}$$

We now vary the gauge parameter. The inductive hypothesis leads immediately to

$$\begin{aligned} \mathcal{J}[J,K+1,M;\xi+\Delta\xi) &= [m+i\hat{\not{D}}'_1][\mathcal{J}^{(2)}[J,K+1,M;\xi) \\ &\quad + \sum_{k=1}^{K}\prod_{l=2}^{k}[m+i\hat{\not{D}}'_l](i\not{\partial}'_{k+1}\Delta_\xi I_M^{(2)})\mathcal{J}^{(k+2)}[J,K+1,M;\xi)]\mathrm{e}^{-\sum_{k,l=1}^{M}\Delta_\xi S_{i\pi}^{(k,l)}+\Delta_\xi I_M}. \end{aligned} \tag{71}$$

Distributing the various derivatives in $\hat{\not{D}}'_1$, the first term on the right-hand side gives

$$\begin{aligned} [m+i\hat{\not{D}}'_1]\mathcal{J}^{(2)}[J,K+1,M;\xi)\mathrm{e}^{-\sum_{k,l=1}^{M}\Delta_\xi S_{i\pi}^{(k,l)}+\Delta_\xi I_M} &= \mathcal{J}[J,K+1,M;\xi)\mathrm{e}^{-\sum_{k,l=1}^{M}\Delta_\xi S_{i\pi}^{(k,l)}+\Delta_\xi I_M} \\ &\quad + \mathcal{J}^{(2)}[J,K+1,M;\xi)\left(i\not{\partial}'_1+\frac{\delta}{\delta\not{J}'_1}\right)\mathrm{e}^{-\sum_{k,l=1}^{M}\Delta_\xi S_{i\pi}^{(k,l)}+\Delta_\xi I_M}. \end{aligned} \tag{72}$$

In the second line, we find the same cancellations between the derivatives as at the start of this section; together, (51) with (66) imply

$$\left(i\not{\partial}'_1+\frac{\delta}{\delta\not{J}'_1}\right)\left(-\sum_{k,l=1}^{M}\Delta_\xi S_{i\pi}^{(k,l)}+\Delta_\xi I_M\right) = i\not{\partial}'_1\Delta_\xi I_M^{(2)}, \tag{73}$$

so that only one contribution from the second line survives. Thus, we recover a result that mirrors (67):

$$[m + i\hat{\mathcal{D}}_1']\mathcal{J}^{(2)}[J, K+1, M; \xi) = [\mathcal{J}[J, K+1, M; \xi) + (i\partial_1' \Delta_\xi I_M^{(2)})\mathcal{J}^{(2)}[J, K+1, M; \xi)]\mathrm{e}^{-\sum_{k,l=1}^{M} \Delta_\xi S_{i\pi}^{(k,l)} + \Delta_\xi I_M}. \quad (74)$$

The nonmultiplicative derivative term combines nicely with the variation produced by the second line of (71) and allows us to write

$$\begin{aligned}
\mathcal{J}[J, K+1, M; \xi + \Delta\xi) &= \mathcal{J}[J, K+1, M; \xi)\mathrm{e}^{-\sum_{k,l=1}^{M} \Delta_\xi S_{i\pi}^{(k,l)} + \Delta_\xi I_M} \\
&\quad + (i\partial_1' \Delta_\xi I_M^{(2)})\mathcal{J}^{(2)}[J, K+1, M; \xi)\mathrm{e}^{-\sum_{k,l=1}^{M} \Delta_\xi S_{i\pi}^{(k,l)} + \Delta_\xi I_M} \\
&\quad + [m + i\hat{\mathcal{D}}_1']\left[\sum_{k=1}^{K}\prod_{l=2}^{k}[m + i\hat{\mathcal{D}}_l'](i\partial_{k+1}' \Delta_\xi I_M^{(2)})\mathcal{J}^{(k+2)}[J, K+1, M; \xi)\right]\mathrm{e}^{-\sum_{k,l=1}^{M} \Delta_\xi S_{i\pi}^{(k,l)} + \Delta_\xi I_M} \\
&= \mathrm{e}^{-\sum_{k,l=1}^{M} \Delta_\xi S_{i\pi}^{(k,l)} + \Delta_\xi I_M}\mathcal{J}[J, K+1, M; \xi) \\
&\quad + \left[\sum_{k=0}^{K}\prod_{l=1}^{k}[m + i\hat{\mathcal{D}}_l'](i\partial_{k+1}' \Delta_\xi I_M^{(2)})\mathcal{J}^{(k+2)}[J, K+1, M; \xi)\right]\mathrm{e}^{-\sum_{k,l=1}^{M} \Delta_\xi S_{i\pi}^{(k,l)} + \Delta_\xi I_M}, \quad (75)
\end{aligned}$$

where the $k = 0$ contribution to the sum comes from the second line of the first expression, and we have made a relabeling to begin the product at $l = 1$ (we also recall that the derivatives $\hat{\mathcal{D}}_l$ act on everything to their right). This shows that $\mathcal{J}[J, K, M; \xi)$ transforms as claimed in (69) for all K. This result can be summarized as showing that the cancellation of derivatives of the $\Delta_\xi S$ between the partial derivative and the $\bar{A\!\!\!/}$ continues to hold for arbitrary N but that there are additional, nonmultiplicative terms in this transformation produced by the presence of the source. We will explain why these do not matter for the LKF transformation of the N-point functions below.

B. Generalized LKF transformation

We can use this immediately to derive the generalized LKF transformation rule for the $(N = 2n)$-point partial amplitude. To do this, we first observe that the γ matrices play no role in determining the functional form of the transformation; we can either factorize them outside of the expectation value or ask that the symbol map reorders the Grassmann variables, η_i, under the path integral accordingly—we only have to ensure that we return to the initial ordering of these matrices by the end of the calculation. Choosing the ordering that corresponds to the propagator, (54), we fix $K = M = n$ and evaluate the transformation just derived on $J = 0$, which gives

$$\begin{aligned}
&\mathcal{S}_\pi(x_1 \ldots x_n; x'_{\pi(1)} \ldots x'_{\pi(n)} | \xi + \Delta\xi) \\
&\quad = \mathrm{e}^{-\sum_{k,l=1}^{n} \Delta_\xi S_{i\pi}^{(k,l)}}\mathcal{S}_\pi(x_1 \ldots x_n; x'_{\pi(1)} \ldots x'_{\pi(n)} | \xi) \\
&\qquad + \left[\sum_{k=0}^{n-1}\prod_{l=1}^{k}[m + i\hat{\mathcal{D}}_l'](i\partial_{k+1}' \Delta_\xi I_n^{(2)})\mathcal{J}^{(k+2)}[J, n, n; \xi)\right] \\
&\qquad \times \mathrm{e}^{-\sum_{k,l=1}^{n} \Delta_\xi S_{i\pi}^{(k,l)} + \Delta_\xi I_n}\big|_{J=0}. \quad (76)
\end{aligned}$$

Now, we assert that $J = 0$ actually kills the second term on the right-hand side of this result. This can be seen by noting that $\Delta_\xi I_n^{(2)}$ is quadratic in J so that we need to apply two functional derivatives to it to obtain something that survives this limit. The resulting expression will depend only on the two variables of these derivatives, neither of which will be the same as the partial derivative, which also acts on $\Delta_\xi I_n^{(2)}$. Thus, the vanishing of this second term when we take $J = 0$ results in Eq. (6),

$$\begin{aligned}
&\mathcal{S}_\pi(x_1 \ldots x_n; x'_{\pi(1)} \ldots x'_{\pi(n)} | \xi + \Delta\xi) \\
&\quad = T_n \mathcal{S}_\pi(x_1 \ldots x_n; x'_{\pi(1)} \ldots x'_{\pi(n)} | \xi), \quad (77)
\end{aligned}$$

where we have followed the notation of [61,62] in defining

$$T_n \coloneqq \mathrm{e}^{-\sum_{k,l=1}^{n} \Delta_\xi S_{i\pi}^{(k,l)}}. \quad (78)$$

Thus, we have arrived at the generalized LKF transformation for the N-point correlation function for spinor QED, which has turned out to be the same transformation as in the scalar case. The key point is that unwanted derivatives of $\Delta_\xi S$ are canceled exactly by similar derivatives produced by contractions between the $\bar{A\!\!\!/}$ of the Dirac operator and the kernels K.

We can add to the discussion in [61,62] with the observation that, after summing over k and l, the complete LKF factor, T_n, is, in fact, *independent* of the permutation, π, that fixes the partial amplitude. Instead, T_n is a function only of the end points of the $(N = 2n)$ worldlines, since the sum forces all of these end points to pair up in all possible combinations. As an important consequence, not only do the partial amplitudes determine the LKF transformation, but they all transform in the same way, leading immediately to the multiplicative result for the propagator itself, with a *global* transformation,

$$S(x_1\ldots x_n;x_1'\ldots x_n'|\xi+\Delta\xi)=T_n S(x_1\ldots x_n;x_1'\ldots x_n'|\xi), \quad (79)$$

where T_n can be determined using, say, the $\Delta_\xi S_{iI}^{(k,l)}$ corresponding to the identity permutation—as claimed in Eq. (7). Note that as a consequence, an analogous statement holds for scalar QED (with the same prefactor that is now promoted to a global multiplicative factor). We refer to the Appendix for an alternative method for deriving the result proven here.

We may carry out a consistency check on this new result. Consider a particular ordering, π, and choose two lines, m and n. If we set $x_n'=x_m'$ and then identify x_n with x_m', then we ought to get the transformation of the $N-2$ point function where line n has been removed. Indeed, one may verify that under this process, there are various cancellations between the sum of terms of the $\Delta_\xi S_i^{(k,l)}$ involving $k,l\in\{m,n\}$ that leave only $\Delta_\xi S_i^{(m,m)}$, while the variations $\Delta_\xi S_i^{(n,l)}$ and $\Delta_\xi S_i^{(l,n)}$ involving line n and other lines $l\neq m$ sum to zero. This turns T_n into T_{n-1} as expected. In particular, for the $N=4$-point function, with lines 1 and 2, the process turns $\Delta_\xi S_i^{(1,1)}+\Delta_\xi S_i^{(1,2)}+\Delta_\xi S_i^{(2,1)}+\Delta_\xi S_i^{(2,2)}\to\Delta_\xi S_i^{(1,1)}$, which recovers the original LKF transformation of the two-point function, (52), as required.[4]

1. Specific examples: Conformal cross ratios

Now that we have arrived at the generalized LKF transformation, it is worthwhile to consider some examples. Since the results coincide with those of scalar QED, this analysis also expands upon the examples given in [61,62]. In general, the complete, or nonperturbative Green functions may have poles, potentially to all orders, in the physical dimension, $D=D_0$, in which cases, we use dimensional regularization, fixing $D=D_0-2\epsilon$. The LKF factor, T_n, should then be taken to all orders in ϵ, which can be a nontrivial task. Here, we restrict our attention to the first nontrivial contributions in various dimensions.

We begin with the case $D_0=4$, also discussed in [61,62]. As noted in deriving (14), expanding $\Delta S_{i\pi}^{(k,l)}$ in powers of ϵ, the $\frac{1}{\epsilon}$ (infrared) pole in the gamma functions cancels between the terms so that T_n is finite in the limit $\epsilon\to 0$. In this case, if we keep only the terms up to order unity, we have

$$\begin{aligned}\Delta_\xi S^{(k,l)}=&-\frac{\Delta\xi e^2}{32\pi^{\frac{D}{2}}}\{\log[(x_k-x_l)^2]\\&-\log[(x_k-x'_{\pi(l)})^2]\log[(x'_{\pi(k)}-x_l)^2]\\&+\log[(x'_{\pi(k)}-x'_{\pi(l)})^2]\}+\mathcal{O}(\epsilon),\end{aligned} \quad (80)$$

[4]We are grateful to an anonymous referee for suggesting that we include this consistency check explicitly here.

so that substituting this into the exponent and expanded to linear order, we see that

$$T_n=\left(\prod_{k,l=1}^{n} r_\pi^{(k,l)}\right)^{\frac{\Delta\xi e^2}{32\pi^2}}+\mathcal{O}(\epsilon), \quad (81)$$

with, as in [61,62], $r_\pi^{(k,l)}$ is the conformal cross ratio corresponding to the end points of the lines with labels k and l:

$$r_\pi^{(k,l)}\equiv\frac{(x_k-x_l)^2(x'_{\pi(k)}-x'_{\pi(l)})^2}{(x_k-x'_{\pi(l)})^2(x'_{\pi(k)}-x_l)^2}. \quad (82)$$

Note that this factor appears regardless of the mass of the spinor particle propagating between these end points (which can occur because this factor does not depend upon the details of the propagation between these points). The original LKF transformation is recovered by setting $n=1$, for which

$$T_1=\left[\frac{(x-x)^2(x'-x')^2}{((x-x')^2)^2}\right]^{\frac{\Delta\xi\alpha}{8\pi}}+\mathcal{O}(\epsilon). \quad (83)$$

Finally, we replace the vanishing numerator by the cut-off $((x_0)^2)^2$ to arrive at (15). Repeating this trick for the arbitrary correlator and including the contributions from all end points, we arrive at the simplification,

$$\prod_{k,l=1}^{n} r_\pi^{(k,l)}=(x_0^2)^{2n}\frac{\prod_{l>k=1}^{n}((x_k-x_l)^2(x_k'-x_l')^2)^2}{\prod_{k,l=1}^{n}((x_k-x_l')^2)^2}, \quad (84)$$

which gives the leading order contribution to the LKF factor independently of the permutation defining the partial amplitude. It corresponds to the product of (regulated) conformal cross ratios of the end points of n lines for all possible pairings of initial and final points.

In $D_0=3$ dimensions, the LKF factor is without poles, and we get an order unity contribution,

$$\begin{aligned}\Delta_\xi S_{i\pi}^{(k,l)}=&-\frac{\Delta\xi e^2}{16\pi}[|x_k-x_l|-|x_k-x'_{\pi(l)}|-|x'_{\pi(k)}-x_l|\\&+|x'_{\pi(k)}-x'_{\pi(l)}|].\end{aligned} \quad (85)$$

Summing over k and l, the LKF exponent can be simplified to

$$\begin{aligned}&-\sum_{k,l=1}^{n}\Delta_\xi S_{i\pi}^{(k,l)}\\&\quad=\frac{\Delta\xi e^2}{8\pi}\left\{\sum_{l>k=1}^{n}[|x_k-x_l|+|x_k'-x_l'|]-\sum_{k,l=1}^{n}|x_k-x_l'|\right\},\end{aligned} \quad (86)$$

which is now manifestly independent of the permutation π. Again, fixing $n = 1$, we find

$$T_1 = \mathrm{e}^{-\frac{\Delta\xi\alpha}{2}|x-x'|} + \mathcal{O}(\epsilon), \tag{87}$$

which has given (13). We repeat that the divergences in loop diagrams would require evaluation of higher order terms in ϵ that we do not consider here.

For the case $D_0 = 2$, $\Delta_\xi S_{i\pi}^{(k,l)}$ again has a pole in ϵ. The singular part is (we introduce an arbitrary mass scale μ for dimensional consistency)

$$\begin{aligned}\Delta_\xi S_{i\pi}^{(k,l)}|_{\epsilon^{-1}} &= \frac{\Delta\xi e^2\mu^2}{32\pi\epsilon}[(x_k - x_l)^2 - (x_k - x'_{\pi(l)})^2 - (x'_{\pi(k)} - x_l)^2 + (x'_{\pi(k)} - x'_{\pi(l)})^2] \\ &= -\frac{\Delta\xi e^2\mu^2}{16\pi\epsilon}(x_k - x'_{\pi(k)}) \cdot (x_l - x'_{\pi(l)}).\end{aligned} \tag{88}$$

Summing this over values of l and k removes the dependence on the permutation, giving

$$-\sum_{k,l=1}^{n} \Delta_\xi S_{i\pi}^{(k,l)}|_{\epsilon^{-1}} = \frac{\Delta\xi e^2\mu^2}{16\pi\epsilon}\left[\sum_{k=1}^{n}(x_k - x'_k)\right]^2. \tag{89}$$

For the original LKF transformation, with $n = 1$, the result is trivial:

$$T_1|_{\epsilon^{-1}} = \mathrm{e}^{\frac{\Delta\xi\alpha\mu^2}{4\epsilon}(x-x')^2}. \tag{90}$$

We also give the finite contribution for this case. There are two contributions to $\Delta_\xi S$,

$$\begin{aligned}\Delta_\xi S_{i\pi}^{(k,l)}|_{\epsilon^0} = \frac{\Delta\xi e^2\mu^2}{32\pi}\{&(1-\gamma_E)(x_k - x'_{\pi(k)}) \cdot (x_l - x'_{\pi(l)}) \\ &+ [(x_k - x_l)^2 \log[\pi\mu^2(x_k - x_l)^2] - (x_k - x'_{\pi(l)})^2 \log[\pi\mu^2(x_k - x'_{\pi(l)})^2] \\ &- (x'_{\pi(k)} - x_l)^2 \log[\pi\mu^2(x'_{\pi(k)} - x_l)^2] + (x'_{\pi(k)} - x'_{\pi(l)})^2 \log[\pi\mu^2(x'_{\pi(k)} - x'_{\pi(l)})^2]]\},\end{aligned} \tag{91}$$

where γ_E is the EulerMascheroni constant. Summing over k and l, we find

$$\begin{aligned}-\sum_{k,l=1}^{n} \Delta_\xi S_{i\pi}^{(k,l)}|_{\epsilon^0} = \frac{\Delta\xi e^2\mu^2}{16\pi}\Bigg\{&(\gamma_E - 1)\left[\sum_{k=1}^{n}(x_k - x'_k)\right]^2 + \sum_{k,l=1}^{n}(x_k - x'_l)^2 \log[\pi\mu^2(x_k - x'_l)^2] \\ &- \sum_{l>k=1}^{n}[(x_k - x_l)^2 \log[\pi\mu^2(x_k - x_l)^2] + (x'_k - x'_l)^2 \log[\pi\mu^2(x'_k - x'_l)^2]]\Bigg\},\end{aligned} \tag{92}$$

in which we have removed the dependence on the permutation π. The simplest case of $n = 1$ gives the original transformation at constant order,

$$T_1|_{\epsilon^0} = \mathrm{e}^{\frac{\Delta\xi\alpha\mu^2}{4}(x-x')^2[\log[\pi\mu^2(x-x')^2]+\gamma_E-1]}. \tag{93}$$

Combining this with (90), it is important to note that this time the pole is *not* canceled when $\Delta_{2-2\epsilon}(0)$ is subtracted, which implies an essential singularity for the LKF transformation, (1), in the limit $\epsilon \to 0$. In a perturbative calculation, the poles, of arbitrary order, would thus need to be taken into account in the transformation. This is consistent with the observation of [107] that in $D = 2$, the pole of the fermion propagator is not gauge invariant in covariant gauges at any finite order in perturbation theory. These considerations, along with the physically interesting aspects of two-dimensional QED, make further studies of this case of both theoretical and practical interest for ongoing and future work.

V. PERTURBATION THEORY

It is clear by now that the LKF transformation is nonperturbative in nature. However, in a practical perturbative calculation, one would like to see how it works order by order in the loop expansion. For this purpose, in this section, we consider a specific fixed loop-order process as an example to illustrate the gauge transformation of the internal photon propagators and represent the

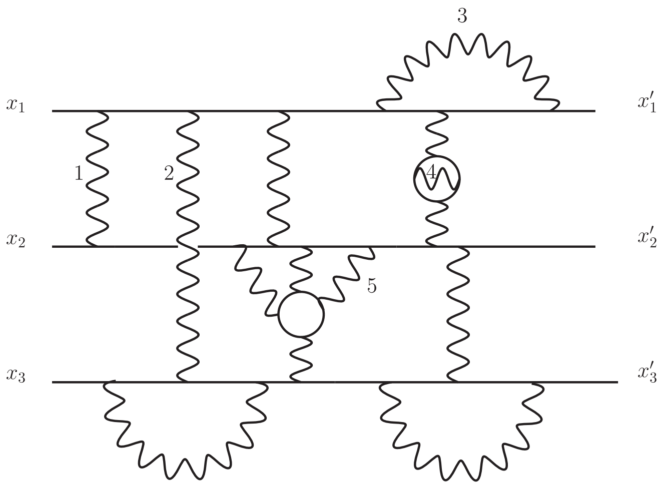
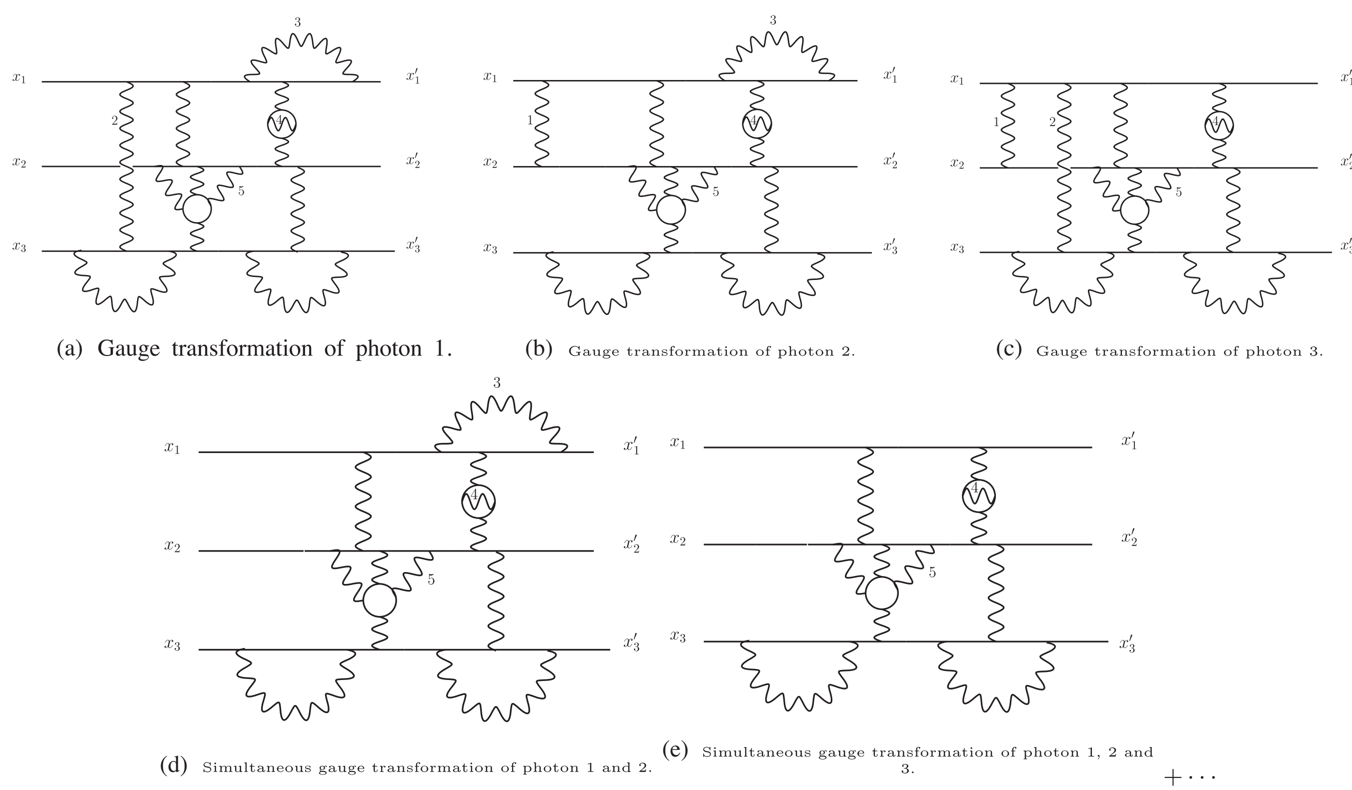


FIG. 1. A Feynman diagram for a typical process with three fermion propagators representing six external particle legs at 12-loop order in configuration space. Here, x_i and x_i' are the end points of the propagators ($i = 1, 2, 3$). The numbers 1–5 indicate some of the photons that could be gauge transformed.

transformation diagrammatically. Since the generalized LKF transformation has turned out to be the same as in the scalar case, we can rework the perturbative discussion in [61,62] for the present case. In particular, although it is possible to obtain gauge dependent higher-loop order contributions from a given amplitude, we restrict attention here to the transformation of terms at a fixed loop order.

For instance, consider the Feynman diagram depicted in Fig. 1 with three electron propagators and 12 loops. It should be understood that we consider the sum of this diagram together with all the ones that differ from it only by *letting photon legs slide along spinor lines*. This is a very complicated process, but here, we are interested in the application of the LKF transformation to some of the internal photons, which are indicated with numbers. We recall that the gauge transformation properties of an amplitude are determined completely by the photons exchanged between two fermion lines or along one fermion line (like photons 1,2,3); in Fig. 1, photons 4 and 5 do not produce a gauge transformation because they start and/or end on an electron loop.

An advantage of our formalism is that we have the freedom to affect a change of gauge parameter on *individual photons* in isolation, which affects the amplitude according to (45); it converts the photon connecting two propagators with end points x_l and x_k into a multiplicative factor of $-\Delta_\xi S_{i\pi}^{(k,l)}$. Thus, the gauge transformation of a photon eliminates that photon and leaves a diagram of lower loop order (the appropriate factor of the coupling constant is contained in $\Delta_\xi S_\pi^{(k,l)}$). To illustrate, this we apply the LKF transformation



(a) Gauge transformation of photon 1.
(b) Gauge transformation of photon 2.
(c) Gauge transformation of photon 3.
(d) Simultaneous gauge transformation of photon 1 and 2.
(e) Simultaneous gauge transformation of photon 1, 2 and 3.

$+\cdots$

FIG. 2. Diagrammatic presentation of gauge transformation of internal photons one by one or of some of them simultaneously using the generalized LKF transformations in perturbation theory.

arising from changing the gauge of the internal photons in Fig. 1 to obtain the gauge variation of this diagram. We denote the value of the diagram by Fig. 1 and its gauge transformation by Δ_ξ Fig. 1, which is built from the value of lower order diagrams (see also Fig. 2) multiplied by the appropriate factors of $\Delta_\xi S$. Then, changing the gauge parameter of the internal photons leads to the gauge transformation of this diagram:

$$\begin{aligned}\Delta_\xi \mathrm{Fig.1} = &(-2\Delta_\xi S_{i\pi}^{(1,2)})\mathrm{Fig.2}a + (-2\Delta_\xi S_{i\pi}^{(1,3)})\mathrm{Fig.2}b + (-\Delta_\xi S_{i\pi}^{(1,1)})\mathrm{Fig.2}c + \cdots \\ &+ (-2\Delta_\xi S_{i\pi}^{(1,2)})(-2\Delta_\xi S_{i\pi}^{(1,3)})\mathrm{Fig.2}d + \cdots \\ &+ (-2\Delta_\xi S_{i\pi}^{(1,2)})(-2\Delta_\xi S_{i\pi}^{(1,3)})(-\Delta_\xi S_{i\pi}^{(1,1)})\mathrm{Fig.2}e + \cdots \\ &+ \vdots\end{aligned} \tag{94}$$

In the above equation, the first line represents the gauge transformation of individual photons, the second line is for the simultaneous transformation of pairs of photons, the last line for the simultaneous gauge transformation of three photons, and so on, which is extremely straightforward using the above LKF rules.

Although this section has focused on perturbation theory in configuration space, it is also possible to transfer the LKF transformations found here to the perturbative expansion in momentum space. This has been achieved for the propagator in scalar and spinor QED in $D = 3$ and $D = 4$ dimensions [38,105,108]; the generalization we have developed here will allow us to apply these techniques to arbitrary correlation functions in future work. In particular, in a further publication, we shall deal with the nontrivial pole structure in the two-dimensional LKF factor in the context of momentum space perturbation theory.

VI. CONCLUSION

We have applied first quantized techniques to determine the transformation of arbitrary fermion correlation functions induced by varying the linear covariant gauge parameter of virtual photons that give loop corrections to the free correlators. These coordinate space transformations generalize the original studies of Landau, Khalatnikov, and Fradkin for the propagator to the general case of the N-point functions of spinor QED and are completely nonperturbative. We recover the original result as a special case with $N = 2$.

The generalized transformations were found by studying the variation induced in partial amplitudes that pair up initial and final points in a particular way. Their variation turned out to be the same as that of their counterparts in scalar QED and corresponds to the introduction of total derivatives in worldline parameter integrals. However, we noted here that the functional form of this variation does not depend upon the ordering implied by the partial amplitude and, as such, factorizes out of the sum over orderings. Moreover, since we explained that virtual fermion loops play no role in determining the LKF transformation, our results found by studying the quenched amplitudes hold unchanged even if such loops are included. We were thus led to a simple, multiplicative transformation for the complete correlators in both scalar and spinor QED, which is the natural generalization of the multiplicative transformation for the propagator.

It was important to check this since the multiplicative form of the transformation could have been broken, even at the level of the partial amplitudes, by the derivative structure of the worldline representation of the correlators, an issue raised in [61]. We have manifested that there is a precise cancellation between all such terms that means that the multiplicative form is maintained after all. In the main text, this was shown using functional methods; a more direct proof is given in the Appendix. We strongly suspect that finding this factorization would be substantially more difficult using standard techniques.

Since the transformation takes the same form as in scalar QED, its application in perturbation theory is the same as in the former case, originally worked out in [61,62] and discussed briefly in Sec. V above. It is also important to note that in spinor QED, renormalization counterterms have the same structure as the bare vertex and propagators, so including these contributions does not produce diagrams with different topology to those considered here (albeit, with some internal photons removed). As long as all diagrams up to some fixed order in α are included, the LKF transformation we have derived will remain unchanged, even if these counterterms are included. In the case of four-dimensional QED, the first order expansion of the multiplicative factor is written in terms of conformal cross ratios of the correlation function arguments. We have added to that work by considering the transformation in two-dimensional QED, wherein there is a divergence that affects the LKF factor to all orders in the perturbative expansion. As such, one can expect poles of arbitrary order to enter the correlation functions after being nonperturbatively gauge transformed.

Current and future work in this context will develop the perturbative application further, in particular, to analyze the momentum space transformation of the propagator for the two-dimensional case as a massive analogue [109] of the Schwinger model [110,111]. Here, there will be the additional difficulty of the pole to treat carefully. Given the extension of the LKF transformation considered here, we shall also be able to examine the momentum space transformation of higher order Green functions in four dimensional QED. Likewise, in lower dimensions, the transformations of the propagator in [48,112] could now be extended to higher order correlation functions.

While we have worked entirely within quantum electrodynamics, the worldline techniques we have applied can be adapted to non-Abelian theories to study the gauge transformation implied by virtual gluons. Moreover, the worldline formalism extends to a gravitational background, which would allow for studies of the diffeomorphism structure of the propagator or correlation functions due to virtual graviton exchange. Likewise, the gauge structure of more complicated objects, such as the propagator in an electromagnetic background or interaction vertices, can be studied with an aim to obtain information about their form factor decomposition. Such work would have application in informing analyzes of the Schwinger-Dyson equations by supplying further restrictions on solutions that incorporate the gauge structure implied by the LKF transformations uncovered here.

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APPENDIX: EXPECTATION VALUES

Here, we define how correlation functions of the quantum background field, $\bar{A}$, are calculated and provide an alternative proof of the cancellation of undesired derivative terms against the contributions from the $\bar{\not{A}}$ terms in the prefactors of (25). Throughout this Appendix, we shall set the value of the charge to $e = 1$.

We define the expectation value $\langle\prod_{i=1}^{N}\bar{A}_{\mu_i}(x_i)\rangle$ according to the Euclidean space path integral over $\bar{A}$ with a particular gauge fixing action $S_{\mathrm{gf}}(\xi) = -\int d^4x(\partial\cdot\bar{A})^2/(2\xi)$ that imposes the covariant linear gauge with parameter ξ:

$$\left\langle\prod_{i=1}^{N}\bar{A}_{\mu_i}(x_i)\right\rangle_{\xi} \coloneqq \int\mathcal{D}\bar{A}(x)\prod_{i=1}^{N}\bar{A}_{\mu_i}(x_i)\mathrm{e}^{-\int d^Dx[-\frac{1}{4}\bar{F}_{\mu\nu}\bar{F}^{\mu\nu}]-S_{\mathrm{gf}}(\xi)}. \tag{A1}$$

In particular, the two-point function reproduces the Green function (20), $\langle\bar{A}_\mu(x)\bar{A}_\nu(x')\rangle_\xi = G_{\mu\nu}(x-x';\xi)$.

As usual, the insertions of the prefactors $\bar{A}_{\mu_i}(x_i)$ can be generated by functional differentiation with respect to a source term, $S[J] = i\int d^DxJ(x)\cdot\bar{A}(x)$. We can take advantage of this to give an alternative derivation of the result given in the main text regarding the cancellation of the unwanted derivatives of $\Delta_\xi S_{i\pi}$ that would otherwise spoil the multiplicative form of the LKF transformations. To this end, we consider

$$\mathcal{K}_\pi(n,N;\xi) \coloneqq \left\langle\prod_{i=1}^{n}\bar{\not{A}}_i(x_i)\prod_{j=1}^{N}K_j^{x'_{\pi(j)}x_j}[\bar{A}]\right\rangle_{\xi};\quad \pi\in S_N, \tag{A2}$$

where we have temporarily ignored the external photons, which play no role in this calculation. The subindices on the $\bar{\not{A}}$ are reminders that the γ matrices must be placed in the correct order according to the product in (54), but it will become clear below that this is not important for now. Expressing the K_j in their path integral representation (38) and generating the $\bar{\not{A}}$ by functional differentiation, we arrive at

$$\mathcal{K}_\pi(n,N;\xi) = \mathrm{symb}^{-1}\left\{\prod_{j=1}^{N}2^{-\frac{D}{2}}\int_0^\infty dT_j\mathrm{e}^{-m^2T_j}\int\mathcal{D}\mathbb{X}(\tau_j,\theta_j)\right.$$
$$\left.\times\,\mathrm{e}^{-\sum_{l=1}^{N}S_0^l[\mathbb{X}_l]}\frac{\delta^n}{\delta\not{J}^n}\langle\mathrm{e}^{i\sum_{i=1}^{N}\int d\tau_i\int d\theta_i\mathbb{D}_i\mathbb{X}_i\cdot\bar{A}(\mathbb{X}_i)+i\int d^DxJ(x)\cdot\bar{A}(x)}\rangle_\xi\right\}, \tag{A3}$$

where we abbreviate $\frac{\delta^n}{\delta\not{J}^n} \coloneqq \frac{(-i)^n\delta^n}{\delta\not{J}(x_1)\cdots\delta\not{J}(x_n)}|_{J=0}$. Completing the square in the final exponent allows for the expectation value to be computed, which supplies

$$\mathcal{K}_\pi(n,N;\xi) = \mathrm{symb}^{-1}\left\{\prod_{j=1}^{N}2^{-\frac{D}{2}}\int_0^\infty dT_j\mathrm{e}^{-m^2T_j}\int\mathcal{D}\mathbb{X}(\tau_j,\theta_j)\mathrm{e}^{-\sum_{l=1}^{N}S_0^l[\mathbb{X}_l]}\frac{\delta^n}{\delta\not{J}^n}\mathrm{e}^{\frac{1}{2}\int d^Dyd^Dy'\mathcal{J}(y)\cdot G(y-y';\xi)\cdot\mathcal{J}(y')}\right\}, \tag{A4}$$

where the current is $\mathcal{J}^\mu(y) = J^\mu(y) + \sum_{i=1}^N \int d\tau_i \int d\theta_i \delta^D(y - \mathbb{X}_i)\mathbb{D}_i \mathbb{X}_i^\mu$. We are not interested in the precise form of this result nor the path integrals over the $\mathbb{X}_i$, but rather how this quantity changes when we vary ξ. As such, we realize the change of gauge directly in the Green function so that the integrand of (A4) changes, according to

$$\frac{\delta^n}{\delta \mathcal{J}^n}\langle \mathrm{e}^{i\sum_{i=1}^N \int d\tau_i \int d\theta_i \mathbb{D}_i \mathbb{X}_i \cdot \bar{A}(\mathbb{X}_i) + i\int d^D x J(x)\cdot \bar{A}(x)}\rangle_{\xi+\Delta\xi} = \sum_{m=0}^{n} \sum_{\mathrm{perm}\{\rho_m\}} \frac{\delta^m_\rho}{\delta \mathcal{J}^m} \mathrm{e}^{\frac{1}{2}\int d^D x \int d^D x' \mathcal{J}(x)\cdot \Delta_\xi G(x-x')\cdot \mathcal{J}(x')} \times \frac{\delta^{n-m}_\rho}{\delta \mathcal{J}^{n-m}} \langle \mathrm{e}^{i\sum_{i=1}^N \int d\tau_i \int d\theta_i \mathbb{D}_i \mathbb{X}_i\cdot \bar{A}(\mathbb{X}_i) + i \int d^D x J(x)\cdot \bar{A}(x)}\rangle_\xi, \tag{A5}$$

where the sum over partitions, $\{\rho_m\}$, counts all ways to choose m variables from $\{1, \ldots n\}$, which appear in the first functional derivative; the remaining $n - m$ are then placed in the second. Here, the exponent of the first term on the right-hand side is derived from the ξ-dependent parts of (20) and can be decomposed as shown in the main text:

$$\frac{1}{2}\int d^D x \int d^D x' \mathcal{J}(x)\cdot \Delta_\xi G(x-x')\cdot \mathcal{J}(x') = -\sum_{i,j=1}^{N} \Delta_\xi S_{i\pi}^{(i,j)} + \Delta_\xi I_N^{(1)} + \Delta_\xi I_N^{(2)}, \tag{A6}$$

where we have used (44) and computed the integrals over the θ_i as in Sec. III B.

The first term in (A6) generates the global exponential factor common to all terms in the Green functions. Meanwhile, the m functional derivatives in (A5) will act on the functions $J(x)$ in the $\Delta_\xi I_N$ of (A6) to produce various terms. Since the derivatives are contracted into γ matrices, the structure is particularly simple. We must choose k pairs and l singletons such that $2k + l = m$. Each pair produces an insertion of the form,

$$\frac{\Delta\xi}{16\pi^{\frac{D}{2}}}\Gamma\left[\frac{D}{2}-2\right]\gamma\cdot\partial_{x_i}\gamma\cdot\partial_{x_j}[(x_i - x_j))^2]^{2-\frac{D}{2}}, \tag{A7}$$

which we recognize as $\not\partial_{x_i}\not\partial_{x_j}\Delta_\xi S_{i\pi}^{(i,j)}$. Similarly, the singletons produce factors,

$$\frac{\Delta\xi}{32\pi^{\frac{D}{2}}}\Gamma\left[\frac{D}{2}-2\right]\sum_{i=1}^{N}\int_0^{T_i} d\tau_i \int d^D x \gamma\cdot\partial_{x_j}\frac{\partial}{\partial\tau_i}[(x_j - x(\tau_i))^2]^{2-\frac{D}{2}}, \tag{A8}$$

which is, of course, $\sum_{i=1}^N \not\partial_{x_j}\Delta_\xi S_{i\pi}^{(i,j)}$. As such, (A5) can be written as

$$\begin{aligned}&\frac{\delta^n}{\delta \mathcal{J}^n}\langle \mathrm{e}^{i\sum_{i=1}^N \int d\tau_i \int d\theta_i \mathbb{D}_i \mathbb{X}_i\cdot \bar{A}(\mathbb{X}_i) + i\int d^D x J(x)\cdot \bar{A}(x)}\rangle_{\xi+\Delta\xi}\\ &= \sum_{m=0}^{n}\sum_{\mathrm{perm}\{\rho_m\}} \frac{\delta^{n-m}_\rho}{\delta \mathcal{J}^{n-m}}\langle \mathrm{e}^{i\sum_{i=1}^N \int d\tau_i \int d\theta_i \mathbb{D}_i \mathbb{X}_i\cdot \bar{A}(\mathbb{X}_i) + i\int d^D x J(x)\cdot \bar{A}(x)}\rangle_\xi \\ &\times \sum_{2k+l=m}\sum_{\sigma\in S_m}\prod_{i=1}^{k}(\not\partial_{x_{\sigma(2i-1)}}\not\partial_{x_{\sigma(2i)}}\Delta_\xi S_{i\pi}^{(\sigma(2i-1),\sigma(2i))})\prod_{j=2k+1}^{m}\left(\sum_{p=1}^{N}\not\partial_{x_{\sigma(j)}}\Delta_\xi S_{i\pi}^{(p,\sigma(j))}\right)\mathrm{e}^{-\sum_{r,s=1}^N \Delta_\xi S_{i\pi}^{(r,s)}},\end{aligned} \tag{A9}$$

where the $\{\sigma\}$ permute the set $\{\rho(1), \ldots \rho(m)\}$. Finally, we note that differentiating $\sum_{r,s=1}^N \Delta_\xi S_{i\pi}^{(r,s)}$ twice with respect to distinct positions, x_1 and x_2, leaves a function only of x_1 and x_2 so that a further derivative with respect to any x_3 kills the result. For this reason, we recognize that the final line of (A9) is precisely the action of m derivatives $\not\partial_x$ on the global exponent. Substituting this into (A3), we arrive at

$$\mathcal{K}_\pi(n, N; \xi + \Delta\xi) = \sum_{m=0}^{n}\sum_{\mathrm{perm}\{\rho_m\}}\left\langle \prod_{j=m+1}^{n}\bar{\not A}(x_{\rho(j)})\prod_{k=1}^{N}K_k^{x'_{\pi(k)}x_k}\right\rangle_\xi \left[\prod_{i=1}^{m} i\not\partial_{x_{\rho(i)}}\right]\mathrm{e}^{-\sum_{l,m=1}^N \Delta_\xi S_{i\pi}^{(l,m)}}. \tag{A10}$$

In fact, we can improve this to give an iterative formula relating changes in the various matrix elements. We define a new "difference" operator, $\blacktriangle_\xi$, which returns the nonmultiplicative terms in the transformations of matrix elements as follows:

$$\blacktriangle_\xi \langle\cdots\rangle_\xi = \langle\cdots\rangle_{\xi+\Delta\xi} - \langle\cdots\rangle_\xi \mathrm{e}^{-\Delta_\xi S}, \tag{A11}$$

where we have denoted $\Delta_\xi S = \sum_{l,m} \Delta_\xi S_{i\pi}^{(l,m)}$, allowing the case that the sum is empty. To give some examples,

$$\blacktriangle_\xi \left\langle \prod_{j=1}^N K_j^{x'_{\pi(j)} x_j}, \right\rangle_\xi = 0 \tag{A12}$$

$$\blacktriangle_\xi \left\langle \bar{\not{A}}_1 \prod_{j=1}^N K_j^{x'_{\pi(j)} x_j} \right\rangle_\xi = \left\langle \prod_{j=1}^N K_j^{x'_{\pi(j)} x_j} \right\rangle_\xi i\not{\partial}_1 \mathrm{e}^{-\sum_{l,m=1}^N \Delta_\xi S_{i\pi}^{(l,m)}}, \tag{A13}$$

$$= \blacktriangle_\xi \left\langle i\not{\partial}_1 \prod_{j=1}^N K_j^{x'_{\pi(j)} x_j} \right\rangle_\xi. \tag{A14}$$

From these relations follows immediately the LKF transformation for the propagator, which, in this notation, takes the form,

$$\blacktriangle_\xi \langle [m + i\not{\partial}' - \bar{\not{A}}(x'_1)] K_1^{x',x} \rangle = 0. \tag{A15}$$

To include derivatives, we note that there is a commutator $[\blacktriangle_\xi, \partial_\mu] = \partial_\mu \mathrm{e}^{-\Delta_\xi S}$, which follows by direct computation from the definition (A11). This allows a nice way of arriving at the last line of (A14).

With this notation and using the properties of $\blacktriangle_\xi$, we can convert (A10) into a stronger statement that is equivalent to the LKFT. This requires moving the derivatives acting on the LKF exponent inside of the expectation value. To achieve this, we first prove a general property: With $\mathcal{A}(\bar{A})$, any function of $\bar{A}$ we have, for $n \geq 1$,

$$\begin{aligned}\left\langle \mathcal{A} \prod_{k=1}^N K_k^{x'_{\pi(k)} x_k} \right\rangle i\not{\partial}'_1 i\not{\partial}'_2 \ldots i\not{\partial}'_n \mathrm{e}^{-\Delta_\xi S} = (-1)^{n-1} \Bigg\{ \blacktriangle_\xi \left\langle i\not{\partial}'_1 i\not{\partial}'_2 \ldots i\not{\partial}'_n \mathcal{A} \prod_{k=1}^N K_k^{x'_{\pi(k)} x_k} \right\rangle \\ - \sum_{m=1}^n \sum_{\{\rho_m\}} (-1)^{m-1} i\not{\partial}'_{\rho(1)} \ldots i\not{\partial}'_{\rho(m)} \blacktriangle_\xi \left\langle i\not{\partial}'_{\rho(m+1)} \ldots i\not{\partial}'_{\rho(n)} \mathcal{A} \prod_{k=1}^N K_k^{x'_{\pi(k)} x_k} \right\rangle \Bigg\},\end{aligned} \tag{A16}$$

where the sum over $\{\rho_m\}$ is over the selection of m out of the n derivatives. Although this very general result follows from repeated application of the commutator, it is quicker to use induction. For $n = 1$, we may repeat the same steps in (A14) with the additional insertion of $\mathcal{A}$, since the commutator holds; the $n = 2$ case, which is needed to check the alternating sign, is discussed below. Assuming then, that the relation holds for n derivatives, we write $\langle \mathcal{A} \prod_{k=1}^N K_k^{x'_{\pi(k)} x_k} \rangle i\not{\partial}'_1 i\not{\partial}'_2 \ldots i\not{\partial}'_{n+1} \mathrm{e}^{-\Delta_\xi S}$ as

$$i\not{\partial}'_{n+1} \left[\left\langle \mathcal{A} \prod_{k=1}^N K_k^{x'_{\pi(k)} x_k} \right\rangle i\not{\partial}'_1 i\not{\partial}'_2 \ldots i\not{\partial}'_n \mathrm{e}^{-\Delta_\xi S} \right] - \left\langle i\not{\partial}'_{n+1} \mathcal{A} \prod_{k=1}^N K_k^{x'_{\pi(k)} x_k} \right\rangle i\not{\partial}'_1 i\not{\partial}'_2 \ldots i\not{\partial}'_n \mathrm{e}^{-\Delta_\xi S}, \tag{A17}$$

where, in the second term, the derivative $\not{\partial}'_{n+1}$ does not act beyond the expectation value. The inductive hypothesis immediately shows that these two terms simply split up contributions, in which, $\not{\partial}'_{n+1}$ is outside or inside of the expectation value, respectively, and hence, give the result for $n + 1$ derivatives.

We can use this immediately in (A10) to move derivatives inside of the expectation values. Then, the LKFT corresponds to the fact that all of the terms coming from the second line of (A16), involving derivatives of the variations, cancel between themselves, and we are left with

$$\blacktriangle_\xi \left\langle \prod_{i=1}^n \bar{\not{A}}_i(x_i) \prod_{j=1}^N K_j^{x'_{\pi(j)} x_j} \right\rangle_\xi = \sum_{m=1}^n \sum_{\{\rho_m\}} (-1)^{m-1} \blacktriangle_\xi \left\langle \prod_{i=1}^m i\not{\partial}'_{\rho(i)} \prod_{j=m+1}^n \bar{\not{A}}_{\rho(j)} \prod_{k=1}^N K_k^{x'_{\pi(k)} x_k} \right\rangle_\xi, \tag{A18}$$

where the sum over permutations sums all possible replacements of m of the $\bar{A}$ with partial derivatives. Indeed, the $n = 1$ case has been given in (A14), and the general case is proven with strong induction as follows.

We suppose the result holds for up to n insertions and use (A10) and (A16) for the case of $n+1$ insertions. This leads straightforwardly to

$$\blacktriangle_\xi\left\langle\prod_{i=1}^{n+1}\bar{\not{A}}_i(x_i)\prod_{j=1}^{N}K_j^{x'_{\pi(j)}x_j}\right\rangle_\xi=\sum_{m=1}^{n+1}\sum_{\{\rho_m\}}(-1)^{m-1}\left[\blacktriangle_\xi\left\langle\prod_{i=1}^{m}i\not{\partial}'_{\rho(i)}\prod_{j=m+1}^{n+1}\bar{\not{A}}_{\rho(j)}\prod_{k=1}^{K}K_k^{x'_{\pi(k)}x_k}\right\rangle_\xi\right.$$
$$\left.-\sum_{p=1}^{m}\sum_{\{\sigma_p\}}\prod_{q=1}^{p}(-1)^{p-1}i\not{\partial}'_{\rho(\sigma(q))}\blacktriangle_\xi\left\langle\prod_{r=p+1}^{m}i\not{\partial}'_{\rho(\sigma(r))}\prod_{j=m+1}^{n+1}\bar{\not{A}}_{\rho(j)}\prod_{k=1}^{K}K_k^{x'_{\pi(k)}x_k}\right\rangle_\xi\right]. \tag{A19}$$

The terms on the first line are precisely the result desired; it remains to show that the sum of terms coming from the second line cancel. To verify this, we consider the sum of all terms that contain $p=s$ derivatives outside of the variation $\blacktriangle_\xi$. The terms involving such derivatives are

$$\sum_{m=s}^{n+1}\sum_{\{\rho_m\}}(-1)^{m+s-1}\sum_{\{\sigma_s\}}\prod_{q=1}^{s}i\not{\partial}'_{\rho(\sigma(q))}\blacktriangle_\xi\left\langle\prod_{r=s+1}^{m}i\not{\partial}'_{\rho(\sigma(r))}\prod_{j=m+1}^{n+1}\bar{\not{A}}_{\rho(j)}\prod_{k=1}^{K}K_k^{x'_{\pi(k)}x_k}\right\rangle_\xi. \tag{A20}$$

Of these terms, when $m=s$, there are no derivatives in the expectation value, and there are n or fewer factors of $\bar{A}$. The inductive hypothesis shows that this term,

$$-\sum_{\{\rho_s\}}\sum_{\{\sigma_s\}}\prod_{q=1}^{s}i\not{\partial}'_{\rho(\sigma(q))}\blacktriangle_\xi\left\langle\prod_{j=s+1}^{n+1}\bar{\not{A}}_{\rho(j)}\prod_{k=1}^{K}K_k^{x'_{\pi(k)}x_k}\right\rangle_\xi, \tag{A21}$$

cancels against the terms that have one or more derivatives inside the brackets. This completes the proof.

The immediate application is to derive the LKF transformation. It is clear that

$$\blacktriangle_\xi\left\langle\prod_{i=1}^{n}[m+i\not{\partial}'_i-\not{A}_i^\gamma-\bar{\not{A}}_i]\prod_{k=1}^{K}K_k^{x'_{\pi(k)}x_k}[A^\gamma+\bar{A}]\right\rangle_\xi=0, \tag{A22}$$

since the signs and the derivatives in the variation of $\blacktriangle_\xi\langle\prod_{i=1}^{n}\bar{\not{A}}_i(x_i)\prod_{j=1}^{N}K_j^{x'_{\pi(j)}x_j}\rangle_\xi$ given in (A18) cancel term by term against the variations with fewer $\bar{\not{A}}$. To illustrate how the cancellation works, we work out the $n=2$ case explicitly, organizing the calculation according to powers of $[m-\not{A}^\gamma]$ whose gauge variation is trivial:

$$\blacktriangle_\xi\langle[m+i\not{\partial}'_1-\not{A}_1^\gamma-\bar{\not{A}}_1][m+i\not{\partial}'_2-\not{A}_2^\gamma-\bar{\not{A}}_2]K_1^{x'_{\pi(1)}x_1}K_2^{x'_{\pi(2)}x_2}\rangle_\xi,$$
$$=[m-\not{A}_1^\gamma][m-\not{A}_2^\gamma]\blacktriangle_\xi\langle K_1^{x'_{\pi(1)}x_1}K_2^{x'_{\pi(2)}x_2}\rangle_\xi, \tag{A23}$$

$$+[m-\not{A}_1^\gamma]\blacktriangle_\xi\langle[i\not{\partial}'_2-\bar{\not{A}}_2]K_1^{x'_{\pi(1)}x_1}K_2^{x'_{\pi(2)}x_2}\rangle_\xi, \tag{A24}$$

$$+[m-\not{A}_2^\gamma]\blacktriangle_\xi\langle[i\not{\partial}'_1-\bar{\not{A}}_1]K_1^{x'_{\pi(1)}x_1}K_2^{x'_{\pi(2)}x_2}\rangle_\xi, \tag{A25}$$

$$+\blacktriangle_\xi\langle[i\not{\partial}'_1-\bar{\not{A}}_1][i\not{\partial}'_2-\bar{\not{A}}_2]K_1^{x'_{\pi(1)}x_1}K_2^{x'_{\pi(2)}x_2}\rangle_\xi. \tag{A26}$$

Now, we already know that $\blacktriangle_\xi\langle K_1^{x'_{\pi(1)}x_1}[A^\gamma+\bar{A}]K_2^{x'_{\pi(2)}x_2}[A^\gamma+\bar{A}]\rangle_\xi=0$, and, in fact, our $n=1$ case from above shows that the second two lines also vanish. For the last line, we begin with the term involving $\bar{\not{A}}_1\bar{\not{A}}_2$:

$$\blacktriangle_\xi\langle\bar{\not{A}}_1\bar{\not{A}}_2K_1^{x'_{\pi(1)}x_1}K_2^{x'_{\pi(2)}x_2}\rangle_\xi=-\blacktriangle_\xi\langle i\not{\partial}'_1i\not{\partial}'_2K_1^{x'_{\pi(1)}x_1}K_2^{x'_{\pi(2)}x_2}\rangle_\xi+\blacktriangle_\xi\langle i\not{\partial}'_1\bar{\not{A}}_2K_1^{x'_{\pi(1)}x_1}K_2^{x'_{\pi(2)}x_2}\rangle_\xi+\blacktriangle_\xi\langle\bar{\not{A}}_1i\not{\partial}'_2K_1^{x'_{\pi(1)}x_1}K_2^{x'_{\pi(2)}x_2}\rangle_\xi, \tag{A27}$$

which cancels the other terms that arise in the final line. The variations therefore sum to zero.

For completeness, we also exhibit how the result (A18), used in this illustration, arises in this simple case. We can use (A10) and the commutator to write

$$\blacktriangle_{\xi}\langle \bar{\not{A}}_1\bar{\not{A}}_2 K_1^{x'_{\pi(1)}x_1} K_k^{x'_{\pi(1)}x_1}\rangle_{\xi} = [\langle \bar{\not{A}}_2 K_1^{x'_{\pi(1)}x_1} K_2^{x'_{\pi(2)}x_2}\rangle_{\xi} i\partial'_1 + \langle \bar{\not{A}}_1 K_1^{x'_{\pi(1)}x_1} K_2^{x'_{\pi(2)}x_2}\rangle_{\xi} i\partial'_2 + \langle K_1^{x'_{\pi(1)}x_1} K_2^{x'_{\pi(2)}x_2}\rangle_{\xi} i\partial'_1 i\partial'_2]\mathrm{e}^{-\Delta_{\xi}S} \tag{A28}$$

$$\begin{aligned} &= \blacktriangle_{\xi}\langle i\partial'_1\bar{\not{A}}_2 K_1^{x'_{\pi(1)}x_1} K_2^{x'_{\pi(2)}x_2}\rangle_{\xi} - i\partial'_1\blacktriangle_{\xi}\langle \bar{\not{A}}_2 K_1^{x'_{\pi(1)}x_1} K_2^{x'_{\pi(2)}x_2}\rangle_{\xi} \\ &\quad + \blacktriangle_{\xi}\langle i\partial'_2\bar{\not{A}}_1 K_1^{x'_{\pi(1)}x_1} K_2^{x'_{\pi(2)}x_2}\rangle_{\xi} - i\partial'_2\blacktriangle_{\xi}\langle \bar{\not{A}}_1 K_1^{x'_{\pi(1)}x_1} K_2^{x'_{\pi(2)}x_2}\rangle_{\xi} \\ &\quad - \blacktriangle_{\xi}\langle i\partial'_1 i\partial'_2 K_1^{x'_{\pi(1)}x_1} K_2^{x'_{\pi(2)}x_2}\rangle_{\xi} + i\partial'_1\blacktriangle_{\xi}\langle i\partial'_2 K_1^{x'_{\pi(1)}x_1} K_2^{x'_{\pi(2)}x_2}\rangle_{\xi} \\ &\quad + i\partial'_2\blacktriangle_{\xi}\langle i\partial'_1 K_1^{x'_{\pi(1)}x_1} K_2^{x'_{\pi(2)}x_2}\rangle_{\xi} - i\partial'_1 i\partial'_2\blacktriangle_{\xi}\langle K_1^{x'_{\pi(1)}x_1} K_2^{x'_{\pi(2)}x_2}\rangle_{\xi}. \end{aligned} \tag{A29}$$

The last two lines can be verified by direct calculation (the last term vanishes, of course) to verify (A16) and (A27). The $n = 1$ case can again be used to cancel all but the three terms that make up (A18). In the course of working out this example in detail, we have also verified the alternating signs that enter the equations (A16) and (A18).

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