Gauge-symmetrization method for energy-momentum tensors in high-order electromagnetic field theories

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For electromagnetic field theories, canonical energy-momentum conservation laws can be derived from the underpinning spacetime translation symmetry according to the Noether procedure. However, the canonical energy-momentum tensors (EMTs) are neither symmetric nor gauge-symmetric (gauge invariant). The Belinfante-Rosenfeld (BR) method is a well-known procedure to symmetrize the EMTs, which also renders them gauge symmetric for first-order field theories. High-order electromagnetic field theories appear in the study of gyrokinetic systems for magnetized plasmas and the Podolsky system for the radiation reaction of classical charged particles. For these high-order field theories, gauge-symmetric EMTs are not necessarily symmetric and vice versa. In the present study, we develop a new gauge-symmetrization method for EMTs in high-order electromagnetic field theories. The Noether procedure is carried out using the Faraday tensor $F_{\mu\nu}$, instead of the 4-potential A_μ, to derive a canonical EMT $T_{\mathrm{N}}^{\mu\nu}$. We show that the gauge-dependent part of $T_{\mathrm{N}}^{\mu\nu}$ can be removed using the displacement-potential tensor $\mathcal{F}^{\sigma\mu\nu} \equiv \mathcal{D}^{\sigma\mu}A^\nu/4\pi$, where $\mathcal{D}^{\sigma\mu}$ is the antisymmetric electric displacement tensor. This method gauge-symmetrizes the EMT without necessarily making it symmetric, which is adequate for applications not involving general relativity. For first-order electromagnetic field theories, such as the standard Maxwell system, $\mathcal{F}^{\sigma\mu\nu}$ reduces to the familiar BR superpotential $\mathcal{S}^{\sigma\mu\nu}$, and the method developed can be used as a simpler procedure to calculate $\mathcal{S}^{\sigma\mu\nu}$ without employing the angular momentum tensor in 4D spacetime. When the electromagnetic system is coupled to classical charged particles, the gauge-symmetrization method for EMTs is shown to be effective as well.

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I. INTRODUCTION

In classical field theories, one can derive canonical energy-momentum tensors (EMTs) $T_{\mathrm{N}}^{\mu\nu}$ from the underpinning spacetime translation symmetry using the Noether procedure [1]. However, for classical systems of electromagnetic field, the canonical EMTs are neither symmetric with respect to tensor indices nor electromagnetic gauge invariant. Gauge dependence is unphysical, and nonsymmetric EMT is not consistent with general relativity. In the present study, we will call an EMT symmetric if it is symmetric with respect to tensor indices, and gauge symmetric if it is gauge invariant. To date, much effort has been focused on symmetrizing the EMTs (with respect to tensor indices), while constructing gauge-symmetric EMTs is oftentimes a challenging task for general systems [2–7].

The first method for symmetrizing EMTs was discovered by Belinfante [8,9] and Rosenfeld [10], who added a divergence-free tensor $\partial_\sigma\mathcal{S}^{\sigma\mu\nu}$ to obtain a symmetric EMT, i.e.,

$$T_{\mathrm{BR}}^{\mu\nu} = T_{\mathrm{N}}^{\mu\nu} + \partial_\sigma\mathcal{S}^{\sigma\mu\nu}, \tag{1}$$

$$\partial_\mu\partial_\sigma\mathcal{S}^{\sigma\mu\nu} = 0. \tag{2}$$

Here, $T_{\mathrm{BR}}^{\mu\nu}$ is Belinfante-Rosenfeld (BR) EMT, and $\mathcal{S}^{\sigma\mu\nu}$ is known as BR superpotential that depends on the angular momentum tensor and is antisymmetric with respect to σ and μ [see Eq. (54)]. General relativity suggests another method to generate symmetric EMTs by varying the action with respect to the spacetime metric [11,12], which was modified by Gotay and Marsden, who employed constraints to define symmetric EMTs [13,14]. The relations between these three types of symmetric EMTs have been discussed in the literature [15–17].

In many systems, including the standard Maxwell system (6), the symmetrization of $T_{\mathrm{N}}^{\mu\nu}$ also renders it gauge symmetric. But for general electromagnetic field theories with high-order field derivations, symmetry with respect to tensor indices in general does not imply gauge symmetry and vice versa. High-order electromagnetic field theories appear in the study of gyrokinetic systems [18–20] for magnetized plasmas and the Podolsky system [21,22] for the radiation reaction of classical charged particles. In the present study, we propose a new method to gauge-symmetrize the canonical EMTs $T_{\rm N}^{\mu\nu}$ in general electromagnetic field theories with high-order field derivations. Our method removes the gauge dependence, but does not necessarily symmetrize the EMTs. In applications that do not involve general relativity, gauge-symmetrized EMTs are adequate.

We first consider the Lagrangian density $\mathcal{L}_F$ which only depends on the Faraday tensor $F_{\mu\nu}$, i.e., $\mathcal{L}_F = \mathcal{L}_F(x^\mu, F_{\mu\nu}, DF_{\mu\nu}, \ldots, D^{(n)}F_{\mu\nu})$. We reformulate the equation of motion for the field by the variational principle with respect to the Faraday tensor $F_{\mu\nu}$, instead of the 4-potential A^μ as in the standard field theory. The Euler-Lagrange (EL) equation is cast into an explicitly gauge-symmetric form. The canonical EMT is then separated into a gauge-invariant part and a gauge-dependent part, the latter of which contains the antisymmetric electric displacement tensor $\mathcal{D}^{\mu\nu}$. We define a superpotential $\mathcal{F}^{\sigma\mu\nu} \equiv \mathcal{D}^{\sigma\mu}A^\nu/4\pi$, called displacement-potential tensor, whose divergence $T_0^{\mu\nu} \equiv D_\sigma \mathcal{F}^{\sigma\mu\nu}$ as a second order tensor is divergence free with respect to the first index, i.e., $D_\mu T_0^{\mu\nu} \equiv 0$. Adding $T_0^{\mu\nu}$ to the canonical EMT $T_{\rm N}^{\mu\nu}$ leads to a gauge-symmetric EMT $T_{\rm GS}^{\mu\nu} = \mathcal{L}_F\eta^{\mu\nu} + \frac{1}{4\pi}\mathcal{D}^{\mu\sigma}F_\sigma^\nu - \Sigma^{\mu\nu}$. Here, the tensor $\Sigma^{\mu\nu}$, defined in Eq. (36), is the gauge invariant. It is simpler to calculate the displacement-potential tensor $\mathcal{F}^{\sigma\mu\nu}$ than the BR superpotential $\mathcal{S}^{\sigma\mu\nu}$, and the former only gauge-symmetrizes the canonical EMT without render it symmetric (with respect to tensor indices). For first-order electromagnetic field theories, such as the standard Maxwell system, $\mathcal{F}^{\sigma\mu\nu}$ reduces to the familiar BR superpotential $\mathcal{S}^{\sigma\mu\nu}$, and the method developed here can be used as a simpler procedure to calculate $\mathcal{S}^{\sigma\mu\nu}$ without employing the angular momentum tensor in 4D spacetime.

In addition, when the electromagnetic system is coupled with classical charged particles, the Lagrangian density is generally written as $\mathcal{L}_F = \mathcal{L}_F(x^\mu, \boldsymbol{X}_a, \dot{\boldsymbol{X}}_a, A_\mu, F_{\mu\nu}, DF_{\mu\nu}, \ldots, D^{(n)}F_{\mu\nu})$. If the 4-potential A_μ is minimally coupled with particle's trajectory and $\mathcal{L}_F$ depends on A_μ linearly, we find that the method is effective as well, even though the Lagrangian density is not gauge symmetric in general.

The paper is organized as follows. In Sec. II, we describe the gauge-symmetrization method for the EMT in a general high-order electromagnetic field theory, and highlight the difference in comparison with the BR method using the example of the Podolsky system [21,22]. Section III shows how the gauge-symmetrization method for the EMT works when the electromagnetic system is coupled with classical charged particles.

II. EXPLICITLY GAUGE-SYMMETRIC CONSERVATION LAWS FOR HIGH-ORDER ELECTROMAGNETIC SYSTEMS

A. Explicitly gauge-symmetric Euler-Lagrange equation

The Lagrangian density of a general electromagnetic system is written as

$$\mathcal{L}_F = \mathcal{L}_F(x^\mu, DA_\mu, \ldots, D^{(n+1)}A_\mu), \tag{3}$$

where $A^\mu = (\varphi, \boldsymbol{A})$ is the 4-potential defined on 4D Minkowski space endowed with a Lorentz metric $\eta_{\mu\nu} = \mathrm{diag}\{1, -1, -1, -1\}$. Here, $A_\mu = \eta_{\mu\nu}A^\nu$ and $D = ((1/c)\partial_t, \boldsymbol{\nabla})$ is the derivative operator over spacetime. The EL equation of the Lagrangian density is

$$E_A^\mu(\mathcal{L}_F) = 0, \tag{4}$$

where the Euler operator E_A^μ of A_μ is defined by

$$E_A^\mu \equiv \sum_{i=1}^{n+1} (-1)^i D_{\mu_1} \cdots D_{\mu_i} \frac{\partial}{\partial(\partial_{\mu_1} \cdots \partial_{\mu_i} A_\mu)}. \tag{5}$$

Note that the Lagrangian density $\mathcal{L}_F$ depends on derivatives of A with respect to the spacetime coordinates up to the $(n+1)$th order. It includes the standard Maxwell system, i.e.,

$$\mathcal{L}_F = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}, \tag{6}$$

as a special case, where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7}$$

is the Faraday tensor. In Eq. (6), $\mathcal{L}_F$ depends only on first-order derivatives of A. High-order electromagnetic field theories appear in the study of gyrokinetic systems [18–20] for magnetized plasmas and radiation reaction for classical charged particles [21,22]. Physics requires that the EL equation (4) is gauge symmetric, i.e., invariant under the gauge transformation $A_\mu \mapsto A_\mu + \partial_\mu f$. In the present study, we assume that $\mathcal{L}_F$ is explicitly gauge symmetric in the form of

$$\mathcal{L}_F = \mathcal{L}_F(x^\mu, F_{\mu\nu}, DF_{\mu\nu}, \ldots, D^{(n)}F_{\mu\nu}). \tag{8}$$

From the variational principle, $\delta\mathcal{A} = \delta \int \mathcal{L}_F d^4x = 0$, we have

$$0 = \delta \int \mathcal{L}_F d^4x = \int [E_F^{\mu\nu}(\mathcal{L}_F)\delta F_{\mu\nu}] d^4x$$
$$= \int [E_F^{\mu\nu}(\mathcal{L}_F)(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu)] d^4x$$
$$= -\int \partial_\mu [2E_F^{[\mu\nu]}(\mathcal{L}_F)]\delta A_\nu d^4x, \tag{9}$$

where the boundary term has been dropped, and $E_F^{\mu\nu}$ denotes the Euler operator for the Faraday tensor $F_{\mu\nu}$ defined by

$$E_F^{\mu\nu}(\mathcal{L}_F) = \frac{\partial \mathcal{L}_F}{\partial F_{\mu\nu}} + \sum_{i=1}^{n} (-1)^i D_{\mu_1} \cdots D_{\mu_i} \frac{\partial \mathcal{L}_F}{\partial \partial_{\mu_1} \cdots \partial_{\mu_i} F_{\mu\nu}}. \tag{10}$$

In Eq. (9), superscript $[\mu\nu]$ represents antisymmetrization with respect to μ and ν, i.e.,

$$E_F^{[\mu\nu]}(\mathcal{L}_F) \equiv \frac{1}{2}[E_F^{\mu\nu}(\mathcal{L}_F) - E_F^{\nu\mu}(\mathcal{L}_F)]. \tag{11}$$

Due to the arbitrariness of δA_ν in Eq. (9), the equation of motion for the system is

$$\partial_\mu \mathcal{D}^{\mu\nu} = 0, \tag{12}$$

where

$$\mathcal{D}^{\mu\nu} \equiv -8\pi E_F^{[\mu\nu]}(\mathcal{L}_F) \tag{13}$$

is the electric displacement tensor.

In Sec. III, we will consider electromagnetic systems coupled with charged particles, and the Lagrangian density $\mathcal{L}$ will depend on 4-potential A_μ i.e.,

$$\mathcal{L} = \mathcal{L}(x^\mu, A_\mu, F_{\mu\nu}, \ldots, D^{(n)} F_{\mu\nu}). \tag{14}$$

Equation (12) then becomes

$$\partial_\mu \mathcal{D}^{\mu\nu} = \frac{4\pi}{c} J_f^\nu, \tag{15}$$

where

$$J_f^\nu \equiv -c \frac{\partial \mathcal{L}_F}{\partial A_\nu} \tag{16}$$

is the free 4-current.

For the standard Maxwell system (6) without free 4-current, the electric displacement tensor is the Faraday tensor, i.e., $\mathcal{D}^{\mu\nu} = F^{\mu\nu}$, and Eq. (12) reduces to Maxwell's equation

$$\partial_\mu F^{\mu\nu} = 0. \tag{17}$$

B. Infinitesimal criterion of symmetry and conservation laws

A continuous symmetry of the action $\mathcal{A}$ is a group of transformation

$$(x^\mu, A^\nu) \mapsto (\tilde{x}^\mu, \tilde{A}^\nu) = g_\epsilon \cdot (x^\mu, A^\nu), \tag{18}$$

such that

$$\int \mathcal{L}_F(\tilde{x}^\mu, \tilde{F}_{\mu\nu}, \tilde{D}\tilde{F}_{\mu\nu}, \ldots, \tilde{D}^{(n)}\tilde{F}_{\mu\nu}) d^4\tilde{x}$$
$$= \int \mathcal{L}_F(x^\mu, F_{\mu\nu}, DF_{\mu\nu}, \ldots, D^{(n)} F_{\mu\nu}) d^4x, \tag{19}$$

where g_ϵ constitutes a continuous group of the transformations parametrized by ϵ [23]. The infinitesimal generator of the transformation group is

$$\boldsymbol{v} := \frac{d}{d\epsilon}\bigg|_0 g_\epsilon \cdot (x^\mu, A^\nu) = \xi^\mu \frac{\partial}{\partial x^\mu} + \phi_\mu \frac{\partial}{\partial A_\mu}. \tag{20}$$

By rewriting the symmetry condition (19) as

$$\frac{d}{d\epsilon}\bigg|_0 \int \mathcal{L}_F(\tilde{x}^\mu, \tilde{F}_{\mu\nu}, \tilde{D}\tilde{F}_{\mu\nu}, \ldots, \tilde{D}^{(n)}\tilde{F}_{\mu\nu}) d^4\tilde{x} = 0, \tag{21}$$

we can derive the following infinitesimal version of the symmetry condition,

$$\mathrm{pr}^{(n+1)}\boldsymbol{v}(\mathcal{L}_F) + \mathcal{L}_F D_\mu \xi^\mu = 0, \tag{22}$$

where $\mathrm{pr}^{(n+1)}\boldsymbol{v}$ is the prolongation of $\boldsymbol{v}$. The standard prolongation formula for $\mathrm{pr}^{(n+1)}\boldsymbol{v}$ can be found in Ref. [23]. In the present study, we rewrite the prolongation formula with respect to $F_{\mu\nu}$, instead of A_μ, as

$$\mathrm{pr}^{(n+1)}\boldsymbol{v} = \boldsymbol{v} + [G_{\sigma\rho} + \xi^\sigma D_\sigma F_{\sigma\rho}] \frac{\partial \mathcal{L}_F}{\partial F_{\sigma\rho}}$$
$$+ \sum_{i=1}^{n} [D_{\mu_1} \cdots D_{\mu_i} G_{\sigma\rho} + \xi^\sigma D_\sigma D_{\mu_1} \cdots D_{\mu_i} F_{\sigma\rho}]$$
$$\times \frac{\partial \mathcal{L}_F}{\partial(\partial_{\mu_1} \cdots \partial_{\mu_i} F_{\sigma\rho})}, \tag{23}$$

where

$$G_{\sigma\rho} = \partial_\sigma Q_\rho - \partial_\rho Q_\sigma \equiv 2\partial_{[\sigma} Q_{\rho]} \tag{24}$$

and

$$Q_\nu = \phi_\nu - \xi^\sigma D_\sigma A_\nu, \tag{25}$$

is a characteristic of the Lie algebra.

Combining Eqs. (12) and (22) generates the conservation law corresponding to the symmetry,

$$D_\mu \left\{ \mathcal{L}_F \xi^\mu - \frac{1}{4\pi} \mathcal{D}^{\mu\nu}(\mathcal{L}_F) Q_\nu + \mathbb{P}^\mu \right\} = 0, \tag{26}$$

where

$$\mathbb{P}^\mu = \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^{j+1} (D_{\mu_{j+1}} \cdots D_{\mu_i} G_{\sigma\rho}) \times \left[D_{\mu_1} \cdots D_{\mu_{j-1}} \frac{\partial \mathcal{L}_F}{\partial(\partial_{\mu_1} \cdots \partial_{\mu_{j-1}} \partial_\mu \partial_{\mu_{j+1}} \cdots \partial_{\mu_i} F_{\sigma\rho})} \right]. \tag{27}$$

The conservation law given by Eq. (26) is not gauge-symmetric in general.

C. Gauge-symmetrization of the canonical EMT

Now we assume the high-order electromagnetic field theory admits the spacetime translation symmetry, i.e.,

$$\frac{\partial \mathcal{L}_F}{\partial x^\mu} = 0, \tag{28}$$

and derive the corresponding energy-momentum conservation law. Because of Eq. (28), the action is invariant under the spacetime translation

$$(x^\mu, A_\nu) \mapsto (x^\mu + \epsilon X_0^\mu, A_\nu), \tag{29}$$

where X_0^μ is 4D constant vector field. The infinitesimal generator $\boldsymbol{v}$, characteristic Q^ν, and $G_{\sigma\rho}$ in Eq. (24) are

$$\boldsymbol{v} = X_0^\mu \frac{\partial}{\partial x^\mu}, \tag{30}$$

$$Q^\nu = -X_0^\nu \partial_\nu A_\sigma, \tag{31}$$

$$G_{\sigma\rho} = -X_0^\nu \partial_\nu F_{\sigma\rho}. \tag{32}$$

The Lagrangian density satisfies the infinitesimal criterion because

$$X_0^\mu \frac{\partial \mathcal{L}_F}{\partial x^\mu} = 0, \tag{33}$$

which implies a conservation law. Substituting Eqs. (30)–(32) into Eq. (26), we obtain the canonical energy-momentum conservation law according the standard Noether procedure,

$$D_\mu T_{\mathrm{N}}^{\mu\nu} = 0, \tag{34}$$

$$T_{\mathrm{N}}^{\mu\nu} = \mathcal{L}_F \eta^{\mu\nu} + \frac{1}{4\pi} \mathcal{D}^{\mu\sigma} \partial^\nu A_\sigma - \Sigma^{\mu\nu}, \tag{35}$$

$$\Sigma^{\mu\nu} = \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^{j+1} (D_{\mu_{j+1}} \cdots D_{\mu_i} \partial^\nu F_{\sigma\rho}) \times \left[D_{\mu_1} \cdots D_{\mu_{j-1}} \frac{\partial \mathcal{L}_F}{\partial(\partial_{\mu_1} \cdots \partial_{\mu_{j-1}} \partial_\mu \partial_{\mu_{j+1}} \cdots \partial_{\mu_i} F_{\sigma\rho})} \right]. \tag{36}$$

In Eq. (34), $T_{\mathrm{N}}^{\mu\nu}$ is the canonical EMT derived from the standard Noether procedure.

Obviously, $T_{\mathrm{N}}^{\mu\nu}$ depends on the gauge as expected. In the expression of $T_{\mathrm{N}}^{\mu\nu}$ given by Eq. (35), the gauge dependence comes from the second term, and the first and third terms are gauge symmetric. Now we show how to gauge-symmetrize $T_{\mathrm{N}}^{\mu\nu}$. Note that because electric displacement tensor $\mathcal{D}^{\sigma\mu}$ is antisymmetric, the following equations hold,

$$D_\mu (D_\sigma \mathcal{F}^{\sigma\mu\nu}) = 0, \tag{37}$$

$$\mathcal{F}^{\sigma\mu\nu} \equiv \frac{1}{4\pi} \mathcal{D}^{\sigma\mu} A^\nu. \tag{38}$$

Here, $\mathcal{F}^{\sigma\mu\nu}$ is a superpotential that is antisymmetric with respect to the first two indices. For easy reference, we will call $\mathcal{F}^{\sigma\mu\nu}$ displacement-potential tensor. The divergence of $\mathcal{F}^{\sigma\mu\nu}$ defines a divergence-free tensor, i.e.,

$$T_0^{\mu\nu} \equiv D_\sigma \mathcal{F}^{\sigma\mu\nu} = -\frac{1}{4\pi} \mathcal{D}^{\mu\sigma} \partial_\sigma A^\nu, \tag{39}$$

where the field equation (12) have been used. When $T_0^{\mu\nu}$ is added to $T_{\mathrm{N}}^{\mu\nu}$, the gauge dependence is removed, i.e.,

$$D_\mu T_{\mathrm{GS}}^{\mu\nu} = 0, \tag{40}$$

$$T_{\mathrm{GS}}^{\mu\nu} \equiv T_{\mathrm{N}}^{\mu\nu} + T_0^{\mu\nu} = \mathcal{L}_F \eta^{\mu\nu} + \frac{1}{4\pi} \mathcal{D}^{\mu\sigma} F_\sigma^{\ \nu} - \Sigma^{\mu\nu}, \tag{41}$$

where $T_{\mathrm{GS}}^{\mu\nu}$ is the gauge-symmetric EMT.

It is worthwhile to mention that we derived the gauge-symmetrized EMT $T_{\mathrm{GS}}^{\mu\nu}$ from the expression of $T_{\mathrm{N}}^{\mu\nu}$ in Eq. (35), which is calculated from the prolongation with respect to $F_{\mu\nu}$. On the other hand, had we started from Eq. (3) and calculated the EMT from the prolongation with respect to A_μ, we would have obtained a canonical EMT in the form of

$$T_{\mathrm{N}}^{\mu\nu} = \mathcal{L}_F \eta^{\mu\nu} - \hat{\Sigma}^{\mu\nu}, \tag{42}$$

where

$$\hat{\Sigma}^{\mu\nu} = \sum_{i=1}^{n+1}\sum_{j=1}^{i}(-1)^{j+1}D_{\mu_{j+1}}\cdots D_{\mu_i}(\partial_\nu A_\sigma) \times\left[D_{\mu_1}\cdots D_{\mu_{j-1}}\frac{\partial\mathcal{L}_F}{\partial(\partial_{\mu_1}\cdots\partial_{\mu_{j-1}}\partial_\mu\partial_{\mu_{j+1}}\cdots\partial_{\mu_i}A_\sigma)}\right]. \tag{43}$$

However, different from the situation in Eq. (35), every term in Eq. (43) is gauge dependent, making the gauge symmetrization difficult, if not impossible.

For the standard Maxwell electromagnetic system specified by Eq. (6), the electric displacement tensor reduces to the Faraday tensor $F^{\sigma\mu}$, and the displacement-potential tensor $\mathcal{F}^{\sigma\mu\nu}$ reduces to $F^{\sigma\mu}A^\nu/4\pi$, coinciding with the tensor used by Blaschke *et al.* for the U(1) gauge theory [24].

D. Comparison with the BR method

As described above, the method proposed in the present study employs displacement-potential tensor $\mathcal{F}^{\sigma\mu\nu}$ to gauge-symmetrize the EMT, while the BR method use the superpotential $\mathcal{S}^{\sigma\mu\nu}$ to symmetrize the EMT. In this subsection, we discuss the difference between the displacement-potential tensor $\mathcal{F}^{\sigma\mu\nu}$ in Eq. (37) and the BR superpotential $\mathcal{S}^{\sigma\mu\nu}$. To calculate $\mathcal{S}^{\sigma\mu\nu}$, we need to first derive the 4D angular momentum conservation laws generated by the Lorentz symmetry. Assume that system is invariant under rotational transformation in 4D spacetime

$$(x^\mu, A^\nu)\mapsto(\tilde{x}^\mu,\tilde{A}^\nu) = (\Lambda_\epsilon^{\mu\sigma}x_\sigma, \Lambda_\epsilon^{\nu\sigma}A_\sigma), \tag{44}$$

where $\{\Lambda_\epsilon^{\mu\sigma}\}$ is one-parameter subgroup of the Lorentz group. The infinitesimal generator $\boldsymbol{v}$, the characteristic Q_ρ, and the term $G_{s\rho}$ are calculated respectively by Eqs. (20), (24) and (25) as

$$\boldsymbol{v} = \frac{d}{d\epsilon}\bigg|_0(\Lambda_\epsilon^{\mu\sigma}x_\sigma, \Lambda_\epsilon^{\nu\sigma}A_\sigma) = (\Omega^{\mu\sigma}x_\sigma, \Omega^{\mu\sigma}A_\sigma), \tag{45}$$

$$Q_\rho = \phi_\rho - \xi^\alpha D_\alpha A_\rho = \Omega_{\rho\alpha}A^\alpha - \Omega_{\alpha\beta}x^\beta D^\alpha A_\rho, \tag{46}$$

$$G_{s\rho} = \Omega_{\rho\alpha}F_s^\alpha - \Omega_{s\alpha}F_\rho^\alpha - \Omega_{\alpha\beta}x^\beta\partial^\alpha F_{s\rho}, \tag{47}$$

where the antisymmetric tensor $\Omega^{\mu\sigma} = [d\Lambda_\epsilon^{\mu\sigma}/d\epsilon]_0$ is the Lie algebra element of the Lorentz group. Substituting Eqs. (45)–(47) into Eq. (26), we obtain the angular momentum conservation law in 4D spacetime,

$$\Omega_{\nu\sigma}D_\mu\{x^\sigma T_{\mathrm{N}}^{\mu\nu} + 2E_F^{[\mu\nu]}(\mathcal{L}_F)A^\sigma + L^{\mu\nu\sigma}\} = 0, \tag{48}$$

which can be rewritten as

$$D_\mu\mathcal{M}^{\mu\nu\sigma} = 0, \tag{49}$$

where

$$\mathcal{M}^{\mu\nu\sigma} = x^\sigma T_{\mathrm{N}}^{\mu\nu} - x^\nu T_{\mathrm{N}}^{\mu\sigma} + S^{\mu\nu\sigma} \tag{50}$$

is the canonical angular momentum tensor. In the above equations,

$$L^{\mu\nu\sigma} = \sum_{i=1}^{n}\sum_{j=1}^{i}(-1)^{j+1}D_{\mu_{j+1}}\cdots D_{\mu_i}(F_\rho^\sigma)D_{\mu_1}\cdots D_{\mu_{j-1}} \times\left[\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\cdots\partial_{\mu_{j-1}}\partial_\mu\partial_{\mu_{j+1}}\cdots\partial_{\mu_i}F_{\rho\nu})} - \frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\cdots\partial_{\mu_{j-1}}\partial_\mu\partial_{\mu_{j+1}}\cdots\partial_{\mu_i}F_{\nu\rho})}\right] - \left[\sum_{i=1}^{n}\sum_{j=1}^{i}(-1)^{j+1}D_{\mu_{j+1}}\cdots D_{\mu_i}(x^\sigma\partial^\nu F_{s\rho})D_{\mu_1}\cdots D_{\mu_{j-1}}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\cdots\partial_{\mu_{j-1}}\partial_\mu\partial_{\mu_{j+1}}\cdots\partial_{\mu_i}F_{s\rho})} - x^\sigma\Sigma^{\mu\nu}\right] \tag{51}$$

$$\frac{1}{2}S^{\mu\nu\sigma} = E_F^{[\mu\nu]}(\mathcal{L}_F)A^\sigma - E_F^{[\mu\sigma]}(\mathcal{L}_F)A^\nu + \Delta^{\mu\nu\sigma}, \tag{52}$$

$$\begin{aligned}\Delta^{\mu\nu\sigma} &\equiv 2L^{\mu[\nu\sigma]}\\ &= \sum_{i=1}^{n}\sum_{j=1}^{i}(-1)^{j+1}D_{\mu_{j+1}}\cdots D_{\mu_i}\bigg\{(F_\rho^\sigma)D_{\mu_1}\cdots D_{\mu_{j-1}}\left[\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\cdots\partial_{\mu_{j-1}}\partial_\mu\partial_{\mu_{j+1}}\cdots\partial_{\mu_i}F_{\rho\nu})} - \frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\cdots\partial_{\mu_{j-1}}\partial_\mu\partial_{\mu_{j+1}}\cdots\partial_{\mu_i}F_{\nu\rho})}\right]\\ &\quad - (F_\rho^\nu)D_{\mu_1}\cdots D_{\mu_{j-1}}\left[\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\cdots\partial_{\mu_{j-1}}\partial_\mu\partial_{\mu_{j+1}}\cdots\partial_{\mu_i}F_{\rho\sigma})} - \frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\cdots\partial_{\mu_{j-1}}\partial_\mu\partial_{\mu_{j+1}}\cdots\partial_{\mu_i}F_{\sigma\rho})}\right]\bigg\}\\ &\quad - \sum_{i=1}^{n}\sum_{j=1}^{i}(-1)^{j+1}D_{\mu_{j+1}}\cdots D_{\mu_i}(x^\sigma\partial^\nu F_{s\rho} - x^\nu\partial^\sigma F_{s\rho})D_{\mu_1}\cdots D_{\mu_{j-1}}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu_1}\cdots\partial_{\mu_{j-1}}\partial_\mu\partial_{\mu_{j+1}}\cdots\partial_{\mu_i}F_{s\rho})} + x^\sigma\Sigma^{\mu\nu} - x^\nu\Sigma^{\mu\sigma},\end{aligned} \tag{53}$$

where the superscript $[\mu\nu]$ denotes antisymmetrization with respect to μ and ν.

The BR superpotential $\mathcal{S}^{\sigma\mu\nu}$ is defined from the tensor $S^{\sigma\mu\nu}$ in Eq. (49) as [8–10]

$$\mathcal{S}^{\sigma\mu\nu} \equiv \frac{1}{2}[S^{\sigma\nu\mu} - S^{\mu\nu\sigma} - S^{\nu\mu\sigma}]. \tag{54}$$

It is clear from Eqs. (51)–(53) that $\mathcal{S}^{\sigma\mu\nu}$ and $\mathcal{F}^{\sigma\mu\nu}$ are related as follows,

$$\mathcal{S}^{\sigma\mu\nu} = \mathcal{F}^{\sigma\mu\nu} + \frac{1}{2}[\Delta^{\sigma\nu\mu} - \Delta^{\mu\nu\sigma} - \Delta^{\nu\mu\sigma}]. \tag{55}$$

In the BR theory, the EMT $T^{\mu\nu}_{\mathrm{BR}}$ defined in Eq. (1) is symmetric with respect to μ and ν because

$$T^{[\mu\nu]}_{\mathrm{BR}} = T^{[\mu\nu]}_{\mathrm{N}} + \partial_\sigma \mathcal{S}^{\sigma[\mu\nu]} = 0, \tag{56}$$

where we have made use of

$$\mathcal{S}^{\sigma[\mu\nu]} = -\frac{1}{2}S^{\sigma\mu\nu}, \qquad T^{[\mu\nu]}_{\mathrm{N}} = \frac{1}{2}\partial_\sigma S^{\sigma\mu\nu}, \tag{57}$$

which can be easily derived from Eqs. (49) and (54). For the classical electromagnetic systems discussed in the present study, $T^{\mu\nu}_{\mathrm{BR}}$ is also gauge symmetric. However, this fact is difficult to establish directly from the definition of $T^{\mu\nu}_{\mathrm{BR}}$ through Eqs. (1), (42), (43), (52) and (53). But it can easily proved from the formalism we developed as follows. Because the tensor $\mathbf{\Delta}$ in Eq. (55) is gauge symmetric, we know that the BR EMT $T^{\mu\nu}_{\mathrm{BR}}$ is gauge symmetric from the fact that $T^{\mu\nu}_{\mathrm{GS}}$ is gauge symmetric and connection between $T^{\mu\nu}_{\mathrm{BR}}$ and $T^{\mu\nu}_{\mathrm{GS}}$ via Eqs. (1), (39), (41), and (55). Our theory clarifies which term implicitly contained in the BR procedure is responsible for the gauge symmetrization, and thus leads to a simpler gauge-symmetrization method. Specifically, it is the displacement-potential tensor $\mathcal{F}^{\sigma\mu\nu}$ that removes the gauge dependence, and the gauge-symmetrization process can be made simpler without using the canonical angular momentum tensor $\mathcal{M}^{\mu\nu\sigma}$ as in the BR method.

Although the BR method always gauge-symmetrizes the classical electromagnetic systems, it fails for systems in which quantum matter fields minimally couple to gauge fields, such as the Proca system. This fact was first studied by Blaschke *et al.* [24] and a gauge-symmetric method for a lowest order field theory was developed. However, for high-order gauge fields, a general method has not been established to date. The result in the present study suggests a possible approach. We can first reformulate the equation of motion with respect to the gauge-strength tensor $F^a_{\mu\nu}$, and convert the particle derivatives for matter fields to covariant derivatives. This will separate the canonical EMT into gauge-symmetric and gauge-dependent parts, and the gauge-symmetrization procedure developed here might be applicable. Since the present study is focused on classical systems, the corresponding method for general quantum field systems will be investigated in the future.

Equation (55) shows that in general $\mathcal{F}^{\sigma\mu\nu}$ is different from $\mathcal{S}^{\sigma\mu\nu}$ when $\Delta^{\sigma\nu\mu}$ is nonvanishing. For a first-order field theory, such as the standard Maxwell system (6), $n = 1$ and the last three terms vanish such that $\mathcal{S}^{\sigma\mu\nu} = \mathcal{F}^{\sigma\mu\nu}$. In this situation, adding $T^{\mu\nu}_0 \equiv D_\sigma \mathcal{F}^{\sigma\mu\nu}$ to $T^{\mu\nu}_{\mathrm{N}}$ will render it both symmetric and gauge-symmetric, and the method developed here can be used as a simpler procedure to calculate the BR superpotential $\mathcal{S}^{\sigma\mu\nu}$ without the necessity to calculate the angular momentum tensor in 4D spacetime.

E. EMT for Podolsky system

As an example of high-order electromagnetic field theory, we consider the Podolsky system [21,22], which was proposed to study the radiation reaction of classical charged particles. The Podolsky Lagrangian density is

$$\mathcal{L}_{\mathrm{Po}} = \frac{1}{8\pi}\left\{\boldsymbol{E}^2 - \boldsymbol{B}^2 + a^2\left[(\nabla\cdot\boldsymbol{E})^2 - \left(\nabla\times\boldsymbol{B} - \frac{1}{c}\partial_t\boldsymbol{E}\right)^2\right]\right\} \tag{58}$$

or in a manifestly covariant form

$$\mathcal{L}_{\mathrm{Po}} = -\frac{1}{16\pi}F_{\sigma\rho}F^{\sigma\rho} - \frac{a^2}{8\pi}\partial_\sigma F^{\sigma\lambda}\partial^\rho F_{\rho\lambda}. \tag{59}$$

The field equation for this system can be easily obtained using Eq. (12) as [22]

$$(1 - a^2\partial_\sigma\partial^\sigma)\partial_\mu F^{\mu\nu} = 0.$$

We substitute the Lagrangian density (59) into Eq. (35) to obtain the canonical EMT

$$\begin{aligned}4\pi T^{\mu\nu}_{\mathrm{N}} = &\left(-\frac{1}{4}F_{\sigma\rho}F^{\sigma\rho} - \frac{a^2}{2}\partial_\sigma F^{\sigma\lambda}\partial^\rho F_{\rho\lambda}\right)\eta^{\mu\nu} \\ &+ [F^{\mu\sigma} - a^2(\partial^\mu\partial_\lambda F^{\lambda\sigma} - \partial^\sigma\partial_\lambda F^{\lambda\mu})]\partial^\nu A_\sigma \\ &+ a^2(\partial^\nu F^\mu_\rho)(\partial_\sigma F^{\sigma\rho}),\end{aligned} \tag{60}$$

where the following equations are used,

$$\frac{\partial}{\partial(\partial_\sigma F_{\mu\nu})}[\partial_\alpha F^{\alpha\lambda}\partial^\rho F_{\rho\lambda}] = 2\eta^{\sigma\mu}\partial_\lambda F^{\lambda\nu}, \tag{61}$$

$$D_\sigma\frac{\partial}{\partial(\partial_\sigma F_{\mu\nu})}[\partial_\alpha F^{\alpha\lambda}\partial^\rho F_{\rho\lambda}] = 2\partial^\mu\partial_\sigma F^{\sigma\nu}, \tag{62}$$

$$E^{\mu\sigma}_F = \frac{\partial\mathcal{L}_{\mathrm{Po}}}{\partial F_{\mu\sigma}} - D_\rho\frac{\partial\mathcal{L}_{\mathrm{Po}}}{\partial\partial_\rho F_{\mu\sigma}} = -\frac{1}{8\pi}F^{\mu\sigma} + \frac{a^2}{4\pi}\partial^\mu\partial_\lambda F^{\lambda\sigma}, \tag{63}$$

$$2E_F^{[\mu\sigma]} = -\frac{1}{4\pi}F^{\mu\sigma} + \frac{a^2}{4\pi}(\partial^\mu\partial_\lambda F^{\lambda\sigma} - \partial^\sigma\partial_\lambda F^{\lambda\mu}), \tag{64}$$

$$\Sigma^{\mu\nu} = (\partial^\nu F_{\sigma\rho})\frac{\partial\mathcal{L}_{\mathrm{Po}}}{\partial(\partial_\mu F_{\sigma\rho})} = -\frac{a^2}{4\pi}\partial^\nu F_{\sigma\rho}[\eta^{\mu\sigma}\partial_\lambda F^{\lambda\rho}]$$
$$= -\frac{a^2}{4\pi}(\partial^\nu F^\mu_\rho)(\partial_\sigma F^{\sigma\rho}). \tag{65}$$

The displacement-potential tensor is

$$\mathcal{F}^{\mu\nu\sigma} \equiv \frac{1}{4\pi}\mathcal{D}^{\sigma\mu}A^\nu$$
$$= \frac{1}{4\pi}[-F^{\mu\sigma} + a^2(\partial^\mu\partial_\lambda F^{\lambda\sigma} - \partial^\sigma\partial_\lambda F^{\lambda\mu})]A^\nu, \tag{66}$$

and

$$4\pi\partial_\sigma\mathcal{F}^{\mu\nu\sigma} = [-F^{\mu\sigma} + a^2(\partial^\mu\partial_\lambda F^{\lambda\sigma} - \partial^\sigma\partial_\lambda F^{\lambda\mu})]\partial_\sigma A^\nu. \tag{67}$$

Adding Eq. (67) to Eq. (60), we obtain the gauge-symmetric EMT,

$$4\pi T^{\mu\nu}_{\mathrm{GS}} = \left[F^{\mu\sigma}F^\nu_\sigma - \frac{1}{4}(F_{\sigma\rho}F^{\sigma\rho})\eta^{\mu\nu}\right] - \frac{a^2}{2}(\partial_\sigma F^{\sigma\lambda}\partial^\rho F_{\rho\lambda})\eta^{\mu\nu}$$
$$- a^2F^\nu_\sigma(\partial^\mu\partial_\rho F^{\rho\sigma}) + a^2F^\nu_\sigma(\partial^\sigma\partial_\rho F^{\rho\mu})$$
$$+ a^2(\partial^\nu F^\mu_\rho)(\partial_\sigma F^{\sigma\rho}). \tag{68}$$

It is easy to see that $T^{\mu\nu}_{\mathrm{GS}}$ for the Podolsky system is not symmetric, i.e., $T^{\mu\nu}_{\mathrm{GS}} \neq T^{\nu\mu}_{\mathrm{GS}}$.

To calculate the BR EMT $T^{\mu\nu}_{\mathrm{BR}}$ for the Podolsky system, we evaluate the $\Delta^{\mu\nu\sigma}$ term in Eq. (54). Using Eq. (53), we have

$$\Delta^{\mu\nu\sigma} = F^\sigma_\rho\left[\frac{\partial\mathcal{L}_{\mathrm{Po}}}{\partial(\partial_\mu F_{\rho\nu})} - \frac{\partial\mathcal{L}_{\mathrm{Po}}}{\partial(\partial_\mu F_{\nu\rho})}\right] - F^\nu_\rho\left[\frac{\partial\mathcal{L}_{\mathrm{Po}}}{\partial(\partial_\mu F_{\rho\sigma})} - \frac{\partial\mathcal{L}_{\mathrm{Po}}}{\partial(\partial_\mu F_{\sigma\rho})}\right]$$
$$= -\frac{a^2}{4\pi}[F^{\mu\sigma}\partial_\rho F^{\rho\nu} - \eta^{\mu\nu}(F^\sigma_\rho\partial_\lambda F^{\lambda\rho}) - F^{\mu\nu}\partial_\rho F^{\rho\sigma} + \eta^{\mu\sigma}F^\nu_\rho(\partial_\lambda F^{\lambda\rho})]. \tag{69}$$

Substituting Eqs. (54) and (69) into Eq. (1), we obtain the BR EMT as

$$4\pi T^{\mu\nu}_{\mathrm{BR}} = 4\pi T^{\mu\nu}_{\mathrm{GS}} + 2\pi[\Delta^{\sigma\nu\mu} - \Delta^{\mu\nu\sigma} - \Delta^{\nu\mu\sigma}]$$
$$= \left[F^{\mu\sigma}F^\nu_\sigma - \frac{1}{4}(F_{\sigma\rho}F^{\sigma\rho})\eta^{\mu\nu}\right] + \frac{a^2}{2}[(\partial_\sigma F^{\sigma\rho})(\partial^\lambda F_{\lambda\rho}) - 2F^\sigma_\rho(\partial_\sigma\partial_\lambda F^{\lambda\rho})]\eta^{\mu\nu}$$
$$+ a^2[F^{\nu\sigma}(\partial_\sigma\partial_\rho F^{\rho\mu}) + F^{\mu\sigma}(\partial_\sigma\partial_\rho F^{\rho\nu}) - (\partial_\sigma F^{\sigma\mu})(\partial_\rho F^{\rho\nu}) - F^\nu_\sigma(\partial^\mu\partial_\rho F^{\rho\sigma}) - F^\mu_\sigma(\partial^\nu\partial_\rho F^{\rho\sigma})]. \tag{70}$$

It is easy to verify that $T^{\mu\nu}_{\mathrm{BR}}$ for the Podolsky system is both symmetric and gauge-symmetric.

III. GAUGE-SYMMETRIC EMTs FOR ELECTROMAGNETIC SYSTEMS COUPLED WITH CLASSICAL CHARGED PARTICLES

For self-consistent electromagnetic systems with free currents, the electromagnetic fields are coupled with charged particles. In this section, we apply the theory established in Sec. II to derive gauge-symmetric EMTs for electromagnetic systems coupled with classical charged particles.

Due to the intrinsic complexity of the dynamics for interaction systems, reduced theoretical models, such as the gyrokinetic models [18–20] for magnetized plasmas, are often adopted. The Lagrangian densities of these systems may not be relativistic covariant and are usually given by "3 + 1" splitting forms. These systems are not written in manifestly covariant forms. However, the equations of motion for the systems are usually gauge invariant. Consequently, for these systems, energy and momentum conservation laws (with respect to split time and space translation symmetries) need to be derived separately. In this section, we demonstrate how the energy and momentum conservation laws can be transformed into gauge-symmetric forms using the "3 + 1" form of Eq. (37), i.e.,

$$\frac{D}{Dt}\left\{\frac{D}{D\boldsymbol{x}}\cdot[\boldsymbol{E_E}(\mathcal{L})\varphi]\right\} + \frac{D}{D\boldsymbol{x}}\cdot\left\{\frac{D}{Dt}[-\boldsymbol{E_E}(\mathcal{L})\varphi]\right\} = 0 \tag{71}$$

and

$$\frac{D}{Dt}\left\{\frac{D}{D\boldsymbol{x}}\cdot\left[-\frac{1}{c}\boldsymbol{E_E}(\mathcal{L})\boldsymbol{A}\right]\right\} + \frac{D}{D\boldsymbol{x}}\cdot\left\{\frac{D}{Dt}\left[\frac{1}{c}\boldsymbol{E_E}(\mathcal{L})\boldsymbol{A}\right]\right\} = 0. \tag{72}$$

A. Weak Euler-Lagrange equation and conservation law

The Lagrangian density of a generic classical electromagnetic field-charge particle system assumes the form of

$$\mathcal{L} = \sum_a \mathcal{L}_a + \mathcal{L}_F, \tag{73}$$

$$\mathcal{L}_a = L_a \delta_a,$$
$$L_a = L_a(x^\mu, \boldsymbol{X}_a, \dot{\boldsymbol{X}}_a; \varphi, \boldsymbol{A}, \boldsymbol{E}, \boldsymbol{B}, D\boldsymbol{E}, D\boldsymbol{B}, \ldots, D^n\boldsymbol{E}, D^n\boldsymbol{B}) \quad (74)$$

where the subscript a labels particles, $\boldsymbol{X}_a$ is its trajectory, $\mathcal{L}_a$ is its Lagrangian density, and $\delta_a \equiv \delta(\boldsymbol{x} - \boldsymbol{X}_a)$. Here, $\delta(x)$ is the Dirac δ-function.

In the "$3+1$" form, the equations of motion for the electromagnetic field are

$$\nabla \cdot \boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L}) = -\frac{\partial \mathcal{L}}{\partial \varphi}, \quad (75)$$

$$-\frac{1}{c}\frac{\partial}{\partial t}[\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})] - \boldsymbol{\nabla} \times [\boldsymbol{E}_{\boldsymbol{B}}(\mathcal{L})] = \frac{\partial \mathcal{L}}{\partial \boldsymbol{A}}. \quad (76)$$

In this study, it is assumed that the particle's trajectory is minimally couple with φ and $\boldsymbol{A}$, and Eqs. (75) and (76) are thus gauge symmetric. Specifically, we assume that $\mathcal{L}$ depends on φ and $\boldsymbol{A}$ only through the term $-q_a\delta_a(\varphi + \boldsymbol{A}\cdot\dot{\boldsymbol{X}}_a/c)$, i.e., the Lagrangian density can be written as

$$\mathcal{L} = \sum_a q_a \delta_a \left[-\varphi + \frac{1}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{X}}_a\right] + \mathrm{GSP}(\mathcal{L}), \quad (77)$$

where "GSP($\mathcal{L}$)" denotes the gauge-symmetric parts of the Lagrangian density $\mathcal{L}$. The right-hand side of Eqs. (75) and (76) are the "$3+1$" form of Eq. (16), the free charge density ρ_f and current density $\boldsymbol{j}_f$, respectively. Using Eq. (77), we have

$$\rho_f = -\frac{\partial \mathcal{L}}{\partial \varphi} = \sum_a q_a \delta_a,$$
$$\boldsymbol{j}_f = c\frac{\partial \mathcal{L}}{\partial \boldsymbol{A}} = \sum_a q_a \dot{\boldsymbol{X}}_a \delta_a. \quad (78)$$

The equation of motion for particles is also derived from the variational principle. However, because particles and field reside on different manifolds, the equation of motion for particles will be the weak EL equation [20,25–27]

$$\boldsymbol{E}_{\boldsymbol{X}_a}(\mathcal{L}) = \frac{D}{D\boldsymbol{x}} \cdot \left(\dot{\boldsymbol{X}}_a \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{X}}_a} - \mathcal{L}_a \boldsymbol{I}\right), \quad (79)$$

where $\boldsymbol{E}_{\boldsymbol{X}_a}$ is the Euler operator for the trajectory of the ath particle,

$$\boldsymbol{E}_{\boldsymbol{X}_a} = \frac{\partial}{\partial \boldsymbol{X}_a} - \frac{d}{dt}\frac{\partial}{\partial \dot{\boldsymbol{X}}_a}. \quad (80)$$

To derive a local conservation law from a symmetry, we need the infinitesimal symmetry criterion for the Lagrangian density. A symmetry of the action $\mathcal{A} \equiv \int \mathcal{L} dt d^3\boldsymbol{x}$ is defined by group transformations

$$(x^\mu, \boldsymbol{X}_a; \varphi, \boldsymbol{A}) \mapsto (\tilde{x}^\mu, \tilde{\boldsymbol{X}}_a; \tilde{\varphi}, \tilde{\boldsymbol{A}}) = g_\epsilon \cdot (x^\mu, \boldsymbol{X}_a; \varphi, \boldsymbol{A}), \quad (81)$$

such that

$$\int \mathcal{L}(\tilde{x}^\mu, \tilde{\boldsymbol{X}}_a, \tilde{\boldsymbol{E}}, \tilde{\boldsymbol{B}}, \ldots, \tilde{D}^n\tilde{\boldsymbol{E}}, \tilde{D}^n\tilde{\boldsymbol{B}}) d\tilde{t} d^3\tilde{\boldsymbol{x}}$$
$$= \int \mathcal{L}(x^\mu, \boldsymbol{X}_a, \boldsymbol{E}, \boldsymbol{B}, \ldots, D^{(n)}\boldsymbol{E}, D^{(n)}\boldsymbol{B}) dt d^3\boldsymbol{x}. \quad (82)$$

The corresponding infinitesimal generator of (81) is

$$\boldsymbol{v} = \xi^\mu \frac{\partial}{\partial x^\mu} + \sum_a \boldsymbol{\theta}_a \cdot \frac{\partial}{\partial \boldsymbol{X}_a} + \phi_0 \frac{\partial}{\partial \varphi} + \boldsymbol{\phi}_{\boldsymbol{A}} \cdot \frac{\partial}{\partial \boldsymbol{A}}. \quad (83)$$

The infinitesimal criterion of the symmetry condition can be derived using the same procedure in Sec. II B,

$$\mathrm{pr}^{(n+1)}\boldsymbol{v}(\mathcal{L}) + \mathcal{L} D_\mu \xi^\mu = 0. \quad (84)$$

The prolongation of $\boldsymbol{v}$ now reads

$$\begin{aligned}
\mathrm{pr}^{(n+1)}\boldsymbol{v}(\mathcal{L}) = \boldsymbol{v} &+ \sum_a \left[(\dot{\boldsymbol{q}}_a + \xi^t \ddot{\boldsymbol{X}}_a) \cdot \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{X}}_a}\right] - [\boldsymbol{\nabla} Q_0 + \xi^\mu D_\mu(\boldsymbol{\nabla}\varphi)] \cdot \frac{\partial \mathcal{L}}{\partial \boldsymbol{E}} \\
&- \cdots - \sum_{i=1}^n [D_{\mu_1} \cdots D_{\mu_i} \boldsymbol{\nabla} Q_{\alpha 0} + \xi^\mu D_\mu D_{\mu_1} \cdots D_{\mu_i}(\boldsymbol{\nabla}\varphi_\alpha)] \cdot \frac{\partial \mathcal{L}}{\partial D_{\mu_1} \cdots D_{\mu_i} \boldsymbol{E}} - [D_t \boldsymbol{Q}_{\boldsymbol{A}} + \xi^\mu D_\mu \boldsymbol{A}_{,t}] \cdot \left(\frac{1}{c}\frac{\partial \mathcal{L}}{\partial \boldsymbol{E}}\right) \\
&- \cdots - \sum_{i=1}^n [D_{\mu_1} \cdots D_{\mu_i} D_t \boldsymbol{Q}_{\boldsymbol{A}} + \xi^\mu D_\mu D_{\mu_1} \cdots D_{\mu_i} \boldsymbol{A}_{,t}] \cdot \frac{\partial \mathcal{L}}{\partial D_{\mu_1} \cdots D_{\mu_i} \boldsymbol{E}} + [\boldsymbol{\nabla} \boldsymbol{Q}_{\boldsymbol{A}} + \xi^\mu D_\mu \boldsymbol{\nabla} \boldsymbol{A}] : \left(\boldsymbol{\varepsilon} \cdot \frac{\partial \mathcal{L}}{\partial \boldsymbol{B}}\right) \\
&+ \cdots + \sum_{i=1}^n [D_{\mu_1} \cdots D_{\mu_i} \boldsymbol{\nabla} \boldsymbol{Q}_{\boldsymbol{A}} + \xi^\mu D_\mu D_{\mu_1} \cdots D_{\mu_i} \boldsymbol{\nabla} \boldsymbol{A}] : \left(\boldsymbol{\varepsilon} \cdot \frac{\partial \mathcal{L}}{\partial D_{\mu_1} \cdots D_{\mu_i} \boldsymbol{B}}\right), \quad (85)
\end{aligned}$$

where

$$\boldsymbol{q}_a = \boldsymbol{\theta}_a - \xi^t \dot{\boldsymbol{X}}_a \tag{86}$$

is another characteristic of $\boldsymbol{v}$ induced by particle's trajectory. To obtain the corresponding conservation law, we transform the infinitesimal criterion into

$$\begin{aligned}&\partial_t\left[\mathcal{L}\xi^t - \frac{1}{c}\boldsymbol{Q}_{\boldsymbol{A}}\cdot\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L}) + \sum_a\left(\boldsymbol{q}_a\cdot\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{X}}_a}\right)\right]\\&\quad+\nabla\cdot[\mathcal{L}\boldsymbol{\kappa} - Q_0\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L}) + \boldsymbol{Q}_{\boldsymbol{A}}\times\boldsymbol{E}_{\boldsymbol{B}}(\mathcal{L})] + D_\mu[\mathbb{P}_1^\mu + \mathbb{P}_2^\mu]\\&\quad+\sum_a[\boldsymbol{q}_a\cdot\boldsymbol{E}_{\boldsymbol{X}_a}(\mathcal{L})] + \left\{\frac{\partial\mathcal{L}}{\partial\varphi} + \nabla\cdot[\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})]\right\}Q_0\\&\quad+\left\{\frac{\partial\mathcal{L}}{\partial\boldsymbol{A}} + \frac{1}{c}\partial_t[\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})] + [\nabla\times\boldsymbol{E}_{\boldsymbol{B}}(\mathcal{L})]\right\}\cdot\boldsymbol{Q}_{\boldsymbol{A}} = 0,\end{aligned} \tag{87}$$

where

$$\begin{aligned}\mathbb{P}_1^\mu = &\sum_{i=1}^n\sum_{j=1}^i(-1)^j D_{\mu_{j+1}}\cdots D_{\mu_i}\left(\nabla Q_0 + \frac{1}{c}D_t\boldsymbol{Q}_{\boldsymbol{A}}\right)\\&\cdot\left[D_{\mu_1}\cdots D_{\mu_{j-1}}\frac{\partial\mathcal{L}}{\partial D_{\mu_1}\cdots D_{\mu_{j-1}}D_\mu D_{\mu_{j+1}}\cdots D_{\mu_i}\boldsymbol{E}}\right],\end{aligned} \tag{88}$$

$$\begin{aligned}\mathbb{P}_2^\mu = &\sum_{i=1}^n\sum_{j=1}^i(-1)^j D_{\mu_{j+1}}\cdots D_{\mu_i}(-\nabla\times\boldsymbol{Q}_{\boldsymbol{A}})\\&\cdot\left[D_{\mu_1}\cdots D_{\mu_{j-1}}\frac{\partial\mathcal{L}}{\partial D_{\mu_1}\cdots D_{\mu_{j-1}}D_\mu D_{\mu_{j+1}}\cdots D_{\mu_i}\boldsymbol{B}}\right].\end{aligned} \tag{89}$$

The last two terms on the left-hand side of Eq. (87) vanish due to Eqs. (75) and (76), but the fourth term does not because of the weak EL equation (79). If the characteristic $\boldsymbol{q}_a$ is independent of $\boldsymbol{x}$, $\boldsymbol{E}$, and $\boldsymbol{B}$, the conservation law of the symmetry is established as

$$\begin{aligned}&\partial_t\left[\mathcal{L}\xi^t - \frac{1}{c}\boldsymbol{Q}_{\boldsymbol{A}}\cdot\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L}) + \sum_a\left(\boldsymbol{q}_a\cdot\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{X}}_a}\right)\right]\\&\quad+\nabla\cdot\left[\mathcal{L}\boldsymbol{\kappa} - Q_0\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L}) + \boldsymbol{Q}_{\boldsymbol{A}}\times\boldsymbol{E}_{\boldsymbol{B}}(\mathcal{L})\right.\\&\quad\left.+\sum_a\left(\dot{\boldsymbol{X}}_a\frac{\partial\mathcal{L}_a}{\partial\dot{\boldsymbol{X}}_a} - \mathcal{L}_a\boldsymbol{I}\right)\cdot\boldsymbol{q}_a\right] + D_\mu[\mathbb{P}_1^\mu + \mathbb{P}_2^\mu] = 0.\end{aligned} \tag{90}$$

B. Gauge-symmetric energy conservation law

We first derive the gauge-symmetric energy conservation law, assuming that the action $\mathcal{A}\equiv\int\mathcal{L}dtd^3\boldsymbol{x}$ is unchanged under the time translation

$$(t,\boldsymbol{x},\boldsymbol{X}_a,\varphi,\boldsymbol{A})\mapsto(t+\epsilon,\boldsymbol{x},\boldsymbol{X}_a,\varphi,\boldsymbol{A}),\qquad \epsilon\in\mathbb{R}. \tag{91}$$

The infinitesimal generator and characteristic are calculated as

$$\boldsymbol{v} = \frac{\partial}{\partial t},\quad \xi^t = 1,\quad \boldsymbol{\kappa} = 0,\quad \boldsymbol{\theta}_a = 0,\quad \phi_0 = \boldsymbol{\phi}_{\boldsymbol{A}} = 0, \tag{92}$$

$$\boldsymbol{q}_a = -\dot{\boldsymbol{X}}_a,\qquad Q_0 = -\varphi_{,t},\qquad \boldsymbol{Q}_{\boldsymbol{A}} = -\boldsymbol{A}_{,t}. \tag{93}$$

And the infinitesimal criterion (84) of the symmetry is

$$\frac{\partial\mathcal{L}}{\partial t} = 0. \tag{94}$$

The corresponding energy conservation law is thus

$$\begin{aligned}&\partial_t\left[\mathcal{L} + \frac{1}{c}\boldsymbol{A}_{,t}\cdot\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L}) - \sum_a\left(\dot{\boldsymbol{X}}_a\cdot\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{X}}_a}\right)\right]\\&\quad+\nabla\cdot\left\{[\varphi_{,t}\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L}) - \boldsymbol{A}_{,t}\times\boldsymbol{E}_{\boldsymbol{B}}(\mathcal{L})]\right.\\&\quad\left.-\sum_a\left(\dot{\boldsymbol{X}}_a\frac{\partial\mathcal{L}_a}{\partial\dot{\boldsymbol{X}}_a} - \mathcal{L}_a\boldsymbol{I}\right)\cdot\dot{\boldsymbol{X}}_a\right\} + D_\mu[\mathbb{P}_1^\mu + \mathbb{P}_2^\mu] = 0,\end{aligned} \tag{95}$$

where

$$\begin{aligned}\mathbb{P}_1^\mu = &\sum_{i=1}^n\sum_{j=1}^i(-1)^j D_{\mu_{j+1}}\cdots D_{\mu_i}\partial_t\boldsymbol{E}\\&\cdot\left[D_{\mu_1}\cdots D_{\mu_{j-1}}\frac{\partial\mathcal{L}}{\partial D_{\mu_1}\cdots D_{\mu_{j-1}}D_\mu D_{\mu_{j+1}}\cdots D_{\mu_i}\boldsymbol{E}}\right],\end{aligned} \tag{96}$$

$$\begin{aligned}\mathbb{P}_2^\mu = &\sum_{i=1}^n\sum_{j=1}^i(-1)^j D_{\mu_{j+1}}\cdots D_{\mu_i}\partial_t\boldsymbol{B}\\&\cdot\left[D_{\mu_1}\cdots D_{\mu_{j-1}}\frac{\partial\mathcal{L}}{\partial D_{\mu_1}\cdots D_{\mu_{j-1}}D_\mu D_{\mu_{j+1}}\cdots D_{\mu_i}\boldsymbol{B}}\right].\end{aligned} \tag{97}$$

The energy density and flux in Eq. (95) are obviously gauge dependent. To gauge-symmetrize the conservation law, we add Eq. (71) to Eq. (95) and obtain,

$$\begin{aligned}&\partial_t\left[\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\varphi}\varphi - \sum_a\left(\dot{\boldsymbol{X}}_a\cdot\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{X}}_a}\right) - \boldsymbol{E}\cdot\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\right]\\&\quad+\nabla\cdot\left\{c\boldsymbol{E}\times\boldsymbol{E}_{\boldsymbol{B}}(\mathcal{L}) + c\frac{\partial\mathcal{L}}{\partial\boldsymbol{A}}\varphi - \sum_a\left(\dot{\boldsymbol{X}}_a\frac{\partial\mathcal{L}_a}{\partial\dot{\boldsymbol{X}}_a} - \mathcal{L}_a\boldsymbol{I}\right)\cdot\dot{\boldsymbol{X}}_a\right\}\\&\quad+D_\mu[\mathbb{P}_1^\mu + \mathbb{P}_2^\mu] = 0.\end{aligned} \tag{98}$$

In deriving Eq. (98), we have rewritten the first and second terms of Eq. (71) as

$$\frac{D}{Dt}\left\{\frac{D}{D\boldsymbol{x}}\cdot[\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\varphi]\right\}=\frac{D}{Dt}\{\nabla\cdot[\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})]\varphi+\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\cdot\nabla\varphi\}$$
$$=\frac{D}{Dt}\left\{-\frac{\partial\mathcal{L}}{\partial\varphi}\varphi+\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\cdot\nabla\varphi\right\},\quad(99)$$

$$\frac{D}{D\boldsymbol{x}}\cdot\left\{\frac{D}{Dt}[-\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\varphi]\right\}$$
$$=\frac{D}{D\boldsymbol{x}}\cdot\left\{-\frac{\partial}{\partial t}[\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})]\varphi-\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\varphi_{,t}\right\}$$
$$=\frac{D}{D\boldsymbol{x}}\cdot\left\{c\nabla\times[\boldsymbol{E}_{\boldsymbol{B}}(\mathcal{L})]\varphi+c\frac{\partial\mathcal{L}}{\partial\boldsymbol{A}}\varphi-\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\varphi_{,t}\right\}$$
$$=\frac{D}{D\boldsymbol{x}}\cdot\left\{-\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\varphi_{,t}-c\nabla\varphi\times\boldsymbol{E}_{\boldsymbol{B}}(\mathcal{L})+c\frac{\partial\mathcal{L}}{\partial\boldsymbol{A}}\varphi\right\},\quad(100)$$

where use has been made of Eqs. (75) and (76). Adding Eqs. (99) and (100) to Eq. (95) leads to Eq. (98).

We now prove that the energy density and flux in Eq. (98) are gauge symmetric. It suffices to show that the following terms

$$s_1\equiv\mathcal{L}-\frac{\partial\mathcal{L}}{\partial\varphi}\varphi-\sum_a\left(\dot{\boldsymbol{X}}_a\cdot\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{X}}_a}\right),$$
$$\boldsymbol{s}_2\equiv c\frac{\partial\mathcal{L}}{\partial\boldsymbol{A}}\varphi-\sum_a\left(\dot{\boldsymbol{X}}_a\frac{\partial\mathcal{L}_a}{\partial\dot{\boldsymbol{X}}_a}-\mathcal{L}_a\boldsymbol{I}\right)\cdot\dot{\boldsymbol{X}}_a\quad(101)$$

are gauge symmetric. Substituting Eq. (77) into the expression of s_1, we have

$$s_1=\sum_a\left[-q_a\varphi\delta_a+\frac{q_a}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{X}}_a\delta_a\right]+\mathrm{GSP}(\mathcal{L})$$
$$-\frac{\partial}{\partial\varphi}\left[\sum_a\left(-q_a\varphi\delta_a+\frac{q_a}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{X}}_a\delta_a\right)\right]\varphi$$
$$-\sum_a\left[\dot{\boldsymbol{X}}_a\cdot\frac{\partial}{\partial\dot{\boldsymbol{X}}_a}\left(-q_a\varphi\delta_a+\frac{q_a}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{X}}_a\delta_a\right)\right]$$
$$=\mathrm{GSP}(\mathcal{L}).\quad(102)$$

Similarly, $\boldsymbol{s}_2$ is also gauge symmetric,

$$\boldsymbol{s}_2=c\frac{\partial\mathcal{L}}{\partial\boldsymbol{A}}\varphi-\sum_a\left(\dot{\boldsymbol{X}}_a\frac{\partial\mathcal{L}_a}{\partial\dot{\boldsymbol{X}}_a}-\mathcal{L}_a\boldsymbol{I}\right)\cdot\dot{\boldsymbol{X}}_a$$
$$=c\frac{\partial}{\partial\boldsymbol{A}}\left[\sum_a\left(-q_a\varphi\delta_a+\frac{q_a}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{X}}_a\delta_a\right)\right]\varphi$$
$$-\sum_a\left\{\dot{\boldsymbol{X}}_a\frac{\partial}{\partial\dot{\boldsymbol{X}}_a}\left(-q_a\varphi\delta_a+\frac{q_a}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{X}}_a\delta_a\right)\right.$$
$$\left.-\left(-q_a\varphi\delta_a+\frac{q_a}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{X}}_a\delta_a\right)\boldsymbol{I}+\mathrm{GSP}(\mathcal{L}_a)\boldsymbol{I}\right\}\cdot\dot{\boldsymbol{X}}_a$$
$$=\sum_a\mathrm{GSP}(\mathcal{L}_a)\dot{\boldsymbol{X}}_a.\quad(103)$$

C. Gauge-symmetric momentum conservation law

We now discuss how to derive a gauge-symmetric momentum conservation law, assuming that the action $\mathcal{A}\equiv\int\mathcal{L}dtd^3\boldsymbol{x}$ of the electromagnetic field-charged particle system is invariant under the space translation

$$(t,x,\boldsymbol{X}_a,\varphi,\boldsymbol{A})\mapsto(t,\boldsymbol{x}+\epsilon\boldsymbol{h},\boldsymbol{X}_a+\epsilon\boldsymbol{h},\varphi,\boldsymbol{A}),\quad\epsilon\in\mathbb{R}.\quad(104)$$

We emphasize that, different from the situation in standard field theories, this symmetry group simultaneously translates both the spatial coordinate $\boldsymbol{x}$ for the field and particle's position $\boldsymbol{X}_a$ [25–27]. The infinitesimal criterion of this symmetry is

$$\frac{\partial\mathcal{L}}{\partial\boldsymbol{x}}+\sum_a\frac{\partial\mathcal{L}}{\partial\boldsymbol{X}_a}=0.\quad(105)$$

From Eq. (104), the infinitesimal generator and its characteristic are

$$\boldsymbol{v}=\boldsymbol{h}\cdot\sum_a\left(\frac{\partial}{\partial\boldsymbol{x}}+\frac{\partial}{\partial\boldsymbol{X}_a}\right),\qquad\xi^t=0,$$
$$\boldsymbol{\kappa}=\boldsymbol{\theta}_a=\boldsymbol{h},\qquad\phi_0=\boldsymbol{\phi}_{\boldsymbol{A}}=0,\quad(106)$$

$$\boldsymbol{q}_a=\boldsymbol{h},\qquad Q_0=-\boldsymbol{h}\cdot\nabla\varphi,\qquad\boldsymbol{Q}_{\boldsymbol{A}}=-\boldsymbol{h}\cdot\nabla\boldsymbol{A}.\quad(107)$$

The corresponding momentum conservation law is obtained by substituting Eqs. (106) and (107) into Eq. (90), i.e.,

$$\partial_t\left[\frac{1}{c}\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\cdot(\nabla\boldsymbol{A})^T+\sum_a\left(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{X}}_a}\right)\right]$$
$$+\nabla\cdot\left\{\mathcal{L}\boldsymbol{I}+\sum_a\left(\dot{\boldsymbol{X}}_a\frac{\partial\mathcal{L}_a}{\partial\dot{\boldsymbol{X}}_a}-\mathcal{L}_a\boldsymbol{I}\right)\right.$$
$$\left.+[\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\nabla\varphi+\boldsymbol{E}_{\boldsymbol{B}}(\mathcal{L})\times(\nabla\boldsymbol{A})^T]\right\}$$
$$+D_\mu[\bar{\boldsymbol{P}}_1^\mu+\bar{\mathbb{P}}_2^\mu]=0,\quad(108)$$

where

$$\bar{\boldsymbol{P}}_1^\mu=\sum_{i=1}^n\sum_{j=1}^i(-1)^jD_{\mu_{j+1}}\cdots D_{\mu_i}\nabla\boldsymbol{E}$$
$$\cdot\left[D_{\mu_1}\cdots D_{\mu_{j-1}}\frac{\partial\mathcal{L}}{\partial D_{\mu_1}\cdots D_{\mu_{j-1}}D_\mu D_{\mu_{j+1}}\cdots D_{\mu_i}\boldsymbol{E}}\right],\quad(109)$$

$$\bar{\boldsymbol{P}}_2^\mu=\sum_{i=1}^n\sum_{j=1}^i(-1)^jD_{\mu_{j+1}}\cdots D_{\mu_i}\nabla\boldsymbol{B}$$
$$\cdot\left[D_{\mu_1}\cdots D_{\mu_{j-1}}\frac{\partial\mathcal{L}}{\partial D_{\mu_1}\cdots D_{\mu_{j-1}}D_\mu D_{\mu_{j+1}}\cdots D_{\mu_i}\boldsymbol{B}}\right].\quad(110)$$

Again, the momentum density and flux in Eq. (108) are gauge dependent. We add Eq. (72) to Eq. (108) to obtain a gauge-symmetric momentum conservation law,

$$\partial_t\left[\frac{1}{c}\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\times\boldsymbol{B}+\frac{1}{c}\frac{\partial\mathcal{L}}{\partial\varphi}\boldsymbol{A}+\sum_a\left(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{X}}_a}\right)\right] + \nabla\cdot\left\{-\frac{\partial\mathcal{L}}{\partial\boldsymbol{A}}\boldsymbol{A}+\sum_a\left(\dot{\boldsymbol{X}}_a\frac{\partial\mathcal{L}_a}{\partial\dot{\boldsymbol{X}}_a}\right)[\mathcal{L}_F-\boldsymbol{B}\cdot\boldsymbol{E}_{\boldsymbol{B}}(\mathcal{L})]\boldsymbol{I} + [-\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\boldsymbol{E}+\boldsymbol{B}\boldsymbol{E}_{\boldsymbol{B}}(\mathcal{L})]\right\}+D_\mu[(\bar{\boldsymbol{P}}_1^\mu+\bar{\boldsymbol{P}}_2^\mu)]=0. \tag{111}$$

In the derivation of Eq. (111), we have rewritten the first and second terms of Eq. (72) as

$$\begin{aligned}\frac{D}{Dt}\left\{\frac{D}{D\boldsymbol{x}}\cdot\left[-\frac{1}{c}\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\boldsymbol{A}\right]\right\} &= \frac{D}{Dt}\left\{-\frac{1}{c}\nabla\cdot[\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})]\boldsymbol{A}-\frac{1}{c}\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\cdot\nabla\boldsymbol{A}\right\}\\ &= \frac{D}{Dt}\left[\frac{1}{c}\frac{\partial\mathcal{L}}{\partial\varphi}\boldsymbol{A}-\frac{1}{c}\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\cdot\nabla\boldsymbol{A}\right],\end{aligned} \tag{112}$$

$$\begin{aligned}\frac{D}{D\boldsymbol{x}}\cdot\left\{\frac{D}{Dt}\left[\frac{1}{c}\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\boldsymbol{A}\right]\right\} &= \frac{D}{D\boldsymbol{x}}\cdot\left\{\frac{1}{c}\frac{\partial}{\partial t}[\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})]\boldsymbol{A}+\frac{1}{c}\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\boldsymbol{A}_{,t}\right\}\\ &= \frac{D}{D\boldsymbol{x}}\cdot\left\{-\nabla\times[\boldsymbol{E}_{\boldsymbol{B}}(\mathcal{L})]\boldsymbol{A}-\frac{\partial\mathcal{L}}{\partial\boldsymbol{A}}\boldsymbol{A}+\frac{1}{c}\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\boldsymbol{A}_{,t}\right\}\\ &= \frac{D}{D\boldsymbol{x}}\cdot\left\{-\boldsymbol{E}_{\boldsymbol{B}}(\mathcal{L})\times\nabla\boldsymbol{A}-\frac{\partial\mathcal{L}}{\partial\boldsymbol{A}}\boldsymbol{A}+\frac{1}{c}\boldsymbol{E}_{\boldsymbol{E}}(\mathcal{L})\boldsymbol{A}_{,t}\right\},\end{aligned} \tag{113}$$

where use has been made of Eqs. (75) and (76). Adding Eqs. (112) and (113) into Eq. (108) gives Eq. (111).

To show that the momentum density and flux in Eq. (111) are gauge symmetric, it suffices to show that the following terms are gauge symmetric,

$$\begin{aligned}\boldsymbol{t}_1 &\equiv \frac{1}{c}\frac{\partial\mathcal{L}}{\partial\varphi}\boldsymbol{A}+\sum_a\left(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{X}}_a}\right),\\ \boldsymbol{t}_2 &\equiv -\frac{\partial\mathcal{L}}{\partial\boldsymbol{A}}\boldsymbol{A}+\sum_a\left(\dot{\boldsymbol{X}}_a\frac{\partial\mathcal{L}_a}{\partial\dot{\boldsymbol{X}}_a}\right).\end{aligned} \tag{114}$$

Substituting Eq. (77) into the expression of $\boldsymbol{t}_1$, we can see that it is gauge symmetric, i.e.,

$$\begin{aligned}\boldsymbol{t}_1 &= \frac{1}{c}\frac{\partial\mathcal{L}}{\partial\varphi}\boldsymbol{A}+\sum_a\left(\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{X}}_a}\right)\\ &= \frac{1}{c}\sum_a\frac{\partial}{\partial\varphi}\left(-q_a\varphi\delta_a+\frac{q_a}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{X}}_a\delta_a\right)\boldsymbol{A}\\ &\quad+\sum_a\frac{\partial}{\partial\dot{\boldsymbol{X}}_a}\left[-q_a\varphi\delta_a+\frac{q_a}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{X}}_a\delta_a+\mathrm{GSP}(\mathcal{L}_a)\right]\\ &= \sum_a\frac{\partial}{\partial\dot{\boldsymbol{X}}_a}[\mathrm{GSP}(\mathcal{L}_a)].\end{aligned} \tag{115}$$

Similarly, $\boldsymbol{t}_2$ is also gauge-symmetric,

$$\begin{aligned}\boldsymbol{t}_2 &= -\frac{\partial\mathcal{L}}{\partial\boldsymbol{A}}\boldsymbol{A}+\sum_a\left(\dot{\boldsymbol{X}}_a\frac{\partial\mathcal{L}_a}{\partial\dot{\boldsymbol{X}}_a}\right)\\ &= -\frac{\partial}{\partial\boldsymbol{A}}\left(-q_a\varphi\delta_a+\frac{q_a}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{X}}_a\delta_a\right)\boldsymbol{A}\\ &\quad+\sum_a\dot{\boldsymbol{X}}_a\frac{\partial}{\partial\dot{\boldsymbol{X}}_a}\left[-q_a\varphi\delta_a+\frac{q_a}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{X}}_a\delta_a+\mathrm{GSP}(\mathcal{L}_a)\right]\\ &= \sum_a\dot{\boldsymbol{X}}_a\frac{\partial}{\partial\dot{\boldsymbol{X}}_a}[\mathrm{GSP}(\mathcal{L}_a)].\end{aligned} \tag{116}$$

IV. CONCLUSION

In this study, we developed a gauge-symmetrization method for the energy and momentum conservation laws in general high-order classical electromagnetic field theories, which appear in the study of gyrokinetic systems [18–20] for magnetized plasmas and the Podolsky system [21,22] for the radiation reaction of classical charged particles. The method only removes the electromagnetic gauge dependence from the canonical EMT derived from the spacetime translation symmetry, without necessarily symmetrizing the EMT with respect to the tensor indices. This is adequate for applications not involving general relativity.

To achieve this goal, we reformulated the EL equation and infinitesimal criterion in terms of the Faraday tensor $F_{\mu\nu}$. The canonical EMT $T_{\mathrm{N}}^{\mu\nu}$ is derived using this formalism, and it was found that the gauge dependent part of $T_{\mathrm{N}}^{\mu\nu}$ can be removed by adding the divergence of the displacement-potential tensor, which is defined as

$$\mathcal{F}^{\sigma\mu\nu}\equiv\frac{1}{4\pi}\mathcal{D}^{\sigma\mu}A^\nu. \tag{117}$$

It was shown that the displacement-potential tensor $\mathcal{F}^{\sigma\mu\nu}$ is related to the well-known BR superpotential $\mathcal{S}^{\sigma\mu\nu}$ as

$$\mathcal{S}^{\sigma\mu\nu}=\mathcal{F}^{\sigma\mu\nu}+\frac{1}{2}[\Delta^{\sigma\nu\mu}-\Delta^{\mu\nu\sigma}-\Delta^{\nu\mu\sigma}], \tag{118}$$

where $\Delta^{\sigma\nu\mu}$ is defined in Eq. (53). Using the example of the Podolsky system [21,22], we show that $\Delta^{\sigma\nu\mu}$ in general is nonvanishing for high-order field theories. For a first-order field theory, such as the standard Maxwell system (6), $\Delta^{\sigma\nu\mu}$ vanishes such that $\mathcal{S}^{\sigma\mu\nu} = \mathcal{F}^{\sigma\mu\nu}$. In the case, the method developed can be used as a simpler procedure to calculate the BR superpotential $\mathcal{S}^{\sigma\mu\nu}$ without the necessity to calculate the angular momentum tensor in 4D spacetime.

Lastly, we applied the method to derive gauge-symmetric EMTs for high-order electromagnetic systems coupled with classical charged particles. Using the "3 + 1" form of Eq. (37), we obtained the explicitly gauge-symmetric energy and momentum conservation laws in a general setting [see Eqs. (98) and (111)].

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