Minimal twist for the Standard Model in noncommutative geometry: The field content

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Noncommutative geometry provides both a unified description of the Standard Model of particle physics together with Einstein-Hilbert action (in Euclidean signature) and some tools to go beyond the Standard Model. In this paper, we extend to the full noncommutative geometry of the Standard Model the twist (in the sense of Connes-Moscovici) initially worked out for the electroweak sector and the free Dirac operator only. Namely, we apply the twist also to the strong interaction sector and the finite part of the Dirac operator. To do so, we are forced to take into account a violation of the twisted first-order condition. As a result, we still obtain the extra scalar field required to stabilize the electroweak vacuum and fit the Higgs mass, but it now has two chiral components. We also get the additive field of 1-forms already pointed out in the electroweak model, but with a richer structure. Finally, we obtain a pair of Higgs doublets, which are expected to combine into a single Higgs doublet in the action formula, as will be investigated in the second part of this work.

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I. INTRODUCTION

Noncommutative geometry [1] (see [2] for a recent review of the various aspects of the field) provides a mathematical framework in which a single action formula yields both the Lagrangian of the Standard Model of fundamental interactions and the Einstein-Hilbert action (in Euclidean signature). As an added value, the Higgs field is obtained on the same footing as the other gauge bosons—as a connection 1-form—but a connection that lives on a slightly generalized notion of space, where points come equipped with an internal structure. Such "spaces" are described by *spectral triples*

$$\mathcal{A}, \quad \mathcal{H}, \quad D \tag{1.1}$$

consisting in an algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ together with an operator D on $\mathcal{H}$ which satisfies a set of axioms [3] guaranteeing that—in case $\mathcal{A}$ is commutative and unital—then there exists a (closed) Riemannian spin manifold $\mathcal{M}$ such that $\mathcal{A}$ coincides with the algebra $C^\infty(\mathcal{M})$ of smooth functions on $\mathcal{M}$. In other terms, a spectral triple with $\mathcal{A}$ commutative does encode all the geometrical information of a (closed) Riemannian spin manifold [4]. These axioms still make sense when $\mathcal{A}$ is noncommutative and provide then a definition of a *noncommutative geometry* as a spectral triple in which the algebra is not necessarily commutative.

The spectral triple of the Standard Model [5] is built upon an "almost-commutative algebra":

$$C^\infty(\mathcal{M}) \otimes \mathcal{A}_{\rm SM}, \tag{1.2}$$

where $\mathcal{M}$ is an even-dimensional closed Riemannian spin manifold and $\mathcal{A}_{\rm SM}$ a noncommutative matrix algebra that encodes the gauge degrees of freedom of the Standard Model. As explained in Ref. [3], this noncommutative algebra provides the points of $\mathcal{M}$ with an internal structure, in such a way that the Standard Model is actually nothing but a pure theory of gravity, on a space that is made slightly noncommutative by multiplying the (infinite-dimensional) commutative algebra $C^\infty(\mathcal{M})$ with the finite-dimensional noncommutative $\mathcal{A}_{\rm SM}$.

After the discovery of the Higgs boson in 2012, it has been noticed in Ref. [6] that an extra scalar field—usually denoted σ—proposed by particle physicists to cure the instability of the electroweak vacuum due to the "low mass of the Higgs" also makes the computation of the Higgs mass (which is not a free parameter in the noncommutative description of the Standard Model) compatible with its experimental value. Various scenarios have been proposed to make this extra scalar field emerge from the mathematical framework of noncommutative geometry, all of them consisting in some modification of one of the axioms, the

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first-order condition (e.g., Refs. [7–13]; see [14] for a recent review).

In this paper, we push forward one of these scenarios, consisting in twisting the spectral triple of the Standard Model. Twists have been introduced by Connes and Moscovici in Ref. [15] with purely mathematical motivations. Later, it has been discovered in Ref. [16] that a very simple twist of the Standard Model produces not only the extra scalar field σ, but also an additive field of 1-form X_μ which turns out to be related with Wick rotation and the transition from the Euclidean to the Lorentzian signature [17,18]. However, in Ref. [16], the twist was applied only to the part of the spectral triple that yields the field σ, namely, the subalgebra of $\mathcal{A}_{\rm SM}$ describing the electroweak interaction and the part of the operator D that contains the Majorana mass of the neutrinos. For simplicity, the subalgebra of $\mathcal{A}_{\rm SM}$ describing the strong interaction was left untouched, and the part of D containing the Yukawa coupling of fermions was not taken into account. In this paper, we extend the twisting procedure to the whole spectral triple of the Standard Model, according to the following lines.

The twist of gauge theories has been investigated in a systematic way in Refs. [19,20], where the twisted version of the first-order condition—introduced by imitation of the nontwisted case in Ref. [16]—has been put onto solid mathematical bases. A notion of *minimal twist* of a spectral triple has also been defined, which consists in making several copies of $\mathcal{A}$ act on $\mathcal{H}$, leaving D untouched. By doing so, one produces models with new bosonic fields, keeping the fermionic content untouched, in agreement with the state of the art of the Standard Model (indeed, the metastability of the electroweak vacuum points toward new scalar fields, but there are no indications of new fermions). A procedure for minimally twisting any real spectral triple is to make two copies of the algebra act independently on the eigenspaces of the grading operator. However, applied to the Standard Model, this does not produce any extra scalar field, as explained in Ref. [21].

That is why in this paper we investigate another minimal twist of the Standard Model that does produce an extra scalar σ. The price to pay is a violation of the twisted first-order condition, which is taken into account following the way pioneered in Ref. [13] and adapted to the twisted case in Ref. [22].

Besides the field content of the Standard Model, we find that the extra scalar σ actually decomposes into two chiral components σ_r and σ_l (Proposition 4.6) which are invariant under a gauge transformation (Proposition 6.6). We also work out the structure of the 1-form field X_μ (Proposition 5.5) and study how it behaves under a gauge transformation (Proposition 6.2). In brief, imposing the same unimodular condition as in the nontwisted case, we find that the anti-self-adjoint part of the (generalized) 1-form generated by the free Dirac operator $\partial\!\!\!/$ yields exactly the bosonic content of the Standard Model as in the nontwisted case. But there is also a self-adjoint part made of two real 1-form fields and one self-adjoint $M_3(\mathbb{C})$-value 1-form field. Altogether, these three fields compose the 1-form field X_μ.

The complete understanding of the physical meaning of these fields passes through the computation of the fermionic and spectral actions and will be the subject of a second paper [21].

The paper is organized as follows. In Sec. II, we recall the basics of the spectral triple of the Standard Model (Sec. II A), make explicit the tensorial notations employed all along the paper (Sec. II B), and use them to write explicitly the Dirac operator, the grading, and the real structure (Sec. II C). Section III deals with the twist. After recalling the procedure of minimal twisting defined in Ref. [19], we apply it to the spectral triple of the Standard Model: The algebra is doubled so as to act independently on the left and right components of Dirac spinors (Sec. III A). The grading and the real structure are the same as in the nontwisted case, and we check explicitly that one of the axioms (the order-zero condition) still holds in the twisted case (Sec. III B), as expected from the general result of Ref. [19]. Section III C is a brief recalling about twisting fluctuations, that is, the way to generate the bosonic fields. The detailed computation of these fluctuations is the subject of Secs. IV and V, which contain the main results of this paper. We first work out the Higgs sector in Sec. IV A. The main result is Proposition 4.4, in which we find two Higgs doublets. The extra scalar field σ is generated in Sec. IV B. Its structure as a doublet of real scalar fields σ_r and σ_l is established in Proposition 4.6. In Sec. V, we compute the twisted fluctuation of the free part $\partial\!\!\!/$ of the Dirac operator. Useful properties of the Dirac matrices with respect to the twist are worked out in Sec. V A. The generalized twisted 1-forms generated by the free Dirac operator are computed in Sec. V B, and the physical degrees of freedom are identified in Sec. V C. The structure of the 1-form field X_μ is summarized in Proposition 5.5 and yields, in Sec. V D, the explicit form of the twisted fluctuation of the free Dirac operator. In Sec. VI, we study how all these fields behave under a gauge transformation. After recalling the basics of gauge transformation for a twisted spectral triple (as stabilized in Ref. [20]), we apply these techniques to the gauge and the 1-form fields in Sec. VI A and to the scalar fields in Sec. VI B. We show in Proposition 6.2 that the bosonic fields transform in the correct way, while the 1-form field is invariant, up to a unitary transformation on the $M_3(\mathbb{C})$-value part. The Higgs doublets as well transform as expected (Proposition 6.5), while the extra scalar field σ is gauge invariant, as shown in Proposition 6.6.

The first section of the Appendix contains notations and generalities on Dirac matrices. In the second section, we write explicitly the components of the twisted fluctuation in terms of the gauge fields (this will be useful in the second part of the paper, to compute the action). In the last section of the Appendix, we check that the twisted first-order condition is only partially verified.

A. Notations and important comments regarding the literature

In the first version of this paper, we erroneously thought the twist we were using was "by grading" and assumed the twisted first-order condition. Actually, the latter is violated only by the off-diagonal part of the internal Dirac operator, and this does not modify the extra scalar field, as explained before Remark 4.7, nor the gauge invariance of the fermionic action, as explained before Proposition 6.6.

We work with one generation of fermions (electron e, neutrino ν_e, and quarks up u and down d). The extension to three generations will be discussed in the second part of the work [23].

All along the paper, we apply the usual rule of contractions of indices in alternate up and down positions. Typically, Greek indices label the coordinates of the manifold.

II. THE NONTWISTED CASE

As a preparation to the twisting, we recall in this section the main features of the spectral description of the Standard Model. Besides the original papers (recalled in the text), the details are extensively discussed in the books [24,25] (for a more physics-oriented presentation).

A. The spectral triple of the Standard Model

The usual spectral triple of the Standard Model [5] is the product of the canonical triple of a (closed) Riemannian spin manifold $\mathcal{M}$ of even dimension m:

$$C^\infty(\mathcal{M}), \quad L^2(\mathcal{M},S), \quad \not{\partial} \tag{2.1}$$

with the finite-dimensional spectral triple (called *internal*)

$$\mathcal{A}_{\mathrm{SM}} = \mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C}), \qquad \mathcal{H}_F = \mathbb{C}^{32n}, \qquad D_F \tag{2.2}$$

that describes the gauge degrees of freedom of the Standard Model. In Eq. (2.1), $C^\infty(\mathcal{M})$ denotes the algebra of smooth functions on $\mathcal{M}$ that acts by multiplication on the Hilbert space $L^2(\mathcal{M},S)$ of square integrable spinors as

$$(f\psi)(x) = f(x)\psi(x) \quad \forall\ f\in C^\infty(\mathcal{M}), \quad \psi\in L^2(\mathcal{M},S), \qquad x\in\mathcal{M}, \tag{2.3}$$

while

$$\not{\partial} = -i\gamma^\mu\nabla_\mu \quad \text{with} \quad \nabla_\mu = \partial_\mu + \omega_\mu \tag{2.4}$$

is the Dirac operator on $L^2(\mathcal{M},S)$ associated with the spin connection ω_μ and the γ^μ's are the Dirac matrices associated with the Riemannian metric g on $\mathcal{M}$:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{I} \quad \forall\ \mu, \qquad \nu = 0, \qquad m-1 \tag{2.5}$$

[$\mathbb{I}$ is the identity operator on $L^2(\mathcal{M},S)$, and we label the coordinates of $\mathcal{M}$ from 0 to $m-1$].

In Eq. (2.2), n is the number of generations of fermions, and D_F is a $32n$ square complex matrix whose entries are the Yukawa couplings of fermions and the coefficients of the Cabibbo-Kobayashi-Maskawa mixing matrix of quarks and of the Pontecorvo-Maki-Nakagawa-Sakata mixing matrix of neutrinos. Details are given in Sec. II C, and the representation of $\mathcal{A}_{\mathrm{SM}}$ on $\mathcal{H}_F$ is in Sec. II B.

The product spectral triple is

$$C^\infty(\mathcal{M})\otimes\mathcal{A}_{\mathrm{SM}}, \qquad \mathcal{H} = L^2(\mathcal{M},S)\otimes\mathcal{H}_F, \qquad D = \not{\partial}\otimes\mathbb{I}_F + \gamma_\mathcal{M}\otimes D_F \tag{2.6}$$

with $\gamma_\mathcal{M}$ the product of the Euclidean Dirac matrices (Appendix 7) and $\mathbb{I}_F$ the identity on $\mathcal{H}_F$.

A spectral triple $(\mathcal{A},\mathcal{H},D)$ is *graded* when the Hilbert space comes equipped with a grading (that is, a self-adjoint operator that squares to $\mathbb{I}$) which anticommutes with D. The spectral triple (2.1) is graded with grading $\gamma_\mathcal{M}$. The internal spectral triple (2.2) is graded, with grading the operator γ_F on $\mathcal{H}_F$ that takes value $+1$ on right particles and left antiparticles and -1 on left particles and right antiparticles. The product spectral triple (2.6) is graded, with grading

$$\Gamma = \gamma_\mathcal{M}\otimes\gamma_F. \tag{2.7}$$

Another important ingredient is the *real structure*, that is, an antilinear operator that squares to $\pm\mathbb{I}$ and commutes or anticommutes with the grading and the operator D (the possible choices define the so-called KO dimension of the spectral triple). For a manifold, the real structure $\mathcal{J}$ is given by the charge conjugation operator. In dimension $m=4$, it satisfies

$$\mathcal{J}^2 = -\mathbb{I}, \qquad \mathcal{J}\not{\partial} = \not{\partial}\mathcal{J}, \qquad \mathcal{J}\gamma_\mathcal{M} = \gamma_\mathcal{M}\mathcal{J}. \tag{2.8}$$

The real structure of the internal spectral triple (2.2) is the antilinear operator J_F that exchanges particles with antiparticles on $\mathcal{H}_F$. It satisfies

$$J_F^2 = \mathbb{I}, \qquad J_F D_F = D_F J_F, \qquad J\gamma_F = -\gamma_F J_F. \tag{2.9}$$

The real structure for the product spectral triple (2.6) is

$$J = \mathcal{J}\otimes J_F. \tag{2.10}$$

For a manifold of dimension $m=4$, it is such that

$$J^2 = -\mathbb{I}, \qquad JD = DJ, \qquad J\Gamma = -\Gamma J. \tag{2.11}$$

The real structure implements an action of the opposite algebra $\mathcal{A}^\circ$ on $\mathcal{H}$, identifying $a^\circ\in\mathcal{A}^\circ$ with Ja^*J^{-1}.

This action is asked to commute with the one of $\mathcal{A}$, yielding the *order-zero condition*

$$[a, b^\circ] = 0 \quad \forall \ a \in \mathcal{A}, \qquad b \in \mathcal{A}^\circ. \tag{2.12}$$

Among the properties of a spectral triple, one particularly relevant for physical models is the first-order condition

$$[[D, b], a^\circ] = 0 \quad \forall \ a, b \in \mathcal{A}. \tag{2.13}$$

B. Representation of the algebra

To describe the action of $\mathcal{A}_{\rm SM} \otimes C^\infty(\mathcal{M})$ on $\mathcal{H}$ in Eq. (2.6), it is convenient to label the $32n$ degrees of freedom of the finite-dimensional Hilbert space $\mathcal{H}_F$ by a multi-index $CI\alpha$ defined as follows.

(i) $C = 0,\ 1$ is for a particle ($C = 0$) or antiparticle ($C = 1$);

(ii) $I = 0; i$ with $i = 1,\ 2,\ 3$ is the leptocolor index: $I = 0$ means lepton, while $I = 1,\ 2,\ 3$ are for the quark, which exists in three colors;

(iii) $\alpha = \dot{1}, \dot{2}; a$ with $a = 1,\ 2$ is the flavor index:

$$\dot{1} = \nu_R, \qquad \dot{2} = e_R, \qquad 1 = \nu_L, \\ 2 = e_L \qquad \text{for leptons}\,(I = 0), \tag{2.14}$$

$$\dot{1} = u_R, \qquad \dot{2} = d_R, \qquad 1 = q_L, \\ 2 = d_L \qquad \text{for quarks}\,(I = i). \tag{2.15}$$

We sometimes use the shorthand notation $\ell_L^a = (\nu_L, e_L)$ for the left-handed neutrino and the associated lepton and $q_L^a = (u_L, d_L)$ for the pair of left-handed quarks.

There are $2 \times 4 \times 4 = 32$ choices of triplet of indices (C, I, α), which is the number of fermions per generation. One should also take into account an extra index $n = 1, 2, 3$ for the generations, but in this paper we work with one generation only and we omit it (we will discuss the number of generations in the computation of the action [21]). So from now on

$$\mathcal{H}_F = \mathbb{C}^{32}. \tag{2.16}$$

An element $\psi \in \mathcal{H} = C^\infty(\mathcal{M}) \otimes \mathcal{H}_F$ is thus a 32-dimensional column vector, in which each component $\psi_{CI\alpha}$ is a Dirac spinor in $L^2(\mathcal{M}, S)$.

Regarding the algebra, unless necessary we omit the symbol of the representation and identify an element $a = (c, q, m)$ in $C^\infty(\mathcal{M}) \otimes \mathcal{A}_{\rm SM}$, where

$$c \in C^\infty(\mathcal{M}, \mathbb{C}), \qquad q \in C^\infty(\mathcal{M}, \mathbb{H}), \\ m \in C^\infty(\mathcal{M}, M_3(\mathbb{C})), \tag{2.17}$$

with its representation as bounded operator on $\mathcal{H}$, that is, a 32-square matrix whose components[1]

$$a_{CI\alpha}^{DJ\beta} \tag{2.18}$$

are smooth functions acting by multiplication on $L^2(\mathcal{M}, S)$ as in Eq. (2.3). Explicitly,[2]

$$a = \begin{pmatrix} Q & \\ & M \end{pmatrix}_C^D, \tag{2.19}$$

where the 16×16 square matrices Q and M have components

$$Q_{I\alpha}^{J\beta} = \delta_I^J Q_\alpha^\beta, \qquad M_{I\alpha}^{J\beta} = \delta_\alpha^\beta M_I^J, \tag{2.20}$$

respectively, where

$$Q_\alpha^\beta = \begin{pmatrix} c & & \\ & \bar{c} & \\ & & q \end{pmatrix}_\alpha^\beta, \qquad M_I^J = \begin{pmatrix} c & \\ & m \end{pmatrix}_I^J. \tag{2.21}$$

Here, the overbar denotes the complex conjugate, m (evaluated at the point x) identifies with its usual representation as 3×3 complex matrices, and the quaternion q (evaluated at x) acts through its representation as 2×2 matrices:

$$\mathbb{H} \ni q(x) = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{C}. \tag{2.22}$$

C. Finite-dimensional Dirac operator, grading, and real structure

With respect to the particle and antiparticle index C, the internal Dirac operator

$$D_F = D_Y + D_M \tag{2.23}$$

decomposes into a diagonal and an off-diagonal part

$$D_Y = \begin{pmatrix} D_0 & \\ & D_0^\dagger \end{pmatrix}_C^D, \qquad D_M = \begin{pmatrix} 0 & D_R \\ D_R^\dagger & 0 \end{pmatrix}_C^D \tag{2.24}$$

containing, respectively, the Yukawa couplings of fermions and the Majorana mass of the neutrino.

[1] D, J, and β are column indices with the same range as the line indices C, I, and α (the position of the indices was slightly different in Ref. [16]; the one adopted here makes the tensorial computation more tractable).

[2] The indices after the closing parenthesis are here to recall that the block entries of $\mathcal{A}$ are labeled by the C, D indices, that is, $a_1^1 = Q$, $a_2^2 = M$, and $a_1^2 = a_2^1 = 0$.

The 16×16 matrices D_0 and D_R are block diagonal with respect to the leptocolor index I:

$$D_0 = \begin{pmatrix} D_0^\ell & & & \\ & D_0^q & & \\ & & D_0^q & \\ & & & D_0^q \end{pmatrix}_I^J,$$

$$D_R = \begin{pmatrix} D_R^\ell & & & \\ & 0_4 & & \\ & & 0_4 & \\ & & & 0_4 \end{pmatrix}_I^J, \qquad (2.25)$$

where we write ℓ for $I = 0$ and q for $I = 1, 2, 3$. Each D_0^I is a 4×4 matrix (in the flavor index α):

$$D_0^I = \begin{pmatrix} 0 & \bar{\mathsf{k}}^{\mathsf{I}} \\ \mathsf{k}^{\mathsf{I}} & 0 \end{pmatrix}_\alpha^\beta \quad \text{where } \mathsf{k}^{\mathsf{I}} \coloneqq \begin{pmatrix} k_u^I & 0 \\ 0 & k_d^I \end{pmatrix}_\alpha^\beta, \qquad (2.26)$$

whose entries are the Yukawa couplings of elementary fermions

$$k_u^I = (\, k_\nu, \quad k_u, \quad k_u, \quad k_u \,),$$
$$k_d^I = (\, k_e, \quad k_d, \quad k_d, \quad k_d \,) \qquad (2.27)$$

(three of them are equal because the Yukawa coupling of quarks does not depend on the color). Similarly, D_R^ℓ is a 4×4 matrix (in the flavor index):

$$D_R^\ell = \begin{pmatrix} k_R & \\ & 0_3 \end{pmatrix}_\alpha^\beta \qquad (2.28)$$

whose only nonzero entry is the Majorana mass of the neutrino.

In tensorial notations, one has

$$D_R = k_R \Xi_{I\alpha}^{J\beta}, \qquad (2.29)$$

where

$$\Xi_\alpha^\beta \coloneqq \begin{pmatrix} 1 & \\ & 0_3 \end{pmatrix}_\alpha^\beta, \qquad \Xi_I^J \coloneqq \begin{pmatrix} 1 & \\ & 0_3 \end{pmatrix}_I^J, \qquad (2.30)$$

and $\Xi_{I\alpha}^{J\beta}$ is a shorthand notation for the tensor $\Xi_I^J \Xi_\alpha^\beta$. Similarly, the internal grading is

$$\gamma_F = \begin{pmatrix} \mathbb{I}_8 & & & \\ & -\mathbb{I}_8 & & \\ & & -\mathbb{I}_8 & \\ & & & \mathbb{I}_8 \end{pmatrix} = \eta_{C\alpha}^{D\beta} \delta_I^J, \qquad (2.31)$$

where the blocks in the matrix act, respectively, on right and left particles and then right and left antiparticles, and we define

$$\eta_\alpha^\beta \coloneqq \begin{pmatrix} \mathbb{I}_2 & \\ & -\mathbb{I}_2 \end{pmatrix}_\alpha^\beta, \qquad \eta_C^D \coloneqq \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}_C^D, \qquad (2.32)$$

and $\eta_{C\alpha}^{D\beta}$ holds for $\eta_C^D \eta_\alpha^\beta$. The internal real structure is

$$J_F = \begin{pmatrix} 0 & \mathbb{I}_{16} \\ \mathbb{I}_{16} & 0 \end{pmatrix}_C^D cc = \xi_C^D \delta_{I\alpha}^{J\beta} cc, \qquad (2.33)$$

where cc denotes the complex conjugation and we define

$$\xi_C^D \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_C^D. \qquad (2.34)$$

III. MINIMAL TWIST OF THE STANDARD MODEL

In the noncommutative geometry description of the Standard Model, the bosonic degrees of freedom are obtained by a so-called fluctuation of the metric, that is, the substitution of the operator D with $D + A + JAJ^{-1}$, where

$$A = \sum_i a_i [D, b_i], \qquad a_i, b_i \in \mathcal{A}, \qquad (3.1)$$

is a generalized 1-form (see [3] for details and the justification of the terminology).

As already noticed in Refs. [5,24], the Majorana mass of the neutrino does not contribute to the bosonic content of the model, for D_M commute with algebra:

$$[\gamma^5 \otimes D_M, a] = 0 \quad \forall\, a \in \mathcal{A}. \qquad (3.2)$$

However, in order to generate the σ field proposed in Ref. [6] to cure the electroweak vacuum instability and solve the problem of the computation of the Higgs mass, one precisely needs to make D_M contribute to the fluctuation.

To do this, a possibility consists in substituting the commutator $[D, a]$ with a twisted commutator

$$[D, a]_\rho \coloneqq Da - \rho(a) D, \qquad (3.3)$$

where ρ is a fixed automorphism of $\mathcal{A}$. This substitution is the base of the definition of *twisted spectral triple* [15] where, instead of asking that $[D, a]$ be bounded for any a (which is one of the axioms of a spectral triple), one requires that there exists an automorphism ρ such that the twisted commutator $[D, a]_\rho$ is bounded for any $a \in \mathcal{A}$. As shown in Ref. [19], starting with a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is almost commutative as in Eq. (1.2), then the only way to build a twisted spectral triple with the same Hilbert space and Dirac operator (which, from a physics point of view, means that one looks for models with the same fermionic content as the Standard Model) is to double the algebra and make them act independently on the left and right components of spinors (following actually an idea of Ref. [26]). All this is detailed in the next section.

A. Algebra and Hilbert space

The algebra $\mathcal{A}$ of the twisted spectral triple of the Standard Model is twice the algebra (2.6):

$$\mathcal{A} = (C^\infty(\mathcal{M}) \otimes \mathcal{A}_{\rm SM}) \otimes \mathbb{C}^2, \tag{3.4}$$

which is isomorphic to

$$(C^\infty(\mathcal{M}) \otimes \mathcal{A}_{\rm SM}) \oplus (C^\infty(\mathcal{M}) \otimes \mathcal{A}_{\rm SM}). \tag{3.5}$$

It acts on the same Hilbert space $\mathcal{H}$ as in the nontwisted case, but now the two copies of $C^\infty(\mathcal{M}) \otimes \mathcal{A}_{\rm SM}$ act independently on the right and left components of spinors. To write this action, it is convenient to view an element of $\mathcal{H}$ as a column vector with $4 \times 32 = 128$ components [4 being the number of components of a usual spinor in $L^2(\mathcal{M}, S)$ for $m = 4$]. To this aim, one introduces two extra indices to label the degrees of freedom of $L^2(\mathcal{M}, S)$.

(i) $s = r, l$ is the chirality index;

(ii) $\dot{s} = \dot{0}, \dot{1}$ denotes the particle ($\dot{0}$) or antiparticle part ($\dot{1}$).

An element a of Eq. (3.5) is a pair of elements of Eq. (2.6), namely,

$$a = (c, c', q, q', m, m') \tag{3.6}$$

with

$$c, c' \in C^\infty(\mathcal{M}, \mathbb{C}), \qquad q, q' \in C^\infty(\mathcal{M}, \mathbb{H}),$$
$$m, m' \in C^\infty(\mathcal{M}, M_3(\mathbb{C})). \tag{3.7}$$

We make (c, q, m) act on the chiral subspace $\mathcal{H}_c$ of $\mathcal{H}$, consisting in particles and antiparticles whose chirality as Dirac spinors coincides with chirality in the internal space, whereas (c', q', m') acts on the antichiral subspace $\mathcal{H}_a$ consisting in particles and particles whose Dirac and internal chiralities do not coincide. The chiral subspace $\mathcal{H}_c$ is the subspace of $\mathcal{H}$ spanned by $r, \alpha = \dot{1}, \dot{2}$ and l, $\alpha = 1, 2$, while $\mathcal{H}_a$ is spanned by $l, \alpha = \dot{1}, \dot{2}$ and $r, \alpha = 1, 2$ (in both cases, C takes both values 1 and 0). In other terms, $a \in \mathcal{A}$ acts as in Eq. (2.19), but now the two 64×64 matrices Q and M are tensor fields of components

$$Q^{\dot{t}tJ\beta}_{\dot{s}sI\alpha} = \delta^{\dot{t}J}_{\dot{s}I} Q^{t\beta}_{s\alpha}, \qquad M^{\dot{t}tJ\beta}_{\dot{s}sI\alpha} = \delta^{\dot{t}}_{\dot{s}} M^{t\beta J}_{s\alpha I}, \tag{3.8}$$

where δ^{tJ}_{sI} denotes the product of the two Kronecker symbols δ^t_s and δ^J_I. Both Q and M still act trivially (i.e., as the identity) on the indices $\dot{s}\,\dot{t}$ but no longer on the chiral indices st. On the latter, the action is given by

$$Q^{t\beta}_{s\alpha} = \begin{pmatrix} (Q_r)^\beta_\alpha & \\ & (Q_l)^\beta_\alpha \end{pmatrix}^t_s,$$
$$M^{t\beta J}_{s\alpha I} = \begin{pmatrix} (M_r)^{\beta J}_{\alpha I} & \\ & (M_l)^{\beta J}_{\alpha I} \end{pmatrix}^t_s, \tag{3.9}$$

with

$$Q_r = \begin{pmatrix} \mathsf{c} & \\ & q' \end{pmatrix}^\beta_\alpha, \qquad Q_l = \begin{pmatrix} \mathsf{c}' & \\ & q \end{pmatrix}^\beta_\alpha, \tag{3.10}$$

and

$$M_r = \begin{pmatrix} \mathsf{m} \otimes \mathbb{I}_2 & 0 \\ 0 & \mathsf{m}' \otimes \mathbb{I}_2 \end{pmatrix}^\beta_\alpha,$$
$$M_l = \begin{pmatrix} \mathsf{m}' \otimes \mathbb{I}_2 & 0 \\ 0 & \mathsf{m} \otimes \mathbb{I}_2 \end{pmatrix}^\beta_\alpha, \tag{3.11}$$

where we denote

$$\mathsf{c} \coloneqq \begin{pmatrix} c & \\ & \bar{c} \end{pmatrix}, \qquad \mathsf{m} \coloneqq \begin{pmatrix} c & \\ & m \end{pmatrix}^J_I,$$
$$\mathsf{c}' \coloneqq \begin{pmatrix} c' & \\ & \bar{c}' \end{pmatrix}, \qquad \mathsf{m}' \coloneqq \begin{pmatrix} c' & \\ & m' \end{pmatrix}^J_I. \tag{3.12}$$

Compared to the usual spectral triple of the Standard Model, $M_{r/l}$ are no longer trivial in the flavor index α.

Remark 3.1.—If we were using the twist-by grading, we should permute m with m' in Eq. (3.11), for on the antiparticle subspace—i.e., $C = 1$—then $\mathcal{H}_c$ is a subspace of the -1 eigenspace of the grading (see also Appendix 7 regarding the twist used in Ref. [16]).

The twist ρ is the automorphism of $\mathcal{A}$ that exchanges the two components of $\mathcal{A}_{\rm SM}$, namely,

$$\rho(c, c', q, q', m, m') = (c', c, q', q, m', m). \tag{3.13}$$

In terms of the representation, one has

$$\rho(a) = \begin{pmatrix} \rho(Q) & \\ & \rho(M) \end{pmatrix}^D_C \tag{3.14}$$

with

$$\rho(Q)^{\dot{t}tJ\beta}_{\dot{s}sI\alpha} = \delta^{\dot{t}J}_{\dot{s}I} \rho(Q)^{t\beta}_{s\alpha}, \qquad \rho(M)^{\dot{t}tJ\beta}_{\dot{s}sI\alpha} = \delta^{\dot{t}}_{\dot{s}} \rho(M)^{t\beta J}_{s\alpha I}, \tag{3.15}$$

where

$$\rho(Q)^{t\beta}_{s\alpha} = \begin{pmatrix} (Q_l)^\beta_\alpha & \\ & (Q_r)^\beta_\alpha \end{pmatrix}^t_s,$$
$$\rho(M)^{t\beta J}_{s\alpha I} = \begin{pmatrix} (M_l)^{\beta J}_{\alpha I} & \\ & (M_r)^{\beta J}_{\alpha I} \end{pmatrix}^t_s. \tag{3.16}$$

In short, the twist amounts to flipping the left and right indices l and r.

B. Grading and real structure

The operators Γ in Eq. (2.7) and J in Eq. (2.10) are the grading and the real structure for the twisted spectral triple, respectively, in the sense defined in Refs. [16,19] [the rule of signs defining the KO dimension is not affected by the twist; that Γ commutes with the representation (3.9) follows from the latter being diagonal but on the α and I indices, where Γ is (block-)diagonal]. In particular, as in the nontwisted case, the real structure implements an action of the opposite algebra $\mathcal{A}^\circ$ on $\mathcal{H}$ that commutes with the one of $\mathcal{A}$. To check this, let us first write down the representation of the opposite algebra.

Proposition 3.2.—For $a \in \mathcal{A}$ as in Eq. (2.19), one has (for $\mathcal{M}$ of dimension 4)

$$JaJ^{-1} = -\begin{pmatrix} \bar{M} & 0 \\ 0 & \bar{Q} \end{pmatrix}_C^D. \tag{3.17}$$

Proof.—From Eqs. (2.10) and (2.33), one has

$$J = \begin{pmatrix} 0 & \mathcal{J}\otimes\mathbb{I}_{16} \\ \mathcal{J}\otimes\mathbb{I}_{16} & 0 \end{pmatrix}_C^D. \tag{3.18}$$

Since $J^{-1} = -J$ by Eq. (2.11), using the representation (2.19) of a, one obtains (omitting $\mathbb{I}_{16}$)

$$JaJ^{-1} = -JaJ = -\begin{pmatrix} 0 & \mathcal{J} \\ \mathcal{J} & 0 \end{pmatrix}_C^E \begin{pmatrix} Q & 0 \\ 0 & M \end{pmatrix}_E^F \begin{pmatrix} 0 & \mathcal{J} \\ \mathcal{J} & 0 \end{pmatrix}_F^D$$
$$= -\begin{pmatrix} \mathcal{J}M\mathcal{J} & 0 \\ 0 & \mathcal{J}Q\mathcal{J} \end{pmatrix}_C^D. \tag{3.19}$$

In addition, $\mathcal{J}$ commutes with the grading $\gamma_\mathcal{M}$ [see Eq. (2.8)], so it is of the form

$$\mathcal{J} = \begin{pmatrix} \mathcal{J}_r & 0 \\ 0 & \mathcal{J}_l \end{pmatrix}_s^t cc, \tag{3.20}$$

where $\mathcal{J}_{r,l}$ are 2×2 matrices carrying the $\dot{s}, \dot{t}$ indices, such that $\mathcal{J}_r\bar{\mathcal{J}}_r = \mathcal{J}_l\bar{\mathcal{J}}_l = -\mathbb{I}_2$. From the explicit form (3.8) of Q and M, one gets (still omitting the indices α and I in which J is trivial)

$$\mathcal{J}Q\mathcal{J} = \begin{pmatrix} \mathcal{J}_r(\delta_{\dot{s}}^{\dot{t}}\bar{Q}_r)\bar{\mathcal{J}}_r & 0 \\ 0 & \mathcal{J}_l(\delta_{\dot{s}}^{\dot{t}}\bar{Q}_l)\bar{\mathcal{J}}_l \end{pmatrix}_s^t$$
$$= \begin{pmatrix} -\delta_{\dot{s}}^{\dot{t}}\bar{Q}_r & 0 \\ 0 & -\delta_{\dot{s}}^{\dot{t}}\bar{Q}_l \end{pmatrix}_s^t = -\bar{Q}, \tag{3.21}$$

$$\mathcal{J}M\mathcal{J} = \begin{pmatrix} \mathcal{J}_r(\delta_{\dot{s}}^{\dot{t}}\bar{M}_r)\bar{\mathcal{J}}_r & 0 \\ 0 & \mathcal{J}_l(\delta_{\dot{s}}^{\dot{t}}\bar{M}_l)\bar{\mathcal{J}}_l \end{pmatrix}_s^t$$
$$= \begin{pmatrix} -\delta_{\dot{s}}^{\dot{t}}\bar{M}_r & 0 \\ 0 & -\delta_{\dot{s}}^{\dot{t}}\bar{M}_l \end{pmatrix}_s^t = -\bar{M}, \tag{3.22}$$

and, hence, the result. ∎

To check the order-zero condition, we denote

$$b = (d, d', p, p', n, n') \tag{3.23}$$

another element of $\mathcal{A}$ with $d, d' \in C^\infty(\mathcal{M}, \mathbb{C})$, $p, p' \in C^\infty(\mathcal{M}, \mathbb{H})$, and $n, n' \in C^\infty(\mathcal{M}, M_3(\mathbb{C}))$. It acts on $\mathcal{H}$ by Eq. (3.24) as

$$b = \begin{pmatrix} R & \\ & N \end{pmatrix}_C^D, \tag{3.24}$$

where R and N are defined as Q and M in Eq. (3.8), with

$$R_r = \begin{pmatrix} \mathsf{d} & \\ & p' \end{pmatrix}_\alpha^\beta, \qquad R_l = \begin{pmatrix} \mathsf{d}' & \\ & p \end{pmatrix}_\alpha^\beta,$$
$$N_r = \begin{pmatrix} \mathsf{n}\otimes\mathbb{I}_2 & \\ & \mathsf{n}'\otimes\mathbb{I}_2 \end{pmatrix}_\alpha^\beta, \qquad N_l = \begin{pmatrix} \mathsf{n}'\otimes\mathbb{I}_2 & \\ & \mathsf{n}\otimes\mathbb{I}_2 \end{pmatrix}_\alpha^\beta. \tag{3.25}$$

Corollary 3.2.1.—The order-zero condition (2.12) holds.

Proof.—By Proposition 3.2, the order-zero condition $[a, JbJ^{-1}] = 0$ for all $a, b \in \mathcal{A}$ is equivalent to $[R, \bar{M}] = 0$ and $[N, \bar{Q}] = 0$. By Eqs. (3.8) and (3.9), one gets (omitting the indices $\dot{s}\,\dot{t}$ on which all actions are trivial)

$$[R, M] = \begin{pmatrix} [\delta_I^J R_r, M_r] & 0 \\ 0 & [\delta_I^J R_l, M_l]. \end{pmatrix}_s^t. \tag{3.26}$$

By Eq. (3.11), one has

$$[\delta_I^J R_r, M_r] = \begin{pmatrix} [\delta_I^J \mathsf{d}, \mathsf{m}\otimes\mathbb{I}_2] & 0 \\ 0 & [\delta_I^J p', \mathsf{m}'\otimes\mathbb{I}_2] \end{pmatrix}_\alpha^\beta, \tag{3.27}$$

which is zero, as can be seen writing $\delta_I^J \mathsf{d} = \mathbb{I}_4 \otimes \mathsf{d}$ and similarly for $[\delta_I^J p', \mathsf{m}'\otimes\mathbb{I}_2]$. The same holds true for $[\delta_I^J R_l, M_l]$. ∎

C. Twisted fluctuation

In the twisted context, fluctuations are similar to Eq. (3.1), replacing the commutator for a twisted one [20]. In addition, if the twisted first-order condition does not hold, one should add a nonlinear term [13,22]. We thus consider the *twisted-covariant* Dirac operator

$$D_A = D + A_{(1)} + \hat{A}_{(1)} + A_{(2)}, \tag{3.28}$$

where

$$A_{(1)} = \sum_i a_i[D, b_i]_\rho, \qquad a_i, b_i \in \mathcal{A}, \tag{3.29}$$

is a twisted (generalized) 1-form and $\hat{A}_{(1)} \coloneqq JA_{(1)}J^{-1}$ is its image by the conjugation with the real structure, while

$$A_{(2)} = \sum_i \hat{a}_i[A_\rho, \hat{b}_i]_{\rho^\circ} \quad \text{with} \quad \hat{a}_i \coloneqq Ja_iJ^{-1} = (a_i^*)^\circ,$$
$$\hat{b}_i \coloneqq JB_iJ^{-1} = (b_i^*)^\circ \tag{3.30}$$

and ρ° denotes the automorphism of the opposite algebra defined as

$$\rho^\circ(a^\circ) \coloneqq (\rho^{-1}(a))^\circ. \tag{3.31}$$

The term $A_{(2)}$ breaks the linearity of the map $A_{(1)} \to D + A_{(1)} + JA_{(1)}J^{-1}$ and vanishes when the twisted first-order condition (A16) holds (this is a straightforward adaptation to the twisted context of the result of Ref. [13]). We need to take it into account for, as explained in Sec. VII, the twisted first-order condition holds only partially.

The twisted 1-form decomposes as the sum $A_{(1)} = A_F + \not{A}$ of two pieces: one that we call the *finite part* of the fluctuation because it comes from the finite-dimensional spectral triple, namely,

$$A_F = \sum_i a_i[\gamma_{\mathcal{M}} \otimes D_F, b_i]_\rho, \qquad a_i, b_i \in \mathcal{A}, \tag{3.32}$$

and another one coming from the manifold part of the spectral triple

$$\not{A} = \sum_i a_i[\not{D}, b_i]_\rho, \qquad a_i, b_i \in \mathcal{A}, \tag{3.33}$$

that we call *gauge part* in the following (terminology will become clear later).

To guarantee that the twisted covariant operator (3.28) is self-adjoint, one assumes that the twisted 1-form $A_{(1)}$ is self-adjoint (Proposition 3.8 in Ref. [22]) (actually, this is not a necessary condition, but requiring $A_{(1)}$ to be self-adjoint makes sense viewing the fluctuation $D \to D_A$ as a three-step process

$$D \to D + A_{(1)} \to D + A_{(1)} + \hat{A}_{(1)} \to D_A \tag{3.34}$$

such that self-adjointness is preserved at each step). This means that, for physical models, we assume that both the gauge $\not{A}$ and the finite A_F parts are self-adjoint.

So far, the construction works for any even-dimension manifold $\mathcal{M}$. To build explicitly the Standard Model, from now on one fixes the dimension of $\mathcal{M}$ to $m = 4$. The grading and the real structure are

$$\gamma_{\mathcal{M}} = \gamma^5 = \gamma_E^0\gamma_E^1\gamma_E^2\gamma_E^3 = \begin{pmatrix} \mathbb{I}_4 & 0 \\ 0 & -\mathbb{I}_4 \end{pmatrix}_s^t = \eta_s^t \delta_{\dot{s}}^{\dot{t}} \tag{3.35}$$

and

$$\mathcal{J} = i\gamma_E^0\gamma_E^2 cc = i\begin{pmatrix} \tilde{\sigma}^2 & 0_2 \\ 0_2 & \sigma^2 \end{pmatrix}_{st} cc = -i\eta_s^t \tau_{\dot{s}}^{\dot{t}} cc, \tag{3.36}$$

respectively, where cc denotes the complex conjugation and we define

$$\tau_{\dot{s}}^{\dot{t}} \coloneqq \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\dot{s}}^{\dot{t}}, \qquad \eta_s^t \coloneqq \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}_s^t. \tag{3.37}$$

For the internal spectral triple, one has

$$\gamma_F = \begin{pmatrix} \mathbb{I}_8 & & & \\ & -\mathbb{I}_8 & & \\ & & -\mathbb{I}_8 & \\ & & & \mathbb{I}_8 \end{pmatrix} = \eta_{C\alpha}^{D\beta}\delta_I^J,$$
$$J_F = \begin{pmatrix} 0 & \mathbb{I}_{16} \\ \mathbb{I}_{16} & 0 \end{pmatrix}_C^D cc = \xi_C^D \delta_{I\alpha}^{J\beta}, \tag{3.38}$$

where the matrix γ_F is written in the basis left and right particles and then left and right antiparticles, and we define

$$\eta_\alpha^\beta \coloneqq \begin{pmatrix} \mathbb{I}_2 & \\ & -\mathbb{I}_2 \end{pmatrix}_\alpha^\beta, \qquad \eta_C^D \coloneqq \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}_C^D,$$
$$\xi_C^D \coloneqq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_C^D \tag{3.39}$$

with $\eta_{C\alpha}^{D\beta}$ holding for $\eta_C^D\eta_\alpha^\beta$. Thus,

$$\Gamma = \gamma_{\mathcal{M}} \otimes \gamma_F = \eta_{sC\alpha}^{tD\beta}\delta_{\dot{s}}^{\dot{t}} \quad \text{and}$$
$$J = J_{\mathcal{M}} \otimes J_F = -i\eta_s^t \tau_{\dot{s}}^{\dot{t}} \xi_D^C \delta_{I\alpha}^{J\beta} cc. \tag{3.40}$$

IV. SCALAR PART OF THE TWISTED FLUCTUATION

The scalar sector of the twisted Standard Model is obtained from the finite part (3.32) of the twisted 1-form, which in turn decomposes into a diagonal part (determined by the Yukawa couplings of fermions)

$$A_Y = \sum_i a_i[\gamma^5 \otimes D_Y, b_i]_\rho \tag{4.1}$$

and an off-diagonal part (determined by the Majorana mass of the neutrino)

$$A_M = \sum_i a_i[\gamma^5 \otimes D_M, b_i]_\rho. \tag{4.2}$$

As shown below, the former produces the Higgs sector, the latter a pair of extra scalar fields.

A. The Higgs sector

We begin with the diagonal part (4.1). We first notice that the $M_3(\mathbb{C})$ part of the algebra (3.4) twist-commutes with $\gamma^5 \otimes D_Y$.

Lemma 4.1.—For any $b \in \mathcal{A}$ as in Eq. (3.24), one has

$$[\gamma^5 \otimes D_Y, b]_\rho = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}_D^C, \tag{4.3}$$

where S has components

$$S^{t\dot{t}J\beta}_{s\dot{s}I\alpha} = \delta^{\dot{t}}_{\dot{s}}(\eta^u_s (D_0)^{J\gamma}_{I\alpha} R^{t\beta}_{s\gamma} - \rho(R)^{u\gamma}_{s\alpha}\eta^t_u (D_0)^{J\beta}_{I\gamma}). \tag{4.4}$$

Proof.—From the explicit forms (2.24) of D_Y and (3.24) of b, one has

$$[\gamma^5 \otimes D_Y, b]_\rho = \begin{pmatrix} [\gamma^5 \otimes D_0, R]_\rho & \\ & [\gamma^5 \otimes D_0^\dagger, N]_\rho \end{pmatrix}_C^D.$$

In the tensorial notation, $S \coloneqq [\gamma^5 \otimes D_0, R]_\rho$ has components

$$S^{t\dot{t}J\beta}_{s\dot{s}I\alpha} = \eta^u_s \delta^{\dot{u}}_{\dot{s}} (D_0)^{K\gamma}_{I\alpha} \delta^{\dot{t}J}_{\dot{u}K} R^{t\beta}_{u\gamma} - \delta^{\dot{u}K}_{\dot{s}I}\rho(R)^{u\gamma}_{s\alpha}\eta^t_u \delta^{\dot{t}}_{\dot{u}}(D_0)^{J\beta}_{K\gamma} \tag{4.5}$$

$$= \delta^{\dot{t}}_{\dot{s}}(\eta^u_s (D_0)^{J\gamma}_{I\alpha} R^{t\beta}_{u\gamma} - \rho(R)^{u\gamma}_{s\alpha}\eta^t_u (D_0)^{J\beta}_{I\gamma}), \tag{4.6}$$

which shows (4.4). To show that

$$[\gamma^5 \otimes D_0^\dagger, N]_\rho = 0, \tag{4.7}$$

let us denote T the left-hand side of the equation above. It has components

$$T^{t\dot{t}J\beta}_{s\dot{s}I\alpha} = \eta^u_s \delta^{\dot{u}}_{\dot{s}} (D_0^\dagger)^{K\gamma}_{I\alpha} \delta^{\dot{t}}_{\dot{u}} N^{t\beta J}_{u\gamma K} - \delta^{\dot{u}}_{\dot{s}}\rho(N)^{u\gamma K}_{s\alpha I}\eta^t_u \delta^{\dot{t}}_{\dot{u}}(D_0^\dagger)^{J\beta}_{K\gamma} \tag{4.8}$$

$$= \delta^{\dot{t}}_{\dot{s}}(\eta^u_s (D_0^\dagger)^{K\gamma}_{I\alpha} N^{t\beta J}_{u\gamma K} - \rho(N)^{u\gamma K}_{s\alpha I}\eta^t_u (D_0^\dagger)^{J\beta}_{K\gamma}) \tag{4.9}$$

$$= \delta^{\dot{t}}_{\dot{s}} \begin{pmatrix} (D_0^\dagger)^{K\gamma}_{I\alpha}(N_r)^{\beta J}_{\gamma K} - (N_l)^{\gamma K}_{\alpha I}(D_0^\dagger)^{J\gamma}_{K\alpha} & 0 \\ 0 & -(D_0^\dagger)^{K\gamma}_{I\alpha}(N_l)^{\beta J}_{\gamma K} + (N_r)^{\gamma K}_{\alpha I}(D_0^\dagger)^{J\beta}_{K\gamma} \end{pmatrix}_s^t. \tag{4.10}$$

Since $(D_0^\dagger)^K_I = \delta^K_I (D_0^I)$ and $(D_0^\dagger)^K_J = \delta^K_J (D_0^J)$ (with no summation on I and J), the upper-left term in Eq. (4.9) is

$$(D_0^I)^\delta_\gamma (N_r)^{\beta J}_{\delta I} - (N_l)^{\gamma J}_{\alpha I}(D_0^J)^\beta_\gamma = \begin{pmatrix} 0 & \bar{\mathsf{k}}^I(\mathsf{n}' \otimes \mathbb{I}_2) \\ \mathsf{k}^I(\mathsf{n} \otimes \mathbb{I}_2) & 0 \end{pmatrix}_\alpha^\beta - \begin{pmatrix} 0 & (\mathsf{n}' \otimes \mathbb{I}_2)\bar{\mathsf{k}}^\mathsf{J} \\ (\mathsf{n} \otimes \mathbb{I}_2)\mathsf{k}^\mathsf{J} & 0 \end{pmatrix}_\alpha^\beta, \tag{4.11}$$

where we omitted the I, J indices on n. One has

$$\mathsf{k}^I(\mathsf{n} \otimes \mathbb{I}_2) = \begin{pmatrix} k^I_u \mathsf{n} & \\ & k^I_d \mathsf{n} \end{pmatrix},$$

$$(\mathsf{n} \otimes \mathbb{I}_2)\mathsf{k}^\mathsf{J} = \begin{pmatrix} \mathsf{n} k^J_u & \\ & \mathsf{n} k^J_d \end{pmatrix}, \tag{4.12}$$

and similarly for the terms in n'. Restoring the indices, one has

$$k^I_u \mathsf{n}^J_I = \begin{pmatrix} k^l_u d & \\ & k^q_u n \end{pmatrix}_I^J, \quad \mathsf{n}^J_I k^J_u = \begin{pmatrix} d k^l_u & \\ & n k^q_u \end{pmatrix}_I^J, \tag{4.13}$$

where we write $k^{I=0}_u = k^l_u$ for the lepton and $k^{I=1,2,3}_u = k^q_u$ for the colored quarks. Again, in the expression above, there is no summation on I and J: $k^I_u \mathsf{n}^J_I$ means the matrix n in which the Ith line is multiplied by k^I_u, while in $\mathsf{n}^J_I k^J_u$ this is the Jth column of n which is multiplied by k^J_u. Therefore,

$$\mathsf{k}^I(\mathsf{n} \otimes \mathbb{I}_2) - (\mathsf{n} \otimes \mathbb{I}_2)\mathsf{k}^J = 0. \tag{4.14}$$

Similarly, $\bar{\mathsf{k}}^I(\mathsf{n}' \otimes \mathbb{I}_2) - (\mathsf{n}' \otimes \mathbb{I}_2)\bar{\mathsf{k}}^J = 0$, so that Eq. (4.11)—that is, the upper-left term in Eq. (4.10)—is zero. The proof that the lower-right term is zero is similar. Hence (4.7) and the result. ■

A similar result holds in the nontwisted case [the computation is similar as above, with $\mathsf{n}' = \mathsf{n}$, so that everything boils down to the single equation (4.14)]. The result, however, is not true if one genuinely generalizes the twist used in Ref. [16]. As explained below, this yields an additional violation of the twisted first-order condition, besides the one required to generate the field σ. That is why we do not use this genuine twist but rather the one presented in Sec. III.

Remark 4.2.—The twist in Ref. [16] was not applied to the $M_3(\mathbb{C})$ part of the algebra. Only $\mathbb{C} \oplus \mathbb{H}$ was doubled, and this yielded an action similar to the one used on the present paper [modulo a change of notations, the representation (4.7) of Ref. [16] coincides with Eq. (3.9)]. A genuine generalization of this twist consists in making two copies of $M_3(\mathbb{C})$ acting independently on the left and right components of spinors; namely, $a \in \mathcal{A}$ acts as in Eq. (3.8), but now $M_{r,l}$ are given by

$$M_r = (\,\mathsf{m}\otimes\mathbb{I}_4\,)^\beta_\alpha, \qquad M_l = (\,\mathsf{m}'\otimes\mathbb{I}_4\,)^\beta_\alpha. \tag{4.15}$$

Then Lemma 4.1 no longer holds, for the lower-right term T is not necessarily zero [on the rhs of Eq. (4.11), the first parentheses now contain only n, and the second only n', so that the cancellation (4.14) is no longer true].

We now compute the 1-forms generated by the Yukawa couplings of the fermions. In order to do so, we extend the action of the automorphism ρ to any polynomial in q, q', p, p', c, c', and d, d'. Namely, ρ "primes" what is unprimed, and vice versa. For instance, $\rho(qp' - c'd) = q'p - cd'$.

Proposition 4.3.—The diagonal part (4.1) of a twisted 1-form is

$$A_Y = \begin{pmatrix} A & \\ & 0 \end{pmatrix}^D_C, \quad \text{where } A = \delta^{\dot{t}J}_{\dot{s}I}\begin{pmatrix} A_r & \\ & A_l \end{pmatrix}^t_s \tag{4.16}$$

with

$$A_r = \begin{pmatrix} & \overline{\mathsf{k}^\mathsf{I}}H_1 \\ H_2\mathsf{k}^\mathsf{I} & \end{pmatrix}^\beta_\alpha,$$
$$A_l = -\begin{pmatrix} & \overline{\mathsf{k}^\mathsf{I}}H_1' \\ H_2'\mathsf{k}^\mathsf{I} & \end{pmatrix}^\beta_\alpha, \tag{4.17}$$

where $H_{i=1,2}$ and $H'_{1,2} = \rho(H_{1,2})$ are quaternionic fields.

Proof.—From Eq. (2.19) and Lemma 4.1, one has $a[\gamma^5\otimes D_Y, b]_\rho = QS$. In components, this gives [using the explicit forms (3.8) of Q, R]

$$A^{t\dot{t}J\beta}_{s\dot{s}I\alpha} = Q^{u\dot{u}K\gamma}_{s\dot{s}I\alpha}\delta^{\dot{t}}_{\dot{u}}[\eta^v_u(D_0)^{J\delta}_{K\gamma}R^{t\beta}_{v\delta} - \rho(R)^{v\delta}_{u\gamma}\eta^t_v(D_0)^{J\beta}_{K\delta}] \tag{4.18}$$

$$= \delta^{\dot{t}J}_{\dot{s}I}Q^{u\gamma}_{s\alpha}[\eta^v_u(D^I_0)^\delta_\gamma R^{t\beta}_{v\delta} - \rho(R)^{v\delta}_{u\gamma}\eta^t_v(D^I_0)^\beta_\delta], \tag{4.19}$$

where we use $\delta^K_I(D_0)^J_K = \delta^J_I(D^I_0)$ (with no summation on I in the last expression). Since Q is diagonal on the chiral indices s, the only nonzero components of A are for $s = t = r$ and $s = t = l$, namely,

$$A^{r\dot{t}J\beta}_{r\dot{s}I\alpha} = \delta^{\dot{t}J}_{\dot{s}I}(A^I_r)^\beta_\alpha \quad \text{with}$$
$$(A^I_r)^\beta_\alpha = (Q_r)^\gamma_\alpha[(D^I_0)^\delta_\gamma(R_r)^\beta_\delta - (R_l)^\delta_\gamma(D^I_0)^\beta_\delta], \tag{4.20}$$

$$A^{l\dot{t}J\beta}_{l\dot{s}I\alpha} = \delta^{\dot{t}J}_{\dot{s}I}(A^I_l)^\beta_\alpha \quad \text{with}$$
$$(A^I_l)^\beta_\alpha = (Q_l)^\gamma_\alpha[-(D^I_0)^\delta_\gamma(R_l)^\beta_\delta + (R_r)^\delta_\gamma(D^I_0)^\beta_\delta]. \tag{4.21}$$

From the explicit expression (3.10), (3.25), and (2.26) of $Q_{r/l}$, $R_{r/l}$, and D^I_0, respectively, one gets

$$Q_rD^I_0R_r = \begin{pmatrix} & \mathsf{c}\overline{\mathsf{k}^\mathsf{I}}p' \\ q'\mathsf{k}^\mathsf{I}\mathsf{d} & \end{pmatrix}^\beta_\alpha,$$
$$Q_rR_lD^I_0 = \begin{pmatrix} & \mathsf{c}\mathsf{d}'\overline{\mathsf{k}^\mathsf{I}} \\ q'p\mathsf{k}^\mathsf{I} & \end{pmatrix}^\beta_\alpha, \tag{4.22}$$

$$Q_lD^I_0R_l = \begin{pmatrix} & \mathsf{c}'\overline{\mathsf{k}^\mathsf{I}}p \\ q\mathsf{k}^\mathsf{I}\mathsf{d}' & \end{pmatrix}^\beta_\alpha,$$
$$Q_lR_rD^I_0 = \begin{pmatrix} & \mathsf{c}'\mathsf{d}\overline{\mathsf{k}^\mathsf{I}} \\ qp'\mathsf{k}^\mathsf{I} & \end{pmatrix}^\beta_\alpha. \tag{4.23}$$

Using that $\mathsf{c}, \mathsf{c}', \mathsf{d}, \mathsf{d}'$ commute with k^I, one has

$$Q_r(D^I_0R_r - R_lD^I_0) = \begin{pmatrix} & \bar{\mathsf{k}}^IH_1 \\ H_2\mathsf{k}^I & \end{pmatrix}^\beta_\alpha,$$
$$-Q_lD^I_0R_l + Q_lR_rD^I_0 = -\begin{pmatrix} & \bar{\mathsf{k}}^IH_1' \\ H_2'\mathsf{k}^I & \end{pmatrix}^\beta_\alpha, \tag{4.24}$$

where

$$H_1 \coloneqq \mathsf{c}(p' - \mathsf{d}'), \qquad H_2 \coloneqq q'(\mathsf{d} - p),$$
$$H_1' \coloneqq \mathsf{c}'(p - \mathsf{d}), \qquad H_2' \coloneqq q(\mathsf{d}' - p'). \tag{4.25}$$

This shows the result. ■

Imposing now self-adjointness as stressed before (3.34) at the beginning of this section, we get the following corollary.

Corollary 4.3.1.—A self-adjoint diagonal twisted 1-form (4.1) is parametrized by two independent scalar quaternionic field H_r and H_l.

Proof.—The twisted 1-form (4.16) is self-adjoint if and only if

$$H_2 = H_1^\dagger \eqqcolon H_r \quad \text{and} \quad H_2' = {H'}_1^\dagger \eqqcolon H_l. \tag{4.26}$$

They are independent as follows from their definition (4.25). ■

Since $\gamma^5\otimes D_Y$ satisfies the twisted first-order condition (Proposition A.8), it does not contribute to the nonlinear term $A_{(2)}$ of the twisted fluctuation. Gathering the results of this section, one thus works out the fields induced by the Yukawa coupling of fermions via a twisted fluctuation of the metric.

Proposition 4.4.—A self-adjoint diagonal fluctuation is

$$D_{A_Y} = \gamma^5\otimes D_Y + A_Y + \widehat{A_Y}$$
$$= \begin{pmatrix} \eta^t_s\delta^{\dot{t}}_{\dot{s}}D_0 + A & \\ & \eta^t_s\delta^{\dot{t}}_{\dot{s}}D_0^\dagger + \bar{A} \end{pmatrix}^D_C, \tag{4.27}$$

where

$$A = \delta^{\dot{t}J}_{\dot{s}I}\begin{pmatrix} A_r & \\ & A_l \end{pmatrix}^t_s$$

is generated by two quaternionic fields H_r and H_l as

$$A_r = \begin{pmatrix} & \bar{\mathsf{k}}^I H_r^\dagger \\ H_r \mathsf{k}^I & \end{pmatrix}^\beta_\alpha,$$
$$A_l = \begin{pmatrix} & \bar{\mathsf{k}}^I H_l^\dagger \\ H_l \mathsf{k}^I & \end{pmatrix}^\beta_\alpha. \qquad (4.28)$$

Proof.—Remembering that $J^{-1} = -J$, Proposition 4.3 yields

$$\begin{aligned}\widehat{A_Y} &= J A_Y J^{-1} \\ &= \begin{pmatrix} 0 & \mathcal{J} \\ \mathcal{J} & 0 \end{pmatrix}^D_C \begin{pmatrix} A & \\ & 0 \end{pmatrix}^D_C \begin{pmatrix} 0 & -\mathcal{J} \\ -\mathcal{J} & 0 \end{pmatrix}^D_C \\ &= \begin{pmatrix} 0 & \\ & -\mathcal{J}A\mathcal{J}^{-1} \end{pmatrix}^D_C. \end{aligned} \qquad (4.29)$$

From the explicit form (3.36) of $\mathcal{J} = -\mathcal{J}$ and (4.16) of A, one obtains (omitting the IJ and $\alpha\beta$ indices in which the real structure J is trivial)

$$\begin{aligned}\mathcal{J}A\mathcal{J}^{-1} &= \eta^u_s \tau^{\dot{u}}_{\dot{s}} \bar{A}^{v\dot{v}}_{u\dot{u}} \eta^t_v \tau^{\dot{t}}_{\dot{v}} = \eta^u_s \tau^{\dot{u}}_{\dot{s}} \begin{pmatrix} \bar{A}_r & 0 \\ 0 & \bar{A}_l \end{pmatrix}^t_s \eta^t_v \tau^{\dot{t}}_{\dot{v}} \\ &= \begin{pmatrix} -\bar{A}_r & 0 \\ 0 & -\bar{A}_l \end{pmatrix}^t_s = -\bar{A}, \end{aligned} \qquad (4.30)$$

where we used (4.16) and write $\tau^{\dot{u}}_{\dot{s}} \delta^{\dot{v}}_{\dot{u}} \tau^{\dot{t}}_{\dot{v}} = -\delta^{\dot{t}}_{\dot{s}}$.

The result follows summing Eq. (4.29) with A_Y given in Proposition 4.3 and D_Y given in Eq. (2.24), then using Corollary 4.3.1 to rename H_r and H_l. ∎

In the nontwisted case, the primed and unprimed quantities are equal, so that one obtains only one quaternionic field $H_r = H_l$, which combines in the action as

$$H \coloneqq H_r + H_l = \begin{pmatrix} \phi_1 & -\bar{\phi}_2 \\ \phi_2 & \bar{\phi}_1 \end{pmatrix}, \qquad (4.31)$$

whose complex components ϕ_1 and ϕ_2 identify with the Higgs doublet. In the twisted case, the complex components $\phi^r_{1,2}$ and $\phi^l_{1,2}$ of H_r and H_l define two scalar doublets

$$\Phi_r \coloneqq \begin{pmatrix} \phi^r_1 \\ \phi^r_2 \end{pmatrix}, \qquad \Phi_l \coloneqq \begin{pmatrix} \phi^l_1 \\ \phi^l_2 \end{pmatrix}, \qquad (4.32)$$

which act, respectively, on the right and on the left part of the Dirac spinors. However, similar to Eq. (4.31), they appear in the fermionic action only through their linear combination $H_r + H_l$ [21]; therefore, there is actually only one physical Higgs doublet in the twisted case as well.

B. The extra scalar field

The computation of the off-diagonal term (4.2) of the finite part of the twisted 1-form is easier than for the diagonal part, because D_M has only one nonzero component.

Proposition 4.5.—The off-diagonal part (4.2) of a twisted 1-form is

$$A_M = \begin{pmatrix} & C \\ D & \end{pmatrix}^D_C, \qquad (4.33)$$

where

$$C = k_R \delta^{\dot{t}}_{\dot{s}} \begin{pmatrix} C_r & \\ & C_l \end{pmatrix}^t_s,$$
$$D = \bar{k}_R \delta^{\dot{t}}_{\dot{s}} \begin{pmatrix} D_r & \\ & D_l \end{pmatrix}^t_s \qquad (4.34)$$

with

$$C_r = D_r = \Xi^{J\beta}_{I\alpha}\sigma, \qquad C_l = D_l = -\Xi^{J\beta}_{I\alpha}\sigma' \qquad (4.35)$$

where σ and σ' are complex fields.

Proof.—Using the explicit form (2.24) of D_M, for a in (2.19) and b in (3.24), one gets

$$\begin{aligned} a[\gamma^5 \otimes D_M, b]_\rho &= \begin{pmatrix} Q & 0 \\ 0 & M \end{pmatrix} \left[\begin{pmatrix} 0 & \gamma^5 \otimes D_R \\ \gamma^5 \otimes D_R^\dagger & 0 \end{pmatrix}, \begin{pmatrix} R & 0 \\ 0 & N \end{pmatrix} \right]_\rho \\ &= \begin{pmatrix} & Q((\gamma^5 \otimes D_R)N - \rho(R)(\gamma^5 \otimes D_R)) \\ M((\gamma^5 \otimes D_R^\dagger)R - \rho(N)(\gamma^5 \otimes D_R^\dagger)) & \end{pmatrix}^D_C. \end{aligned} \qquad (4.36)$$

With D_R given in Eq. (2.29), one computes the upper-right component C of the matrix above:

$$C^{\dot{t}\dot{t}J\beta}_{s\dot{s}I\alpha} = Q^{u\dot{u}K\gamma}_{s\dot{s}I\alpha}[k_R \eta^v_u \delta^{\dot{v}}_{\dot{u}} \Xi^{L\delta}_{K\gamma} N^{t\dot{t}J\beta}_{v\dot{v}L\delta} - k_R \rho(R)^{v\dot{v}L\delta}_{u\dot{u}K\gamma} \eta^t_v \delta^{\dot{t}}_{\dot{v}} \Xi^{J\beta}_{L\delta}]. \qquad (4.37)$$

Since Q and N are diagonal in the s index and proportional to $\delta^{\dot{t}}_{\dot{s}}$, the nonzero components of C are

$$(C_r)^{J\beta}_{I\alpha} = k_R \delta^{\dot{t}}_{\dot{s}} (Q_r)^{K\gamma}_{I\alpha} [\Xi^{L\delta}_{K\gamma} (N_r)^{J\beta}_{L\delta} - (R_l)^{L\delta}_{K\gamma} \Xi^{J\beta}_{L\delta}], \qquad (4.38)$$

$$(C_l)^{J\beta}_{I\alpha} = k_R\delta^{\dot{t}}_{\dot{s}}(Q_l)^{K\gamma}_{I\alpha}[-\Xi^{L\delta}_{K\gamma}(N_l)^{J\beta}_{L\delta} + (R_r)^{L\delta}_{K\gamma}\Xi^{J^\beta}_{L\delta}]. \tag{4.39}$$

Explicitly, from the formula (3.10) for $Q_{r/l}$ and (3.25) of $R_{r/l}$ and $N_{r/l}$, one gets

$$\begin{aligned} Q_r(\Xi N_r - R_l\Xi) &= \begin{pmatrix} \mathsf{c}\delta^J_I & \\ & q'\delta^j_I \end{pmatrix}^\beta_\alpha \left(\begin{pmatrix} \Xi^J_I & \\ & 0_3 \end{pmatrix}^\beta_\alpha \begin{pmatrix} \mathsf{n}\otimes\mathbb{I}_2 & \\ & \mathsf{n}'\otimes\mathbb{I}_2 \end{pmatrix}^\beta_\alpha - \begin{pmatrix} \mathsf{d}'\delta^J_I & \\ & p\delta^J_I \end{pmatrix}^\beta_\alpha \begin{pmatrix} \Xi^J_I & \\ & 0_3 \end{pmatrix}^\beta_\alpha\right) \\ &= \begin{pmatrix} \mathsf{c}\delta^J_I & \\ & q'\delta^j_I \end{pmatrix}^\beta_\alpha \begin{pmatrix} \Xi^J_I d - d'\Xi^J_I & \\ & 0_3 \end{pmatrix}^\beta_\alpha = \begin{pmatrix} c(d-d')\Xi^J_I & \\ & 0_3 \end{pmatrix}^\beta_\alpha = \sigma\Xi^{J\beta}_{\alpha I} \end{aligned}$$

and, similarly,

$$Q_l(-\Xi N_l + R_r\Xi) = -\sigma'\Xi^{J\beta}_{\alpha I}, \tag{4.40}$$

where we define the scalar fields

$$\sigma \coloneqq c(d-d'), \qquad \sigma' \coloneqq c'(d'-d). \tag{4.41}$$

Similarly, one computes that the lower-left component D of (4.36) has nonzero components

$$\begin{aligned} D_r &= \bar{k}^R\delta^{\dot{t}}_{\dot{s}}M_r(\Xi R_r - N_l\Xi) = \bar{k}^R\delta^{\dot{t}}_{\dot{s}}\Xi^{\beta J}_{\alpha I}c(d-d') \\ &= \bar{k}^R\delta^{\dot{t}}_{\dot{s}}\Xi^{\beta J}_{\alpha I}\sigma, \\ D_l &= \bar{k}^R\delta^{\dot{t}}_{\dot{s}}M_l(-\Xi R_l + N_r\Xi) = \bar{k}^R\delta^{\dot{t}}_{\dot{s}}\Xi^{\beta J}_{\alpha I}c'(-d'+d) \\ &= -\bar{k}^R\delta^{\dot{t}}_{\dot{s}}\Xi^{\beta J}_{\alpha I}\sigma'. \end{aligned} \tag{4.42}$$

■

An off-diagonal 1-form A_M is self-adjoint if and only if $D_r^\dagger = C_r$ and $D_l^\dagger = C_l$, that is,

$$\sigma = \bar{\sigma}, \qquad \sigma' = \bar{\sigma}'. \tag{4.43}$$

The part of the twisted fluctuation induced by the Majorana mass of the neutrino is then easily obtained, taking into account, however, the contribution of D_M to the nonlinear term $A_{(2)}$, since $\gamma^5\otimes D_M$ violates the twisted first-order condition (cf. Proposition A.8).

Proposition 4.6.—An off-diagonal fluctuation is parametrized by two independent real scalar fields σ_r and σ_l:

$$\begin{aligned} D_{A_M} &= \gamma^5\otimes D_M + A_M + \widehat{A_M} + A_{M(2)} \\ &= \delta^{\dot{t}}_{\dot{t}}\begin{pmatrix} 0 & \eta^t_s D_0 + k_R\Xi^{J\beta}_{I\alpha}\bar{\Sigma}^t_s \\ \eta^t_s D_0^\dagger + \bar{k}_R\Xi^{J\beta}_{I\alpha}\Sigma^t_s & 0 \end{pmatrix}^D_C, \end{aligned} \tag{4.44}$$

where

$$\Sigma = \begin{pmatrix} \sigma_r & \\ & \sigma_l \end{pmatrix}^t_s. \tag{4.45}$$

Proof.—As in the proof of Proposition 4.4, one has

$$\widehat{A_M} = JA_MJ^{-1} = \begin{pmatrix} 0 & -\mathcal{J}D\mathcal{J} \\ -\mathcal{J}C\mathcal{J} & 0 \end{pmatrix}^D_C \tag{4.46}$$

with

$$\mathcal{J}C\mathcal{J}^{-1} = \eta^u_s\tau^{\dot{u}}_{\dot{s}}\bar{C}^{v\dot{v}}_{u\dot{u}}\eta^t_v\tau^{\dot{t}}_{\dot{v}} = -\bar{C} \tag{4.47}$$

and similarly for D. Hence,

$$A_M + \widehat{JA_M} = \begin{pmatrix} 0 & C+\bar{D} \\ \bar{C}+D & 0 \end{pmatrix}^D_C. \tag{4.48}$$

The nonlinear term is (omitting the summation index)

$$A_{M(2)} = \hat{a}[A_M, \hat{b}]_{\rho^\circ}. \tag{4.49}$$

By Proposition 3.2 and the explicit form (4.33) of A_M, one gets

$$\hat{a}[A_M,\hat{b}]_{\rho^\circ} = -\begin{pmatrix} \bar{M} & 0 \\ 0 & \bar{Q} \end{pmatrix}^D_C \begin{pmatrix} 0 & \overline{\rho(N)}C - C\bar{R} \\ \overline{\rho(R)}D - D\bar{N} & \end{pmatrix}^D_C, \tag{4.50}$$

where we use $\rho^\circ(\hat{b}) = \rho^\circ((b^*)^\circ) = (\rho^{-1}(b^*))^\circ = (\rho(b)^*)^\circ = \widehat{\rho(b)}$, which follows from the definition (3.31) of ρ° together with the regularity condition $\rho(a^*) = (\rho^{-1}(a))^*$ satisfied by ρ. From Eqs. (4.34) and (3.24),

$$\begin{aligned} C\bar{R} &= k_R\delta^{\dot{t}}_{\dot{s}}\Xi^{J\beta}_{I\alpha}\begin{pmatrix} \bar{d}\sigma & \\ & -\bar{d}'\sigma' \end{pmatrix}^t_s, \\ \overline{\rho(N)}C &= k_R\delta^{\dot{t}}_{\dot{s}}\Xi^{J\beta}_{I\alpha}\begin{pmatrix} \bar{d}'\sigma & \\ & -\bar{d}\sigma' \end{pmatrix}^t_s, \end{aligned} \tag{4.51}$$

$$\begin{aligned} \overline{\rho(R)}D &= \overline{k_R}\delta^{\dot{t}}_{\dot{s}}\Xi^{J\beta}_{I\alpha}\begin{pmatrix} \bar{d}'\sigma & \\ & -\bar{d}\sigma' \end{pmatrix}^t_s, \\ D\bar{N} &= \overline{k_R}\delta^{\dot{t}}_{\dot{s}}\Xi^{J\beta}_{I\alpha}\begin{pmatrix} \bar{d}\sigma & \\ & -\bar{d}'\sigma' \end{pmatrix}^t_s. \end{aligned} \tag{4.52}$$

Remembering Eq. (4.41), one obtains

$$-\bar{M}(\overline{\rho(N)}C - C\bar{R}) = k_R\delta^{\dot{i}}_{\dot{s}}\Xi^{J\beta}_{I\alpha}\begin{pmatrix}\bar{c}(\bar{d}-\bar{d}')\sigma & \\ & \overline{c'}(\bar{d}-\bar{d}')\sigma'\end{pmatrix}^t_s$$
$$= k_R\delta^{\dot{i}}_{\dot{s}}\Xi^{J\beta}_{I\alpha}\begin{pmatrix}|\sigma|^2 & \\ & -|\sigma'|^2\end{pmatrix}^t_s, \quad (4.53)$$

$$-\bar{Q}(\bar{R}'D - D\bar{N}) = \overline{k_R}\delta^{\dot{i}}_{\dot{s}}\Xi^{J\beta}_{I\alpha}\begin{pmatrix}\bar{c}(\bar{d}-\bar{d}')\sigma & \\ & \overline{c'}(\bar{d}-\bar{d}')\sigma'\end{pmatrix}^t_s$$
$$= \begin{pmatrix}|\sigma|^2 & \\ & -|\sigma'|^2\end{pmatrix}^t_s. \quad (4.54)$$

Hence,

$$A_{(2)} = \delta^{\dot{i}}_{\dot{s}}\Xi^{J\beta}_{I\alpha}\begin{pmatrix}0 & k_R\\ \overline{k_R} & 0\end{pmatrix}^D_C\begin{pmatrix}|\sigma|^2 & \\ & -|\sigma'|^2\end{pmatrix}^t_s. \quad (4.55)$$

The explicit form of Σ follows from Eqs. (4.34) and (4.35), defining

$$\sigma_r = \bar{\sigma} + \sigma + |\sigma|^2 \quad \text{and} \quad \sigma_l = -\bar{\sigma}' - \sigma' - |\sigma'|^2.$$

∎

The nonlinear term does not modify the nature of the extra scalar field σ. It simply modifies the relation between the components σ_r and σ_l and the elements of the algebra defining the twisted 1-form, introducing the terms $|\sigma|^2$ and $|\sigma'|^2$ in the equation above.

Remark 4.7.—The field σ is chiral, in the sense it has two independent components σ_r and σ_l. The one initially worked out in Ref. [16] was not chiral. This is because, in the latter case, one does not double $M_3(\mathbb{C})$ and identifies the complex component of m with the complex component of Q_r. This means that the component d' of N_l identifies with the component d of R_r, so that Eqs. (4.40) and (4.42) vanish, that is, $C_l = D_r = 0$. Similarly, the component c' of M_l becomes c, so that $D_l = C_r$. One thus retrieves the formula (4.32) of Ref. [16] (in which the roles of c and d have been interchanged). However, forcing the identification of the (nondoubled) $M_3(\mathbb{C})$ component with one of the (doubled) component of $\mathbb{C}$ is actually not compatible with the twist, as explained in greater detail in Ref. [21]. This problem is resolved in the present paper, where $M_3(\mathbb{C})$ is doubled and there is a minimal violation of the twisted first-order condition.

As an illustration that the self-adjointness of the 1-form is not necessary to get a self-adjoint twisted fluctuation (see Sec. III C), notice that in the proposition above D_{A_M} is self-adjoint regardless of the self-adjointness of A_M. As well, one does not need to assume that A_M is self-adjoint to ensure that the fields σ_r and σ_l are real.

V. GAUGE PART OF THE TWISTED FLUCTUATION

In this section, we compute the twisted fluctuation induced by the free part $D\!\!\!/ = \partial\!\!\!/ \otimes \mathbb{I}_F$ of the Dirac operator (2.6), that is,

$$D\!\!\!/ + A\!\!\!/ + JA\!\!\!/J^{-1}, \quad (5.1)$$

where $A\!\!\!/$ is the twisted 1-form (3.33) induced by $D\!\!\!/$, that we call in the following a *free 1-form*. As will be checked in Sec. VI, the components of this form are the gauge fields of the model. There is no nonlinear term $A\!\!\!/_{(2)}$, for $D\!\!\!/$ does verify the twisted first-order condition, as shown in Proposition A.8.

A. Dirac matrices and twist

We begin by recalling some useful relations between the Dirac matrices and the twist.

Lemma 5.1.—If an operator $\mathcal{O}$ on $L^2(\mathcal{M}, S)$ twist commutes with the Dirac matrices,

$$\gamma^\mu\mathcal{O} = \rho(\mathcal{O})\gamma^\mu \quad \forall \ \mu \quad (5.2)$$

for some automorphism ρ of $\mathcal{B}(\mathcal{H})$ and commutes the spin connection ω_μ, then

$$[\partial\!\!\!/, \mathcal{O}]_\rho = -i\gamma^\mu\partial_\mu\mathcal{O}. \quad (5.3)$$

Proof.—One has

$$[\gamma^\mu\nabla_\mu, \mathcal{O}]_\rho = [\gamma^\mu\partial_\mu, \mathcal{O}]_\rho + [\gamma^\mu\omega_\mu, \mathcal{O}]_\rho. \quad (5.4)$$

On the one side, the Leibniz rule for the differential operator ∂_μ together with Eq. (5.2) yields

$$\begin{aligned}[\gamma^\mu\partial_\mu, \mathcal{O}]_\rho\psi &= \gamma^\mu\partial_\mu\mathcal{O}\psi - \rho(\mathcal{O})\gamma^\mu\partial_\mu\psi\\ &= \gamma^\mu(\partial_\mu\mathcal{O})\psi + \gamma^\mu\mathcal{O}\partial_\mu\psi - \rho(\mathcal{O})\gamma^\mu\partial_\mu\psi\\ &= \gamma^\mu(\partial_\mu\mathcal{O})\psi.\end{aligned}$$

On the other side, by Eq. (5.2),

$$[\gamma^\mu\omega_\mu, \mathcal{O}]_\rho = \gamma^\mu\omega_\mu\mathcal{O} - \rho(\mathcal{O})\gamma^\mu\omega_\mu = \gamma^\mu[\omega^\mu, \mathcal{O}] \quad (5.5)$$

vanishes by hypothesis. Hence, the result. ∎

This lemma applies, in particular, to the components Q and M of the representation of the algebra $\mathcal{A}$ in Eq. (2.19). The slight difference is that these components do not act on $L^2(\mathcal{M}, S)$ but on $L^2(\mathcal{M}, S) \otimes \mathbb{C}^{32}$. With a slight abuse of notation, we write

$$\gamma^\mu Q \coloneqq (\gamma^\mu \otimes \mathbb{I}_{16})Q, \qquad \partial_\mu Q \coloneqq (\partial_\mu \otimes \mathbb{I}_{16})Q \quad (5.6)$$

and similarly for M.

Corollary 5.1.1.—One has

$$\gamma^\mu Q = \rho(Q)\gamma^\mu, \qquad [\partial\!\!\!/, Q]_\rho = -i\gamma^\mu\partial_\mu Q, \tag{5.7}$$

$$\gamma^\mu M = \rho(M)\gamma^\mu, \qquad [\partial\!\!\!/, M]_\rho = -i\gamma^\mu\partial_\mu Q. \tag{5.8}$$

Proof.—From Eq. (3.9) and omitting the internal indices (on which the action of $\gamma^\mu \otimes \mathbb{I}_{16}$ is trivial), one checks from the explicit form (A2) of the Euclidean Dirac matrices that

$$\gamma^\mu_E Q - \rho(Q)\gamma^\mu_E = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde\sigma^\mu & 0 \end{pmatrix}_s^t \begin{pmatrix} Q_r & 0 \\ 0 & Q_l \end{pmatrix}_s^t - \begin{pmatrix} Q_l & 0 \\ 0 & Q_r \end{pmatrix}_s^t \begin{pmatrix} 0 & \sigma^\mu \\ \tilde\sigma^\mu & 0 \end{pmatrix}_s^t = 0. \tag{5.9}$$

The same holds true for the curved Dirac matrices (A4), by linear combination.

The commutation with the spin connection follows by remembering that the latter is

$$\omega_\mu = \Gamma^{\rho\nu}_\mu \gamma_\rho\gamma_\nu = \Gamma^{\rho\nu}_\mu \begin{pmatrix} \sigma_\mu\tilde\sigma_\nu & 0 \\ 0 & \tilde\sigma_\mu\sigma_\nu \end{pmatrix}_s^t \tag{5.10}$$

and so commutes with Q, which is diagonal in the s, t indices and trivial in the $\dot s$, $\dot t$ indices. ∎

B. Free 1-form

With the previous results, it is not difficult to compute a free 1-form (3.33).

Lemma 5.2.—A free 1-form is

$$A\!\!\!/ = -i\gamma^\mu A_\mu \quad \text{with} \quad A_\mu = \begin{pmatrix} Q_\mu & 0 \\ 0 & M_\mu \end{pmatrix}_C^D, \tag{5.11}$$

where we use notations similar to Eq. (5.6), with

$$Q_\mu \coloneqq \sum_i \rho(Q_i)\partial_\mu R_i, \qquad M_\mu = \sum_i \rho(M_i)\partial_\mu N_i \tag{5.12}$$

for Q_i (M_i) and R_i (N_i) the components of a_i (b_i), respectively, as in Eqs. (2.19) and (3.24).

Proof.—Omitting the summation index i, one has

$$\begin{aligned} A\!\!\!/ = a[D\!\!\!\!/, b]_\rho &= \begin{pmatrix} Q & 0 \\ 0 & M \end{pmatrix}_C^D \begin{pmatrix} [\partial\!\!\!/, R]_\rho & 0 \\ 0 & [\partial\!\!\!/, N]_\rho \end{pmatrix}_C^D \\ &= -i\begin{pmatrix} Q & 0 \\ 0 & M \end{pmatrix}_C^D \begin{pmatrix} \gamma^\mu\partial_\mu R & 0 \\ 0 & \gamma^\mu\partial_\mu N \end{pmatrix}_C^D \\ &= -i\gamma^\mu \begin{pmatrix} \rho(Q)\partial_\mu R & 0 \\ 0 & \rho(M)\partial_\mu N \end{pmatrix}_C^D, \end{aligned} \tag{5.13}$$

where the last equalities follow from Corollary 5.1.1. Restoring the index i, one gets the result. ∎

By computing explicitly the components of $A\!\!\!/$, one finds that a free 1-form is parametrized by two complex fields c^r_μ and c^l_μ, two quaternionic fields q^r_μ and q^l_μ, and two $M_3(\mathbb{C})$-valued fields m^r_μ and m^l_μ.

Proposition 5.3.—The components Q_μ and M_μ of $A\!\!\!/$ in Eq. (5.11) are, respectively,

$$Q_\mu = \delta^{\dot iJ}_{\dot sI} \begin{pmatrix} Q^r_\mu & \\ & Q^l_\mu \end{pmatrix}_s^t, \qquad M_\mu = \delta^{\dot i}_{\dot s} \begin{pmatrix} M^r_\mu & \\ & M^l_\mu \end{pmatrix}_s^t, \tag{5.14}$$

where

$$Q^r_\mu = \begin{pmatrix} \mathsf{c}^r_\mu & \\ & q^r_\mu \end{pmatrix}_\alpha^\beta, \qquad Q^l_\mu = \begin{pmatrix} \mathsf{c}^l_\mu & \\ & q^l_\mu \end{pmatrix}_\alpha^\beta \tag{5.15}$$

for

$$\mathsf{c}^r_\mu = \begin{pmatrix} c^r_\mu & \\ & \bar c^r_\mu \end{pmatrix}$$

and

$$\mathsf{c}^l_\mu = \begin{pmatrix} c^l_\mu & \\ & \bar c^l_\mu \end{pmatrix}$$

and

$$M^r_\mu = \begin{pmatrix} \mathsf{m}^r_\mu \otimes \mathbb{I}_2 & 0 \\ 0 & \mathsf{m}^l_\mu \otimes \mathbb{I}_2 \end{pmatrix}_\alpha^\beta, \qquad M^l_\mu = \begin{pmatrix} \mathsf{m}^l_\mu \otimes \mathbb{I}_2 & 0 \\ 0 & \mathsf{m}^r_\mu \otimes \mathbb{I}_2 \end{pmatrix}_\alpha^\beta \tag{5.16}$$

for

$$\mathsf{m}^r_\mu = \begin{pmatrix} c^r_\mu & \\ & m^r_\mu \end{pmatrix}_I^J$$

and

$$\mathsf{m}^l_\mu = \begin{pmatrix} c^l_\mu & \\ & m^l_\mu \end{pmatrix}_I^J.$$

The complex, quaternionic, and $M_3(\mathbb{C})$-value fields $c^{r/l}_\mu$, $q^{r/l}_\mu$, and $m^{r/l}_\mu$, respectively, are defined in the proof.

Proof.—The form (5.14) and (5.15) of the components of $A\!\!\!/$ follows calculating explicitly Eq. (5.12) using Eqs. (3.9)–(3.12) for Q_i and M_i and Eq. (3.25) for R_i and N_i. Omitting the i index, one finds

$$Q^r_\mu = Q_l\partial_\mu R_r, \qquad Q^l_\mu = Q_r\partial_\mu R_l,$$
$$M^r_\mu = M_l\partial_\mu N_r, \qquad M^l_\mu = M_r\partial_\mu N_l. \tag{5.17}$$

The first two equations yield Eq. (5.15) with

$$c^r_\mu = c'\partial_\mu d, \qquad c^l_\mu = c\partial_\mu d',$$
$$q^r_\mu = q\partial_\mu p', \qquad q^l_\mu = q'\partial_\mu p, \tag{5.18}$$

and the last two yield $\mathsf{m}^r_\mu = \mathsf{m}'\partial_\mu \mathsf{n}$ and $\mathsf{m}^l_\mu = \mathsf{m}\partial_\mu \mathsf{n}'$, from which Eq. (5.16) follows with

$$m^r_\mu = m'\partial_\mu n, \qquad m^l_\mu = m\partial_\mu n'.$$

■

Corollary 5.3.1.—A free 1-form $\not{A}$ is self-adjoint if and only if

$$c^l_\mu = -\bar{c}^r_\mu, \qquad q^l_\mu = -(q^r_\mu)^\dagger, \qquad m^l_\mu = -(m^r_\mu)^\dagger. \tag{5.19}$$

Proof.—From Lemma 5.2 and Corollary 5.1.1, using that ρ is a $*$-automorphism,[3] one has

$$\not{A}^\dagger = i(A^\mu)^\dagger\gamma^\mu = i\gamma^\mu\rho(A^\mu)^\dagger, \tag{5.20}$$

so $\not{A}$ is self-adjoint if and only if $\gamma^\mu(\rho(A_\mu)^\dagger + A_\mu) = 0$. Since A_μ is diagonal from the s and t indices, the sum $\Delta_\mu \coloneqq \rho(A_\mu)^\dagger + A_\mu$ is also diagonal with components $\Delta^{r/l}_\mu$. Thus,

$$\gamma^\mu\Delta_\mu = \begin{pmatrix} 0 & \sigma^\mu\Delta^l_\mu \\ \tilde{\sigma}^\mu\Delta^r_\mu & \end{pmatrix}^t_s. \tag{5.21}$$

If this is zero, then for any γ^ν

$$\gamma^\nu\gamma^\mu\Delta_\mu = \begin{pmatrix} \sigma^\nu\tilde{\sigma}^\mu\Delta^r_\mu & 0 \\ 0 & \tilde{\sigma}^\nu\sigma^\mu\Delta^l_\mu \end{pmatrix} = 0. \tag{5.22}$$

A_μ—and, hence, Δ_μ—being trivial in $\dot{s}$ and $\dot{t}$, and since $\mathrm{Tr}\tilde{\sigma}^\mu\sigma^\nu = 2\delta_{\mu\nu}$, the partial trace on the $\dot{s}$ and $\dot{t}$ indices of the expression above yields $\Delta^r_\mu = \Delta^l_\mu = 0$. Therefore, $\gamma^\mu(\rho(A_\mu)^\dagger + A_\mu) = 0$ implies

$$\rho(A_\mu)^\dagger = -A_\mu. \tag{5.23}$$

The converse is obviously true. Consequently, $\not{A}_\mu$ is self-adjoint if and only if Eq, (5.23) holds true.

From Eq. (5.11), this is equivalent to $\rho(Q_\mu)^\dagger = -Q_\mu$ and $\rho(M_\mu)^\dagger = -M_\mu$; that is, from Eq. (5.14),

$$(Q^l_\mu)^\dagger = -Q^r_\mu \quad \text{and} \quad (M^l_\mu)^\dagger = -M^r_\mu. \tag{5.24}$$

This is equivalent to Eq. (5.19). ■

C. Identification of the physical degrees of freedom

To identify the physical fields, one follows the nontwisted case [5] and separates the real from the imaginary parts. We thus define two real fields $a_\mu = \mathrm{Re}c^r_\mu$ and $B_\mu = -\frac{2}{g_1}\mathrm{Im}c^r_\mu$ (g_1 is a real constant, and the signs are such to match the notations of Ref. [24]; see Remark 5.6), so that

$$c^r_\mu = a_\mu - i\frac{g_1}{2}B_\mu, \qquad c^l_\mu = -\bar{c}^r_\mu = -a_\mu - i\frac{g_1}{2}B_\mu. \tag{5.25}$$

Moreover, we denote w_μ and $-\frac{g_2}{2}W^k$ for $k = 1, 2, 3$ the real components of the quaternionic field q^r_μ on the basis $\{\mathbb{I}_2, i\sigma_j\}$ of the (real) algebra of quaternions (with g_2 another real constant), so that

$$q^r_\mu = w_\mu\mathbb{I}_2 - i\frac{g_2}{2}W^k_\mu\sigma_k,$$
$$q^l_\mu = -(q^r_\mu)^\dagger = -w_\mu\mathbb{I}_2 - i\frac{g_2}{2}W^k_\mu\sigma_k. \tag{5.26}$$

Finally, we write m^r_μ as the sum of a self-adjoint part $g_\mu = \frac{1}{2}(m^r_\mu + m^{r\dagger}_\mu)$ and an anti-self-adjoint part $\frac{1}{2}(m^r_\mu - m^{r\dagger}_\mu)$. We denote V^0_μ and $\frac{g_3}{2}V^m_\mu$ the real-field components of the latter on the basis $\{i\mathbb{I}_3, i\lambda_m\}$ of the (real) vector space of anti-self-adjoint 3×3 complex matrices (with $\{\lambda_m, m = 1\ldots 8\}$ the Gell-Mann matrices and g_3 a real constant), so that

$$m^r_\mu = g_\mu + iV^0_\mu\mathbb{I}_3 + i\frac{g_3}{2}V^m_\mu\lambda_m, \tag{5.27}$$

$$m^l_\mu = -(m^r_\mu)^\dagger = -g_\mu + iV^0_\mu\mathbb{I}_3 + i\frac{g_3}{2}V^m_\mu\lambda_m. \tag{5.28}$$

The cancellation of anomalies is imposed requiring the unimodularity condition

$$\mathrm{Tr}A_\mu = 0. \tag{5.29}$$

This yields the same condition as in the nontwisted case.

Proposition 5.4.—The unimodularity condition for a self-adjoint free 1-form yields

$$V^0_\mu = \frac{g_1}{6}B_\mu. \tag{5.30}$$

Proof.—From Proposition 5.3, one gets $\mathrm{Tr}A_\mu = \mathrm{Tr}Q_\mu + \mathrm{Tr}M_\mu$. On the one side (neglecting the $\dot{s}$ and I indices),

$$\mathrm{Tr}Q_\mu = \mathrm{Tr}Q^r_\mu + \mathrm{Tr}Q^l_\mu$$
$$= c^r_\mu + \bar{c}^r_\mu + \mathrm{Tr}q^r_\mu + c^l_\mu + \bar{c}^l_\mu + \mathrm{Tr}q^l_\mu \tag{5.31}$$

[3] In a twisted spectral triple, the automorphism is not necessarily involutive. What is asked is the regularity condition $\rho(a^*) = (\rho^{-1}(a))^*$. In our case, since $\rho^{-1} = \rho$, the latter is equivalent to ρ being a $*$-automorphism.

vanishes by Eq. (5.19), when one notices that $\mathrm{Tr}q^\dagger = \mathrm{Tr}q$ for any quaternion q. On the other side,

$$\begin{aligned}\mathrm{Tr}M_\mu &= \mathrm{Tr}M^r_\mu + \mathrm{Tr}M^l_\mu = 4\mathrm{Tr}\{\mathsf{m}\}^r_\mu + 4\mathrm{Tr}\{\mathsf{m}\}^l_\mu \\ &= 4(c^r_\mu + \mathrm{Tr}m^r_\mu + c^l_\mu + \mathrm{Tr}m^l_\mu) \\ &= 4(-ig_1 B_\mu + 6iV^0_\mu), \end{aligned} \tag{5.32}$$

where we use $c^r_\mu + c^l_\mu = -ig_1 B_\mu$ and $m^r_\mu + m^l_\mu = 2iV^0_\mu \mathbb{I}_3 + 2ig_3 V^m_\mu \lambda_m$, remembering then that the Gell-Mann matrices are traceless. Hence, Eq. (5.29) is equivalent to Eq. (5.30). ∎

Let us summarize the results of this section in the following.

Proposition 5.5.—A unimodular self-adjoint free 1-form $\mathcal{A}\!\!\!/$ is parametrized by

(i) two real 1-form fields a_μ and w_μ and a self-adjoint $M_3(\mathbb{C})$-value field g_μ,

(ii) a $\mathfrak{u}(1)$-value field iB_μ, a $\mathfrak{su}(2)$-value field iW_μ, and a $\mathfrak{su}(3)$-value field iV_μ.

Proof.—Collecting the previous results, denoting $W_\mu \coloneqq W^k_\mu \sigma_k$ and $V_\mu \coloneqq V^m_\mu \lambda_m$, one has

$$c^r_\mu = a_\mu - i\frac{g_1}{2}B_\mu, \qquad c^l_\mu = -a_\mu - i\frac{g_1}{2}B_\mu, \tag{5.33}$$

$$q^r_\mu = w_\mu \mathbb{I}_2 - i\frac{g_2}{2}W_\mu, \qquad q^l_\mu = -w_\mu \mathbb{I}_2 - i\frac{g_2}{2}W_\mu, \tag{5.34}$$

$$\begin{aligned} m^r_\mu &= g_\mu + i\left(\frac{g_1}{6}B_\mu \mathbb{I}_3 + \frac{g_3}{2}V_\mu\right), \\ m^l_\mu &= -g_\mu + i\left(\frac{g_1}{6}B_\mu \mathbb{I}_3 + \frac{g_3}{2}V_\mu\right). \end{aligned} \tag{5.35}$$

On the one side, a_μ and w_μ are in $C^\infty(\mathcal{M},\mathbb{R})$ and $g_\mu = g_\mu{}^\dagger$ is in $C^\infty(\mathcal{M}, M_3(\mathbb{C}))$. On the other side, since B_μ is real, $iB_\mu \in C^\infty(\mathcal{M}, i\mathbb{R})$ is a $\mathfrak{u}(1)$-value field. The Pauli matrices span the space of traceless 2×2 self-adjoint matrices; thus, the field iW_μ takes a value in the set of anti-self-adjoint such matrices, that is, $\mathfrak{su}(2)$. Finally, the real span of the Gell-Mann matrices is the space of traceless self-adjoint elements of $M_3(\mathbb{C})$; hence, iV_μ is a $\mathfrak{su}(3)$-value field. ∎

In the nontwisted case, the primed and unprimed quantities in Eq. (5.18) and the next equation are equal, meaning that the right and left components of the fields (5.33)–(5.35) are equal; hence,

$$a_\mu = w_\mu = g_\mu = 0. \tag{5.36}$$

That the twisting produces some extra 1-form fields has already been pointed out for manifolds in Ref. [19] and for electrodynamic in Ref. [18]. Actually, such a field (improperly called vector field) appeared initially in the twisted version of the Standard Model presented in Ref. [16], but its precise structure—a collection of three self-adjoint fields a_μ, w_μ, and g_μ, each associated with a gauge field of the Standard Model—had not been worked out there.

In the minimal twist of electrodynamics, there is only one such field [associated with the $U(1)$ gauge symmetry]. By studying the fermionic action, it gets interpreted as an energy-momentum 4-vector in Lorentzian signature. Whether such an interpretation still holds for a_μ, w_μ, and g_μ will be investigated in a forthcoming paper [21].

Remark 5.6.—In the nontwisted case, the fields B_μ, W_μ, and V_μ coincide with those of the spectral triple of the Standard Model. More precisely, within the conditions of Eq. (5.36), then

(i) our $c^r_\mu = c^l_\mu$ coincides with $-i\Lambda_\mu$ of Sec. 15.4 in Ref. [24].[4] The self-adjointness condition (5.19) then implies that Λ_μ is real, in agreement with Ref. [24]. Then $B_\mu = \frac{2}{g_1}\Lambda_\mu$ as defined in Ref. [24] coincides with our $B_\mu = -i\frac{2}{g_1}c^r_\mu = -i\frac{2}{g_1}c^l_\mu$ as defined in Eq. (5.25).

(ii) Our $q^r_\mu = q^l_\mu$ coincides with $-iQ_\mu$ of Sec. 15.4 in Ref. [24]. The self-adjointness condition (5.19) then implies that Q_μ is self-adjoint, in agreement with Ref. [24]. Then $W_\mu = \frac{2}{g_2}Q_\mu$ as defined in Ref. [24] coincides with our $W_\mu = W^k_\mu\sigma_k = i\frac{2}{g_2}q^r_\mu = i\frac{2}{g_2}q^l_\mu$ in Eq. (5.26).

(iii) The identification of our V_μ with the one of the nontwisted case is made after Proposition 5.8.

Remark 5.7.—If one does not impose the self-adjointness of $\mathcal{A}\!\!\!/$, then one obtains two copies of the bosonic contents of the Standard Model, acting independently on the right and left components of Dirac spinors. Whether this may yield physically meaningful models should be investigated elsewhere (considering to remove also the self-adjointness of the finite part of the fluctuation).

D. Twisted fluctuation of the free Dirac operator

We now compute the free part (5.1) of the twisted fluctuation.

Proposition 5.8.—A twisted fluctuation of the free Dirac operator $\partial\!\!\!/$ is $D_Z = \partial\!\!\!/ + Z$, where

$$Z = \mathcal{A}\!\!\!/ + J\mathcal{A}\!\!\!/ J^{-1} = -i\gamma^\mu \begin{pmatrix} Z^\mu & 0 \\ 0 & \bar{Z}^\mu \end{pmatrix}_C^D \quad \text{with}$$
$$Z_\mu = \gamma^5 \otimes X_\mu + \mathbb{I}_4 \otimes iY_\mu, \tag{5.37}$$

in which X_μ and Y_μ are self-adjoint $\mathcal{A}_{\mathrm{SM}}$-value tensor fields on $\mathcal{M}$ with components

$$(X_\mu)^{\dot{2}J}_{\dot{1}I} = (X_\mu)^{\dot{1}J}_{\dot{2}I} = (Y_\mu)^{\dot{2}J}_{\dot{1}I} = (Y_\mu)^{\dot{1}J}_{\dot{2}I} = 0, \tag{5.38}$$

[4] Beware that $\partial\!\!\!/_M$ in the formula of Λ is $i\gamma^\mu\partial_\mu$ [24], so that $\Lambda = \Lambda_\mu\gamma^\mu$ is the $U(1)$ part of $-\mathcal{A}\!\!\!/$, meaning that Λ_μ is the $U(1)$ part of iA_μ.

and

$$(X_\mu)^{\dot{1}J}_{\dot{1}I} = (X_\mu)^{\dot{2}J}_{\dot{2}I} = \begin{pmatrix} 2a_\mu & \\ & a_\mu \mathbb{I}_3 + g_\mu \end{pmatrix}^J_I, \tag{5.39}$$

$$(Y_\mu)^{\dot{1}J}_{\dot{1}I} = \begin{pmatrix} 0 & \\ & -\frac{2g_1}{3} B_\mu \mathbb{I}_3 - \frac{g_3}{2} V_\mu \end{pmatrix}, \qquad (Y_\mu)^{\dot{2}J}_{\dot{2}I} = \begin{pmatrix} g_1 B_\mu & \\ & \frac{g_1}{3} B_\mu \mathbb{I}_3 - \frac{g_3}{2} V_\mu \end{pmatrix}, \tag{5.40}$$

$$(X_\mu)^{bJ}_{aI} = \begin{pmatrix} \delta^b_a (w_\mu - a_\mu) & \\ & \delta^b_a w_\mu \mathbb{I}_3 - g_\mu \end{pmatrix}^J_I, \tag{5.41}$$

$$(Y_\mu)^{bJ}_{aI} = \begin{pmatrix} \delta^b_a \frac{g_1}{2} B_\mu - \frac{g_2}{2} (W_\mu)^b_a & \\ & -\delta^b_a \left(\frac{g_1}{6} B_\mu \mathbb{I}_3 + \frac{g_3}{2} V_\mu\right) - \frac{g_2}{2} (W_\mu)^b_a \mathbb{I}_3 \end{pmatrix}^J_I. \tag{5.42}$$

Proof.—With $J = -J^{-1}$ as defined in Eq. (3.18), one has

$$J A\!\!\!/ J^{-1} = -J(-i\gamma^\mu A_\mu) J^{-1} = -iJ\gamma^\mu A_\mu J^{-1}$$
$$= i\gamma^\mu J A_\mu J^{-1} = i\gamma^\mu \begin{pmatrix} \mathcal{J} M_\mu \mathcal{J}^{-1} & 0 \\ 0 & \mathcal{J} Q_\mu \mathcal{J}^{-1} \end{pmatrix}^D_C,$$

where we use that J is antilinear and anticommutes with γ^μ (Lemma A.7). Noticing that $\mathcal{J} M_\mu \mathcal{J}^{-1} = -\bar{M}_\mu$ and $\mathcal{J} Q_\mu \mathcal{J}^{-1} = -\bar{Q}_\mu$ [this is shown as in Eqs. (3.22) and (3.21), respectively], one obtains

$$Z_\mu = Q_\mu + \bar{M}_\mu. \tag{5.43}$$

Explicitly,

$$Z_\mu = \begin{pmatrix} Z^r_\mu & \\ & Z^l_\mu \end{pmatrix},$$

where, using the explicit forms (5.15) and (5.16) of Q^r_μ and M^r_μ,

$$Z^r_\mu = \delta^{\dot{t}J}_{\dot{s}I} Q^r_\mu + \delta^{\dot{t}}_{\dot{s}} \bar{M}^r_\mu = \delta^{\dot{t}}_{\dot{s}} \begin{pmatrix} \mathsf{c}^r_\mu \delta^J_I + \delta^{\dot{b}}_{\dot{a}} \bar{\mathsf{m}}^r_\mu & \\ & q^r_\mu \delta^J_I + \delta^b_a \bar{\mathsf{m}}^l_\mu \end{pmatrix}^\beta_\alpha \tag{5.44}$$

and

$$Z^l_\mu = \delta^{\dot{t}J}_{\dot{s}I} Q^l_\mu + \delta^{\dot{t}}_{\dot{s}} \bar{M}^l_\mu. \tag{5.45}$$

The components of the matrix in the rhs of Eq. (5.44) are

$$(Z^r_\mu)^{\dot{b}J}_{\dot{a}I} = \mathsf{c}^r_\mu \delta^J_I + \delta^{\dot{b}}_{\dot{a}} \bar{\mathsf{m}}^r_\mu = \begin{pmatrix} c^r_\mu \delta^J_I + \bar{\mathsf{m}}^r_\mu & \\ & \bar{c}^r_\mu \delta^J_I + \bar{\mathsf{m}}^r_\mu \end{pmatrix}^{\dot{b}}_{\dot{a}} \tag{5.46}$$

with $(Z^r_\mu)^{\dot{2}J}_{\dot{1}I} = (Z^r_\mu)^{\dot{1}J}_{\dot{2}I} = 0$ and, using Proposition 5.5,

$$(Z^r_\mu)^{\dot{1}J}_{\dot{1}I} = c^r_\mu \delta^J_I + \bar{\mathsf{m}}^r_\mu = \begin{pmatrix} 2a_\mu & \\ & \left(a_\mu - i\frac{g_1}{2} B_\mu\right)\mathbb{I}_3 + g_\mu - i\left(\frac{g_1}{6} B_\mu \mathbb{I}_3 + \frac{g_3}{2} V_\mu\right) \end{pmatrix}^J_I =: (X^r_\mu)^{\dot{1}J}_{\dot{1}I} + i(Y^r_\mu)^{\dot{1}J}_{\dot{1}I},$$

$$(Z^r_\mu)^{\dot{2}J}_{\dot{2}I} = \bar{c}^r_\mu \delta^J_I + \bar{\mathsf{m}}^r_\mu = \begin{pmatrix} 2a_\mu + ig_1 B_\mu & \\ & \left(a_\mu + i\frac{g_1}{2} B_\mu\right)\mathbb{I}_3 + g_\mu - i\left(\frac{g_1}{6} B_\mu \mathbb{I}_3 + \frac{g_3}{2} V_\mu\right) \end{pmatrix}^J_I =: (X^r_\mu)^{\dot{2}J}_{\dot{2}I} + i(Y^r_\mu)^{\dot{2}J}_{\dot{2}I};$$

and

$$(Z^r_\mu)^{bJ}_{aI} = q^r_\mu \delta^J_I + \delta^b_a \bar{\mathsf{m}}^l_\mu = \begin{pmatrix} (q^r_\mu)^1_1 \delta^J_I + \bar{\mathsf{m}}^l_\mu & (q^r_\mu)^2_1 \delta^J_I \\ (q^r_\mu)^1_2 \delta^J_I & \bar{q}^{r\,2}_{\mu 2} \delta^J_I + \bar{\mathsf{m}}^l_\mu \end{pmatrix}^b_a \tag{5.47}$$

with

$$
\begin{aligned}
(Z_\mu^r)_{aI}^{aJ} &= (q_\mu^r)_a^a \delta_I^J + \bar{\mathsf{m}}_\mu^l \\
&= \begin{pmatrix} w_\mu - i\frac{g_2}{2}(W_\mu)_a^a - a_\mu + i\frac{g_1 B_\mu}{2} & \\ & \left(w_\mu - i\frac{g_2}{2}(W_\mu)_a^a\right)\mathbb{I}_3 - g_\mu - i\left(\frac{g_1 B_\mu}{6}\mathbb{I}_3 + \frac{g_3}{2}V_\mu\right) \end{pmatrix}_I^J \\
&=: (X_\mu^r)_{aI}^{aJ} + i(Y_\mu^r)_{aI}^{aJ},
\end{aligned}
$$

$$
(Z_\mu^r)_a^{b\neq a} = (q_\mu^r)_a^b \delta_I^J = \begin{pmatrix} -i\frac{g_2}{2}(W_\mu)_a^b & \\ & -i\frac{g_2}{2}(W_\mu)_a^b \mathbb{I}_3 \end{pmatrix}_I^J = (X_\mu^r)_{aI}^{bJ} + i(Y_\mu^r)_{aI}^{bJ}.
$$

The matrices X_μ^r and Y_μ^r defined by the equations above are self-adjoint (notice that W_μ as defined in Proposition 5.5 is self-adjoint) and such that

$$Z_\mu^r = X_\mu^r + iY_\mu^r. \tag{5.48}$$

The self-adjointness condition (5.24) applied to Eq. (5.45) yields

$$Z_\mu^l = -(Z_\mu^r)^\dagger = -X_\mu^r + iY_\mu^r. \tag{5.49}$$

In other terms, $Z_\mu^l = X_\mu^l + iY_\mu^l$ with

$$X_\mu^l = -X_\mu^r, \qquad Y_\mu^l = Y_\mu^r. \tag{5.50}$$

Redefining $X_\mu := X_\mu^r = -X_\mu^l$, $Y_\mu := Y_\mu^r = Y_\mu^l$, one obtains the result. ∎

We collect the components of Z in Appendix 7. There, we also make explicit that iY_μ coincides exactly with the gauge fields of the Standard Model [including the $\mathfrak{su}(3)$ gauge field V_μ]. Thus, the twist does not modify the gauge content of the model. What it does is to add the self-adjoint part X_μ whose action on spinors breaks chirality. As shown in the next section, this field is invariant under a gauge transformation.

VI. GAUGE TRANSFORMATIONS

A gauge transformation is implemented by an action of the group $\mathcal{U}(\mathcal{A})$ of unitary elements of $\mathcal{A}$, both on the Hilbert space and on the Dirac operator. On a twisted spectral triple, these actions have been worked out in Refs. [17,20] and consist in a twist of the original formula of Connes [3], later generalized without the first-order condition in Ref. [13]. Explicitly, on the Hilbert space, the fermion fields transform under the adjoint action of $\mathcal{U}(\mathcal{A})$ induced by the real structure, namely,

$$\psi \to Adu\psi := u\psi u = uu^\circ\psi = uJu^*J^{-1}\psi, \quad u \in \mathcal{U}. \tag{6.1}$$

On the other hand, the twisted-covariant Dirac operator D_A (3.28) transforms under the twisted conjugate action of Ad u:

$$D_A \to \mathrm{Ad}\rho(u) D_A \mathrm{Ad}u^*. \tag{6.2}$$

By Proposition 4.2 in Ref. [22], the operator D_A, viewed as a function of the components a_i and b_i of the twisted 1-form $A = A_{(1)} = \sum_i a_i[D, b_i]$, transforms under a gauge transformation in the operator D_{A^u}, where

$$A^u := \rho(u)[D, u^*]_\rho + \rho(u)Au^*. \tag{6.3}$$

This is the twisted version of the law of transformation of generalized 1-forms in ordinary spectral triples, which, in turn, is a noncommutative generalization of the law of transformation of the gauge potential in ordinary gauge theories.

To write down the transformation $A \to A^u$, we need the explicit form of a unitary u of $\mathcal{A}$. The latter is a pair of functions on $\mathcal{M}$ with a value in

$$\mathcal{U}(\mathbb{C}) \times \mathcal{U}(\mathbb{H}) \times \mathcal{U}(M_3(\mathbb{C})) \simeq U(1) \times SU(2) \times U(3). \tag{6.4}$$

Namely,

$$u = (e^{i\alpha}, e^{i\alpha'}, \boldsymbol{q}, \boldsymbol{q}', \boldsymbol{m}, \boldsymbol{m}') \tag{6.5}$$

with

$$
\begin{aligned}
&\alpha, \alpha' \in C^\infty(\mathcal{M}, \mathbb{R}), \qquad \boldsymbol{q}, \boldsymbol{q}' \in C^\infty(\mathcal{M}, SU(2)), \\
&\boldsymbol{m}, \boldsymbol{m}' \in C^\infty(\mathcal{M}, U(3)).
\end{aligned} \tag{6.6}
$$

It acts on $\mathcal{H}$ as

$$u = \begin{pmatrix} \mathfrak{A} & \\ & \mathfrak{B} \end{pmatrix}_C^D, \tag{6.7}$$

where, following Eqs. (3.8)–(3.12), one has $\mathfrak{A}_{\dot{s}sI\alpha}^{\dot{t}tJ\beta} = \delta_{\dot{s}I}^{\dot{t}J}\mathfrak{A}_{s\alpha}^{t\beta}$ and $\mathfrak{B}_{\dot{s}sI\alpha}^{\dot{t}tJ\beta} = \delta_{\dot{s}}^{\dot{t}}\mathfrak{B}_{s\alpha I}^{t\beta J}$ with

$$\mathfrak{A}^{t\beta}_{s\alpha} = \begin{pmatrix} (\mathfrak{A}_r)^\beta_\alpha & \\ & (\mathfrak{A}_l)^\beta_\alpha \end{pmatrix}^t_s,$$
$$\mathfrak{B}^{t\beta J}_{s\alpha I} = \begin{pmatrix} (\mathfrak{B}_r)^{\beta J}_{\alpha I} & \\ & (\mathfrak{B}_l)^{\beta J}_{\alpha I} \end{pmatrix}^t_s, \tag{6.8}$$

in which

$$\mathfrak{A}_r = \begin{pmatrix} \boldsymbol{\alpha} & \\ & \boldsymbol{q}' \end{pmatrix}^\beta_\alpha, \qquad \mathfrak{A}_l = \begin{pmatrix} \boldsymbol{\alpha}' & \\ & \boldsymbol{q} \end{pmatrix}^\beta_\alpha, \tag{6.9}$$

and

$$\mathfrak{B}_r = \begin{pmatrix} \mathsf{m}\otimes\mathbb{I}_2 & 0 \\ 0 & \mathsf{m}'\otimes\mathbb{I}_2 \end{pmatrix}^\beta_\alpha,$$
$$\mathfrak{B}_l = \begin{pmatrix} \mathsf{m}'\otimes\mathbb{I}_2 & 0 \\ 0 & \mathsf{m}\otimes\mathbb{I}_2 \end{pmatrix}^\beta_\alpha, \tag{6.10}$$

where we denote

$$\boldsymbol{\alpha} \coloneqq \begin{pmatrix} e^{i\alpha} & \\ & e^{-i\alpha} \end{pmatrix}, \qquad \mathsf{m} \coloneqq \begin{pmatrix} e^{i\alpha} & \\ & \boldsymbol{m} \end{pmatrix}^J_I,$$
$$\boldsymbol{\alpha}' \coloneqq \begin{pmatrix} e^{i\alpha'} & \\ & e^{-i\alpha'} \end{pmatrix}, \qquad \mathsf{m}' \coloneqq \begin{pmatrix} e^{i\alpha'} & \\ & \boldsymbol{m}' \end{pmatrix}^J_I. \tag{6.11}$$

A. Gauge sector

A twisted gauge transformation (6.2) does not necessarily preserve the self-adjointness of the Dirac operator (because the action of the unitary is twisted on the left, not on the right). Equivalently, A^u in Eq. (6.3) is not necessarily self-adjoint, even though one starts with a self-adjoint A.

This may seem as a weakness of the twisted case, since in the nontwisted case self-adjointness is preserved. Actually, the possibility to lose self-adjointness allows one to implement Lorentz symmetry and yields—at least for electrodynamics [18]—an interesting interpretation of the component X_μ of the free fluctuation Z of Proposition 5.8 as a four-vector energy impulsion.

However, regarding the gauge part of the Standard Model which—as shown below—is fully encoded in the component iY_μ of Z, it is rather natural to ask the self-adjointness of the free 1-form $\not{A}$ to be preserved. This reduces the choice of unitaries to pair of elements of Eq. (6.4) equal up to a constant.

Proposition 6.1.—A unitary u whose action (6.3) preserves the self-adjointness of any unimodular self-adjoint free 1-form $\not{A}$ is given by Eq. (6.5) with

$$\alpha' = \alpha + K, \qquad \boldsymbol{q} = \boldsymbol{q}', \qquad \boldsymbol{m}' = \boldsymbol{m}. \tag{6.12}$$

The components (5.11) of $\not{A}$ then transform as

$$c^r_\mu \to c^r_\mu - i\partial_\mu\alpha, \qquad c^l_\mu \to c^l_\mu - i\partial_\mu\alpha, \tag{6.13}$$

$$q^r_\mu \to \boldsymbol{q}q^r_\mu\boldsymbol{q}^\dagger + \boldsymbol{q}(\partial_\mu\boldsymbol{q}^\dagger), \quad q^l_\mu \to \boldsymbol{q}q^l_\mu\boldsymbol{q}^\dagger + \boldsymbol{q}(\partial_\mu\boldsymbol{q}^\dagger), \tag{6.14}$$

$$m^r_\mu \to \boldsymbol{m}m^r_\mu\boldsymbol{m}^\dagger + \boldsymbol{m}(\partial_\mu\boldsymbol{m}^\dagger), \qquad m^l_\mu \to \boldsymbol{m}m^l_\mu\boldsymbol{m}^\dagger + \boldsymbol{m}(\partial_\mu\boldsymbol{m}^\dagger). \tag{6.15}$$

Proof.—From Corollary 5.1.1, one has [with the same abuse of notations (5.6), now with $\mathbb{I}_{32}$]

$$\not{A}^u = \rho(u)([\not{D}, u^*]_\rho + \not{A}u^*)$$
$$= -i\gamma^\mu(u(\partial_\mu u^*) + uA_\mu u^*). \tag{6.16}$$

Using the explicit forms (6.7) of u and (5.11) of A_μ, one finds

$$\not{A}^u = -i\gamma^\mu \begin{pmatrix} \mathfrak{A}(\partial_\mu\mathfrak{A}^\dagger) + \mathfrak{A}Q_\mu\mathfrak{A}^\dagger & 0 \\ 0 & \mathfrak{B}(\partial_\mu\mathfrak{B}^\dagger) + \mathfrak{B}M_\mu\mathfrak{B}^\dagger \end{pmatrix}^D_C, \tag{6.17}$$

meaning that a gauge transformation is equivalent to the transformation

$$Q_\mu \to \mathfrak{A}(\partial_\mu\mathfrak{A}^\dagger) + \mathfrak{A}Q_\mu\mathfrak{A}^\dagger,$$
$$M_\mu \to \mathfrak{B}(\partial_\mu\mathfrak{B}^\dagger) + \mathfrak{B}M_\mu\mathfrak{B}^\dagger. \tag{6.18}$$

From Eqs. (5.15) and (5.16), these equations are equivalent to

$$c^r_\mu \to e^{i\alpha}\partial_\mu e^{-i\alpha} + c^r_\mu = c^r_\mu - i\partial_\mu\alpha,$$
$$c^l_\mu \to c^l_\mu - i\partial_\mu\alpha', \tag{6.19}$$

$$q^r_\mu \to \boldsymbol{q}'q^r_\mu\boldsymbol{q}'^\dagger + \boldsymbol{q}'(\partial_\mu\boldsymbol{q}'^\dagger),$$
$$q^l_\mu \to \boldsymbol{q}q^l_\mu\boldsymbol{q}^\dagger + \boldsymbol{q}(\partial_\mu\boldsymbol{q}^\dagger), \tag{6.20}$$

$$m^r_\mu \to \boldsymbol{m}m^r_\mu\boldsymbol{m}^\dagger + \boldsymbol{m}(\partial_\mu\boldsymbol{m}^\dagger),$$
$$m^l_\mu \to \boldsymbol{m}'m^l_\mu\boldsymbol{m}'^\dagger + \boldsymbol{m}'(\partial_\mu\boldsymbol{m}'^\dagger). \tag{6.21}$$

For any unitary operator $\mathfrak{q}$, one has that $\mathfrak{q}(\partial_\mu\mathfrak{q}^\dagger) = \mathfrak{q}[\partial_\mu, \mathfrak{q}^\dagger]$ is anti-Hermitian (∂_μ being anti-Hermitian as well). Hence, beginning with a self-adjoint $\not{A}$ as in Eq. (5.19), requiring that $\not{A}^u$ be self-adjoint is equivalent to

$$\partial_\mu\alpha' = \partial_\mu\alpha, \tag{6.22}$$

$$\boldsymbol{q}q^l_\mu\boldsymbol{q}^\dagger + \boldsymbol{q}(\partial_\mu\boldsymbol{q}^\dagger) = \boldsymbol{q}'q^l_\mu\boldsymbol{q}'^\dagger + \boldsymbol{q}'(\partial_\mu\boldsymbol{q}'^\dagger), \tag{6.23}$$

$$\boldsymbol{m}'m^l_\mu\boldsymbol{m}'^\dagger + \boldsymbol{m}'(\partial_\mu\boldsymbol{m}'^\dagger) = \boldsymbol{m}m^l_\mu\mathsf{m}^\dagger + \boldsymbol{m}(\partial_\mu\boldsymbol{m}^\dagger). \tag{6.24}$$

In particular, for q^l_μ the identity, the second of these equations yields $\boldsymbol{q}(\partial_\mu\boldsymbol{q}^\dagger) = \boldsymbol{q}'(\partial_\mu\boldsymbol{q}'^\dagger)$ for any $\boldsymbol{q}, \boldsymbol{q}'$.

Hence, for any q^l_μ one has $\boldsymbol{q} q^l_\mu \boldsymbol{q}^\dagger = \boldsymbol{q}' q^l_\mu \boldsymbol{q}'^\dagger$. This means that $\boldsymbol{q}'^\dagger \boldsymbol{q}$ is in the center of $\mathbb{H}$. Being a unitary, $\boldsymbol{q}'^\dagger \boldsymbol{q}$ is thus the identity. So $\boldsymbol{q} = \boldsymbol{q}'$. Similarly, one gets that $\boldsymbol{m}'^\dagger \boldsymbol{m}$ is in the center of $M_3(\mathbb{C})$, that is, a multiple of the identity. Being unitary, $\boldsymbol{m}'^\dagger \boldsymbol{m}$ can be only the identity; hence, $\boldsymbol{m}' = \boldsymbol{m}$. Thus, (6.19)—(6.21) yield the result. ■

These transformations of the components of the free 1-form induce the following transformations of the physical fields defined in Eqs. (5.33)—(5.35).

Proposition 6.2.—Under a twisted gauge transformation that preserve the self-adjointness of a unimodular free 1-form, the physical fields a_μ and w_μ are invariant, g_μ undergoes an algebraic (i.e., nondifferential) transformation

$$g_\mu \to \boldsymbol{n} g_\mu \boldsymbol{n}^\dagger, \tag{6.25}$$

and the gauge fields transform as in the Standard Model:

$$B_\mu \to B_\mu + \frac{2}{g_1}\partial_\mu \alpha, \tag{6.26}$$

$$W_\mu \to \boldsymbol{q} W_\mu \boldsymbol{q}^\dagger + \frac{2i}{g_2}\boldsymbol{q}(\partial_\mu \boldsymbol{q}^\dagger), \tag{6.27}$$

$$V_\mu \to \boldsymbol{n} V_\mu \boldsymbol{n}^\dagger - \frac{2i}{g_3}\boldsymbol{n}(\partial_\mu \boldsymbol{n}^\dagger), \tag{6.28}$$

where $\boldsymbol{n} = (\det \boldsymbol{m})^{-1/3}\boldsymbol{m}$ is the $SU(3)$ part of $\boldsymbol{m}$.

Proof.—Applying the gauge transformations (6.13)–(6.15) to the physical fields defined through Eqs. (5.33)–(5.35), one obtains

$$\pm a_\mu - i\frac{g_1}{2}B_\mu \to \pm a_\mu - i\left(\frac{g_1}{2}B_\mu + \partial_\mu \alpha\right), \tag{6.29}$$

$$\pm w_\mu \mathbb{I}_2 - i\frac{g_2}{2}W_\mu \to \pm w_\mu \mathbb{I}_2 - i\left(\frac{g_2}{2}\boldsymbol{q} W_\mu \boldsymbol{q}^\dagger + i\boldsymbol{q}(\partial_\mu \boldsymbol{q}^\dagger)\right), \tag{6.30}$$

$$\begin{aligned} &\pm g_\mu + i\left(\frac{g_1}{6}B_\mu \mathbb{I}_3 + \frac{g_3}{2}V_\mu\right) \\ &\quad \to \pm \boldsymbol{m} g_\mu \boldsymbol{m}^\dagger + i\left(\frac{g_1}{6}B_\mu \mathbb{I}_3 + \frac{g_3}{2}\boldsymbol{m} V_\mu \boldsymbol{m}^\dagger - i\boldsymbol{m}(\partial_\mu \boldsymbol{m}^\dagger)\right), \end{aligned} \tag{6.31}$$

where the anti-self-adjointness of $\boldsymbol{q}(\partial_\mu \boldsymbol{q}^\dagger)$ and $\boldsymbol{m}(\partial_\mu \boldsymbol{m}^\dagger)$ guarantees that the rhs of Eqs. (6.30) and (6.31) is split into a self-adjoint and anti-self-adjoint part. The first two equations above yield Eqs. (6.26) and (6.27). Writing $\boldsymbol{m} = e^{i\theta}\boldsymbol{n}$ with $e^{i\theta} = (\det m)^{1/3}$ and $\boldsymbol{n} \in SU(3)$, then the right-hand side of Eq. (6.31) becomes

$$\pm \boldsymbol{n} g_\mu \boldsymbol{n}^\dagger + i\left(\left(\frac{g_1}{6}B_\mu - \partial_\mu \theta\right)\mathbb{I}_3 + \frac{g_3}{2}\boldsymbol{n} V_\mu \boldsymbol{n}^\dagger - i\boldsymbol{n}\partial_\mu \boldsymbol{n}^\dagger\right), \tag{6.32}$$

where we use $\boldsymbol{m}\partial_\mu \boldsymbol{m}^\dagger = -i\partial_\mu \theta + \boldsymbol{n}\partial_\mu \boldsymbol{n}^\dagger$. Requiring the unimodularity condition to be gauge invariant forces one to identify $-\theta$ with $\frac{\alpha}{3}$, thus reducing the gauge group $U(3)$ to $SU(3)$. This yields Eqs. (6.25) and (6.28). ■

Remark 6.3.—If one does not impose that the twisted gauge transformation preserves self-adjointness, then the left and right components of spinors transform independently. As explained in Remark 5.7, the viability of such models should be explored elsewhere.

B. Scalar sector

We now study the gauge transformation (6.3) of the scalar part of the twisted $A_Y + A_M$ of the twisted 1-form computed in Sec. IV, beginning with the Yukawa part A_Y in Eq. (4.1).

Lemma 6.4.—Let u be a unitary of $\mathcal{A}$ as in Eq. (6.5). One has

$$\begin{aligned} A_Y^u &= \rho(u)[\gamma^5 \otimes D_Y, u^\dagger]_\rho + \rho(u) A_Y u^\dagger \\ &= \begin{pmatrix} A^u & \\ & 0 \end{pmatrix}_C^D, \end{aligned} \tag{6.33}$$

where

$$A^u = \delta^{\dot{t}J}_{\dot{s}I}\begin{pmatrix} (A^u)_r & \\ & (A^u)_l \end{pmatrix}_s^t, \tag{6.34}$$

with

$$(A^u)_r = \begin{pmatrix} 0 & \bar{\mathsf{k}}^{\mathsf{l}}(\boldsymbol{\alpha}'(H_1 + \mathbb{I})\boldsymbol{q}'^\dagger - \mathbb{I}) \\ (\boldsymbol{q}(H_2 + \mathbb{I})\boldsymbol{\alpha}^\dagger - \mathbb{I})\mathsf{k}^{\mathsf{l}} & 0 \end{pmatrix}, \tag{6.35}$$

$$(A^u)_l = -\begin{pmatrix} 0 & \bar{\mathsf{k}}^{\mathsf{l}}(\boldsymbol{\alpha}(H_1' + \mathbb{I})\boldsymbol{q}^\dagger - \mathbb{I}) \\ (\boldsymbol{q}'(H_2' + \mathbb{I})\boldsymbol{\alpha}'^\dagger - \mathbb{I})\mathsf{k}^{\mathsf{l}} & 0 \end{pmatrix}, \tag{6.36}$$

where $H_{1,2}$ are the components of A_Y and α, α', q, q' those of u.

Proof.—From the formula (4.16) of A_Y and (6.7) and (6.8) of u, one gets

$$\rho(u) A_Y u^\dagger = \begin{pmatrix} \rho(\mathfrak{A}) A \mathfrak{A}^\dagger & \\ & 0 \end{pmatrix}_C^D,$$

$$\text{where } \rho(\mathfrak{A}) A \mathfrak{A}^\dagger = \delta^{\dot{t}J}_{\dot{s}I}\begin{pmatrix} \mathfrak{A}_l A_r \mathfrak{A}_r^\dagger & \\ & \mathfrak{A}_r A_l \mathfrak{A}_l^\dagger \end{pmatrix}_s^t, \tag{6.37}$$

where, using Eqs. (4.17) and (6.9),

$$\mathfrak{A}_l A_r \mathfrak{A}_r^\dagger = \begin{pmatrix} \boldsymbol{\alpha}' & \\ & \boldsymbol{q} \end{pmatrix} \begin{pmatrix} & \bar{\mathsf{k}}^{\mathsf{l}} H_1 \\ H_2 \mathsf{k}^{\mathsf{l}} & \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}^\dagger & \\ & \boldsymbol{q}'^\dagger \end{pmatrix} = \begin{pmatrix} & \bar{\mathsf{k}}^{\mathsf{l}} \boldsymbol{\alpha}' H_1 \boldsymbol{q}'^\dagger \\ \boldsymbol{q} H_2 \alpha^\dagger \mathsf{k}^{\mathsf{l}} & \end{pmatrix}_\alpha^\beta, \tag{6.38}$$

$$\mathfrak{A}_r A_l \mathfrak{A}_l^\dagger = -\begin{pmatrix} \boldsymbol{\alpha} & \\ & \boldsymbol{q}' \end{pmatrix} \begin{pmatrix} & \bar{\mathsf{k}}^{l} H_1' \\ H_2' \mathsf{k}^{\mathsf{l}} & \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha}'^\dagger & \\ & \boldsymbol{q}^\dagger \end{pmatrix} = -\begin{pmatrix} & \bar{\mathsf{k}}^{\mathsf{l}} \boldsymbol{\alpha} H_1' \boldsymbol{q}^\dagger \\ \boldsymbol{q}' H_2' \boldsymbol{\alpha}'^\dagger \mathsf{k}^{\mathsf{l}} & \end{pmatrix}_\alpha^\beta, \tag{6.39}$$

where we used that k^{l} and $\bar{\mathsf{k}}^{\mathsf{l}}$ commute with $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ and their conjugates.

The computation of the twisted commutator part in Eq. (6.33) is similar to that of A_Y in Proposition 4.3, with $a_i = \rho(u)$ and $b_i = u^\dagger$ for u as in Eq. (6.5), that is,

$$\rho(u)[\gamma^5 \otimes D_Y, u^\dagger]_\rho = \begin{pmatrix} \mathfrak{U} & \\ & 0 \end{pmatrix}_C^D, \tag{6.40}$$

where

$$\mathfrak{U} = \delta_{\dot{s}I}^{\dot{t}J} \begin{pmatrix} \mathfrak{U}_r & \\ & \mathfrak{U}_l \end{pmatrix}_s^t \quad \text{with} \quad \mathfrak{U}_r = \begin{pmatrix} & \bar{\mathsf{k}}^{\mathsf{l}} \mathfrak{H}_1 \\ \mathfrak{H}_2 \mathsf{k}^{\mathsf{l}} & \end{pmatrix}_\alpha^\beta,$$

$$\mathfrak{U}_l = -\begin{pmatrix} & \bar{\mathsf{k}}^{\mathsf{l}} \mathfrak{H}_1' \\ \mathfrak{H}_2' \mathsf{k}^{\mathsf{l}} & \end{pmatrix}_\alpha^\beta, \tag{6.41}$$

in which $\mathfrak{H}_{i=1,2}$ and $\mathfrak{H}'_{1,2} = \rho(\mathfrak{H}_{1,2})$ are given by Eq. (4.25) with [remembering Eq. (6.11)]

$$\begin{aligned} &\mathsf{c} = \boldsymbol{\alpha}', \qquad \mathsf{c}' = \boldsymbol{\alpha}, \qquad q = \boldsymbol{q}', \qquad q' = \boldsymbol{q} \quad \text{and} \\ &\mathsf{d} = \boldsymbol{\alpha}^\dagger, \qquad \mathsf{d}' = \boldsymbol{\alpha}'^\dagger, \qquad p = \boldsymbol{q}^\dagger, \qquad p' = \boldsymbol{q}'^\dagger; \end{aligned} \tag{6.42}$$

that is,

$$\begin{aligned} &\mathfrak{H}_1 = \boldsymbol{\alpha}'(\boldsymbol{q}'^\dagger - \boldsymbol{\alpha}'^\dagger), \qquad \mathfrak{H}_2 = \boldsymbol{q}(\boldsymbol{\alpha}^\dagger - \boldsymbol{q}^\dagger) \quad \text{and} \\ &\mathfrak{H}_1' = \boldsymbol{\alpha}(\boldsymbol{q}^\dagger - \boldsymbol{\alpha}^\dagger), \qquad \mathfrak{H}_2' = \boldsymbol{q}'(\boldsymbol{\alpha}'^\dagger - \boldsymbol{q}'^\dagger). \end{aligned} \tag{6.43}$$

Thus, one obtains Eq. (6.33) with

$$A^u = \mathfrak{U} + \rho(\mathfrak{A}) A \mathfrak{A}^\dagger = \delta_{\dot{s}I}^{\dot{t}J} \begin{pmatrix} \mathfrak{U}_r + \mathfrak{A}_l A_r \mathfrak{A}_r^\dagger & \\ & \mathfrak{U}_l + \mathfrak{A}_r A_l \mathfrak{A}_l^\dagger \end{pmatrix}_s^t. \tag{6.44}$$

From Eqs. (6.38) and (6.41), one obtains the explicit forms of $(A^u)^r$ and $(A^u)^l$:

$$(A^u)_r := \mathfrak{U}_r + \mathfrak{A}_l A_r \mathfrak{A}_r^\dagger = \begin{pmatrix} & \bar{\mathsf{k}}^{\mathsf{l}}(\mathfrak{H}_1 + \boldsymbol{\alpha}' H_1 \boldsymbol{q}'^\dagger) \\ (\mathfrak{H}_2 + \boldsymbol{q} H_2 \alpha^\dagger)\mathsf{k}^{\mathsf{l}} & 0 \end{pmatrix}, \tag{6.45}$$

$$(A^u)_l := \mathfrak{U}_l + \mathfrak{A}_r A_l \mathfrak{A}_l^\dagger = -\begin{pmatrix} & \bar{\mathsf{k}}^{\mathsf{l}}(\mathfrak{H}_1' + \boldsymbol{\alpha} H_1' \boldsymbol{q}^\dagger) \\ (\mathfrak{H}_2' + \boldsymbol{q}' H_2' \boldsymbol{\alpha}'^\dagger)\mathsf{k}^{\mathsf{l}} & 0 \end{pmatrix}. \tag{6.46}$$

The final result follows substituting $\mathfrak{H}_{1,2}$ with their explicit formulas (6.43). ■

A unitary u that preserves the self-adjointness of the unimodular free 1-form (Proposition 6.1) also preserves the self-adjointness of A_Y if, and only if, $K = 0$. Indeed, in that case u is twist invariant (i.e., $\boldsymbol{q}' = \boldsymbol{q}$ and $\boldsymbol{\alpha}' = \boldsymbol{\alpha}$), and one easily checks that for a self-adjoint A_Y (that is, $H_1^\dagger = H_2 = H_r$ and $H_1'^\dagger = H_2' = H_l$ by Corollary 4.3.1) then A_Y^u is self-adjoint as well. If $K \neq 0$, then H_1 and H_2 undergo different gauge transformations, forbidding A_Y^u to be self-adjoint. For this reason, from now on we take $K = 0$. With this caveat, the gauge transformation of Lemma 6.4 then reads as a law of transformation of the complex components (4.32) of the quaternionic fields H_r and H_l.

Proposition 6.5.—Let A_Y be a self-adjoint diagonal 1-form parametrized by two quaternionic field H_r and H_l. Under a gauge transformation induced by a twist-invariant unitary $u = (\alpha, \alpha, \boldsymbol{q}, \boldsymbol{q}, m, m)$, the components $\phi_{1,2}^r$ and $\phi_{1,2}^l$ of H_r and H_l transform, respectively, as

$$\begin{aligned} \begin{pmatrix} \phi_1^r + 1 \\ \phi_2^r \end{pmatrix} &\to \boldsymbol{q} \begin{pmatrix} \phi_1^r + 1 \\ \phi_2^r \end{pmatrix} e^{-i\alpha}, \\ \begin{pmatrix} \phi_1^l + 1 \\ \phi_2^l \end{pmatrix} &\to \boldsymbol{q} \begin{pmatrix} \phi_1^l + 1 \\ \phi_2^l \end{pmatrix} e^{-i\alpha}. \end{aligned} \tag{6.47}$$

Proof.—A_Y being self-adjoint means that Eq. (4.26) holds. A twist-invariant unitary satisfies Eq. (6.12) with $K = 0$. Under these conditions, comparing the formula (4.17) of A_Y with its gauge transformed counterpart (6.35) and (6.36), one finds that the fields H_r and H_l undergo the same transformation:

$$\begin{aligned} H_r &\to \boldsymbol{q}(H_r + \mathbb{I})\boldsymbol{\alpha}^\dagger - \mathbb{I}, \\ H_l &\to \boldsymbol{q}(H_l + \mathbb{I})\boldsymbol{\alpha}^\dagger - \mathbb{I}. \end{aligned} \tag{6.48}$$

Written in components (4.32), with q_{ij} the components of $\mathbf{q}$, these equations read

$$\begin{aligned} \phi_1^r &\to q_{11}(\phi_1^r + 1)e^{-i\alpha} + q_{12}\phi_2^r e^{-i\alpha} - 1, \\ \phi_2^r &\to q_{21}\alpha(\phi_1^r + 1)e^{-i\alpha} + q_{22}\phi_2^r e^{-i\alpha}, \end{aligned} \tag{6.49}$$

and similarly for $\phi_{1,2}^l$. In matricial form, these equations are nothing but Eq. (6.47). ■

The transformations (6.47) are similar to those of the Higgs doublet in the Standard Model (see, e.g., Proposition 11.5 in Ref. [25]). In the twisted version of the Standard Model, we thus obtain two Higgs fields, acting independently on the left and right components of the Dirac spinors. However, as we already mentioned, the two have no individual physical meaning on their own, since they appear in the fermionic action only through the linear combination $h = (H_r + H_l)/2$. Therefore, there is actually only one physical Higgs doublet in the twisted case as well.

In conclusion, we check that the scalar field σ is gauge invariant. As explained below Eq. (6.2), this invariance is not affected by the nonlinear term and is encoded within the transformation

$$A_M \to A_M^u = \rho(u)[\gamma^5 \otimes D_M, u^\dagger]_\rho + \rho(u) A_M u^\dagger. \quad (6.50)$$

Proposition 6.6.—Under a gauge transformation induced by a twist-invariant unitary u, the real fields σ_r and σ_l parameterizing a self-adjoint off-diagonal fluctuation (Proposition 4.6) are invariant.

Proof.—The result amounts to showing that A_M is invariant under Eq. (6.50). Since $u = \rho(u)$ by hypothesis, the twisted commutator in Eq. (6.3) coincides with the usual one $[\gamma^5 \otimes D_M, u^\dagger]$ which is zero by Eq. (3.2). The explicit forms (4.33) of A_M and (6.7) of u yield

$$u A_M u^\dagger = \begin{pmatrix} & \mathfrak{A} C \mathfrak{B}^\dagger \\ \mathfrak{B} D \mathfrak{A}^\dagger & \end{pmatrix}. \quad (6.51)$$

From Eq. (4.35), one checks that $\mathfrak{A} C \mathfrak{B}^\dagger$ has components (omitting the global factor $k_R \delta_s^i$ and δ_I^J per $\mathfrak{A}_{r/l}$)

$$\mathfrak{A}_r C_r \mathfrak{B}_r^\dagger = \sigma \mathfrak{A}_r \Xi_{I\alpha}^{J\beta} \mathfrak{B}_r^\dagger = \sigma \Xi_{I\alpha}^{J\beta}, \quad (6.52)$$

$$\mathfrak{A}_l C_l \mathfrak{B}_l^\dagger = -\sigma' \mathfrak{A}_l \Xi_{I\alpha}^{J\beta} \mathfrak{B}_l^\dagger = -\sigma' \Xi_{I\alpha}^{J\beta}, \quad (6.53)$$

where we use the explicit forms (6.9)–(6.11) of $\mathfrak{A}$ and $\mathfrak{B}$ to get $\mathfrak{A}_r \Xi_{I\alpha}^{J\beta} \mathfrak{B}_r^\dagger = e^{i\alpha} \Xi_{I\alpha}^{J\beta} e^{-i\alpha} = \Xi_{I\alpha}^{J\beta}$, and similarly for Eq. (6.53). Hence, $u A_M u^\dagger = A_M$, and the result. ∎

VII. CONCLUSION

We have worked out the field content of a twisted version of the spectral triple of the Standard Model. The physical meaning of these fields will be made precise by the computation of the fermionic action in the second part of this work [23], as well as the possibility of gauge transformations induced by non-twist-invariant unitaries and their relation with Lorentzian signature.

As shown in Ref. [21], the twisted first-order condition needs to be violated in order to generate the extra scalar field σ. This forbids one to apply the twist by grading of Ref. [20], since the latter always preserves this condition. However, this violation has no real importance, being reabsorbed in the definition of the components of σ. In this sense, the model presented here is the one that minimally violates the twisted first-order condition.

APPENDIX: MORE EXPLICIT COMPUTATIONS

1. Dirac matrices and real structure

Let $\sigma_{j=1,2,3}$ be the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (A1)$$

In four-dimensional Euclidean space, the Dirac matrices (in chiral representation) are

$$\gamma_E^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \tilde{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma_E^5 \coloneqq \gamma_E^1 \gamma_E^2 \gamma_E^3 \gamma_E^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \quad (A2)$$

where, for $\mu = 0, j$, we define

$$\sigma^\mu \coloneqq \{\mathbb{I}_2, -i\sigma_j\}, \qquad \tilde{\sigma}^\mu \coloneqq \{\mathbb{I}_2, i\sigma_j\}. \quad (A3)$$

On a (non-necessarily flat) Riemannian spin manifold, the Dirac matrices are linear combinations of the Euclidean ones:

$$\gamma^\mu = e_\mu^\alpha \gamma_E^\alpha, \quad (A4)$$

where $\{e_\mu^\alpha\}$ are the vierbein, which are real fields on $\mathcal{M}$. These Dirac matrices are no longer constant on $\mathcal{M}$. This is a general result of spin geometry that the charge conjugation commutes with the spin derivative (see, e.g., Proposition 4.18 in Ref. [25]). For the sake of completeness, we check it explicitly for a four-dimensional Riemannian manifold.

Lemma A.7.—The real structure satisfies

$$\mathcal{J}\gamma^\mu = -\gamma^\mu \mathcal{J}, \quad \mathcal{J}\omega_\mu^s = \omega_\mu^s \mathcal{J}, \quad \mathcal{J}\nabla_\mu^s = +\nabla_\mu^s \mathcal{J}. \quad (A5)$$

Proof.—Let us first show that $\mathcal{J}$ anticommutes with the Euclidean Dirac matrices,

$$\{\mathcal{J}, \gamma_E^\mu\} = 0. \quad (A6)$$

From the explicit forms (2.33) of $\mathcal{J}$, this is equivalent to

$$\gamma_E^0 \gamma_E^2 \bar{\gamma}_E^\mu = -\gamma_E^\mu \gamma_E^0 \gamma_E^2, \quad (A7)$$

which is true for $\mu = 0, 2$, since then $\bar{\gamma}_E^\mu = \gamma_E^\mu$ anticommutes with $\gamma_E^0 \gamma_E^2$, and is also true for $\mu = 1, 2$, in which case $\bar{\gamma}_E^\mu = -\gamma_E^\mu$ commutes with $\gamma_E^0 \gamma_E^2$.

Since the spin connection is a real linear combination of products of two Euclidean Dirac matrices, it commutes with $\mathcal{J}$. The latter, having constant components, commutes with ∂_μ and, hence, also with the spin covariant derivative ∇_μ.

These results hold as well in the curved case, for then one has from Eq. (A4)

$$\{\mathcal{J},\gamma^\mu\} = e^\alpha_\mu\{\mathcal{J},\gamma^\alpha_E\} = 0. \tag{A8}$$

■

2. Components of the gauge sector of the twisted fluctuation

The components of the free twisted fluctuation of Proposition 5.8 are $Z^r_\mu = \delta^{\dot{t}}_{\dot{s}}(Z^r_\mu)^{J\beta}_{I\alpha}$ given by (we invert the order of the leptocolor and flavor indices in order to make the comparison with the nontwisted case easier)

$$(Z_\mu)^{0\dot{1}}_{0\dot{1}} = 2a_\mu, \tag{A9}$$

$$(Z_\mu)^{0\dot{2}}_{0\dot{2}} = 2a_\mu + ig_1B_\mu, \tag{A10}$$

$$(Z_\mu)^{0b}_{0a} = \delta^b_a(w^\mu - a_\mu) + i\left(\delta^b_a\frac{g_1B_\mu}{2} - \frac{g_2}{2}(W_\mu)^b_a\right), \tag{A11}$$

$$(Z_\mu)^{j\dot{1}}_{i\dot{1}} = (a_\mu\delta^j_i + (g_\mu)^j_i) - i\left(\frac{2g_1B_\mu}{3}\delta^j_i + \frac{g_3}{2}(V_\mu)^j_i\right), \tag{A12}$$

$$(Z_\mu)^{j\dot{2}}_{i\dot{2}} = (a_\mu\delta^j_i + (g_\mu)^j_i) + i\left(\frac{g_1B_\mu}{3}\delta^j_i - \frac{g_3}{2}(V_\mu)^j_i\right), \tag{A13}$$

$$(Z_\mu)^{jb}_{ia} = (\delta^b_a w_\mu\delta^j_i - (g_\mu)^j_i) - i\left(\delta^b_a\left(\frac{g_1B_\mu}{6}\delta^j_i + \frac{g_3}{2}(V_\mu)^j_i\right) + \frac{g_2}{2}(W_\mu)^b_a\delta^j_i\right), \tag{A14}$$

$$(Z_\mu)^{J\dot{2}}_{I\dot{1}} = (Z^r_\mu)^{J\dot{1}}_{I\dot{2}} = 0. \tag{A15}$$

One then checks that

$$i\begin{pmatrix}(Y_\mu)^{j\dot{1}}_{i\dot{1}} & & \\ & (Y_\mu)^{j\dot{2}}_{i\dot{2}} & \\ & & (Y_\mu)^{j\dot{b}}_{i\dot{a}}\end{pmatrix}^\beta_\alpha = \begin{pmatrix}-i\left(\frac{2g_1B_\mu}{3}\delta^j_i + \frac{g_3}{2}(V_\mu)^j_i\right) & & \\ & i\left(\frac{g_1B_\mu}{3}\delta^j_i - \frac{g_3}{2}(V_\mu)^j_i\right) & \\ & & -i\left(\delta^b_a\left(\frac{g_1B_\mu}{6}\delta^j_i + \frac{g_3}{2}(V_\mu)^j_i\right) + \frac{g_2}{2}(W_\mu)^b_a\delta^j_i\right)\end{pmatrix}^\beta_\alpha$$

coincides with the matrix $\mathbb{A}^q_\mu$ of the nontwisted case [Eq. (1.733) in Ref. [24]], while

$$i\begin{pmatrix}(Y_\mu)^{0\dot{1}}_{0\dot{1}} & & \\ & (Y_\mu)^{0\dot{2}}_{0\dot{2}} & \\ & & (Y_\mu)^{0\dot{b}}_{0\dot{a}}\end{pmatrix}^\beta_\alpha = \begin{pmatrix}0 & & \\ & ig_1B_\mu & \\ & & i\left(\delta^b_a\frac{g_1B_\mu}{2} - \frac{g_2}{2}(W_\mu)^b_a\right)\end{pmatrix}^\beta_\alpha$$

coincides with the matrix $\mathbb{A}^l_\mu$ [Eq. (1.734) in Ref. [24]].

3. Twisted first-order condition

For a twisted spectral triple, there is a natural twisted version of the first-order condition (2.13) that was introduced in Ref. [16] and whose mathematic pertinence has been investigated in detail in Refs. [19,20], namely,

$$[[D,b]_\rho, a^\circ]_{\rho^\circ} = 0, \qquad a,b\in\mathcal{A}, \tag{A16}$$

where $\rho^\circ\in\mathrm{Aut}\mathcal{A}^\circ$ is the automorphism of the opposite algebra $\mathcal{A}^\circ$ induces in Eq. (3.31) by the twisting automorphism $\rho\in\mathrm{Aut}\mathcal{A}$.

Proposition A.8.—The free part $\partial\!\!\!/\otimes\mathbb{I}_F$ and the diagonal part $\gamma^5\otimes D_Y$ of the Dirac operator satisfy the twisted first-order condition (A16), while the off-diagonal part $\gamma^5\otimes D_M$ violates it.

Proof.—For $\partial\!\!\!/\otimes\mathbb{I}_F$, using Eq. (3.17) and Corollary 5.1.1, one gets

$$[[\partial\!\!\!/\otimes\mathbb{I}_F, b]_\rho, JaJ^{-1}]_{\rho^\circ} = i\gamma^\mu\begin{pmatrix}[\partial_\mu R,\bar{M}] & \\ & [\partial_\mu N,\bar{Q}]\end{pmatrix}^D_C. \tag{A17}$$

The top-left entry reads (omitting the s and $\dot{s}$ indices for simplicity)

$$[\partial_\mu R,\bar{M}] = \begin{pmatrix}\partial_\mu\mathsf{d}[\mathbb{I}_4,\bar{\mathsf{m}}]^J_I & \\ & \partial_\mu p'[\mathbb{I}_4,\bar{\mathsf{m}}']^J_I\end{pmatrix}^\beta_\alpha = 0. \tag{A18}$$

Similarly, one shows that $[\partial_\mu N,\bar{Q}] = 0$; hence, $\partial\!\!\!/\otimes\mathbb{I}_F$ satisfies the twisted first-order condition.

For the diagonal part $\gamma^5\otimes D_Y$, Lemma 4.1 and Eq. (3.17) yield

$$[[\gamma^5 \otimes D_Y, b]_\rho, JaJ^{-1}]_{\rho^\circ} = -\begin{pmatrix} [S, \bar{M}]_{\rho^\circ} & 0 \\ 0 & 0 \end{pmatrix}_C^D. \tag{A19}$$

In tensorial notation,

$$[S, \bar{M}]_{\rho^\circ} = \delta_{\dot{s}}^{\dot{t}}[\eta_s^u (D_0)_{I\alpha}^{J\gamma} R_{u\gamma}^{t\beta}, \bar{M}_{s\alpha I}^{t\beta J}]_{\rho^\circ} - \delta_{\dot{s}}^{\dot{t}}[\rho(R)_{s\alpha}^{u\gamma}\eta_u^t (D_0)_{I\gamma}^{J\beta}, \bar{M}_{s\alpha I}^{t\beta J}]_{\rho^\circ}. \tag{A20}$$

The right-hand side of Eq. (A20) is (omitting the indices $\dot{s}$, α, and I)

$$\begin{aligned} &\begin{pmatrix} D_0 R_r & \\ & -D_0 R_l \end{pmatrix}_s^t \begin{pmatrix} \bar{M}_r & \\ & \bar{M}_l \end{pmatrix}_s^t - \begin{pmatrix} \bar{M}_l & \\ & \bar{M}_r \end{pmatrix}_s^t \begin{pmatrix} D_0 R_r & \\ & -D_0 R_l \end{pmatrix}_s^t \\ &- \begin{pmatrix} R_l D_0 & \\ & -R_r D_0 \end{pmatrix}_s^t \begin{pmatrix} \bar{M}_r & \\ & \bar{M}_l \end{pmatrix}_s^t + \begin{pmatrix} \bar{M}_l & \\ & \bar{M}_r \end{pmatrix}_s^t \begin{pmatrix} R_l D_0 & \\ & -R_r D_0 \end{pmatrix}_s^t \\ &= \begin{pmatrix} D_0 R_r \bar{M}_r - \bar{M}_l D_0 R_r - R_l D_0 \bar{M}_r + \bar{M}_l R_l D_0 & \\ & -D_0 R_l \bar{M}_l + \bar{M}_r D_0 R_l + R_r D_0 \bar{M}_l - \bar{M}_r R_r D_0 \end{pmatrix}. \end{aligned} \tag{A21}$$

From the explicit form (2.26) of D_0, (3.11) of $M_{l/r}$, and (3.25) of $R_{r/l}$, one checks that

$$\begin{aligned} D_0 R_r \bar{M}_r = \bar{M}_l D_0 R_r &= \begin{pmatrix} 0 & \bar{\mathsf{k}} p' \bar{\mathsf{m}}' \\ \mathsf{k}\mathsf{d}\bar{\mathsf{m}} & 0 \end{pmatrix}, \\ R_l D_0 \bar{M}_r = \bar{M}_l R_l D_0 &= \begin{pmatrix} 0 & \bar{\mathsf{k}}\,\bar{\mathsf{m}}'\,\mathsf{d}' \\ \mathsf{k}\bar{\mathsf{m}} p & 0 \end{pmatrix}, \end{aligned} \tag{A22}$$

so that the upper-left term in Eq. (A21) is zero. The same is true for the lower-right term; hence, $[S, \bar{M}] = 0$.

This shows that Eq. (A19) vanishes, which is equivalent to the proposition.

For the off-diagonal part $\gamma^5 \otimes D_M$, one has (omitting the s and $\dot{s}$ indices for simplicity)

$$[\gamma^5 \otimes D_M, b]_\rho = \begin{pmatrix} 0 & \gamma^5 \Xi_{I\alpha}^{J\beta} k_R (d - d') \\ \gamma^5 \Xi_{I\alpha}^{J\beta} \bar{k}_R (d - d') & \end{pmatrix}_C^D, \tag{A23}$$

and, hence,

$$[[\gamma^5 \otimes D_M, b]_\rho, J\bar{a}J^{-1}]_{\rho^\circ} = -\begin{pmatrix} 0 & \gamma^5 \Xi_{I\alpha}^{K\gamma} k_R (d - d') Q_{K\gamma}^{J\beta} - \rho(M_{I\alpha}^{K\gamma}) \gamma^5 \Xi_{K\gamma}^{J\beta} k_R (d - d') \\ \cdots & 0 \end{pmatrix}_C^D, \tag{A24}$$

whose top-right entry reads

$$\gamma^5 \Xi_{I\alpha}^{K\gamma} k_R (d - d') Q_{K\gamma}^{J\beta} - \rho(M_{I\alpha}^{K\gamma}) \gamma^5 \Xi_{K\gamma}^{J\beta} k_R (d - d') = k_R \delta_{\dot{s}}^{\dot{t}} \Xi_{I\alpha}^{J\beta} \begin{pmatrix} \sigma + \sigma' & \\ & -(\sigma + \sigma') \end{pmatrix}_s^t \tag{A25}$$

and is nonzero. ∎

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