Emergent black hole thermodynamics from monodromy

Alexandra B. Chanson,[1,*] Jacob Ciafre,[1,†] and Maria J. Rodriguez[1,2,‡]

[1]*Department of Physics, Utah State University, 4415 Old Main Hill Road, Utah 84322, USA*
[2]*Instituto de Física Teórica UAM/CSIC, Universidad Autónoma de Madrid, Cantoblanco, 28049 Madrid, Spain*

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We argue that the equations of motion of quantum field theories in curved backgrounds encode new underlying black hole thermodynamic relations. We define new entropy variation relations. These "emerge" through the monodromies that capture the infinitesimal changes in the black hole background produced by the field excitations. This raises the possibility of new thermodynamic relations defined as independent sums involving entropies, temperatures, and angular velocities defined at every black hole horizon. We present explicit results for the sum of all horizon entropy variations for general rotating black holes, in both asymptotically flat and asymptotically anti–de Sitter spacetimes in four and higher dimensions. The expressions are universal and in most cases add up to zero. We also find that these thermodynamic summation relations apply in theories involving multicharge black holes.

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I. INTRODUCTION

Quantum field theory in curved backgrounds is expected to provide an accurate description of quantum phenomena occurring near—and inside of—black holes. The simplest form of the equations of motion are embodied in the Klein-Gordon (KG) equation, which describes a massless scalar field in a curved background. Surprisingly, the KG equations encode not only the analytic structure of the background geometries but also new underlying black hole thermodynamic relations. In this paper we will describe the emergence of a new thermodynamic identity from the KG equation, a constraint on the sum of the variational horizon entropies, and identify new universal thermodynamic relations for black holes.

The discovery of the thermodynamic behavior of black holes has given rise to most of our present physical insights into the quantum nature in the strong field regime. Most famously, in 1973, Bardeen, Carter, and Hawking [1] provided a general proof of the first laws of thermodynamics of black holes with the intensive quantities defined at the black hole event horizon r_+. Yet, eternal black hole solutions contain a much richer geometrical horizon structure which includes Cauchy horizons (in Kerr spacetimes, the so-called inner event horizon r_-). More recently, it was shown in [2] that there is a universal "geometrical first law of thermodynamics" for a Cauchy horizon. And, while physically unobservable, the fact that the inner horizon obeys standard thermodynamic relations is an indication of the validity of the conformal field theory description [3,4]. This additional horizon structure appears to play an important role in the precise description of generic microstates [5]. It is natural then to inquire whether, analogously, a geometrical thermodynamic law holds for every (real or complex valued) horizon in more general classes of black holes. In what follows, we shall just refer to zeros of the radial function as horizons, regardless of whether these are real, imaginary, or complex.

In this paper, we consider the geometrical relation of black hole thermodynamics:

$$dE = T_i dS_i + \sum_k \Omega_i^{(k)} dJ_{(k)} + \sum_l \Phi_i^l dQ_l, \qquad (1)$$

Here S_i is the entropy, and the extensive quantities—the total mass or energy E, the angular momenta $J_{(k)}$, and the total charge Q_l—are the Komar charges. The corresponding intensive quantities—the temperature T_i, the angular velocities $\Omega_i^{(k)}$, and electromagnetic potential Φ_i^l—are defined at each black hole horizon. The i subscript represents the r_i black hole horizon where the thermodynamic quantity was defined, the index $k = 1, 2, \ldots, [\frac{d-1}{2}]$ represents the independent planes of rotation in a d spacetime and l the number of electric/magnetic charges. An explicit examination of a fairly extensive number of black hole solutions reveals that the geometrical law of thermodynamics (1) is indeed universal for all black hole horizons.

Many discussions in the literature of KG wave equations in curved spacetimes with more than one Killing horizon

*a.chanson@aggiemail.usu.edu
†jakeciafre@outlook.com
‡maria.rodriguez@usu.edu
maria.rodriguez@csic.es

note that these will have dominant contributions associated with poles at each of these zeros [3–6]. One can therefore expect that the thermodynamics associated with each horizon will play a role in governing the properties of the black hole at the microscopic level. Remarkably, we find that the geometrical first law of thermodynamics (1) for every black hole horizon emerges from the KG equation (via monodromies). We argue that the variations with respect to the conserved charges of the black hole entropies δS_i are directly proportional to the monodromies α_i around each horizon:

$$\delta S_i = \frac{(d-2)}{4}\mathcal{A}_{d-2}\alpha_i, \tag{2}$$

where $\mathcal{A}_{d-2} = 2\pi^{(d-1)/2}/\Gamma((d-1)/2)$ is the area of a unit $(d-2)$ sphere. The entropy variation relation (2) follows from a rather remarkable extension of the ideas in [2] for the black hole event horizon, which is recovered when, e.g., $d=4$ for the Kerr black hole solution

$$\delta S_+ = 4\pi\tilde{\alpha}_+, \quad \text{where } \tilde{\alpha}_+ = \frac{(\omega - \Omega_+ m)}{4\pi T_+}. \tag{3}$$

The scalar field excitations produce infinitesimal changes in the black hole mass $\omega = \delta M$ and angular momentum $m = \delta J$; these associations reproduce (1) for the Kerr black hole. Having verified (1) explicitly for all horizons, we can confirm that the monodromies have an important physical interpretation; namely they represent the variation in the entropy (2). We find that these infinitesimal relations do not imply global identities, such as Smarr relations.

Critically, the KG equation can be reduced to a Fuchsian-type radial ansatz. This equation encodes the monodromy coefficient at infinity α_∞ of points at $r=\infty$. Employing a similar argument as in (2), we can therefore also define a new monodromy/entropy variation relation at infinity, which is $\delta S_\infty = \frac{(d-2)}{4}\mathcal{A}_{d-2}((\mathcal{K}-1)-\alpha_\infty)$ with a constant parameter $\mathcal{K}$.

Subsequently we find another remarkable thermodynamic feature for black hole solutions emerging from the KG equation: we find that the summation of all monodromies—generally, a Fuchs(-type) relation—defines a new thermodynamic relation of the entropies, namely

$$\sum_i \delta S_i = \delta S_\infty. \tag{4}$$

We have explicitly verified this relation for an extensive list of black hole solutions. These include Schwarzschild, Kerr, d-dimensional Schwarzschild and Myers-Perry (MP) [7], Bañados-Teitelboim-Zanelli (BTZ) [8], d-dimensional Schwarzschild and Kerr-(A)dS [9–11], Reissner-Nordstrom (RN) [12], Kerr-Newman [13], Kerr-Newman-Anti-deSitter (AdS), d-dimensional RN-(A)dS, $d=5$ min gauged Supergravity (SUGRA) [14], and $d=6$ gauged SUGRA [15] black hole solutions.

In all cases, except for Kerr and Kerr-Newman,[1] S_∞ seems independent of the extensive quantities for all black holes. Therefore, $\delta S_\infty = 0$ and the relation (4) leads to

$$\delta\left(\sum_i S_i\right) = 0, \tag{5}$$

and the following new universal relations for black holes:

$$\sum_i \frac{1}{T_i} = 0, \qquad \sum_i \frac{\Omega_i^{(k)}}{T_i} = 0. \tag{6}$$

It is also possible to verify, by considering charged black hole solutions, that

$$\sum_i \frac{\Phi_i^l}{T_i} = 0. \tag{7}$$

Our results are summarized in Table I.

The paper proceeds as follows. Section II includes an overview of the KG equation, our results for the monodromies, and the Fuchs(-type) relation for all black holes listed in Table I. Section III details the AdS and Flat calculations for the Scharzschild and Kerr geometries. Section IV contains the explicit physical parameters for general asymptotically flat or AdS black holes solutions—which are in some cases neutral, charged, or rotating—in d-spacetime dimensions; parameters are defined at every black hole horizon and individually shown to be consistent with the first law of thermodynamics for all black hole horizons. In each case we also assess the relation (5)–(7) employing purely thermodynamical quantities. In Sec. V, we prove the new relations (6) and (7) that follow from the first law of thermodynamics and the properties of the $\sum_i S_i$. Finally, in Sec. VI, concluding remarks are presented.

II. OVERVIEW

One can explore the geometry of a neutral black hole by considering small perturbations of the background. One of the simplest possibilities is a minimally coupled scalar, i.e., a massless scalar field that satisfies the KG equation,

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) = 0. \tag{8}$$

The solutions to this equation can be presented in such a way that the Killing symmetries deriving from stationarity and the axial symmetries are satisfied. Then the wave function can be written as

[1] The Kerr black hole solution does partially break the condition for the sum of the entropies being independent of the extensive quantities. Namely, $\sum_i S_i = 4\pi M^2$, and thus $\sum_i \frac{1}{T_i} \neq 0$. Further details can be found in Sec. IVA.

TABLE I. This table summarizes new thermodynamic relations for a copious number of black hole solutions in $d \geq 3$ spacetime dimensions. The results include the confirmation of the emergent relation for the sum of the entropy variations of each black hole horizon $\sum_i \delta S_i = \delta S_\infty$ as well as new thermodynamic relations. These new relationships, jointly with the properties of the $\sum_i S_i$, are a by-product of the geometrical relation of thermodynamics for all black hole horizons (1). The ✓ represent the thermodynamic relations that are fulfilled, while the ✗ signal those that fail to obey the relations. The horizon radii r_i are defined by the radial function $\Delta(r_i) = 0$.

Black hole	$\sum_i \delta S_i = \delta S_\infty$	$\sum_i T_i^{-1} = 0$	$\sum_i \Omega_i^a / T_i = 0$	$\sum_i \Phi_i^a / T_i = 0$
Schwarzschild	✓	✗	…	…
Kerr [16]	✓	✗	✓	…
Reissner-Nordstrom (RN)	✓	✗	…	✗
Kerr-Newman (KN) [13]	✓	✗	✓	✗
Schwarzschild $d > 4$ [17]	✓	✓	…	…
Myers-Perry [7]	✓	✓	✓	…
BTZ [8]	✓	✓	✓	…
Schw-(A)dS	✓	✓	…	…
Kerr-(A)dS [9]	✓	✓	✓	…
Schw-(A)dS$_d$	✓	✓	…	…
Kerr-(A)dS$_d$ [10,18]	✓	✓	✓	…
RN$_d$ [12]	✓	✓	…	✓
RN-(A)dS	✓	✓	…	✓
RN-(A)dS$_d$	✓	✓	…	✓
KN-(A)dS [12]	✓	✓	✓	✓
$5d$ gauged SUGRA [14]	✓	✓	✓	✓
$6d$ gauged SUGRA [15]	✓	✓	✓	✓

$$\Phi(t, r, \theta, \phi_k) = \exp^{-i\omega t + \sum_k i m_k \phi_k} T(\theta) R(r), \qquad (9)$$

by means of the time coordinate t, the radial coordinate r, the polar coordinate θ, together with $[(d-1)/2]$ azimuthal angular coordinates ϕ_k. From the black hole backgrounds (given in, e.g., Sec. III) it is straightforward to write out equation (8) explicitly. For black hole solutions to Einstein's equation of general relativity (GR) $R_{\mu\nu} = \Lambda g_{\mu\nu}$ with $\Lambda = -(d-1)/L^2$, in Boyer-Lindquist coordinates, the wave equation can be reduced in the radial r plane to a second order differential equation (ODE) with singularities $r_1, \ldots, r_\mathcal{K}$ and the point at infinity $r = \infty$. It is useful to define an "evenness" integer $\epsilon = (d-1) \bmod 2$ which is 1 for even d and 0 for odd d to express

$$\mathcal{K} = d + 1 - \epsilon. \qquad (10)$$

For black holes,[2] while $r_1, \ldots, r_\mathcal{K}$ are regular singular points, $r = \infty$ can be either a regular or an irregular singular point. For instance, in d-spacetime dimensions we find that the point $r = \infty$ is an irregular singular point in asymptotically flat black holes ($\Lambda = 0$) and is regular for asymptotically AdS black holes ($\Lambda \neq 0$)—see Appendix A for details. This distinction between regular and irregular singular points might seem artificial, but their implications for $R(r)$ are starkly different. Regular singular points have series expansion around, say, r_i of the form

$$R(r) = (r - r_i)^{i\alpha_i}[1 + O(r - r_i)], \qquad (11)$$

whereas the asymptotic expansion for the solutions, e.g., around $r = \infty$, one must also include exponential factors in the series expansion

$$R(r) = e^{i\omega r} r^{i\lambda_\infty - (d-2)/2}[1 + O(r^{-1})], \qquad (12)$$

where λ_∞ is the eigenvalue of the formal monodromy that we will refer to as *fake monodromy.*

Having identified the singular points, we turn to the Fuchs relation, which describes a direct relationship between the asymptotic exponents α_i (i.e., the *monodromies*) of formal series solutions (11) of Fuchsian linear differential equations. We argue that the radial ODE for AdS black holes has all regular singularities (including $r = \infty$) and is therefore a Fuchsian-type equation with a corresponding Fuchs relation. As we will now show, the Fuchs relation relating the sum of the monodromies around each regular singular point of the ODE is exactly the relation of the sum of the variation of the black hole entropies defined at each horizon (4). Furthermore, we will establish as a limiting case a Fuchs-type relation for the asymptotically flat black holes. The new Fuchs-type

[2] We focus on the nonextremal black holes in this paper where $r_i \neq r_j$. The extremal black hole cases will be studied elsewhere.

relation also gives account to the sum of the variation of the black hole entropies defined at each horizon that is found from purely thermodynamical relations. In this sense, we find a new *emergent black hole thermodynamic relation from monodromies relations*.

Understanding the Fuchs relation will suffice to illustrate the essence of our approach to the sum of entropy variations, so we focus first on asymptotically AdS black holes in Sec. II A, discussing the added complications of irregular singular points for asymptotically flat black holes in Sec. II B.

A. ODE with all regular singular points: Fuchsian equation

Here we give the fundamentals of our approach to the sum of entropy variations for asymptotically AdS black holes in all dimensions. We begin by defining the second-order Fuchsian equation with all $\mathcal{K}$ regular singularities at $r = r_i$, where $i = 1, 2, \ldots, \mathcal{K}$, and at ∞. This is given by

$$\frac{d^2R}{dr^2} + p(r)\frac{dR}{dr} + q(r)R = 0, \tag{13}$$

with rational functions of the form

$$p(r) = \sum_{i=1}^{\mathcal{K}} \frac{1-\alpha_i}{(r-r_i)}, \qquad q(r) = \sum_{i-1}^{\mathcal{K}} \frac{q_i}{(r-r_i)}. \tag{14}$$

A brief review can be found in [19]. This representation of Eq. (13), which we will refer to as the *monodromy frame* choice, corresponds to a setup for a Fuchsian equation with $\lim_{r\to r_i} q(r)(r-r_i)^2 = 0$ where one of the two monodromy exponent parameters (around the regular singular point) is null. This particular frame gives us the simplest form for the monodromies to establish the link with the entropy variations as we explain below. While other frame choices are possible, we found that alternative frames as defined in [20] give the same results with slightly more involved relations.

When the differential equation (13) has regular singularities over $P_1(\mathbb{C})$, its exponents obey the so-called Fuchs relation [20]:

$$\sum_{i=1}^{\mathcal{K}} \alpha_i = (\mathcal{K}-1) - \alpha_\infty. \tag{15}$$

Note that at $r = r_i$, the indicial equation is

$$\alpha_i(\alpha_i - 1) + p_0\alpha_i + q_0 = 0, \tag{16}$$

where

$$p_0 = \lim_{r\to r_i}(r-r_i)p(r), \qquad q_0 = \lim_{r\to r_i}(r-r_i)^2 q(r). \tag{17}$$

For (13), one finds that $p_0 = 1 - \alpha_i$ and $q_0 = 0$, so that the $\mathcal{K}$-nontrivial monodromy exponent parameters at the finite singularities r_i are α_i. The monodromy exponent parameter α_∞ relative to the $r = \infty$ regular singular point is given by (11) transformed by $x = r^{-1}$ at $x = 0$.

Having established our conventions, we proceed to present our general approach for AdS black holes in d-spacetime dimensions. An examination of the radial equation derived from the KG equation in AdS black hole backgrounds in all dimensions shows that the ODE contains all regular singular points, including asymptotic infinity (p_0, q_0 are finite at every singular point). It is therefore a Fuchsian equation and its parameters are subject to the Fuchs relation (15). To implement the Fuchs relation we set the frame that simplifies the analysis by bringing the ODE to form (13). The parameter $\mathcal{K}$ corresponds to the total number of horizons in the black hole solution.

Extending the ideas in [2], we argue that at every (regular singular point) black hole horizon in (15) we can identify the monodromies with the entropy variations as in Eq. (2).

The Fuchs relation then encodes an emergent relation on the sum of entropy variations

$$\sum_{i=1}^{\mathcal{K}} \alpha_i = (\mathcal{K}-1) - \alpha_\infty \quad \to \quad \sum_{i=1}^{\mathcal{K}} \delta S_i = \delta S_\infty, \tag{18}$$

where we defined $\delta S_\infty = \frac{(d-2)}{4}\mathcal{A}_{d-2}((\mathcal{K}-1) - \alpha_\infty)$. As we will illustrate explicitly in Sec. III, the variation $\delta S_\infty = 0$ for asymptotically AdS black holes. Therefore, replacing these emergent relations between the entropy variations and monodromies, the Fuchs relation (15) can be interpreted as the sum of the variation of entropies

$$\sum_{i=1}^{\mathcal{K}} \delta S_i = 0 \qquad \text{for AdS} - \text{black holes}. \tag{19}$$

We verified this new emerging thermodynamic relation for the sum of the variations of entropies in Sec. IV employing purely thermodynamic definitions. Having identified this relation, we were able to find and verify the sum of the entropies being independent of all intrinsic or extrinsic parameters

$$\delta\left(\sum_{i=1}^{\mathcal{K}} S_i\right) = 0, \tag{20}$$

which follows (19) and other universal relations (6) for black holes such as

$$\sum_{i=1}^{\mathcal{K}} \frac{1}{T_i} = 0, \qquad \sum_{i=1}^{\mathcal{K}} \frac{\Omega_i^{(k)}}{T_i} = 0. \tag{21}$$

We present a proof for the latter relations in Sec. V.

B. ODE with one irregular singular point: Fuchs-type relation

In contrast with the black hole solutions in curved spacetimes, the radial equation derived from the KG equation in asymptotically flat black hole backgrounds is irregular at asymptotic infinity. This ODE is not given by a Fuchsian equation, yet we can still find a Fuchs-type relation that holds for asymptotically flat black holes. Asymptotically flat black hole solutions of GR result from taking a limit of vanishing cosmological constant $\Lambda \to 0$. Equivalently, since $\Lambda = -(d-1)/L^2$, the vanishing cosmological constant limit corresponds to $L \to \infty$. The Fuchs-type relation that we find for asymptotically flat spacetimes (with one irregular singularity at $r = \infty$) follows from taking the vanishing cosmological constant limit ($L \to \infty$ in our case) in Eq. (15):

$$\lim_{L\to\infty} \left(\sum_{i=1}^{\mathcal{K}} \alpha_i - (\mathcal{K} - 1) + \alpha_\infty \right) = 0, \tag{22}$$

$$\sum_{i=1}^{\mathcal{K}-2} \alpha_i^{\text{flat}} - \lambda_\infty = 0. \tag{23}$$

We will refer to (23) as the Fuchs-type relation for asymptotically flat black holes containing one irregular singular point at ∞. This relation results from $\mathcal{K} - 2$ monodromies remaining finite

$$\lim_{L\to\infty} \alpha_i = \alpha_i^{\text{flat}}, \tag{24}$$

around the $\mathcal{K} - 2$ singularities that, in the $L \to \infty$ limit, remain (finite) regular singular points. The other two regular singular points converge to $r = \infty$, and, interestingly, the limiting sum of their corresponding monodromies is precisely the fake monodromy λ_∞ as defined in (12),

$$\lim_{L\to\infty} \alpha_{\mathcal{K}-1} + \alpha_{\mathcal{K}} = -\lambda_\infty, \tag{25}$$

We can implement the relation (23) to asymptotically flat black holes (such as Kerr or Myers-Perry black hole solutions). The total number of horizons in the black hole solution in this case is $\bar{\mathcal{K}} = d - 1 - \epsilon$ with $\epsilon = 0,\ 1$, respectively, for odd or even d-dimensional spacetimes (or alternatively, in terms of the $\mathcal{K}$, $\bar{\mathcal{K}} = \mathcal{K} - 2$). As in the previous section, employing Eq. (2) relating the monodromies to the entropy variations at each black hole horizon (which are always regular singularities) we find through (2) that

$$\sum_{i=1}^{\mathcal{K}-2} \alpha_i^{\text{flat}} - \lambda_\infty = 0 \qquad \to \qquad \sum_{i=1}^{\mathcal{K}-2} \delta S_i^{\text{flat}} = \delta S_\infty^{\text{flat}}, \tag{26}$$

where $\delta S_\infty^{\text{flat}} = \frac{(d-2)}{4} \mathcal{A}_{d-2} \lambda_\infty$. We therefore argue that a new universal thermodynamic relation involving the sum of the variation of entropies emerges from the Fuchs-type relation (23) for asymptotically flat black hole solutions with $\bar{\mathcal{K}}$ horizons:

$$\sum_{i=1}^{\bar{\mathcal{K}}} \delta S_i^{\text{flat}} = \delta S_\infty^{\text{flat}} \qquad \text{for asymptotically flat black holes.} \tag{27}$$

As we will explicitly show in Appendix A, the fake monodromy for asymptotically flat black holes is

$$\text{For Asympt. Flat}_4\colon\ \lambda_\infty \to 4M\omega, \tag{28}$$

$$\text{For Asympt. Flat}_{d>4}\colon\ \lambda_\infty \to 0. \tag{29}$$

We verified in these cases that the new emerging thermodynamic relation for the sum of the variations of entropies (27) employing solely the physical thermodynamic parameters—see Sec. IV for details. A new set of thermodynamic relations (6) also arises for asymptotically flat black holes with $\lambda_\infty = 0$,

$$\sum_{i=1}^{\bar{\mathcal{K}}} \frac{1}{T_i} = 0, \qquad \sum_{i=1}^{\bar{\mathcal{K}}} \frac{\Omega_i^{(k)}}{T_i} = 0 \tag{30}$$

that follow from (27). Section V contains further details and proofs.

III. MONODROMIES AND EMERGENT THERMODYNAMICS

This section focuses on the study of a scalar field in the d-dimensional (asymptotically AdS and flat) Schwarzschild and Kerr black hole background, but the methods are readily extendable to a broad class of physically relevant situations. We will first revisit the wave equation of the probes with particular emphasis on the machinery to compute the (finite and fake) monodromies.

A. AdS$_d$ Schwarzschild black hole

To set up our notation and conventions, we start by reviewing aspects of the geometry of a d-dimensional Schwarzschild black hole with mass M. In Boyer-Lindquist coordinates, we have

$$ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 d\Omega_{d-2}^2, \tag{31}$$

where $\Delta = 1 - \frac{2M}{r^{d-3}} + \frac{r^2}{L^2} = (r^{d-3}L^2)^{-1}\prod_{i=1}^{\mathcal{K}-1}(r-r_i)$ with $\mathcal{K} = d+1-\epsilon$.

1. Wave equation

The Klein-Gordon equation for a massless scalar is (8) and, using (9), makes the equation separable. When the separation constant is set to zero, the radial equation for $R(r)$ is given by

$$\frac{1}{r^{d-2}}\partial_r(r^{d-2}\Delta\partial_r R(r)) + \frac{\omega^2 R(r)}{\Delta} = 0. \tag{32}$$

When choosing

$$R(r) = \prod_{i=1}^{\mathcal{K}-1}(r-r_i)^{-i\tilde{\alpha}_i}\tilde{R}(r), \quad \text{with} \quad \tilde{\alpha}_i = \frac{\omega}{\Delta'(r_i)}, \tag{33}$$

the ODE for $\tilde{R}(r)$ is a Fuchsian equation of the form (13) with (14) given by

$$p(r) = \frac{2r^{d-2}\Delta\partial_r P + P\partial_r(r^{d-2}\Delta)}{r^{d-2}\Delta P},$$
$$q(r) = \frac{\partial_r^2 P}{P} + \frac{\partial_r(r^{d-2}\Delta)(\partial_r P)}{r^{d-2}\Delta P} + \frac{\omega^2}{\Delta^2}, \tag{34}$$

where $P = \prod_{i=1}^{\mathcal{K}-1}(r-r_i)^{-i\tilde{\alpha}_i}$. In turn, the monodromy exponents $\{\alpha_0, \alpha_i, \alpha_\infty\}$ with $i = 1, 2, \ldots, \mathcal{K}-1$ in the formal series solutions (11) for $\tilde{R}(r)$ obey the Fuchs relation

$$\sum_{i=0}^{\mathcal{K}} \alpha_i = (\mathcal{K}-1) - \alpha_\infty. \tag{35}$$

2. Monodromies and regular singular points

One can identify the nature of the singular points, for example, by following the steps in Sec. II and Appendix A. As it turns out in this case, all singularities in the ODE for $\tilde{R}(r)$ are regular singular points. In addition to ∞, the $\mathcal{K}$ finite regular singularities are located at $r = \{0, r_i\}$. The monodromies that we find via the indicial equation are, respectively,

$$\alpha_\infty = \mathcal{K}-1, \qquad \alpha_0 = 0,$$
$$\alpha_i = 2\tilde{\alpha}_i \quad \text{for } i = 1, 2, \ldots., \mathcal{K}-1. \tag{36}$$

An alternative way of determining the monodromy of the singularity at, say, $r = 0$, would be to substitute a series expansion (11) and study the behavior of the ODE near these points. And, similarly, substituting the series expansion (11) and changing $r \to x^{-1}$ in (32) for $\tilde{R}(r)$ we find that the monodromy around $x = 0$ ($r = \infty$).

3. Sum of entropy variations

Having computed the monodromies we can verify that these obey the Fuchs relation (15). We find, replacing the relation (2) between the monodromies and the entropy variations in the Fuchs relation, that the sum of the monodromy parameters is equivalent to the sum of the variations of the entropies:

$$\sum_{i=1}^{\mathcal{K}}\alpha_i = 0, \quad \to \quad \sum_{i=1}^{\mathcal{K}}\delta S_i = 0, \quad \to \quad \delta\left(\sum_{i=1}^{\mathcal{K}} S_i\right) = 0, \tag{37}$$

where $S_i = (\mathcal{A}_{d-2}/4)r_i^{d-2}$ is the entropy computed at each horizon of the AdS$_d$ Schwarzschild black hole. We further verified this new entropy bound employing purely thermodynamic relations in Sec. IV. In agreement with the previous result in $d = 4$, Eq. (2) becomes $S_+ = 4\pi\tilde{\alpha}_+$ as found in [2].

B. Schwarzschild black hole in d dimensions

We begin by briefly describing the geometry and radial part of the wave equation for an asymptotically flat d-dimensional Schwarzschild black hole describing a generic asymptotically flat static black hole with mass M. The line element is of the form (31) where the function $\Delta \to \bar{\Delta} = 1 - \frac{2M}{r^{d-3}} = (r^{d-3})^{-1}\prod_{i=1}^{\bar{\mathcal{K}}-1}(r-r_i)$ and $\bar{\mathcal{K}} = d-1-\epsilon$.

1. Wave equation

The massless scalar Klein-Gordon equation in the background of a d-dimensional Schwarzschild black hole solution is separable. Employing the ansatz (9), and setting the separation constant to zero, the radial equation for the function $R(r)$ becomes

$$\frac{1}{r^{d-2}}\partial_r(r^{d-2}\bar{\Delta}\partial_r R(r)) + \frac{\omega^2 R(r)}{\bar{\Delta}} = 0. \tag{38}$$

As we did in the previous section, we choose a frame (33) with $\mathcal{K} \to \bar{\mathcal{K}}$ to find the corresponding functions $p(r)$ and $q(r)$ which become (34) with $\Delta \to \bar{\Delta}$.

2. Fake monodromies and irregular singular points

As it turns out in this case, the singularities in the ODE (38) are

$$r = \{0, (2M)^{1/(d-3)}e^{\mathrm{i}(2\pi(i-1))/(d-3)}, \infty\},$$
$$\text{where } i = 1, 2, \ldots, d-3. \tag{39}$$

Employing the procedure described in Appendix A, we find that all but ∞ are regular singularities. A way of determining the monodromy of the regular points of the ODE would be to solve the indicial equation (16).

Following the previously described steps for regular singular points, we find that

$$\alpha_0^{\rm flat} = 0, \qquad \alpha_i^{\rm flat} = \frac{2\omega}{\bar{\Delta}'(r_i)}. \tag{40}$$

In order to compute the monodromy λ_∞ around the irregular singular point $r = \infty$, we analyze the ODE changing $r \to x^{-1}$. We present the details to compute the fake monodromy exponent λ_∞ in Appendix A. Our results are summarized in (28).

It is easily verified that these flat spacetime monodromies $\{\alpha_i^{\rm flat}, \lambda_\infty\}$ can also be found from those in AdS-Schwarzschild backgrounds in taking the flat spacetime limit, $L \to \infty$, while keeping the other physical quantities fixed. For example in $d = 4$, the nonvanishing monodromies (36) yield

$$\lim_{L\to\infty} \alpha_+ = \alpha_+^{\rm flat}, \qquad \lim_{L\to\infty} \alpha_2 + \alpha_3 = -4M\omega = -\lambda_\infty. \tag{41}$$

3. Sum of entropy variations

In general, the monodromy exponents in ODEs that have an irregular singular point do not satisfy a Fuchs relation. However, in the present flat case, taking the sum of the monodromy coefficients found in the preceding subsection leads to a relation that is consistent with the Fuchs-type relation. Plugging these expressions into (23) and employing the relation (2) between the monodromies and the entropy variations

$$\sum_{i=0}^{d-3} \alpha_i = \lambda_\infty,$$
$$\to \sum_{i=0}^{d-3} \delta S_i = \delta S_\infty^{\rm flat} \to \delta\left(\sum_{i=0}^{d-3} S_i\right) = \begin{cases} 8\pi M\delta M & d = 4 \\ 0 & d > 4 \end{cases}, \tag{42}$$

where $S_i = (\mathcal{A}_{d-2}/4) r_i^{d-2}$ is the entropy computed at each horizon of the d-dimensional Schwarzschild black hole. This new entropy bound is consistent with purely thermodynamic relations in Sec. IV.

C. Kerr-AdS black hole

We start by reviewing aspects of a four-dimensional AdS-Kerr black hole with mass M employing the notation in [4]. In $d = 4$ AdS spacetime, using Boyer-Lindquist coordinates, the corresponding line element is

$$ds^2 = \frac{\Sigma}{\Delta} dr^2 - \frac{\Delta}{\Sigma}(dt - \frac{a}{\Xi}\sin^2\theta d\phi)^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta}{\Sigma}\sin^2\theta\left(\frac{(r^2+a^2)}{\Xi} d\phi - a dt\right)^2, \tag{43}$$

where

$$\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta, \quad \Xi = 1 - \frac{a^2}{l^2}, \quad \Sigma = r^2 + a^2\cos^2\theta, \tag{44}$$

$$\Delta = (r^2 + a^2)\left(1 + \frac{r^2}{L^2}\right) - 2Mr = \frac{1}{L^2}\prod_{i=1}^{4}(r - r_i). \tag{45}$$

Further, we define the event horizons as the zeros of the Δ function: here, Δ has four roots, two are real ($r_\pm$) and two are imaginary ($r_{3,4}$). Therefore $\mathcal{K}_{\rm AdSKerr} = 4$. The analytic value of these roots, additional to particular properties of these roots, can be found in Appendix B.

1. Wave equation

Using the process illustrated in Appendix A and the results found in [4], the differential equation for the radial ansatz is found to be

$$\left[\partial_r \Delta \partial_r + \sum_i \frac{(r_i^2 + a^2)^2}{\Delta'(r_i)} \frac{(\omega - \Omega_i m)^2}{r - r_i} - L^2 \Xi \omega^2 + \frac{a^2 m^2}{L^2}\right] \times R_{\rm AdS}(r) = K_{l,\rm AdS} R_{\rm AdS}(r), \tag{46}$$

where $\Omega_i = \frac{a}{r_i^2 + a^2}(1 + \frac{r_i}{L^2})$. Here $K_{l,\rm AdS}$ is the angular coupling constant, and to simplify the calculations it will be set to zero here. For the form of the angular ansatz, see [4]. Next, we follow the procedure of (33)

$$R(r) = \prod_{i=1}^{4}(r - r_i)^{-i\tilde{\alpha}_i}\tilde{R}(r),$$
$$\text{with} \quad \tilde{\alpha}_i = \frac{r_i^2 + a^2}{\Delta'(r_i)}(\omega - \Omega_i m). \tag{47}$$

It can easily be verified that in this case

$$\sum_{i=1}^{4} \tilde{\alpha}_i = 0. \tag{48}$$

Then, with $\Delta^* = -L^2\Xi\omega^2 + \frac{a^2 m^2}{L^2}$ and some algebra, Eq. (46) becomes

$$\left[\partial_r^2 + \left(\sum_{i=1}^{4}\frac{1 - 2i\tilde{\alpha}_i}{r - r_i}\right)\partial_r - \sum_i^4 \sum_{j\neq i}^4 \frac{\tilde{\alpha}_i(i + \tilde{\alpha}_j)}{(r - r_i)(r - r_j)} + \sum_{i=1}^{4}\frac{\tilde{\alpha}_i^2}{(r - r_i)^2}\left(\frac{L^2\Delta'(r_i)}{\prod_{j\neq i}(r - r_j)} - 1\right) + \frac{\Delta^*}{\Delta}\right]\tilde{R}(r) = 0. \tag{49}$$

The above expression is, in the notation of (13), $\lim_{r\to r_i} q(r)(r - r_i)^2 = 0$. Further, here, $\lim_{r\to r_i} p(r)(r - r_i) = 1\text{–}2i\tilde{\alpha}_i$.

2. Monodromies and sum of entropy variations

The indicial equation determining the monodromies α_i in the $\tilde{R}$ frame reads

$$\alpha_i(\alpha_i - 2\tilde{\alpha}_i) = 0$$
$$\Rightarrow \alpha_i = \{0, 2\tilde{\alpha}_i\} = \left\{0, \frac{2(r_i^2 + a^2)}{\Delta'(r_i)}(\omega - \Omega_i m)\right\}. \quad (50)$$

where to be consistent with the definition in (11) the coefficients $\alpha_i \in \mathbb{R}$. By mapping $r \to x^{-1}$ and analyzing $x = 0$, it can be proven independently that $\alpha_\infty = \{-\sum_{i=1}^4 \tilde{\alpha}_i, 3 - \sum_{i=1}^4 \tilde{\alpha}_i\}$. Considering the relation (48) that was previously identified, we conclude that

$$\alpha_\infty = \{0, 3\}. \quad (51)$$

Our results are in agreement with $\alpha_\infty = \mathcal{K}_{\mathrm{AdSKerr}} - 1$ and the limiting AdS-Schwarschild results (36). It is worth noting that similar results hold in more exotic AdS (dS) spacetimes: for an analysis of dS spacetime, see [21], and for an analysis of NUT spacetimes, see [22]. Using the same associations as (3)

$$\sum_{i=1}^4 \alpha_i = 0, \quad \to \quad \sum_{i=1}^4 \delta S_i = 0, \quad \to \quad \delta\left(\sum_{i=1}^4 S_i\right) = 0, \quad (52)$$

where S_i is the entropy computed at the r_i horizon. Additionally, this new bound is reverified using purely thermodynamic relations in Sec. IV.

D. Kerr black hole

In this section we will consider a Kerr black hole with mass M and angular momentum $J = Ma$. Using Boyer-Lindquist coordinates, for this asymptotically flat spacetime we find that

$$ds^2 = \frac{\Sigma}{\bar{\Delta}}dr^2 - \frac{\bar{\Delta}}{\Sigma}(dt - a\sin^2\theta d\phi)^2 + \Sigma d\theta^2$$
$$+ \frac{\sin^2\theta}{\Sigma}((r^2 + a^2)d\phi - adt)^2, \quad (53)$$

where $\bar{\Delta} = r^2 + a^2 - 2Mr = (r - r_-)(r - r_+)$ and $\Sigma = r^2 + a^2\cos^2\theta$. As above, the event horizon radii are the zeros of the $\bar{\Delta}$ function; here, $r_\pm = M \pm \sqrt{M^2 - a^2}$. Therefore, $\bar{\mathcal{K}} = 2$.

1. Wave equation

Using the process illustrated in Appendix A and [3], the radial ansatz is found to be

$$\left[\partial_r \bar{\Delta} \partial_r + (r_+ - r_-)\left(\frac{\tilde{\alpha}_+^2}{r - r_+} - \frac{\tilde{\alpha}_-^2}{r - r_-}\right) + \Delta^*\right] R(r)$$
$$= K_l R(r), \quad (54)$$

where

$$\tilde{\alpha}_\pm = \frac{\Delta'(r_\pm)}{(r_\pm^2 + a^2)}(\omega - \Omega_\pm m), \quad (55)$$

the function $\Delta^* = (r^2 + 2M(r + 2M))\omega^2$, and K_l is the angular coupling constant. For the analysis of the angular ansatz, see [3]. We can again follow the transform of (33) and $K_l = 0$, leading to an analogue of (49):

$$\left[\partial_r^2 + \left(\sum_{i=1}^2 \frac{1 - 2i\tilde{\alpha}_i}{r - r_i}\right)\partial_r - \frac{2\tilde{\alpha}_+\tilde{\alpha}_- + i(\tilde{\alpha}_+ + \tilde{\alpha}_-)}{\bar{\Delta}}\right.$$
$$\left. + \sum_{i=1, j\neq i}^2 \frac{\tilde{\alpha}_i^2}{(r - r_i)^2}\left(\frac{r_i - r_j}{r - r_j} - 1\right) + \frac{\bar{\Delta}^*}{\bar{\Delta}}\right]\bar{R}(r) = 0. \quad (56)$$

As in (13), $\lim_{r\to r_\pm} q(r)(r - r_\pm)^2 = 0$. Further, here, $\lim_{r\to r_\pm} p(r)(r - r_\pm) = 1\text{–}2i\tilde{\alpha}_\pm$. Then, the indicial equation reads

$$\alpha_i^{\mathrm{flat}}(\alpha_i{}^{\mathrm{flat}} - 2\tilde{\alpha}_\pm) = 0$$
$$\Rightarrow \alpha_i^{\mathrm{flat}} = \{0, 2\tilde{\alpha}_i\} = \left\{0, \frac{2\Delta'(r_\pm)}{(r_\pm^2 + a^2)}(\omega - \Omega_\pm m)\right\}, \quad (57)$$

where $\alpha_i^{\mathrm{flat}} \in \mathbb{R}$. Using the mapping $r \to x^{-1}$ and a Frobenius expansion about $x = 0$ it can be directly shown that $\lambda_\infty = 4M\omega$. Additionally, an analysis similar to that done in Appendix A is also possible. For more analysis of the irregular singularity, including a discussion of its unique scattering properties and the Stoke's phenomenon, see [23].

2. Fake monodromies and sum of entropy variations

From (57), we find the Fuchs-type relation $\alpha_+^{\mathrm{flat}} + \alpha_-^{\mathrm{flat}} = 4M\omega$. It is also possible to show the monodromic sum evolves under $L \to \infty$ such as (23). In fact, the Kerr-AdS monodromies (50) in the limit yield

$$\lim_{L\to\infty} \alpha_\pm \to \alpha_\pm^{\mathrm{flat}}, \qquad \lim_{L\to\infty} \alpha_{3,4} \to -2M\omega \pm i\omega \lim_{L\to\infty} L, \quad (58)$$

and the Fuchs relation

$$\lim_{L\to\infty}\left(\sum_{i=1}^4 \alpha_i\right) = 0 \quad \to \quad \alpha_+^{\mathrm{flat}} + \alpha_-^{\mathrm{flat}} - 4M\omega = 0. \quad (59)$$

Letting $\lambda_\infty \coloneqq -\lim_{L\to\infty}(\alpha_3 + \alpha_4) = 4M\omega$ and the identifications in (2) we find

$$\alpha_+^{\mathrm{flat}} + \alpha_-^{\mathrm{flat}} = \lambda_\infty \quad \to \quad \delta(S_+^{\mathrm{flat}} + S_-^{\mathrm{flat}}) = \delta S_\infty^{\mathrm{flat}}, \quad (60)$$

where $S_{\pm}^{\rm flat}$ is the entropy computed at the $r_{\pm}$ horizon. Additionally, this new bound is reverified using purely thermodynamic relations in Sec. IV.

IV. THERMODYNAMIC IDENTITIES OF BLACK HOLES

We now turn to the study of the geometrical relation of thermodynamics for every black hole horizon (1) and the novel thermodynamic identities of black holes (5)–(7) from a purely thermodynamic perspective. Our focus is on thermodynamics properties of black hole solutions that have smooth horizons with spherical topologies. To establish a well rounded catalog for the thermodynamic identities we shall present explicitly several examples in this section. Some of these results were previously assessed in literature. We indicate the references accordingly in each case.

In general, these equations involve the physical parameters defined at each black hole horizon. It is convenient to define the intensive quantities employing the Arnowitt-Deser-Misner (ADM) formalism in which the line element is

$$ds^2 = -N^2 dt^2 + \gamma_{ab}(dx^a + N^a dt)(dx^b + N^b dt), \quad (61)$$

with x^a spatial directions, and $N(x^a)$ and $N^b(x^a)$ are the lapse function and the shift vector, respectively. The intensive variables are intrinsic to each horizon that will be indicated by the subscripts.

Note that with this foliation of spacetime, the black hole horizons r_i (real or complex) are at $N^2 = 0$. As in [24], the angular potentials and temperatures for each horizon are defined:

$$\Omega_i^{(k)} = -N^k|_{r_i}, \qquad T_i = \frac{1}{4\pi}\frac{(N^2)'}{\sqrt{g_{rr}N^2}}\bigg|_{r_i}, \quad (62)$$

where $k = 1, 2, \ldots, [\frac{d-1}{2}]$ represents the independent planes of rotation in d dimensions. Indeed, when evaluated at the black hole event horizon r_+ we recover the Hawking temperature T_+.

In the ensuing analysis we will find the entropy sums over all horizons. In almost all cases the relation is independent of the extensive parameters, which in turn implies the addition of the variations of the entropies to vanish. In all cases we find a perfect agreement with the corresponding results of the entropy and monodromy relations analyzed in previous sections. From each of these examples, new thermodynamic identities involving the sum of the intensive quantities are obtained.

A. Kerr black hole

The radii function of a Kerr black hole [16] of mass M and angular momentum $J = Ma$ satisfying $R_{\mu\nu} = 0$ is defined by the function $\Delta(r) = r^2 + a^2 - 2Mr$. The outer $r_+ = M + \sqrt{M^2 - a^2}$ and inner event horizons $r_- = M - \sqrt{M^2 - a^2}$ are located at $\Delta(r_{\pm}) = 0$. The corresponding physical parameters are given by

$$T_{\pm} = \frac{\Delta'(r_i)}{8\pi M r_{\pm}}, \quad S_{\pm} = \pi(r_{\pm}^2 + a^2), \quad \Omega_{\pm} = \frac{a}{r_{\pm}^2 + a^2}, \quad (63)$$

corresponding to Hawking's temperature, the entropy, the angular momentum, and the angular velocity defined at black hole horizons. In this case we checked that the above physical parameters obey the relation (1). The sum of the horizon areas is

$$\sum_{i=1}^{2} S_i = 4\pi M^2. \quad (64)$$

The variation of this expression is

$$\delta\left(\sum_{i=1}^{2} S_i\right) = 8\pi M \delta M, \quad (65)$$

consistent with Sec. III. Note also that

$$\sum_{i=1}^{2}\frac{1}{T_i} = 8\pi M, \qquad \sum_{i=1}^{2}\frac{\Omega_i}{T_i} = 0. \quad (66)$$

These can be viewed as a consequence of the first law of thermodynamics at each horizon r_i. Treatments of some notable cases are given in the following subsections and a completely analogous story holds for general black hole solutions, as we prove in Sec. V.

B. Kerr-AdS black hole

The metric of the four-dimensional Kerr-AdS black hole [9], satisfying $R_{\mu\nu} = -3L^{-2}g_{\mu\nu}$ is asymptotic to AdS_4 in a rotating frame, with angular velocity $\Omega_\infty = -aL^{-2}$. The radii function is determined by $\Delta = (r^2 + a^2)(1 + r^2 L^{-2}) - 2Mr$. The horizons are located at $\Delta(r_i) = 0$ with $i = 1, 2, \ldots, 4$. The physical parameters corresponding to Hawking's temperature, the entropy, and the angular velocity of the horizon (as measured in the asymptotically rotating frame) are given by

$$T_i = \frac{\Delta'(r_i)}{4\pi(r_i^2 + a^2)}, \qquad S_i = \frac{\pi(r_i^2 + a^2)}{\Xi},$$
$$\Omega_i = \frac{a(1 + r_+^2/L^2)}{r_+^2 + a^2}, \quad (67)$$

where $\Xi = 1 - a^2/L^2$ and Ω_i is the angular velocity measured relative to a *rotating* observer at infinity. In [25] the physical mass E and angular momentum J of the AdS-Kerr black hole solution were computed at the boundary (infinity) via the Komar integrals

$$E = \frac{M}{\Xi^2}, \qquad J = Ea. \tag{68}$$

It is straightforward to verify that these quantities obey the geometrical law of thermodynamics (1). Further properties among the horizons of the Kerr-AdS black hole solution can be found in Appendix B. Using these, the sum of the entropies at every horizon is independent of the physical parameters

$$\sum_{i=1}^{4} S_i = -2\pi L^2, \tag{69}$$

and, also

$$\sum_{i=1}^{4} \frac{\Omega_i}{T_i} = 0, \qquad \sum_{i=1}^{4} \frac{1}{T_i} = 0. \tag{70}$$

In the study of this system obeying the KG equation, we found in Sec. III C the thermodynamic relation (5) from the Fuchs relation. Given the sum of entropies relation (69), one can compute its variation (keeping L fixed) and thence obtain the associated thermodynamic relation $\delta \sum S_i = 0$ consistent with previous results. A completely analogous story holds for all the AdS black holes in all dimensions. See Secs. IV C and IV D 3 for details.

C. BTZ black hole

The mass, angular momentum, and entropy of the BTZ black hole [8] are

$$M = \frac{r_+^2 + r_-^2}{L^2}, \qquad J = \frac{2r_+ r_-}{L}, \qquad S_i = 4\pi r_i, \tag{71}$$

and the Hawking temperature T_i and angular velocity Ω_i are

$$T_i = \frac{r_+^2 - r_-^2}{2\pi L^2 r_i}, \qquad \Omega_i = \frac{J}{2r_i^2}. \tag{72}$$

For this black hole solution, the roots of the radii equation

$$\frac{J^2}{4r^2} + \frac{r^2}{L^2} - M = 0 \tag{73}$$

are found to be $r_i = \{\pm r_+, \pm r_-\}$ with $r_\pm = \sqrt{\frac{L(ML \pm \sqrt{ML^2 - J^2})}{2}}$.

Just as in the previous cases, we compute the sum of all entropies defined at each horizon r_i

$$\sum_{i=1}^{4} S_i = 0. \tag{74}$$

Considering the intensive quantities we further find

$$\sum_{i=1}^{4} \frac{\Omega_i}{T_i} = 0, \qquad \sum_{i=1}^{4} \frac{1}{T_i} = 0. \tag{75}$$

D. Higher-dimensional black holes

In this paper we are also interested in higher-dimensional black holes. The physical parameters for each $d \geq 5$ black hole solution are described in the following subsections. Employing these quantities, we are able to establish new thermodynamic relations.

1. Myers-Perry black hole in d = 5

The metric of the five-dimensional rotating black hole, satisfying $R_{\mu\nu} = 0$, was found in [7]. The function

$$\Delta(r) = \frac{1}{r^2}(r^2 + a_1^2)(r^2 + a_2^2) - 2M \tag{76}$$

defines the horizon radii located at $\Delta(r_i) = 0$ with $i = 1, \ldots, 4$. While the four roots are real, only two are positive roots and correspond to the outer and inner event horizons. Hawking's temperature, the entropy, both angular momenta, the angular velocities of the horizon, and physical mass parameters are, respectively, given by[3]

$$T_i = \frac{r_i^2 \Delta'(r_i)}{4\pi(r_i^2 + a_1^2)(r_i^2 + a_2^2)}, \qquad S_i = \frac{\pi^2 (r_i^2 + a_1^2)(r_i^2 + a_2^2)}{2r_i},$$
$$J_{(k)} = \frac{\pi M a_k}{2}, \qquad \Omega_i^{(k)} = \frac{a_k}{r_i^2 + a_k^2}, \qquad E = \frac{3\pi}{4} M, \tag{77}$$

where $k = 1, 2$. The geometrical thermodynamical relation (1) for the above quantities is satisfied as well as the following relations:

$$\sum_{i=1}^{4} S_i = 0 \tag{78}$$

and

$$\sum_{i=1}^{4} \frac{\Omega_i^{(k)}}{T_i} = 0, \qquad \sum_{i=1}^{4} \frac{1}{T_i} = 0. \tag{79}$$

2. Myers-Perry black holes

The extension of Einstein's gravity for asymptotically flat rotating black holes in $d \geq 5$ spacetime dimensions is shown in [10,11]. The $\mathcal{K} = d - 1 - \epsilon$ horizons can be found by the following radial equation:

[3] In particular, for $r_\pm$ we can write the temperature as $T_\pm = \frac{r_\pm^4 - a_1^2 a_2^2}{2\pi r_\pm (r_\pm^2 + a_1^2)(r_\pm^2 + a_2^2)}$.

$$\Delta(r) = r^{\epsilon-2} \prod_{k=1}^{[(d-1)/2]} (r^2 + a_k^2) - 2M = 0, \quad (80)$$

where $\epsilon = (d-1) \bmod 2$. As in [25] the entropy is defined by

$$S_i = \frac{\mathcal{A}_{d-2}}{4r_i^{1-\epsilon}} \prod_{k=1}^{[(d-1)/2]} (r_i^2 + a_k^2). \quad (81)$$

The Hawking temperatures T_i, angular velocities $\Omega_i^{(k)}$, mass E, and angular momenta $J_{(k)}$ are given, respectively, by

$$T_i = \frac{1}{2\pi}\left(r_i \sum_{k=1}^{[(d-1)/2]} \frac{1}{r_i^2 + a_k^2} - \frac{2-\epsilon}{2r_i}\right),$$
$$\Omega_i^{(k)} = \frac{a_k}{r_i^2 + a_k^2}, \quad (82)$$

$$E = \frac{M\mathcal{A}_{d-2}}{4\pi}\left(\frac{d-2}{2}\right), \qquad J_{(k)} = \frac{M a_k \mathcal{A}_{D-2}}{4\pi}, \quad (83)$$

where $i = 1, \ldots, \mathcal{K}$ includes all horizons. For the above quantities, we find that geometrical thermodynamical relation (1) is satisfied as well as the following relations:

$$\sum_{i=1}^{\mathcal{K}} S_i = 0 \quad (84)$$

and

$$\sum_{i=1}^{\mathcal{K}} \frac{\Omega_i^{(k)}}{T_i} = 0, \qquad \sum_{i=1}^{\mathcal{K}} \frac{1}{T_i} = 0. \quad (85)$$

Some of the results in this subsection for the entropy sum were derived in [26].

3. Kerr-AdS black holes in $d \geq 5$

The extension of Einstein's gravity for rotating black holes that are asymptotically AdS in $d \geq 5$ dimensions is shown in [10,11].

The horizons can be found by the following radial equation:

$$\Delta(r) = r^{\epsilon-2}(1 + r^2 L^{-2}) \prod_{k=1}^{[(d-1)/2]} (r^2 + a_k^2) - 2M = 0, \quad (86)$$

where $\epsilon = (d-1) \bmod 2$.

As in [25] the entropy is defined by

$$S_i = \frac{\mathcal{A}_{d-2}}{4r_i^{1-\epsilon}} \prod_{k=1}^{[(d-1)/2]} \frac{r_i^2 + a_k^2}{1 - a_k^2 L^{-2}}. \quad (87)$$

The Hawking temperatures T_i, angular velocities $\Omega_i^{(k)}$, mass E, and angular momenta $J_{(k)}$ are given, respectively, by

$$T_i = \frac{1}{2\pi}\left(r_i(1 + r_i^2 L^{-2}) \sum_{k=1}^{[(d-1)/2]} \frac{1}{r_i^2 + a_k^2} - \frac{2 - \epsilon(1 - r_i^2 L^{-2})}{2r_i}\right), \quad (88)$$

$$\Omega_i^{(k)} = \frac{(1 + r_i^2 L^{-2}) a_k}{r_i^2 + a_k^2}, \quad (89)$$

$$E = \frac{M\mathcal{A}_{d-2}}{4\pi \prod_{k=1}^{[(d-1)/2]} (1 - a_k^2 L^{-2})}\left(\sum_{i=1}^{[(d-1)/2]} \frac{1}{1 - a_k^2 L^{-2}} - \frac{1-\epsilon}{2}\right), \quad (90)$$

$$J_{(k)} = \frac{M a_k \mathcal{A}_{d-2}}{4\pi(1 - a_k^2 L^{-2}) \prod_{k=1}^{[(d-1)/2]} (1 - a_k^2 L^{-2})}, \quad (91)$$

where i includes all horizons. The geometrical thermodynamical relation (1) for the above quantities is satisfied as well as the following relations:

$$\sum_{i=1}^{\mathcal{K}} S_i = \begin{cases} 0 & \text{d is odd} \\ \frac{L^{d-2}}{2}\mathcal{A}_{d-2} & \text{d is even} \end{cases} \quad (92)$$

and

$$\sum_{i=1}^{\mathcal{K}} \frac{\Omega_i^{(k)}}{T_i} = 0, \qquad \sum_{i=1}^{\mathcal{K}} \frac{1}{T_i} = 0. \quad (93)$$

Some of the results in this subsection for the entropy sum were derived in [26,27].

E. Charged black holes

We now turn to the study of the sum of entropies of charged holes. It is convenient to first define the relevant thermodynamic quantities. Given a black hole solution one can also compute new relations involving the intensive physical parameter such as (6) and (7).

1. Reissner-Nordstrom black hole

The metric of a four-dimensional static, charged black hole in the usual Weyl coordinates is presented [28,29] where the vanishing of the function

$$\Delta(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{94}$$

determines the horizon radii $\Delta(r_i) = 0$. These are trivially

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{95}$$

The entropy, Hawking's temperature, and electric potential parameters are, respectively, given by

$$S_\pm = \pi r_\pm^2, \qquad T_\pm = \frac{r_\pm^2 - Q^2}{4\pi r_\pm^3}, \qquad \Phi_\pm = \frac{Q}{r_\pm}. \tag{96}$$

The geometrical law of thermodynamics (1) for the above quantities is satisfied as well as the following relations:

$$\sum_{i=1}^{2} S_i = 2\pi(2M^2 - Q^2) \tag{97}$$

and

$$\sum_{i=1}^{2} \frac{\Phi_i}{T_i} = 4\pi Q, \qquad \sum_{i=1}^{2} \frac{1}{T_i} = 8\pi M. \tag{98}$$

2. Reissner-Nordstrom-AdS black hole

The solution representing a four-dimensional static, charged black hole in AdS is presented in [28,29]. The function

$$\Delta(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{3L^2} \tag{99}$$

defines the horizon radii located at $\Delta(r_i) = 0$ with $i = 1, \ldots, 4$. The entropy, Hawking's temperature, and electric potential parameters are, respectively, given by

$$S_i = \pi r_i^2, \qquad T_i = \frac{3r_i^4 L^{-2} + r_i^2 - Q^2}{4\pi r_i^3}, \qquad \Phi_i = \frac{Q}{r_i}. \tag{100}$$

The geometrical law of thermodynamics (1) for the above quantities is satisfied as well as the following relations:

$$\sum_{i=1}^{4} S_i = -2\pi L^2 \tag{101}$$

and

$$\sum_{i=1}^{4} \frac{\Phi_i}{T_i} = 0, \qquad \sum_{i=1}^{4} \frac{1}{T_i} = 0, \tag{102}$$

3. Reissner-Nordstrom black hole in $d \geq 5$

A charged, nonrotating, asymptotically flat black hole in $d \geq 5$ dimensions is described in detail in [30]. The radii function in this case can be written as

$$\Delta(r) = 1 - \frac{2\mu}{r^{d-3}} + \frac{q^2}{r^{2(d-3)}}. \tag{103}$$

Integral to defining thermodynamic properties are the outermost and innermost radii,

$$r_\pm^{d-3} = \mu \pm \sqrt{\mu^2 - q^2}, \tag{104}$$

where

$$\mu = \frac{8\pi M}{\mathcal{A}_{d-2}(d-2)}, \qquad q^2 = \frac{2Q^2}{(d-2)(d-3)}. \tag{105}$$

Thus the ADM mass and charge can be constructed,

$$M = \frac{\mathcal{A}_{d-2}(d-2)}{16\pi}(r_+^{d-3} + r_-^{d-3}),$$
$$Q = \sqrt{\frac{(d-3)(d-2)}{2}}(r_+ r_-)^{\frac{d-3}{2}}. \tag{106}$$

The entropy, Hawking temperature, and electric potential defined at the outermost and innermost horizons can be generalized:

$$S_i = \frac{\mathcal{A}_{d-2}}{4} r_i^{d-2}, \qquad T_\pm = \frac{d-3}{4\pi r_\pm}\left[1 - \left(\frac{r_\mp}{r_\pm}\right)^{d-3}\right],$$
$$\Phi_i = \frac{\mathcal{A}_{d-2} Q}{4\pi(d-3) r_i^{d-3}}. \tag{107}$$

The geometrical relation of thermodynamics (1) for the above quantities is satisfied as well as the following relations for $d > 4$:

$$\sum_{i=1}^{2d-6} S_i = 0 \tag{108}$$

and

$$\sum_{i=1}^{2d-6} \frac{\Phi_i}{T_i} = 0, \qquad \sum_{i=1}^{2d-6} \frac{1}{T_i} = 0. \tag{109}$$

4. Reissner-Nordstrom-AdS black hole in $d \geq 5$

The asymptotically AdS static charged black hole in d dimensions is described in detail in [31,32] with a radii function,

$$\Delta(r) = 1 - \frac{2\mu}{r^{d-3}} + \frac{q^2}{r^{2(d-3)}} + \frac{2r^2}{(d-2)(d-1)L^2}, \quad (110)$$

where μ and q have the same definitions of (105). In this case, Hawking's temperature, entropy, and electric potential defined at the respective black hole (BH) horizons located at r_i can be generalized:

$$T_i = \frac{\frac{2r_i^{2d+2}(d-1)}{L^2} + (d-2)(d-3)r_i^{2d} - \frac{32\pi^2 Q^2 r_i^6}{\mathcal{A}_{d-2}^2}}{4\pi(d-2)r_i^{2d+1}}, \quad (111)$$

$$S_i = \frac{\mathcal{A}_{d-2} r_i^{d-2}}{4}, \qquad \Phi_i = \sqrt{\frac{d-2}{2(d-3)}}\frac{Q}{r_i^{d-3}}. \quad (112)$$

Note that the geometrical relation of thermodynamics (1) for the above quantities is satisfied as well as the following relations for $d \geq 4$:

$$\sum_{i=1}^{2(d-2)} S_i = \begin{cases} 0 & \text{d is odd} \\ \frac{1}{2}\left(\frac{2-d}{2}\right)^{(d-2)/2}\mathcal{A}_{d-2}L^{d-2} & \text{d is even} \end{cases} \quad (113)$$

and

$$\sum_{i=1}^{2(d-2)} \frac{\Phi_i}{T_i} = 0, \qquad \sum_{i=1}^{2(d-2)} \frac{1}{T_i} = 0. \quad (114)$$

5. Kerr-Newman black hole

The metric of a four-dimensional (asymptotically flat) rotating, charged black hole was found in [12]. The function that defines the horizons is

$$\Delta(r) = (r^2 + a^2) - 2Mr + Q^2 \quad (115)$$

when $\Delta(r_\pm) = 0$ and $r_\pm = M \pm \sqrt{M^2 - (a^2 + Q^2)}$. The entropy, Hawking's temperature, angular velocity, and electric potential that are, respectively, given by

$$S_\pm = 4\pi(r_\pm^2 + a^2), \qquad T_\pm = \frac{r_\pm^2 - (a^2 + Q^2)}{4\pi r_\pm (r_\pm^2 + a^2)},$$
$$\Omega_\pm = \frac{a}{a^2 + r_\pm^2}, \qquad \Phi_\pm = \frac{Q r_\pm}{a^2 + r_\pm^2}. \quad (116)$$

Employing these quantities the geometrical law of thermodynamics (1) and the following relations can be verified:

$$\sum_{i=1}^{2} S_i = 2\pi(2M^2 - Q^2) \quad (117)$$

and

$$\sum_{i=1}^{2} \frac{\Phi_i}{T_i} \neq 0, \qquad \sum_{i=1}^{2} \frac{\Omega_i}{T_i} = 0, \qquad \sum_{i=1}^{2} \frac{1}{T_i} \neq 0. \quad (118)$$

6. Kerr-Newman-AdS black hole

The metric of a four-dimensional rotating, charged black hole in AdS is presented in [12]. The function

$$\Delta(r) = (r^2 + a^2)\left(1 + \frac{r^2}{L^2}\right) - 2Mr + Q^2 \quad (119)$$

defines the horizon radii located at $\Delta(r_i) = 0$ with $i = 1, \ldots, 4$. The mass, angular momentum, entropy, Hawking's temperature, angular velocity, and electric potential parameters are, respectively, given by

$$E = \frac{M}{(1 - aL^{-2})^2}, \qquad J = \frac{aM}{(1 - aL^{-2})^2},$$
$$S_i = 4\pi\frac{(r_i^2 + a^2)}{(1 - \frac{a^2}{L^2})}, \quad (120)$$

$$T_i = \frac{r_i\left(1 + \frac{a^2}{L^2} + 3\frac{r_i^2}{L^2} - \frac{a^2+Q^2}{r_i^2}\right)}{4\pi(r_i^2 + a^2)}, \qquad \Omega_i = \frac{a\left(1 - \frac{a^2}{L^2}\right)}{a^2 + r_i^2},$$
$$\Phi_i = \frac{Q r_i}{a^2 + r_i^2}. \quad (121)$$

The geometrical law of thermodynamics (1) for the above quantities is satisfied as well as the following relations:

$$\sum_{i=1}^{4} S_i = -8\pi L^2 \quad (122)$$

and

$$\sum_{i=1}^{4} \frac{\Phi_i}{T_i} = 0, \qquad \sum_{i=1}^{4} \frac{\Omega_i}{T_i} = 0, \qquad \sum_{i=1}^{4} \frac{1}{T_i} = 0. \quad (123)$$

7. Nonextremal rotating black holes in minimal $d = 5$ gauged supergravity

The solution of a nonextremal rotating black hole in minimal $d = 5$ gauged supergravity is presented in [14]. The coordinate choice is a Boyer-Lindquist type of $(t, r, \theta, \phi, \psi)$. The function

$$\Delta(r) = \frac{(r^2 + a_1^2)(r^2 + a_2^2)(1 + L^{-2}r^2) + Q^2 + 2a_1 a_2 Q}{r^2} - 2M \quad (124)$$

defines the horizon radii located at $\Delta(r_i) = 0$ with $i = 1, \ldots, 6$. The entropy, Hawking's temperature, angular velocities, and electric potential parameters are, respectively, given by

$$
\begin{aligned}
S_i &= \frac{\pi^2[(r_i^2 + a_1^2)(r_i^2 + a_2^2) + a_1 a_2 Q]}{2(1 - a_1^2 L^{-2})(1 - a_2^2 L^{-2}) r_i}, \\
T_i &= \frac{r_i^4[(1 + L^{-2}(2r_i^2 + a_1^2 + a_2^2)] - (a_1 a_2 + Q)^2}{2\pi r_i[(r_i^2 + a_1^2)(r_i^2 + a_2^2) + a_1 a_2 Q]}, \\
\Omega_i^{(1)} &= \frac{a_1(r_i^2 + a_2^2)(1 + L^{-2} r_i^2) + a_2 Q}{(r_i^2 + a_1^2)(r_i^2 + a_2^2) + a_1 a_2 Q}, \\
\Omega_i^{(2)} &= \frac{a_2(r_i^2 + a_1^2)(1 + L^{-2} r_i^2) + a_1 Q}{(r_i^2 + a_1^2)(r_i^2 + a_2^2) + a_1 a_2 Q}, \\
\Phi_i &= (\ell^\mu A_\mu)_i,
\end{aligned} \tag{125}
$$

where

$$
\begin{aligned}
A_i &= \frac{\sqrt{3} Q}{r_i^2 + a_1^2 \cos^2\theta + a_2^2 \sin^2\theta} \\
&\quad \times \left(\frac{1 - a_1^2 L^{-2} \cos^2\theta - a_2^2 L^{-2} \sin^2\theta}{(1 - a_1^2 L^{-2})(1 - a_2^2 L^{-2})} dt \right. \\
&\quad \left. - \frac{a_1 \sin^2\theta}{1 - a_1^2 L^{-2}} d\phi - \frac{a_2 \cos^2\theta}{1 - a_2^2 L^{-2}} d\psi \right), \\
\ell &= \frac{\partial}{\partial t} + \Omega_i^{(1)} \frac{\partial}{\partial \phi} + \Omega_i^{(2)} \frac{\partial}{\partial \psi}.
\end{aligned} \tag{126}
$$

The generalized first law of thermodynamics (1) takes the form

$$
dE = T_i dS + \Omega_i^{(1)} dJ_{(1)} + \Omega_i^{(2)} dJ_{(2)} + \Phi_i dQ, \tag{127}
$$

where the mass and angular momenta are defined, respectively, as

$$
E = \frac{\pi L^4[M(3L^4 - (a_1^2 + a_2^2)L^2 - a_1^2 a_2^2) - 2a_1 a_2 Q(a_1^2 + a_2^2 - 2L^2)]}{4(a_1^2 - L^2)^2 (L^2 - a_2^2)^2}, \tag{128}
$$

$$
J_{(1)} = \frac{\pi[2a_1 M + Q a_2(1 + a_1^2 L^{-2})]}{4(1 - a_1^2 L^{-2})^2 (1 - a_2^2 L^{-2})}, \qquad J_{(2)} = \frac{\pi[2a_2 M + Q a_1(1 + b^2 L^{-2})]}{4(1 - a_2^2 L^{-2})^2 (1 - a_1^2 L^{-2})}. \tag{129}
$$

Our results show that the relation of the entropy sum vanishes,

$$
\sum_{i=1}^{6} S_i = 0, \tag{130}
$$

and the relations between the intensive quantities yield

$$
\sum_{i=1}^{6} \frac{\Phi_i}{T_i} = 0, \qquad \sum_{i=1}^{6} \frac{\Omega_i^{(1,2)}}{T_i} = 0, \qquad \sum_{i=1}^{6} \frac{1}{T_i} = 0. \tag{131}
$$

8. Charged rotating black holes in $d = 6$ gauged supergravity

The metric of a six-dimensional rotating, charged black hole in $d = 6$ gauged supergravity is presented in [15]. The function

$$
\begin{aligned}
\Delta(r) &= (r^2 + a_1^2)(r^2 + a_2^2) \\
&\quad + L^{-2}[r(r^2 + a_1^2) + Q][r(r^2 + a_2^2) + Q] - 2Mr
\end{aligned} \tag{132}
$$

defines the horizon radii located at $\Delta(r_i) = 0$ with $i = 1, \ldots, 6$. The entropy, angular velocities, electric potential, and Hawking's temperature are, respectively, given by

$$
\begin{aligned}
S_i &= \frac{2\pi^2[(r_i^2 + a_1^2)(r_i^2 + a_2^2) + Q r_i]}{3(1 - a_1^2 L^{-2})(1 - a_2^2 L^{-2})}, \\
\Omega_i^{(1)} &= \frac{a_1[(1 + L^{-2} r_i^2)(r_i^2 + a_2^2) + Q r_i L^{-2}]}{(r_i^2 + a_1^2)(r_i^2 + a_2^2) + Q r_i}, \\
\Omega_i^{(2)} &= \frac{a_2[(1 + L^{-2} r_i^2)(r_i^2 + a_1^2) + Q r_i L^{-2}]}{(r_i^2 + a_1^2)(r_i^2 + a_2^2) + Q r_i}, \\
\Phi_i &= \frac{Q r_i}{\pi} \frac{(1 - a_1^2 L^{-2})(1 - a_2^2 L^{-2})}{(r_i^2 + a_1^2)(r_i^2 + a_2^2) + Q r_i},
\end{aligned} \tag{133}
$$

and

$$
T_i = \frac{2(1 + L^{-2} r_i^2) r_i^2 (2r_i^2 + a_1^2 + a_2^2)}{4\pi r_i[(r_i^2 + a_1^2)(r_i^2 + a_2^2) + Q r_i]} + \frac{4Q L^{-2} r_i^3 - (1 - L^{-2} r_i^2)(r_i^2 + a_1^2)(r_i^2 + a_2^2) - Q^2 L^{-2}}{4\pi r_i[(r_i^2 + a_1^2)(r_i^2 + a_2^2) + Q r_i]}, \tag{134}
$$

$$E=\frac{\pi\left[2M\left(\frac{1}{(1-a_1^2L^{-2})}+\frac{1}{(1-a_2^2L^{-2})}\right)+Q\left(1+\frac{(1-a_1^2L^{-2})}{(1-a_2^2L^{-2})}+\frac{(1-a_2^2L^{-2})}{(1-a_1^2L^{-2})}\right)\right]}{3(1-a_1^2L^{-2})(1-a_2^2L^{-2})}, \tag{135}$$

$$J_{(1)}=\frac{\pi a_1(2M+(1-a_2L^{-2})Q)}{3(1-a_1L^{-2})^2(1-a_2L^{-2})},\qquad J_{(2)}=\frac{\pi a_2(2M+(1-a_1L^{-2})Q)}{3(1-a_1L^{-2})(1-a_2L^{-2})^2}. \tag{136}$$

For the above quantities, the geometrical thermodynamical law (1) is satisfied as well as the following relations:

$$\sum_{i=1}^{6}S_i=0 \tag{137}$$

and

$$\sum_{i=1}^{6}\frac{\Phi_i}{T_i}=0,\qquad \sum_{i=1}^{6}\frac{\Omega_i^{(j)}}{T_i}=0,\qquad \sum_{i=1}^{6}\frac{1}{T_i}=0. \tag{138}$$

V. GENERAL THERMODYNAMIC RELATIONS FOR BLACK HOLES

In this section we will show the new thermodynamic relations follow from the mechanical law of black hole horizons and the properties of the sum of the entropies. Our starting point is the first law for all black hole horizons (1) for a black hole solution with r_i horizons (including all horizons). This property for all black hole horizons can be written as

$$dS_i=\frac{1}{T_i}dE-\sum_k\frac{\Omega_i^{(k)}}{T_i}dJ_{(k)}-\sum_l\frac{\Phi_i^l}{T_i}dQ_l. \tag{139}$$

Now, adding all these equations together for every black hole horizon leads to the expression

$$\sum_i dS_i=\left(\sum_i\frac{1}{T_i}\right)dE-\sum_a\left(\sum_i\frac{\Omega_i^a}{T_i}\right)dJ_a-\sum_b\left(\sum_i\frac{\Phi_i^b}{T_i}\right)dQ_b. \tag{140}$$

The left-hand side is

$$\sum_i dS_i=d\left(\sum_i S_i\right), \tag{141}$$

such that when $\sum_i S_i$ is independent of the extensive quantities,

$$\sum_i S_i\neq f(E,J_a,Q_b)\rightarrow\sum_i\frac{1}{T_i}=0,$$
$$\sum_i\frac{\Omega_i^a}{T_i}=0,\qquad\sum_i\frac{\Phi_i^b}{T_i}=0. \tag{142}$$

Therefore, together with the universal property of a first law for every black hole horizon, it is only necessary to identify the functional dependence of the sum of the entropies of every horizon in the solution $\sum S_i$ to single out the thermodynamic relations the solution will obey. For black hole solutions, such as for the Schwarzschild, Kerr, Reissner-Nordstrom, and Kerr-Newman black hole solutions, where $\sum_i S_i=f(E,Q)$ we can infer that

$$\sum_i S_i=f(E,Q)\rightarrow\sum_i\frac{\Omega_i^a}{T_i}=0, \tag{143}$$

while

$$\sum_i\frac{1}{T_i}\neq 0,\qquad\sum_i\frac{\Phi_i^b}{T_i}\neq 0. \tag{144}$$

VI. DISCUSSION

We have verified that the equations of motion of quantum field theories in curved backgrounds, more precisely the KG equation in black hole backgrounds, encode important black hole thermodynamic relations. The universality of the entropy variation relations was established for a large class of black holes, and dimensions, in both asymptotically flat and asymptotically anti–de Sitter spacetimes. The monodromies capture the infinitesimal changes in the black hole background produced by the field excitations. This emergent link between monodromies and entropies results in a thermodynamic identity for the sum of all horizon entropy variations. This raises the possibility of further thermodynamic relations defined as independent sums of temperatures and angular velocities defined at every black hole horizon. Their structure may be interpreted as conditions for thermodynamical equilibrium.[4] In fact, for a fixed value of the black hole mass, all black hole systems would seem to be in thermodynamic equilibrium

[4]We thank the referee for this comment.

giving rise to a relation between the thermodynamic systems defined at each horizon.

The origin of these relations can be put down to the fact that $\sum_i S_i \neq f(E, J_k, Q_l)$. Our explicit results indicate that black hole solutions with at least one imaginary horizon obey all the thermodynamics relations (6) and (7). We emphasize that the thermodynamic summation relations apply in theories involving multicharge black holes, including black hole solutions in gauged supergravities. For example, the charged rotating black holes in minimal $d = 5$ and $d = 6$ gauged supergravity are consistent with

$$\delta\left(\sum_{i=1}^{\mathcal{K}} S_i\right) = 0. \tag{145}$$

From the more formal perspective, we have here worked out the details for the link between the monodromies and entropies for charged black holes. Nevertheless, our thermodynamic analysis contains robust evidence to argue that similar results will be found. A way to formally verify this proposal is to consider a field interacting with electromagnetism through the equations of motion for a massless charged scalar. We leave this analysis for the future.

Note that the KG equation for extremal asymptotically flat black hole solutions contain two irregular singular points (the event horizon and infinity). This is in contrast with nonextremal asymptotically flat black hole solutions containing only one irregular singular point at $r = \infty$. The emergence of thermodynamic relations is also expected and will be studied elsewhere.

Black hole solutions in GR in higher dimensions also contain more exotic solutions such as black rings, bicycling black rings, black branes, and black strings. It would be interesting to analyze the first law of all black hole horizons and the emergent thermodynamic relations from monodromies for all these other cases. Further tests will include those for black hole solutions in alternatives of GR, e.g., Gauss-Bonnet or $f(R)$ theories.

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APPENDIX A: REGULAR AND IRREGULAR SINGULAR POINTS FOR BLACK HOLES

Here we elaborate some subtleties that arise in the computations of the fake monodromy λ_∞. Section II A focused on regular singular points. In order to determine the fake monodromy around an irregular singular point, such as $r = \infty$ in this paper, we consider an ODE of the form

$$\frac{d^2R}{dr^2} + p(r)\frac{dR}{dr} + q(r)R = 0. \tag{A1}$$

Then the following definition can be applied to determine the nature of any singularity. *Definition*: Point a is an ordinary point when functions $p(r)$ and $q(r)$ are analytic at $r = a$. When the functions $p(r)$, $q(r)$ each have poles on, e.g., $r = a$, we call a singular point a to be *regular* if either $p(r)$ or $q(r)$ diverges as $r \to a$ but

$$\lim_{r\to a}(r-a)p(r) = \text{finite} = p_0,$$
$$\lim_{r\to a}(r-a)^2 q(r) = \text{finite} = q_0. \tag{A2}$$

Otherwise, we call it *irregular.*

By performing a suitable coordinate transformation to a new variable $u(r) = f(r)R(r)$, for some function $f(r)$ to be determined, we may write this differential equation (A1) in the language of [33] as

$$u'' + \tilde{q}(r)u = 0, \quad \text{with} \quad \tilde{q}(r) = q_0 + \frac{q_1}{r} + \frac{q_2}{r^2} + \cdots \tag{A3}$$

around the irregular singular point $r = \infty$. Using Eq. (3) in [33] Sec. III.2,

$$u(r) = e^{\pm\sqrt{q_0}r} r^{i\lambda_\infty}\left(1 + \sum_{n\geq 1}\frac{a_n}{r^n}\right), \tag{A4}$$

where the fake monodromy is determined by $\lambda_\infty = \frac{\pm q_1}{2i\sqrt{q_0}}$. This prescription for computing λ_∞ is employed throughout this paper. We present the explicit computations for the fake λ_∞ (at the irregular singular point $r = \infty$) for a d-dimensional Schwarzschild black hole back to back with the asymptotically AdS black hole cousin (with all regular singular points, including $r = \infty$).

Example: $d \geq 4$ Schwarzschild black hole (asymptotically flat and AdS)

Consider first an AdS_d Schwarzschild black hole:

$$ds^2 = -\Delta dt^2 + \Delta^{-1}dr^2 + r^2 d\Omega, \tag{A5}$$

$$\text{where } \Delta = 1 - \frac{2M}{r^{d-3}} + \frac{r^2}{L^2}. \tag{A6}$$

Then,

$$g_{\mu\nu} = \begin{pmatrix} -\Delta & & \\ & \Delta^{-1} & \\ & & r^2[\Sigma]_{ij} \end{pmatrix}. \tag{A7}$$

Here, $[\Sigma]_{ij}$ is the surface submetric. By dimensional analysis, it is apparent that, if $\dim\Sigma = n$: $\det\Sigma = r^{2n}f(\phi_\alpha)$, where $f(\phi_\alpha)$ is some function of the generalized angles [in fact, using generalized polar coordinates: $f(\phi_\alpha) = \prod_{i=1}^{n-2} \sin^{n-1-i}(\phi_i)$]. Because it is a strictly projected submetric, $n = d-2$, then, $\det g = -r^{2d-4} f(\phi_\alpha)$.

We now turn to the Klein-Gordon equation

$$K_g[\Phi] \equiv \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu)[\Phi] = 0, \tag{A8}$$

$$\frac{1}{r^{d-2}\sqrt{f(\phi_\alpha)}} \partial_\mu (r^{d-2} \sqrt{f(\phi_\alpha)} g^{\mu\nu} \partial_\nu)[\Phi] = 0, \tag{A9}$$

$$\begin{aligned} \frac{1}{r^{d-2}\sqrt{f(\phi_\alpha)}} (\partial_t [r^{d-2} \sqrt{f(\phi_\alpha)} (-\Delta^{-1} \partial_t)] \\ + \partial_r [r^{d-2} \sqrt{f(\phi_\alpha)} (\Delta \partial_r)] \\ + \partial_i [r^{d-2} \sqrt{f(\phi_\alpha)} (\Sigma^{ij} \partial_j)])[\Phi] = 0. \end{aligned} \tag{A10}$$

Thus, invoking the separation of variables, let $\Phi(t, r, \phi_\alpha) = W(t) R(r) \prod_{i=1}^{n-2} \theta_i(\phi_i)$, so that

$$\frac{K_g[\Phi]}{\Phi} = 0, \tag{A11}$$

$$\begin{aligned} \frac{-\Delta^{-1} \partial_t^2 [W(t)]}{W(t)} + \frac{\partial_r [r^{d-2} \Delta \partial_r R]}{r^{d-2} R(r)} \\ + \frac{\partial_i (\sqrt{f(\theta_\alpha)} \Sigma^{ij} \partial_j \prod_{i=1}^{n-2} \theta_i(\phi_i))}{\sqrt{f(\theta_\alpha)} \prod_{i=1}^{n-2} \theta_i(\phi_i)} = 0. \end{aligned} \tag{A12}$$

Then, imposing eigenvalues,

$$\frac{-\Delta^{-1} \partial_t^2 [W(t)]}{W(t)} = -\omega^2 \Leftrightarrow W(t) = W_0 e^{\pm i\omega t}, \tag{A13}$$

$$\begin{aligned} \frac{1}{\sqrt{f(\theta_\alpha)} \prod_{i=1}^{n-2} \theta_i(\phi_i)} \partial_i \left(\sqrt{f(\theta_\alpha)} \Sigma^{ij} \partial_j \prod_{i=1}^{n-2} \theta_i(\phi_i) \right) \\ = \sum K_{L_i}, \end{aligned} \tag{A14}$$

so that we may rewrite the radial ansatz as

$$\frac{\omega^2}{\Delta} + \frac{1}{r^{d-2} R(r)} \partial_r [r^{d-2} \Delta \partial_r R] + \sum K_{L_i} = 0, \tag{A15}$$

$$\text{or} \quad \left[\frac{1}{r^{d-2}} \partial_r [r^{d-2} \Delta \partial_r R] + \frac{\omega^2}{\Delta} + \sum K_{L_i} \right] R(r) = 0. \tag{A16}$$

The transform

$$R(r) = (r^{d-2} - 2Mr)^{-1/2} u(r) \tag{A17}$$

eliminates the first order term, yielding

$$\partial_r^2 u(r) + \left(\frac{\partial_r (r^{d-2} \Delta \partial_r (r^{d-2} - 2Mr)^{-1/2})}{r^{d-2} \Delta (r^{d-2} - 2Mr)^{-1/2}} + \frac{\omega^2}{\Delta^2} \right) u(r) = 0. \tag{A18}$$

In the above form, the asymptotic waveform is more apparent, albeit with a local, functional frequency $\partial_r^2 u(r) + \bar{\omega}^2(r) u(r) = 0$,

$$\bar{\omega}^2 = -\frac{\partial_r (r^{d-2} \Delta)((d-2) r^{d-3} - 2M)}{2 r^{d-2} \Delta (r^{d-2} - 2Mr)} - \frac{(d-2)(d-3)(r^{d-2} - 2Mr) r^{d-4} - 3((d-2) r^{d-3} - 2M)^2}{2 (r^{d-2} - 2Mr)^2} + \frac{\omega^2}{\Delta^2} \tag{A19}$$

or

$$\bar{\omega}^2 = -\frac{(\frac{d}{L^2} + \frac{d-2}{r^2} - \frac{2M}{r^{d-1}})((d-2) - \frac{2M}{r^{d-3}})}{2r^2 (\frac{1}{L^2} + \frac{1}{r^2} - \frac{2M}{r^{d-1}})(1 - \frac{2M}{r^{d-3}})} - \frac{(d-2)(d-3)(1 - \frac{2M}{r^{d-3}}) - 3((d-2) - \frac{2M}{r^{d-3}})^2}{2r^2 (1 - \frac{2M}{r^{d-3}})^2} + \frac{\omega^2}{r^4 (\frac{1}{L^2} + \frac{1}{r^2} - \frac{2M}{r^{d-1}})^2}. \tag{A20}$$

For AdS (or finite L), asymptotically expanding at $r \to \infty$

$$\begin{aligned} \bar{\omega}^2 \approx & -\frac{L^2}{2r^2} \left(\frac{d}{L^2} + \frac{d-2}{r^2} - \frac{2M}{r^{d-1}} \right) \left((d-2) - \frac{2M}{r^{d-3}} \right) \left[\sum_{n=0} \left(\frac{r^{d-3} - 2M}{r^{d-1}} \right)^n l^{2n} \right] \left[\sum_{n=0} \left(\frac{2M}{r^{d-3}} \right)^n \right] \\ & - \frac{1}{2r^2} \left[(d-2)(d-3) \left(1 - \frac{2M}{r^{d-3}} \right) - 3 \left((d-2) - \frac{2M}{r^{d-3}} \right)^2 \right] \left[\sum_{n=1} n \left(\frac{2M}{r^{d-3}} \right)^{n-1} \right] \\ & + \frac{L^4 \omega^2}{r^4} \left[\sum_{n=1} n \left(\frac{r^{d-3} - 2M}{r^{d-1}} \right)^{n-1} L^{2n-2} \right] \end{aligned} \tag{A21}$$

$$\approx \frac{L^4 \omega^2}{r^4} \left(1 + O\left(\frac{1}{r^2} \right) \right) + \frac{(d-2)(d-3)}{2r^2} \left(1 + O\left(\frac{1}{r} \right) \right). \tag{A22}$$

Thus, using (A1), we see that $p_0 = 0$ and $q_0 = \frac{(d-2)(d-3)}{2}$. Therefore, $r \to \infty$ is a regular singular point in AdS_d, $\forall\, d$.

Returning to (A20) and considering the flat case, $\lim_{L\to\infty} \bar{\omega}^2 = \bar{\omega}^2_{\text{flat}}$:

$$\bar{\omega}^2_{\text{flat}} = \frac{\omega^2}{(1-\frac{2M}{r^{d-3}})^2} - \frac{(d-2)(d-3)(1-\frac{2M}{r^{d-3}}) - 2((d-2)-\frac{2M}{r^{d-3}})^2}{2r^2(1-\frac{2M}{r^{d-3}})^2}$$
$$\approx \left[\omega^2 - \frac{1}{2r^2}\left((d-2)(d-3)\left(1-\frac{2M}{r^{d-3}}\right) - 2\left((d-2)-\frac{2M}{r^{d-3}}\right)^2\right)\right]\sum_{n=1} n\left(\frac{2M}{r^{d-3}}\right)^{n-1}. \tag{A23}$$

Clearly, due to the constant term, $\bar{\omega}^2$ fails the criteria in (A1): $\lim_{r\to\infty} r^2\bar{\omega}^2 = \infty$. Thus, we must analyze the point $r \to \infty$ as irregular. As described in [33], the ODE is of the form (A3). Here $q_0 = -\omega^2\ \forall\, d$. For $d = 4$, $q_1 = 4\omega^2 M^2$, while $\forall\, d \neq 4$, $q_1 = 0$. Using [33] Sec. III.2 Eq. (6), we find

$$\tilde{\lambda}_\infty^{(d=4)} = \pm 2M\omega, \qquad \tilde{\lambda}_\infty^{d>4} = 0. \tag{A24}$$

Then, in the $u(r)$ frame we find the asymptotic solutions to be

$$u(r) = e^{\pm i\omega r}\cdot r^{\pm i\tilde{\lambda}_\infty}\left(1 + O\left(\frac{1}{r}\right)\right).$$

Equivalently, we can determine the asymptotic expansion in the R frame by employing the transform (A17) between the different frames

$$R(r) = e^{\pm i\omega r}\cdot r^{\pm i\tilde{\lambda}_\infty - (d-2)/2}\left(1 + O\left(\frac{1}{r}\right)\right). \tag{A25}$$

In the case of $d = 4$, we find

$$R(r) = e^{\pm i\omega r}\cdot r^{\pm 2iM\omega - 1}\left(1 + O\left(\frac{1}{r}\right)\right). \tag{A26}$$

Note that in the R frame, one finds two (rather than one) values of the fake monodromy $\pm\tilde{\lambda}_\infty$. As we explain in Sec. II, the frame that simplifies the identification between the monodromies and entropy variations is a frame in which one of the monodromies vanishes at each of the singularities. This complication is an artifact of the coordinate system, where having two monodromies at each point may result in ambiguities. We argue that this complication does not arise in the $\tilde{R}$ frame where the ODE is of the form (13) with (14). Moreover, for consistency in implementing the Fuchs-type relation the monodromies have to be determined in the same frame. The $\tilde{R}$ frame was the preferred choice in the previous sections to find the monodromies α_i at the finite regular singular points. Inverting the relation (33)

$$\tilde{R}(r) = \prod_{i=1}^{\bar{\mathcal{K}}-1}(r-r_i)^{+i\tilde{\alpha}_i}R(r), \quad \text{with} \quad \tilde{\alpha}_i = \frac{\omega}{\bar{\Delta}'(r_i)}. \tag{A27}$$

To be explicit, the associated asymptotic expansions at $r = \infty$ in the $\tilde{R}$ frame are

$$\tilde{R}(r) = e^{-i\omega r}\cdot r^{-(d-2)/2}\left(1 + O\left(\frac{1}{r}\right)\right), \tag{A28}$$

$$\tilde{R}(r) = e^{+i\omega r}\cdot r^{+i2\tilde{\lambda}_\infty - (d-2)/2}\left(1 + O\left(\frac{1}{r}\right)\right). \tag{A29}$$

Within this frame $\lambda_\infty = 2\tilde{\lambda}_\infty$, the *fake* monodromy, contributes to the Fuchs-type relation at $r = \infty$ and, equivalently, constrains the variational entropy sum over nite horizons. To summarize, the fake monodromies in the $\tilde{R}$ frame are

$$\text{For Asympt, Flat}_4\colon\ \lambda_\infty \to 4M\omega, \tag{A30}$$

$$\text{For Asympt, Flat}_{d>4}\colon\ \lambda_\infty \to 0. \tag{A31}$$

APPENDIX B: PROPERTIES OF KERR-AdS HORIZONS

We briefly describe a few useful properties of the Kerr-AdS black hole solution: (1) The function $\Delta(r) = 0$ has four roots $r_1 = r_+$, $r_2 = r_-$, r_3, r_4 and can be written as

$$\Delta(r) = (r-r_+)(r-r_-)(r-r_3)(r-r_4)/L^2. \tag{B1}$$

(2) The roots of $\Delta(r) = 0$ are related in the following way:

$$\sum_{i=1}^{4} r_i = 0, \qquad \prod_{i=1}^{4} r_i = L^2 a^2, \tag{B2}$$

$$\prod(r_i^2 + a^2) = (2MaL^2)^2,$$
$$\prod(1 + r_i^2/L^2) = (2M/L)^2. \tag{B3}$$

(3) This implies that we can rewrite the parameters as

$$M = \frac{(L^2 + r_-^2)(L^2 + r_+^2)(r_- + r_+)}{2L^2(L^2 - r_- r_+)}, \qquad a^2 = \frac{r_- r_+ (L^2 + r_-^2 + r_- r_+ + r_+^2)}{L^2 - r_- r_+}, \tag{B4}$$

$$r_3 = -\frac{1}{2}\left(r_- + r_+ + \sqrt{\frac{4L^4 + L^2(3r_-^2 + 2r_- r_+ + 3r_+^2) + r_- r_+ (r_- + r_+)^2}{r_- r_+ - L^2}}\right), \tag{B5}$$

$$r_4 = -\frac{1}{2}\left(r_- + r_+ - \sqrt{\frac{4L^4 + L^2(3r_-^2 + 2r_- r_+ + 3r_+^2) + r_- r_+ (r_- + r_+)^2}{r_- r_+ - L^2}}\right). \tag{B6}$$

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