

Geometrized effective-one-body formalism for extreme-mass-ratio limits: Generic orbits

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(Received 4 March 2021; accepted 28 June 2021; published 19 July 2021)

Compact objects inspiraling into supermassive black holes, known as extreme-mass-ratio inspirals, are an important source for future space-borne gravitational-wave detectors. When constructing waveform templates, usually the adiabatic approximation is employed to treat the compact object as a test particle for a short duration, and the radiation reaction is reflected in the changes of the constants of motion. However, the mass of the compact object should have contributions to the background. In the present paper, employing the effective-one-body formalism, we analytically calculate the trajectories of a compact object around a massive Kerr black hole with generally three-dimensional orbits and express the fundamental orbital frequencies in explicit forms. In addition, by constructing an approximate “constant” similar to the Carter constant, we transfer the dynamical quantities such as energy, angular momentum, and the “Carter constant” to the semilatus rectum, eccentricity, and orbital inclination with mass-ratio corrections. The linear mass-ratio terms in the formalism may not be sufficient for accurate waveforms, but our analytical method for solving the equations of motion could be useful in various approaches to building waveform models.

DOI: [10.1103/PhysRevD.104.024050](https://doi.org/10.1103/PhysRevD.104.024050)

I. INTRODUCTION

One of the most promising and rewarding sources of gravitational waves for low-frequency, space-based gravitational-wave (GW) detectors—such as the future Laser Interferometer Space Antenna (LISA) [1], Taiji [2] and Tian-Qin [3]—are the so-called extreme-mass-ratio inspirals (EMRIs) [4–6], which are compact objects (COs) such as neutron stars or stellar-mass (or stellar origin) black holes (SOBHs) inspiraling into supermassive black holes (SMBHs) in the mass range 10^5 – $10^7 M_\odot$.

The majority of observed EMRI events are expected to be SOBH-SMBH mergers; this is partly due to mass segregation concentrating heavier BHs in the Galactic center and partly because their louder intrinsic amplitude enables them to be detected out to greater distances. The small CO is usually approximated as a test particle, and over the short orbital time scale follows a nearly geodesic trajectory in the background metric of the SMBH. The system radiates GWs at harmonics of the geodesic

frequencies, so the GW frequency spectrum encodes details of the instantaneous geodesic trajectory.

Though the signals from EMRIs are usually very weak, after one year of observation the signal-to-noise ratio may be enough to be detected [4]. To detect this kind of long duration signals, the requirement on the accuracy of waveform templates is that the dephasing should be less than a few radians after 10^5 cycles [5,7].

Nowadays, there are several kinds of EMRI template. The first kind uses Teukolsky equations [8] and treats COs as test particles (omitting the mass in their conservation dynamics part) and thus they just extract a snapshot waveform; however, accurate Teukolsky-based waveforms are computationally expensive to generate. Another kind uses the post-Newtonian (PN) fluxes approximation to describe the evolution of orbital parameters with gravitational radiation reaction, in particular the so-called kludge waveforms [9–11]. The third kind considers the correction due to the small mass by using the effective-one-body (EOB) formalism, but does not consider the spin of the small object, such as in circular orbits with PN waveforms [12,13] and the Teukolsky-based waveforms [14,15]. The kludge models can generate the waveforms quickly for three-dimensional (3D) orbits, but they take the COs as test

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particles in the orbital motion even though they include the radiation reaction.

If the time scale of GW radiation is much larger than that of the orbital motion, then at a given instant the properties of motion can be calculated from a conservative equation $\ddot{x}^\mu + {}^B\Gamma_{\rho\sigma}^\mu \dot{x}^\rho \dot{x}^\sigma = 0 + O(\nu) + O(\nu^2) + \dots$. Here the label “B” represents the background field of the massive body. In popular EMRI template models, like AAK [9], NK [10], AAK [11], XSPEG [16] etc., the mass-ratio corrections in the conservative dynamics are omitted, i.e., the right-hand side of the equation is exactly zero. This is just the geodesic equation of a test particle around a massive black hole. The test particle–adiabatic approximation (adiabatic model) makes the problem much simpler, but induces errors in the GW simulation. As in Eq. (74) of Ref. [17], the mass-ratio expansion to the phase evolution is

$$\phi(t) = \frac{1}{\nu} [\phi_{(0)} + \nu\phi_{(1)} + O(\nu^2)]. \quad (1.1)$$

An EMRI model that captures the leading-order term $\phi_{(0)}$ is called an adiabatic model. The calculation of $\phi_{(0)}$ requires only the averaged dissipative first-order self-force (in the mass-ratio expansion) that is fully equivalent to the fluxes, and some literature states that in general it may be accurate enough for detecting EMRI signals using LISA [18]. The next-to-leading-order term $\phi_{(1)}$ is called the post-1 adiabatic term, which includes both the dissipative and conservative pieces of the first-order self-force as well as the averaged dissipative piece of the second-order self-force. The post-1 adiabatic phase is required for the parameter estimation of EMRIs [18]. In the present paper, we mainly focus on the conservation dynamics part of EMRI, and try to analytically solve the equation of motion which includes the first-order mass-ratio corrections. The mass-ratio terms we use may not be enough for final EMRI waveforms, so the aim of this paper is not to construct the waveform template, but rather to demonstrate the method for the analytical solution of orbits. Our method should be useful once there are some more precise self-force corrections.

The EOB formalism, by including the mass-ratio corrections up to a certain order in the PN expansion, can well describe the dynamical evolution of binary black holes [19,20], and is widely used to construct the waveform templates for LIGO [21–28]. Most of these models only consider the circular-orbit cases. An eccentric EOB numerical relativity waveform template for spinning black holes was developed [29], but the orbits were not geometrized and the orbital parameters were not well defined. Recently, an analytically eccentric EOB formalism for Schwarzschild BHs was given [30]. In a previous work [31], we presented an analytically equatorial-eccentric EOB formalism for spinning cases in the extreme-mass-ratio limit. Note that the EOB formalism’s correction in the extreme-mass-ratio limit has not been guaranteed. However, as stated in

Ref. [32], the extreme-mass-ratio limit plays a pivotal role in the EOB development, especially for what concerns waveforms and fluxes, which can be informed by and compared with numerical results [33–44]. In addition, the EOB formalism with the extreme-mass-ratio limit should be improved due to a lot of works dedicated to analytically calculating gravitational self-force terms and providing comparisons with numerical results [45–51].

It is well known that the orbits of EMRIs could be highly eccentric [5] with orbital inclination (orbital plane precession), and the supermassive black hole in the center should be spinning in general. In the present paper, we extend the previous work by Hinderer and Babak [30] and ourselves [31] to the inclined-eccentric orbits in the Kerr background with the extreme-mass-ratio limit. We analytically transfer the original EOB dynamical equations to geometric kinetic motion with the semilatus rectum p , eccentricity e , and orbital inclination ι as the orbital parameters, together with three phase variables associated with the spatial geometry of the radial, azimuthal, and polar motion denoted by (ξ, ϕ, χ) . Because of the extremely small mass ratio, we omit the spin of the effective small body, and thus the very complicated spin-spin coupling terms disappear but maintain enough accuracy.

An important feature of the dynamics of an extreme-mass-ratio binary system in a bounded inclined-eccentric orbit is that the orbit can be characterized by three frequencies: the radial frequency ω_r associated with the libration between the apoapsis and periapsis, the polar frequency ω_θ associated with the libration between θ_{\min} and $\pi - \theta_{\min}$, and the azimuthal rotational frequency ω_ϕ . Once these three frequencies and orbital parameters are obtained, the frequency of GWs can be obtained and may be encoded in the Teukolsky equation to get accurate waveforms [14,15,52–56].

The organization of this paper is as follows. The basic knowledge of the EOB formalism is introduced in the following section. In Sec. III, we reparametrize the original spinning EOB dynamical description to a geometric formalism in the more efficient reparametrized terms of $(p, e, \iota, \xi, \chi, \phi)$. We analytically express the fundamental frequencies in three integrals with two parameters: ξ and χ . In particular, we investigate the influence of the mass ratio on the detection of EMRIs. We also give the general forms for the evolution of orbital parameters with gravitational radiation reaction. Finally, the last section contains our conclusions and discussions.

Throughout this paper we use geometric units $G = c = 1$, the units of time and length are the mass of system M , and the units of linear and angular momentum are μ and μM , respectively, where μ is the reduced mass of the effective body.

II. EFFECTIVE-ONE-BODY HAMILTONIAN

The EOB formalism was originally introduced in Refs. [19,20] to describe the evolution of a binary system.

We start by considering an EMRI system with a central Kerr black hole m_1 and inspiraling object m_2 (we assume that it is nonspinning for simplicity, $m_2 \ll m_1$). For the moment, we neglect the radiation-reaction effects and focus on purely geodesic motion. The conservative orbital dynamics is derived via Hamilton's equations using the EOB Hamiltonian

$$H_{\text{EOB}} = M\sqrt{1 + 2\nu(\hat{H}_{\text{eff}} - 1)}, \quad (2.1)$$

where $M = m_1 + m_2$, $\nu = m_1 m_2 / M^2$, $\mu = \nu M$, and the reduced effective Hamiltonian $\hat{H}_{\text{eff}} = H_{\text{eff}} / \mu$. The deformed Kerr metric is given by [57]

$$g^{tt} = -\frac{\Lambda_t}{\Delta_t \Sigma}, \quad (2.2a)$$

$$g^{rr} = \frac{\Delta_r}{\Sigma}, \quad (2.2b)$$

$$g^{\theta\theta} = \frac{1}{\Sigma}, \quad (2.2c)$$

$$g^{\phi\phi} = \frac{1}{\Lambda_t} \left(-\frac{\tilde{\omega}_{\text{fd}}^2}{\Delta_t \Sigma} + \frac{\Sigma}{\sin^2 \theta} \right), \quad (2.2d)$$

$$g^{t\phi} = -\frac{\tilde{\omega}_{\text{fd}}}{\Delta_t \Sigma}. \quad (2.2e)$$

The quantities Σ , Δ_t , Δ_r , Λ_t , and $\tilde{\omega}_{\text{fd}}$ in Eqs. (2.2a)–(2.2e) are given by

$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad (2.3a)$$

$$\Delta_t = r^2 \left[A(u) + \frac{a^2}{M^2} u^2 \right], \quad (2.3b)$$

$$\Delta_r = \Delta_t D^{-1}(u), \quad (2.3c)$$

$$\Lambda_t = (r^2 + a^2)^2 - a^2 \Delta_t \sin^2 \theta, \quad (2.3d)$$

$$\tilde{\omega}_{\text{fd}} = 2aMr + \omega_1^{\text{fd}} \nu \frac{aM^3}{r} + \omega_2^{\text{fd}} \nu \frac{Ma^3}{r}, \quad (2.3e)$$

where $a = |\mathbf{S}_{\text{Kerr}}|/M$ is the effective Kerr parameter and $u = M/r$. The values of ω_1^{fd} and ω_2^{fd} given by a preliminary comparison of the EOB model with numerical relativity results are about -10 and 20 [58,59], and the metric potentials A and D for the EOB model are given as follows.

The log-resummed, calibrated A potential is given by the expression from Appendix A of Ref. [60],

$$A(u) = \Delta_u - \frac{a^2}{M^2} u^2, \quad (2.4)$$

$$\Delta_u = \bar{\Delta}_u (\Delta_0 \nu + \nu \log(\Delta_5 u^5 + \Delta_4 u^4 + \Delta_3 u^3 + \Delta_2 u^2 + \Delta_1 u + 1) + 1), \quad (2.5)$$

with

$$\bar{\Delta}_u = \frac{a^2 u^2}{M^2} + \frac{1}{(K\nu - 1)^2} + \frac{2u}{K\nu - 1}, \quad (2.6a)$$

$$\Delta_5 = (K\nu - 1)^2 \left[\frac{64}{5} \log(u) + \left(-\frac{1}{3} a^2 (\Delta_1^3 - 3\Delta_1 \Delta_2 + 3\Delta_3) - \frac{\Delta_1^5 - 5\Delta_1^3 \Delta_2 + 5\Delta_1^2 \Delta_3 + 5\Delta_1 \Delta_2^2 - 5\Delta_2 \Delta_3 - 5\Delta_4 \Delta_1}{5(K\nu - 1)^2} + \frac{\Delta_1^4 - 4\Delta_1^2 \Delta_2 + 4\Delta_1 \Delta_3 + 2\Delta_2^2 - 4\Delta_4}{2K\nu - 2} + \frac{2275\pi^2}{512} + \frac{128\gamma}{5} - \frac{4237}{60} + \frac{256 \log(2)}{5} \right) \right], \quad (2.6b)$$

$$\Delta_4 = \frac{1}{96} [8(6a^2(\Delta_1^2 - 2\Delta_2)(K\nu - 1)^2 + 3\Delta_1^4 + \Delta_1^3(8 - 8K\nu) - 12\Delta_1^2 \Delta_2 + 12\Delta_1(2\Delta_2 K\nu - 2\Delta_2 + \Delta_3)) + 48\Delta_2^2 - 64(K\nu - 1)(3\Delta_3 - 47K\nu + 47) - 123\pi^2(K\nu - 1)^2], \quad (2.6c)$$

$$\Delta_3 = -a^2 \Delta_1 (K\nu - 1)^2 - \frac{\Delta_1^3}{3} + \Delta_1^2 (K\nu - 1) + \Delta_1 \Delta_2 - 2(K\nu - 1)(\Delta_2 - K\nu + 1), \quad (2.6d)$$

$$\Delta_2 = \frac{1}{2} (\Delta_1 (\Delta_1 - 4K\nu + 4) - 2a^2 \Delta_0 (K\nu - 1)^2), \quad (2.6e)$$

$$\Delta_1 = -2(\Delta_0 + K)(K\nu - 1), \quad (2.6f)$$

$$\Delta_0 = K(K\nu - 2), \quad (2.6g)$$

where K is a calibration parameter tuned to numerical-relativity simulations whose most recently updated value was determined in Eq. (4.8) of Ref. [61],

$$K = 267.788\nu^3 - 126.687\nu^2 + 10.2573\nu + 1.7336. \quad (2.6h)$$

The D potential is

$$D^{-1}(u) = 1 + \log[D_{\text{Taylor}}] \\ D_{\text{Taylor}} = 1 + 6\nu u^2 + 2\nu u^3 (26 - 3\nu). \quad (2.7a)$$

III. GEOMETRIZATION OF THE CONSERVATIVE EOB DYNAMICS

The EOB Hamiltonian in Eq. (2.1) is canonically transformed and subsequently mapped to an effective Hamiltonian H_{eff} describing a particle of effective

mass μ and effective spin $S_* = aM(m_2/m_1)$, moving in a deformed Kerr metric of mass M and effective spin $S_{\text{Kerr}} = aM$, and the effective Hamiltonian is given by [57,62,63]

$$H_{\text{eff}} = H_{\text{NS}} + H_S - \frac{\nu}{2r^3} S_*^2, \quad (3.1)$$

where the first part is just the Hamiltonian of a nonspinning particle in the deformed Kerr metric with the mass-ratio correction. When $\nu \rightarrow 0$, this part goes back to the test-particle limit in Kerr spacetime. The forms of H_S and S_* are quite trivial; readers can refer to Ref. [62] for details. For EMRIs, due to the small mass ratio $\nu \sim 10^{-7}$ – 10^{-4} , as we have shown in Ref. [31] the last two terms in Eq. (3.1) can be neglected without loss of accuracy because they are two orders less than the mass ratio. The Hamiltonian equations for the orbital motion are

$$\frac{d\mathbf{r}}{dt} = \frac{\partial H_{\text{EOB}}}{\partial \mathbf{P}}, \quad \frac{d\mathbf{P}}{dt} = -\frac{\partial H_{\text{EOB}}}{\partial \mathbf{r}}. \quad (3.2)$$

The above differential equations are coupling each other, especially for radial r and polar θ equations. Though the

numerical integral can give accurate trajectories, analytical ones with geometrized parameters will be valuable for revealing the properties of motion. In the section, we follow the procedure of the test-particle case, decouple the motion in the r and θ directions, and give a geometrized formalism to replace the dynamical equation (3.2).

A. Semi-Carter “constant” in deformed Kerr spacetime

The effective one-body dynamics was given by a Hamilton-Jacobi equation of the form [64]

$$g^{\alpha\beta} P_\alpha P_\beta + \frac{Q_4 M^2 P_r^4}{r^2 \mu^2} + \mu^2 = 0. \quad (3.3)$$

The function $Q_4 = 2(4 - 3\nu)\nu$ [65] represents a nongeodesic term that appears at 3PN order. We omit this term in the following calculations because it is next-to-leading order in the mass ratio and the above equation cannot be separated if we retain this term. We use the deformed Kerr metric components (2.2a)–(2.2e) to bring Eq. (3.3) into the concrete form

$$\begin{aligned} -\mu^2 \Sigma = & -\frac{(r^2 + a^2)^2 - a^2 \sin^2 \theta (r^2 A(u) + a^2)}{r^2 A(u) + a^2} P_t^2 - 2 \frac{\tilde{\omega}_{\text{fd}}}{r^2 A(u) + a^2} P_t P_\phi + \Delta_r P_r^2 \\ & + P_\theta^2 + \frac{1}{(r^2 + a^2)^2 - a^2 \sin^2 \theta (r^2 A(u) + a^2)} \left(\frac{(r^2 + a^2 \cos^2 \theta)^2}{\sin^2 \theta} - \frac{\tilde{\omega}_{\text{fd}}^2}{r^2 A(u) + a^2} \right) P_\phi^2, \end{aligned} \quad (3.4)$$

where P_r , P_ϕ , and P_θ are the canonical radial, azimuthal, and polar angular momentum. From the symmetries we immediately obtain two constants of motion corresponding to the conservation of energy, H_{eff} , and angular momentum about the symmetry axis, L_z ; thus, we have

$$P_t = -H_{\text{eff}}, \quad (3.5a)$$

$$P_\phi = L_z. \quad (3.5b)$$

Then, Eq. (3.4) becomes

$$\begin{aligned} -\Sigma = & -\frac{(r^2 + a^2)^2 - a^2 \sin^2 \theta (r^2 A(u) + a^2)}{r^2 A(u) + a^2} \hat{H}_{\text{eff}}^2 + 2 \frac{\tilde{\omega}_{\text{fd}}}{r^2 A(u) + a^2} \hat{H}_{\text{eff}} \hat{L}_z + \Delta_r \hat{P}_r^2 \\ & + \hat{P}_\theta^2 + \frac{1}{(r^2 + a^2)^2 - a^2 \sin^2 \theta (r^2 A(u) + a^2)} \left(\frac{(r^2 + a^2 \cos^2 \theta)^2}{\sin^2 \theta} - \frac{\tilde{\omega}_{\text{fd}}^2}{r^2 A(u) + a^2} \right) \hat{L}_z^2, \end{aligned} \quad (3.6)$$

where we have defined the reduced momenta $\hat{P}_r = P_r/\mu$, $\hat{P}_\theta = P_\theta/\mu$, and $\hat{L}_z = L_z/\mu$. It is convenient to rewrite this expression as

$$\begin{aligned} -\Sigma = & -\frac{(a\hat{L}_z - (r^2 + a^2)\hat{H}_{\text{eff}})^2}{r^2 A(u) + a^2} + \frac{(\hat{L}_z - a \sin^2 \theta \hat{H}_{\text{eff}})^2}{\sin^2 \theta} + \Delta_r \hat{P}_r^2 + \hat{P}_\theta^2 \\ & + 2 \frac{\tilde{\omega}_{\text{fd}} + ar^2(A(u) - 1)}{r^2 A(u) + a^2} \hat{H}_{\text{eff}} \hat{L}_z - \frac{\tilde{\omega}_{\text{fd}}^2 - a^2 r^4 (A(u) - 1)^2}{(r^2 A(u) + a^2)((r^2 + a^2)^2 - a^2 \sin^2 \theta (r^2 A(u) + a^2))} \hat{L}_z^2, \end{aligned} \quad (3.7)$$

and solving this equation by separation of variables gives

$$\begin{aligned} & \left(\frac{\hat{L}_z}{\sin \theta} - a \hat{H}_{\text{eff}} \sin \theta \right)^2 + \hat{P}_\theta^2 + a^2 \cos^2 \theta \\ & = \frac{(a \hat{L}_z - (r^2 + a^2) \hat{H}_{\text{eff}})^2}{r^2 A(u) + a^2} - \Delta_r \hat{P}_r^2 - r^2 - 2 \frac{\tilde{\omega}_{\text{fd}} + ar^2(A(u) - 1)}{r^2 A(u) + a^2} \hat{H}_{\text{eff}} \hat{L}_z + F(r, \theta) \hat{L}_z^2, \end{aligned} \quad (3.8)$$

where

$$F(r, \theta) = \frac{\tilde{\omega}_{\text{fd}}^2 - a^2 r^4 (A(u) - 1)^2}{(r^2 A(u) + a^2)((r^2 + a^2)^2 - a^2 \sin^2 \theta (r^2 A(u) + a^2))}. \quad (3.9)$$

In the test-particle limit $\nu \rightarrow 0$, $A(u) \rightarrow 1 - 2u$, the r - θ coupling function $F(r, \theta) \rightarrow 0$, and both sides must be equal to a new constant of the motion since their Poisson brackets with the Hamiltonian are equal to zero, and thus we can obtain the reduced Carter constant \hat{K} expressed in terms of (θ, \hat{P}_θ) like Eq. (49) of Ref. [66],

$$\hat{K} = \hat{P}_\theta^2 + a^2 \cos^2 \theta + \left(\frac{\hat{L}_z}{\sin \theta} - a \sin \theta \hat{H}_{\text{eff}} \right)^2. \quad (3.10)$$

There is another definition of the Carter constant, $\hat{Q} \equiv \hat{K} - (\hat{L}_z - a \hat{H}_{\text{eff}})^2$, which vanishes for equatorial orbits ($\theta = \pi/2$).

If we consider the cases of the inspirals with nonzero mass ratio, $F(r, \theta)$ should no longer be ignored, then the r and θ motion cannot be decoupled anymore. Now, let us see if there is an approximation $F(r, \theta) \approx F(r)$. It is convenient to rewrite the function $F(r, \theta)$ as

$$F(r, \theta) = \frac{\tilde{\omega}_{\text{fd}}^2 - a^2 r^4 (A(u) - 1)^2}{(r^2 A(u) + a^2)(r^2 + a^2)^2} \times \frac{1}{1 - \frac{a^2 \sin^2 \theta (r^2 A(u) + a^2)}{(r^2 + a^2)^2}}. \quad (3.11)$$

It can be easily noticed that $\frac{a^2 \sin^2 \theta (r^2 A(u) + a^2)}{(r^2 + a^2)^2}$ is a 2PN correction and we abort it first, and thus the first-order Maclaurin series of $F(r, \theta)$ is obtained as

$$F(r, \theta) \simeq G(r) + \frac{a^2 \sin^2 \theta}{(r^2 + a^2)^4} (\tilde{\omega}_{\text{fd}}^2 - a^2 r^4 (A(u) - 1)^2), \quad (3.12)$$

where

$$G(r) = \frac{\tilde{\omega}_{\text{fd}}^2 - a^2 r^4 (A(u) - 1)^2}{(r^2 A(u) + a^2)(r^2 + a^2)^2}, \quad (3.13)$$

and through 3PN order [20,65]

$$A(u) = 1 - 2u + \nu \left(2u^3 + \left(\frac{94}{3} - \frac{41\pi^2}{32} \right) u^4 \right). \quad (3.14)$$

Inserting it into Eq. (3.12) [truncated at the $\mathcal{O}(\nu)$ term] leads to

$$\begin{aligned} F(r, \theta) \simeq & G(r) + \frac{4\nu a^4 M^4 \sin^2 \theta}{(r^2 + a^2)^4} \\ & \times \left(2 + \left(\frac{94}{3} - \frac{41\pi^2}{32} \right) u + \omega_1^{\text{fd}} + \omega_2^{\text{fd}} \frac{a^2}{M^2} \right), \end{aligned} \quad (3.15)$$

where the second term as a function of both r and θ is just 8PN with a mass ratio and can be safely ignored. We replace $F(r, \theta)$ by $G(r)$ to decouple r and θ so that Eq. (3.8) can be separated. As shown in Fig. 1, the error between $F(r, \theta)$ and $G(r)$ mainly depends on the value of r and is maximal when $\theta = \pi/2$ (the equatorial plane). One can see that even very close to the horizon ($\xi \rightarrow 0$ for the orange

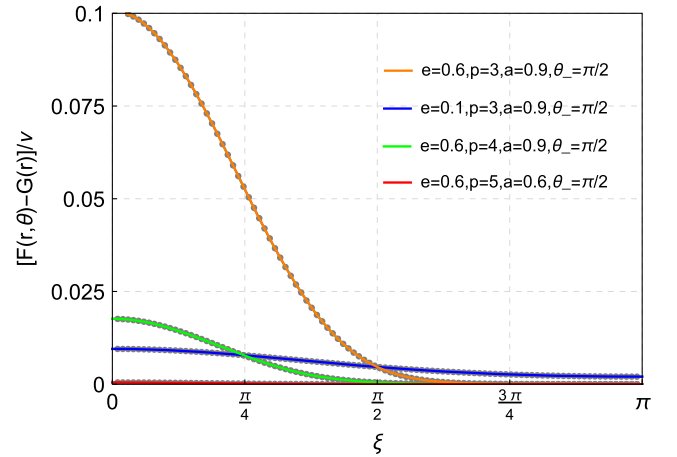


FIG. 1. Error of the approximation of the coupling function $F(r, \theta)$. The solid line and points represent $\nu = 10^{-4}$ and 10^{-6} , respectively. It is a very close approximation to substitute the r function $G(r)$ for $F(r, \theta)$. Even if we chose the extreme orbital parameters near LSO (for the orange line, $p_s = 2.96M$), the error is less than 0.1ν .

TABLE I. Comparison of the maximum \hat{P}_θ (\hat{P}_θ at $\theta = \pi/2$) between the analytical formalism and numerical integration.

$\nu = 0.01$	Analytical max \hat{P}_θ	Numerical max \hat{P}_θ
$p = 8, e = 1/3, a = 0.99, \theta_{\min} = \pi/4$	2.2909944716	2.2909945165
$p = 4, e = 1/3, a = 0.99, \theta_{\min} = \pi/4$	1.8126129568	1.8126323323
$p = 8, e = 1/3, a = 0.1, \theta_{\min} = \pi/4$	2.5258065572	2.5258065544
$p = 8, e = 1/3, a = 0.99, \theta_{\min} = \pi/10$	3.2445223467	3.2445223751

line) the error is still one order smaller than the mass ratio. Otherwise, the term we aborted is two or more orders less than the mass ratio. We believe that replacing $F(r, \theta)$ with $G(r)$ retains enough accuracy for EMRIs.

Now we can obtain the approximate reduced Carter constant $\hat{\mathcal{K}}$ in terms of (θ, \hat{P}_θ) as same as Eq. (3.10) and the other expression in terms of (r, \hat{P}_r)

$$\hat{\mathcal{K}} = \frac{[a\hat{L}_z - (r^2 + a^2)\hat{H}_{\text{eff}}]^2}{r^2A(u) + a^2} - \Delta_r \hat{P}_r^2 - r^2 - 2 \frac{\tilde{\omega}_{\text{fd}} + ar^2(A(u) - 1)}{r^2A(u) + a^2} \hat{H}_{\text{eff}} \hat{L}_z + G(r) \hat{L}_z^2. \quad (3.16)$$

By using these constants of motion, the angular momentum \hat{P}_θ and \hat{P}_r can be expressed as

$$\hat{P}_\theta^2 = \hat{Q} - \cos^2\theta \left[a^2(1 - \hat{H}_{\text{eff}}^2) + \frac{\hat{L}_z^2}{\sin^2\theta} \right], \quad (3.17a)$$

$$\hat{P}_r^2 = \frac{[a\hat{L}_z - (r^2 + a^2)\hat{H}_{\text{eff}}]^2 - (r^2A(u) + a^2)[r^2 + \hat{\mathcal{K}} + 2 \frac{\tilde{\omega}_{\text{fd}} + ar^2(A(u) - 1)}{r^2A(u) + a^2} \hat{H}_{\text{eff}} \hat{L}_z - G(r) \hat{L}_z^2]}{(r^2A(u) + a^2)^2 D^{-1}(u)}. \quad (3.17b)$$

In Eq. (3.17a), if $\hat{P}_\theta = 0$, the polar motion is at the turning points $\theta = \theta_{\min}$ or $\pi - \theta_{\min}$. Then, we get the relation between the semi-Carter “constant” and the orbital inclination,

$$\hat{Q} = \cos^2\theta_{\min} \left[a^2(1 - \hat{H}_{\text{eff}}^2) + \frac{\hat{L}_z^2}{\sin^2\theta_{\min}} \right]. \quad (3.18)$$

This semi-Carter constant has the same form as in the test-particle case, but with mass-ratio corrections hidden in H_{eff} and L_z , which will be given explicitly later.

There is another straightforward way to get \mathbf{P} by numerically solving the Hamiltonian equations (3.2).

We then compare the analytical equation (3.17a) and the numerical results (also including the spin terms like H_s) and show the results in Table I. We find that our analytical approximations are very close to the numerical results, even for a mass ratio as large as 0.01, which proves that our approximation works well for EMRIs or even intermediate-mass-ratio inspirals.

B. Reparametrization of the energy and angular momentum

Solving Eqs. (3.3) and (3.5a) for H_{eff} yields the effective Hamiltonian associated with the deformed Kerr metric,

$$H_{\text{eff}} = \frac{g^{t\phi}}{g^{tt}} P_\phi + \frac{1}{\sqrt{-g^{tt}}} \sqrt{\mu^2 + \left[g^{\phi\phi} - \frac{(g^{t\phi})^2}{g^{tt}} \right] P_\phi^2 + g^{rr} P_r^2 + g^{\theta\theta} P_\theta^2}. \quad (3.19)$$

The energy of the system is given by

$$E = H_{\text{EOB}}, \quad (3.20)$$

which implies the relation

$$\hat{H}_{\text{eff}}(E) = 1 + \frac{1}{2\nu} \left(\frac{E^2}{M^2} - 1 \right). \quad (3.21)$$

The constants of motion and the dynamical equations in the last subsection can be written in terms of the

geometrized orbital elements: the semilatus rectum p , eccentricity e , and orbital inclination i . This will make the description of the system more intuitive. For an eccentric orbit, there exist apastron and periastron points which can be expressed as

$$r_1 = \frac{pM}{1-e}, \quad r_2 = \frac{pM}{1+e}, \quad (3.22)$$

where r_1, r_2 are the turning points of the radial motion. θ_{\min} and $\pi - \theta_{\min}$ is the turning points of polar motion, and $i \equiv \pi/2 - \theta_{\min}$ defines the so-called orbital inclination (see Fig. 2). These turning points are computed by solving the radial and polar equations of motion for $\dot{r} = 0, \dot{\theta} = 0$. By setting the radial and polar equations of motion equal to zero with $P_r = P_\theta = 0$, evaluating Eq. (3.21) at (r_1, θ_{\min}) and (r_2, θ_{\min}) leads to

$$\hat{L}_z^2 = \frac{(a_1 - a_2)^2(b_1^2 + b_2^2) - (b_1^2 - b_2^2)(b_1^2 c_1 - b_2^2 c_2) - 2(a_1 - a_2)b_1 b_2 \sqrt{(a_1 - a_2)^2 - (b_1^2 - b_2^2)(c_1 - c_2)}}{[(a_1 - a_2)^2 - (b_1^2 c_1 - b_2^2 c_2)]^2} \quad (3.23a)$$

$$\frac{E^2}{M^2} = 1 + 2\nu \left(a_1 \hat{L}_z + \sqrt{\frac{c_1 \hat{L}_z^2 + 1}{b_1}} - 1 \right), \quad (3.23b)$$

where the coefficients are

$$a_1 = \frac{\tilde{\omega}_{fd1}}{\Lambda_{r1}}, \quad (3.24a)$$

$$a_2 = \frac{\tilde{\omega}_{fd2}}{\Lambda_{r2}}, \quad (3.24b)$$

$$b_1 = \sqrt{\frac{\Sigma_1 \Delta_{r1}}{\Lambda_{r1}}}, \quad (3.24c)$$

$$b_2 = \sqrt{\frac{\Sigma_2 \Delta_{r2}}{\Lambda_{r2}}}, \quad (3.24d)$$

$$c_1 = \frac{\Sigma_1}{(1 - \cos^2 \theta_{\min}) \Lambda_{r1}}, \quad (3.24e)$$

$$c_2 = \frac{\Sigma_2}{(1 - \cos^2 \theta_{\min}) \Lambda_{r2}}, \quad (3.24f)$$

in which the subscripts 1 and 2 mean that the function is to be evaluated at (r_1, θ_{\min}) and (r_2, θ_{\min}) ,

$$\Sigma_1 = a^2 \cos^2 \theta_{\min} + r_1^2, \quad (3.25a)$$

$$\Sigma_2 = a^2 \cos^2 \theta_{\min} + r_2^2, \quad (3.25b)$$

$$\Delta_{r1} = a^2 + A(r_1)r_1^2, \quad (3.25c)$$

$$\Delta_{r2} = a^2 + A(r_2)r_2^2, \quad (3.25d)$$

$$\Lambda_{r1} = (a^2 + r_1^2)^2 - a^2 \sin^2 \theta_{\min} \Delta_{r1}, \quad (3.25e)$$

$$\Lambda_{r2} = (a^2 + r_2^2)^2 - a^2 \sin^2 \theta_{\min} \Delta_{r2}, \quad (3.25f)$$

$$\tilde{\omega}_{fd1} = \frac{a^3 \nu \omega_2}{r_1} + \frac{a \nu \omega_1}{r_1} + 2a r_1, \quad (3.25g)$$

$$\tilde{\omega}_{fd2} = \frac{a^3 \nu \omega_2}{r_2} + \frac{a \nu \omega_1}{r_2} + 2a r_2. \quad (3.25h)$$

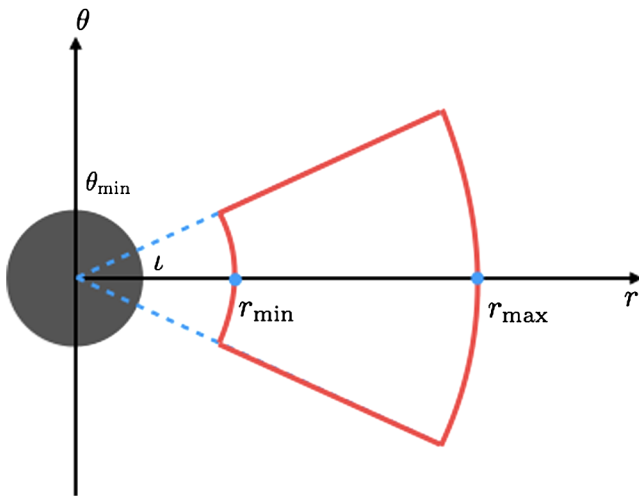


FIG. 2. Diagram of 3D motion of a CO around a Kerr black hole. The area encircled by the solid red line is the section of trajectories on the r - θ plane. Here, $r_{\max/\min}$ is just $r_{1/2}$ in Eq. (3.22).

The above formalism for a Kerr black hole is much more complicated than the Schwarzschild ones in Ref. [30]. Obviously, for the test-particle limit $\nu \rightarrow 0$, the above results will go back to the geodesic motion of a test particle in Kerr spacetime.

C. Determination of the fundamental frequencies

Since in our approximation the equations of motion in the deformed Kerr spacetime are separable in the coordinates r , θ , and ϕ , the action variables can be calculated from cyclic integrals over the spatial conjugate momenta in the Boyer-Lindquist coordinate representation:

$$J_r = \frac{1}{2\pi} \oint P_r dr, \quad (3.26)$$

$$J_\theta = \frac{1}{2\pi} \oint P_\theta d\theta, \quad (3.27)$$

$$J_\phi = \frac{1}{2\pi} \oint P_\phi d\phi = L_z. \quad (3.28)$$

The standard procedure of determining fundamental frequencies is to find the explicit form of the Hamiltonian in the action-angle representation, $H^{(\text{aa})}$, and calculate the frequencies from the partial derivatives with respect to the radial and polar action variables J_k [67]:

$$\mu\omega_k = \frac{\partial H^{(\text{aa})}}{\partial J_k}. \quad (3.29)$$

Unfortunately, Eqs. (3.26) and (3.27) cannot be solved analytically, and so they do not admit an explicit inversion. However, according to the Schmidt method [68], Eq. (3.29) can be solved even without knowing the functional form of the Hamiltonian $H^{(\text{aa})}$ if the theorem on implicit functions is employed.

Let $P_\beta^{(\text{aa})} = f_\beta^{(\text{aa})}(-\mu^2/2, H_{\text{eff}}, L_z, Q)$ be the momenta given by $P_0^{(\text{aa})} = p_t = -H_{\text{eff}}$ and $P_k^{(\text{aa})} = J_k$. If we denote the Jacobian matrix of f by Df , then, by the theorem on implicit functions, $Df \cdot D(f^{-1}) = Df \cdot (Df)^{-1} = I$, provided that f is nonzero and the Jacobian does not vanish [69]. As $-\mu^2/2$ is the invariant value of the Hamiltonian, we can substitute $-\mu^2/2 = g^{\alpha\beta} P_\alpha P_\beta / 2 = H^{(\text{aa})}(-H_{\text{eff}}, J_k)$. In addition, two rows of the Jacobian matrix are trivial due to the identities $P_0^{(\text{aa})} = -H_{\text{eff}}$ and $J_\phi = L_z$. For simplicity, we use the symbol H to denote the Hamiltonian $H^{(\text{aa})}$ below. Thus, the equation $Df \cdot D(f^{-1}) = I$ reads

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ \frac{\partial J_r}{\partial H} & \frac{\partial J_r}{\partial H_{\text{eff}}} & \frac{\partial J_r}{\partial L_z} & \frac{\partial J_r}{\partial Q} \\ \frac{\partial J_\theta}{\partial H} & \frac{\partial J_\theta}{\partial H_{\text{eff}}} & \frac{\partial J_\theta}{\partial L_z} & \frac{\partial J_\theta}{\partial Q} \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -\frac{\partial H}{\partial H_{\text{eff}}} & \frac{\partial H}{\partial J_r} & \frac{\partial H}{\partial J_\theta} & \frac{\partial H}{\partial J_\phi} \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\partial Q}{\partial H_{\text{eff}}} & \frac{\partial Q}{\partial J_r} & \frac{\partial Q}{\partial J_\theta} & \frac{\partial Q}{\partial J_\phi} \end{pmatrix} = I. \quad (3.30)$$

The above matrix equation is valid if and only if the orbit is nonequatorial [68]. We have discussed the condition of equatorial orbits in our previous work [31]. Then, it can be

split into four nontrivial sets of linear equations in the eight unknowns $-\frac{\partial H}{\partial H_{\text{eff}}}$, $\frac{\partial H}{\partial J_k}$, $-\frac{\partial Q}{\partial H_{\text{eff}}}$, and $\frac{\partial Q}{\partial J_k}$:

$$-A \cdot \frac{\partial}{\partial H_{\text{eff}}} \begin{pmatrix} H \\ Q \end{pmatrix} = \begin{pmatrix} 2W(r_1, r_2) \\ 2a^2 H_{\text{eff}} U(\theta_{\min}, \pi/2) \end{pmatrix}, \quad (3.31)$$

$$A \cdot \frac{\partial}{\partial J_r} \begin{pmatrix} H \\ Q \end{pmatrix} = \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}, \quad (3.32)$$

$$A \cdot \frac{\partial}{\partial J_\theta} \begin{pmatrix} H \\ Q \end{pmatrix} = \begin{pmatrix} 0 \\ \pi \end{pmatrix}, \quad (3.33)$$

$$A \cdot \frac{\partial}{\partial J_\phi} \begin{pmatrix} H \\ Q \end{pmatrix} = \begin{pmatrix} 2Z(r_1, r_2) \\ 2L_z V(\theta_{\min}, \pi/2) \end{pmatrix}, \quad (3.34)$$

with the coefficient matrix

$$A = \begin{pmatrix} 2\pi \frac{\partial J_r}{\partial H} & 2\pi \frac{\partial J_r}{\partial Q} \\ \pi \frac{\partial J_\theta}{\partial H} & \pi \frac{\partial J_\theta}{\partial Q} \end{pmatrix} = \begin{pmatrix} 2Y(r_1, r_2) & -X(r_1, r_2) \\ 2a^2 U(\theta_{\min}, \pi/2) & T(\theta_{\min}, \pi/2) \end{pmatrix}, \quad (3.35)$$

where r_1 and r_2 are the turning points of radial motion, and $X(r_1, r_2)$, $Y(r_1, r_2)$, $Z(r_1, r_2)$, and $W(r_1, r_2)$ are radial integrals defined by

$$X(r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{\Delta_r P_r}, \quad (3.36a)$$

$$Y(r_1, r_2) = \int_{r_1}^{r_2} \frac{r^2 dr}{\Delta_r P_r}, \quad (3.36b)$$

$$Z(r_1, r_2) = - \int_{r_1}^{r_2} \frac{\partial P_r}{\partial L_z} dr, \quad (3.36c)$$

$$W(r_1, r_2) = \int_{r_1}^{r_2} \frac{\partial P_r}{\partial H_{\text{eff}}} dr. \quad (3.36d)$$

$T(\theta_{\min}, \pi/2)$, $U(\theta_{\min}, \pi/2)$, and $V(\theta_{\min}, \pi/2)$ are polar integrals defined by

$$T(\theta_{\min}, \pi/2) = \int_{\theta_{\min}}^{\pi/2} \frac{d\theta}{P_\theta}, \quad (3.37a)$$

$$U(\theta_{\min}, \pi/2) = \int_{\theta_{\min}}^{\pi/2} \frac{\cos^2 \theta}{P_\theta} d\theta, \quad (3.37b)$$

$$V(\theta_{\min}, \pi/2) = \int_{\theta_{\min}}^{\pi/2} \frac{\cot^2 \theta}{P_\theta} d\theta. \quad (3.37c)$$

The above radial functions X , Y , Z , W and polar functions T , U , V are not proper integrals because the integrated

functions are divergent at the turning points r_1 , r_2 and θ_{\min} , $\pi - \theta_{\min}$. Thus, we define ξ by the equation $r = pM/(1 + e \cos \xi)$, where p is called the semilatus rectum and e is the eccentricity of the orbit, and we define χ with $\cos^2 \theta = \cos^2 \theta_{\min} \cos^2 \chi$. As ξ varies from 0 to 2π as r goes through a complete cycle, χ varies from 0 to 2π as θ oscillates through its full range of motion. Then, we transform X , Y , Z , W and T , U , V into well-behaved integrals,

$$X = \int_0^\pi \frac{ep \sin \xi}{(1 + e \cos \xi)^2} \frac{d\xi}{\Delta_r P_r}, \quad (3.38a)$$

$$Y = \int_0^\pi \frac{ep \sin \xi}{(1 + e \cos \xi)^2} \frac{r^2 d\xi}{\Delta_r P_r}, \quad (3.38b)$$

$$Z = - \int_0^\pi \frac{ep \sin \xi}{(1 + e \cos \xi)^2} \frac{\partial P_r}{\partial L_z} d\xi, \quad (3.38c)$$

$$W = \int_0^\pi \frac{ep \sin \xi}{(1 + e \cos \xi)^2} \frac{\partial P_r}{\partial H_{\text{eff}}} d\xi, \quad (3.38d)$$

which transforms Eq. (3.17a) into

$$\beta^2(z^2 - z_+^2)(z^2 - z_-^2) = P_\theta^2(1 - z^2), \quad (3.39)$$

where $\beta^2 = a^2(\mu^2 - H_{\text{eff}}^2)$, and z_\pm^2 are the two roots of the equation $P_\theta = 0$ when substituting $\cos \theta = z$ into P_θ , and they are given by

$$z_- = \cos \theta_{\min}, \quad (3.40)$$

$$z_+ = \frac{L_z^2 + Q + \beta + \sqrt{(L_z^2 + Q + \beta)^2 - 4\beta Q}}{2\beta}. \quad (3.41)$$

In this way, Schmidt derived the following expressions [68] for the polar integrals:

$$T = \frac{1}{\beta z_+} K(k), \quad (3.42a)$$

$$U = \frac{z_+}{\beta} [K(k) - E(k)], \quad (3.42b)$$

$$V = \frac{z_+}{\beta} [\Pi(z_-^2, k) - K(k)], \quad (3.42c)$$

where $k = z_-^2/z_+^2$, and $K(k)$, $E(k)$, and $\Pi(z_-^2, k)$ are, respectively, the complete elliptical integrals of the first, second, and third kind,

$$K(k) = \int_0^{\pi/2} \frac{d\chi}{\sqrt{1 - k \sin^2 \chi}}, \quad (3.43a)$$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k \sin^2 \chi} d\chi, \quad (3.43b)$$

$$\Pi(z_-^2, k) = \int_0^{\pi/2} \frac{d\chi}{(1 - z_-^2 \sin^2 \chi) \sqrt{1 - k \sin^2 \chi}}. \quad (3.43c)$$

The solutions of the above systems of equations for $-\frac{\partial H}{\partial H_{\text{eff}}}$ and $\frac{\partial H}{\partial J_k}$ are given by

$$-\frac{\partial H}{\partial H_{\text{eff}}} = \frac{K(k)W + a^2 z_+^2 E[K(k) - E(k)]X}{K(k)Y + a^2 z_+^2 [K(k) - E(k)]X}, \quad (3.44)$$

$$\frac{\partial H}{\partial J_r} = \frac{\pi K(k)}{K(k)Y + a^2 z_+^2 [K(k) - E(k)]X}, \quad (3.45)$$

$$\frac{\partial H}{\partial J_\theta} = \frac{\pi \beta z_+ X}{2\{K(k)Y + a^2 z_+^2 [K(k) - E(k)]X\}}, \quad (3.46)$$

$$\frac{\partial H}{\partial J_\phi} = \frac{K(k)Z + L_z [\Pi(z_-^2, k) - K(k)]X}{K(k)Y + a^2 z_+^2 [K(k) - E(k)]X}. \quad (3.47)$$

From Eqs. (3.29) and (3.44)–(3.47), the coordinate-time frequencies ω_r , ω_θ , and ω_ϕ are obtained,

$$\omega_r = \frac{\pi K(k)}{K(k)W + a^2 z_+^2 E[K(k) - E(k)]X}, \quad (3.48)$$

$$\omega_\theta = \frac{\pi \beta z_+ X}{2\{K(k)W + a^2 z_+^2 E[K(k) - E(k)]X\}}, \quad (3.49)$$

$$\omega_\phi = \frac{K(k)Z + L_z [\Pi(z_-^2, k) - K(k)]X}{K(k)W + a^2 z_+^2 E[K(k) - E(k)]X}. \quad (3.50)$$

The above equations have the same forms as the test particle ones given in Ref. [68]. However, the mass-ratio corrections have to be encoded in each variable. To explore the influence of the mass ratio on these frequencies, we demonstrate the relative frequency shift $(\omega(\nu) - \omega_0)/\omega_0$ (where the subscript 0 means the test-particle case $\nu \rightarrow 0$) with varied orbital parameters.

In Fig. 3, we can see that the three frequency shifts due to the mass-ratio corrections are one order larger than the mass ratio, especially when p approaches p_s (the edge of the last stable orbit). The effective spin terms we omitted at the beginning of this section have much smaller magnitudes. This proves again that the approximation we made is reasonable for EMRIs.

Figure 4 demonstrates the relations of frequency shifts due to mass ratios with eccentricities. Interestingly, the azimuthal and polar frequency shifts increase when the eccentricity grows, but the radial frequency error is not sensitive to the eccentricity [about $O(\nu)$]. When e becomes extreme, the shift decreases.

The variation of frequency shifts also depends on the orbital inclination. In Fig. 5, for small semilatus rectum, the frequency errors due to the mass ratio grow very fast when the orbital inclination i increases (i.e., $\theta_{\min} \rightarrow 0$). However,

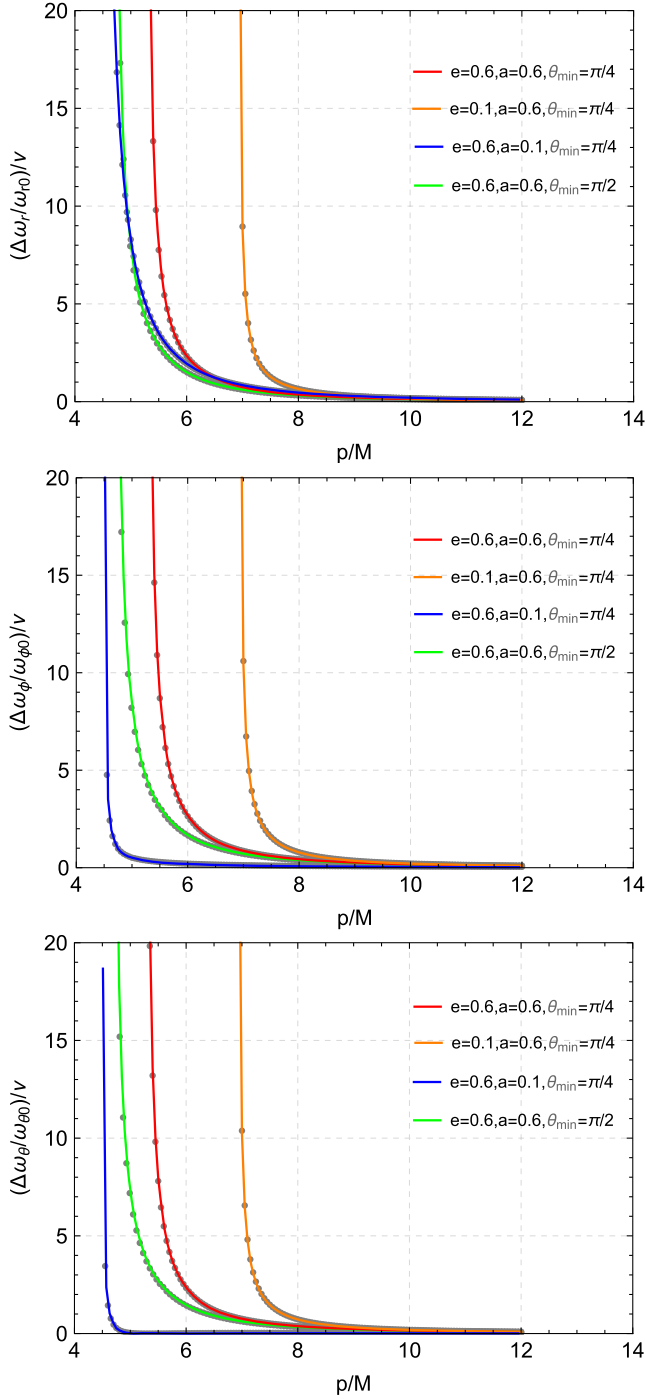


FIG. 3. Frequency shifts $\frac{\Delta\omega}{\nu\omega_0}$ vs semilatus rectum p in the cases of various a, ν, e, θ_{\min} . The solid line and points represent $\nu = 10^{-4}$ and 10^{-6} , respectively.

for large p , the frequency shifts are not very sensitive to the orbital inclination.

Finally, Fig. 6 shows the frequency shifts vs the effective Kerr parameter a (approaches to the Kerr parameter of the central black hole when $\nu \rightarrow 0$). When a becomes smaller, the last stable orbit (LSO) is farther away from the black hole (i.e., p_s becomes larger). Then, for the fixed $p = 6$, the

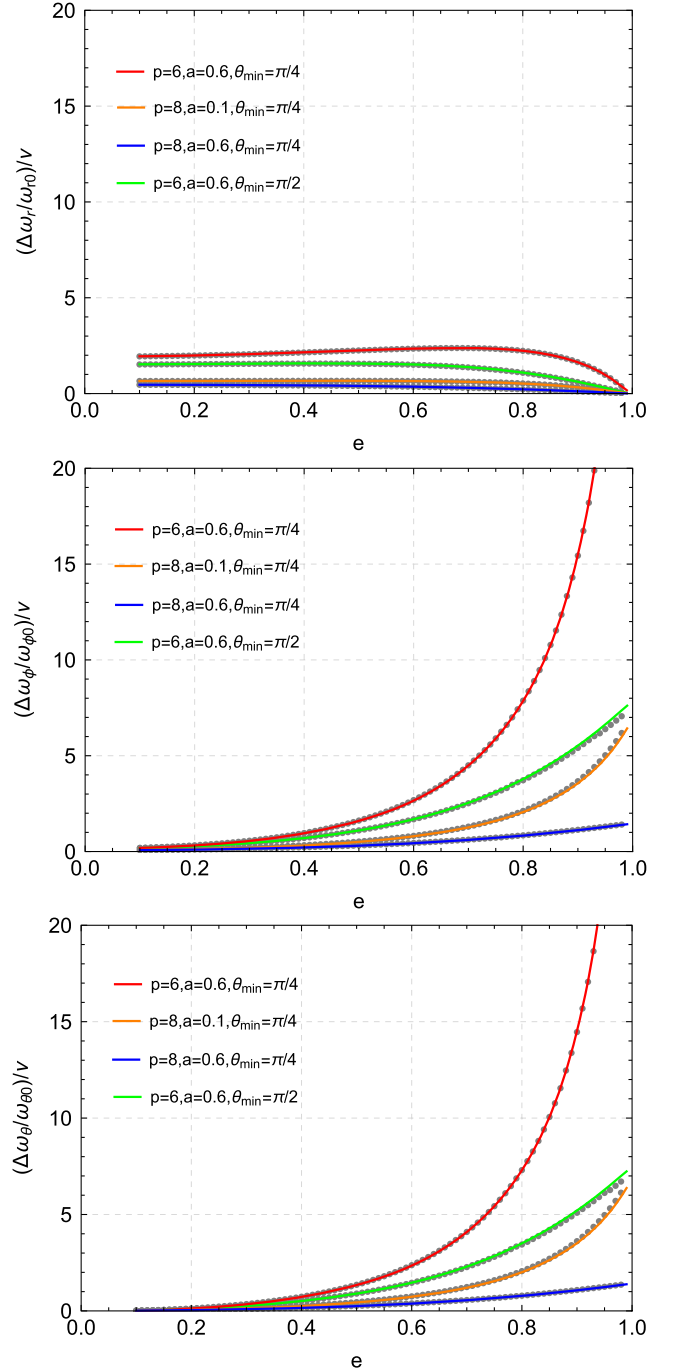


FIG. 4. Frequency shifts $\frac{\Delta\omega}{\nu\omega_0}$ vs eccentricity e in the cases of various a, ν, p, θ_{\min} . The solid line and points represent $\nu = 10^{-4}$ and 10^{-6} , respectively.

orbit is closer to the LSO as a decreases, so we can see that the frequency shifts increase. When $p = 8$, which is just a little larger than the radius of the innermost stable circular orbit ($r_{\text{isco}} = 6$ for a nonspinning BH), the dependence is no longer obvious.

All of the above results state that the mass-ratio correction has a substantial influence on the orbital frequencies,

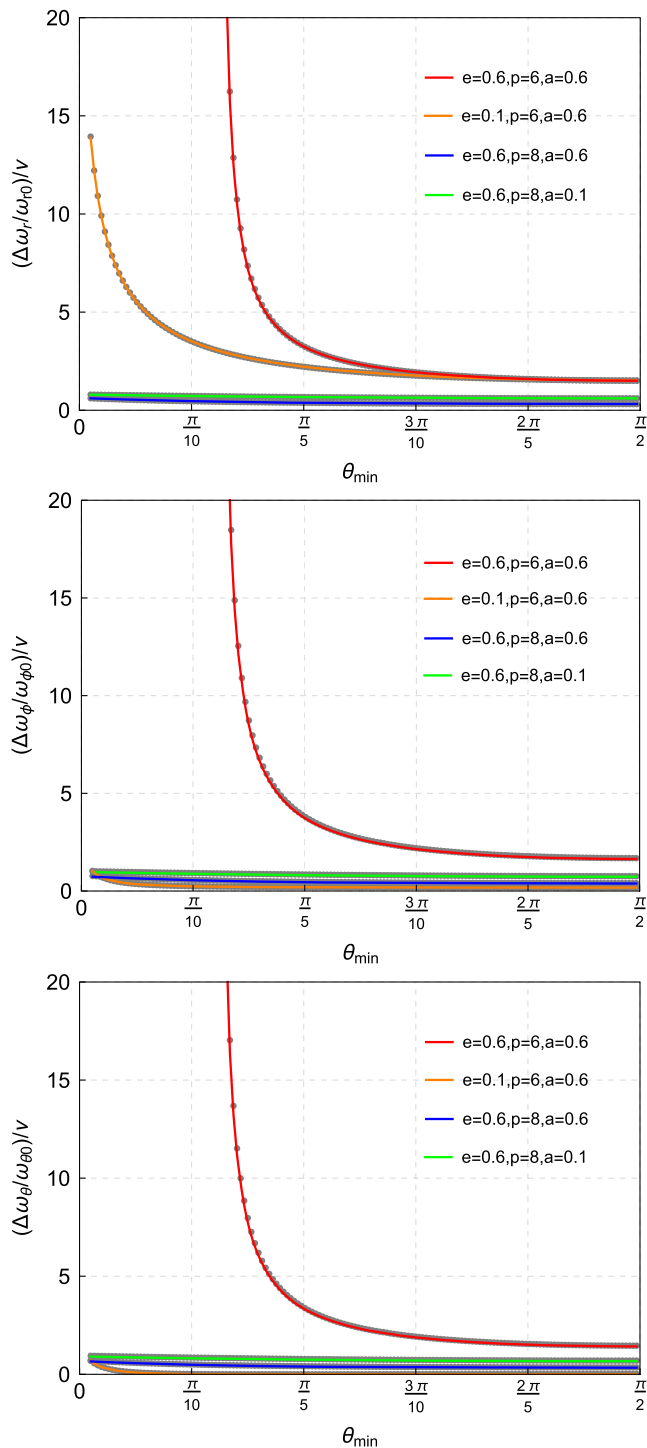


FIG. 5. Frequency shifts $\frac{\Delta\omega}{\nu\omega_0}$ vs the minimal polar angle θ_{\min} in the cases of various a , ν , p , e . The solid line and points represent $\nu = 10^{-4}$ and 10^{-6} , respectively.

even for the extreme-mass-ratio limit. This means that in the construction of the waveform templates for EMRIs, the mass ratio needs to be included in the orbital calculations. However, as we stated in the Introduction, the frequency shifts in Figs. 3–6 are not guaranteed quantitatively.

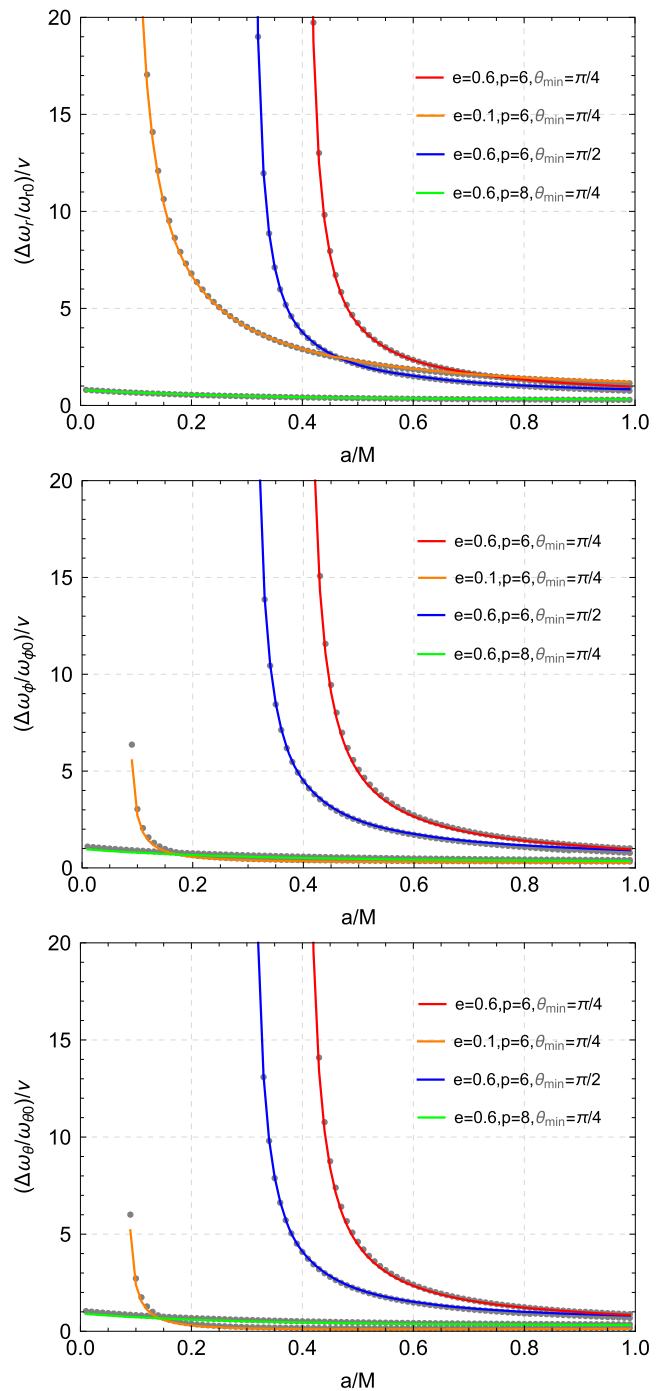


FIG. 6. Frequency shifts $\frac{\Delta\omega}{\nu\omega_0}$ vs spin a in the cases of various ν , p , e , θ_{\min} . The solid line and points represent $\nu = 10^{-4}$ and 10^{-6} , respectively.

Comparison of innermost stable circular orbit (ISCO) shifts between the EOB and gravitational self-force (GSF) in Ref. [70] shows obvious deviation, but still hints that the frequency shift is at about the same order of mass ratio, which coincides with our results in Figs. 3–6. The ISCO shifts in Ref. [70] were calculated based on an earlier version of the EOB potential; we recalculated the ISCO

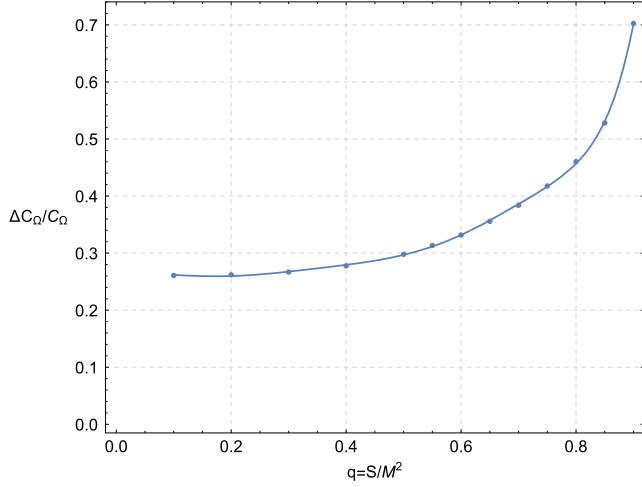


FIG. 7. Relative difference of the ISCO frequency shift $\Delta C_{\Omega}/C_{\Omega} := 1 - C_{\Omega}^{\text{EOB}}/C_{\Omega}^{\text{GSF}}$ between the EOB and GSF methods [70].

shifts based on the updated potential which is used in this work [60], and we found that the results improve a lot (see Fig. 7). If the spin of a SMBH is not extreme (below 0.8, based on Fig. 7), the EOB’s ISCO frequency shifts exhibit less than a 50% difference from the GSF ones. This may support our results in Figs. 3–6 which qualitatively state that the influence of the mass ratio on the conservative dynamics of EMRIs cannot be ignored and give qualitative magnitudes. In other words, the test-particle approximation may not be enough for the EMRI waveform simulation, and we expect more accurate self-force corrections.

D. Orbits of conservative dynamics

We find evolution equations for r , ϕ , and χ with the forms

$$\dot{r} = \frac{\partial E}{\partial P_r} = -\frac{g^{rr}\hat{P}_r}{E/M(g^{tt}\hat{H}_{\text{eff}} - g^{t\phi}\hat{L}_z)}, \quad (3.51a)$$

$$\dot{\phi} = \frac{\partial E}{\partial P_{\phi}} = \frac{g^{t\phi} - [g^{tt}g^{\phi\phi} - (g^{t\phi})^2]\frac{\hat{L}_z}{g^{tt}\hat{H}_{\text{eff}} - g^{t\phi}\hat{L}_z}}{g^{tt}E/M}, \quad (3.51b)$$

$$\dot{\theta} = \frac{\partial E}{\partial P_{\theta}} = -\frac{g^{\theta\theta}\hat{P}_{\theta}}{E/M(g^{tt}\hat{H}_{\text{eff}} - g^{t\phi}\hat{L}_z)}, \quad (3.51c)$$

where \hat{P}_r , \hat{P}_{θ} have been analytically obtained in Eqs. (3.17a)–(3.17b), and \hat{H}_{eff} , \hat{L}_z have been given in Eq. (3.23). The metric components are Eqs. (2.2a)–(2.2e). Due to the definitions of ξ , χ , all the variables in terms of r and θ can be transfer to the functions of ξ , χ . Finally, the above equations can now be expressed in terms of only the variables (ξ, χ) and orbital parameters (p, e, θ_{\min}) or (p, e, ι) ,

$$\begin{aligned} \dot{\xi} &= -\frac{(1 + e \cos \xi)^2}{epM \sin \xi} \frac{g^{rr}\hat{P}_r}{E/M(g^{tt}\hat{H}_{\text{eff}} - g^{t\phi}\hat{L}_z)} \\ &= \Xi(p, e, \iota, \xi, \chi), \end{aligned} \quad (3.52a)$$

$$\dot{\chi} = -\frac{g^{\theta\theta} \sqrt{(a^2(1 - \hat{H}_{\text{eff}}^2))(z_+^2 - z_-^2 \cos^2 \chi)}}{E/M(g^{tt}\hat{H}_{\text{eff}} - g^{t\phi}\hat{L}_z)} = \Theta(p, e, \iota, \xi, \chi), \quad (3.52b)$$

$$\dot{\phi} = \frac{g^{t\phi} - [g^{tt}g^{\phi\phi} - (g^{t\phi})^2]\frac{\hat{L}_z}{g^{tt}\hat{H}_{\text{eff}} - g^{t\phi}\hat{L}_z}}{g^{tt}E/M} = \Phi(p, e, \iota, \xi, \chi). \quad (3.52c)$$

The detailed expressions of Ξ , Θ , and Φ can be directly obtained, but they are too long to be written here. Solving the above ordinary differential equations by numerical integration, we can get ξ , χ , and ϕ associated with coordinate time t , i.e., the orbital motion. Projecting the Boyer-Lindquist coordinates onto a spherical coordinate grid, we can define the corresponding Cartesian coordinate system,

$$\tilde{x} = \frac{p \cos \phi \sqrt{1 - z_-^2 \cos^2 \chi}}{1 + e \cos \xi}, \quad (3.53a)$$

$$\tilde{y} = \frac{p \sin \phi \sqrt{1 - z_-^2 \cos^2 \chi}}{1 + e \cos \xi}, \quad (3.53b)$$

$$\tilde{z} = \frac{pz_- \cos \chi}{1 + e \cos \xi}. \quad (3.53c)$$

Combining the equations of the motion (3.52a)–(3.52c), we can plot the orbits of conservative dynamics in the Cartesian coordinates (see Fig. 8).

Like the zoom-whirl orbits [71] in r - ϕ components, from Fig. 8 there are similar zoom-whirl phenomena in the r - θ directions. Different orbital parameters with a corresponding prograde separatrix have different polar angular periods. We can imagine that as the orbit gradually approaches the separatrix, the test particle will spend a more considerable amount of its orbital “life” (in both the azimuthal and polar directions) close to the periastron.

E. Evolution of orbital parameters under radiation reaction

The description of geodesic motion around BHs is based on the semilatus rectum p , the eccentricity e , and the orbital inclination ι , together with three phase variables associated with the spatial geometry of the radial, azimuthal, and polar motion denoted by (ξ, ϕ, χ) . ξ was already defined by expressing the radial motion as

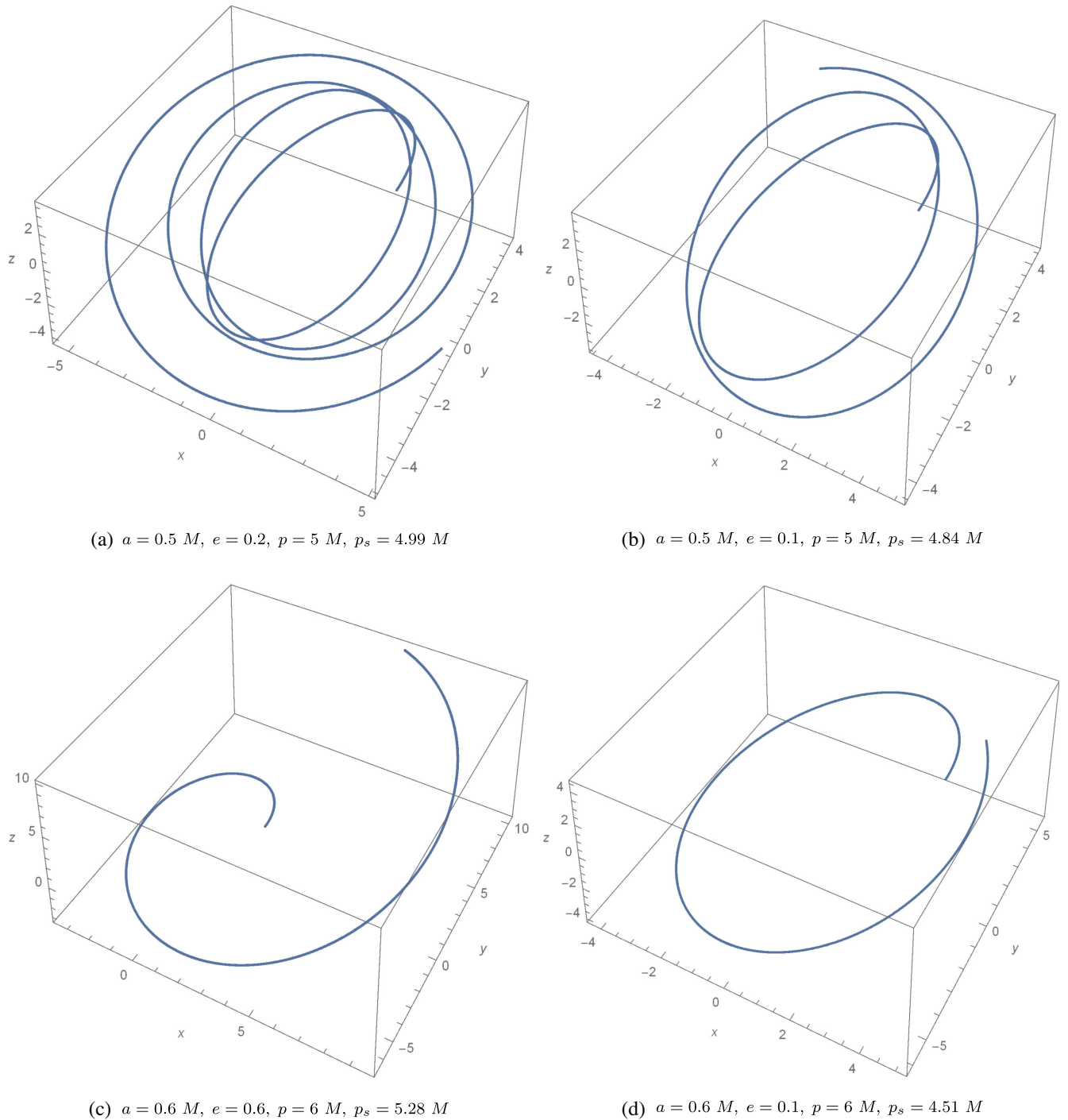


FIG. 8. Four orbits of different initial parameters. We set $\nu = 10^{-4}$, $\theta_{\min} = \pi/4$ in all four orbits and just plot half of the orbital period from the apastron to the periastron.

$$r = \frac{pM}{1 + e \cos \xi}. \quad (3.54)$$

$$\dot{\xi} = \frac{(1 + e \cos \xi)^2}{epM \sin \xi} \dot{r} - \frac{1 + e \cos \xi}{ep \sin \xi} \dot{p} + \frac{\cot \xi}{e} \dot{e}. \quad (3.55)$$

Taking the derivative of Eq. (3.54), we get the evolution equation for the phase variable ξ ,

For a conservative system, $\dot{p} = \dot{e} = 0$. However, if we take the radiation reaction of GWs into account, the rate of change of the energy, the reduced angular momentum, and the Carter constant are given by

$$\frac{dE}{dt} = \frac{\partial E}{\partial p} \dot{p} + \frac{\partial E}{\partial e} \dot{e} + \frac{\partial E}{\partial \iota} \dot{\iota}, \quad (3.56a)$$

$$\frac{d\hat{L}_z}{dt} = \frac{\partial \hat{L}_z}{\partial p} \dot{p} + \frac{\partial \hat{L}_z}{\partial e} \dot{e} + \frac{\partial \hat{L}_z}{\partial \iota} \dot{\iota} \quad (3.56b)$$

$$\frac{d\hat{Q}}{dt} = \frac{\partial \hat{Q}}{\partial p} \dot{p} + \frac{\partial \hat{Q}}{\partial e} \dot{e} + \frac{\partial \hat{Q}}{\partial \iota} \dot{\iota}. \quad (3.56c)$$

From Eqs. (3.56a)–(3.56c), we express the evolution of (p, e, ι) using the energy and angular momentum fluxes of gravitational radiation,

$$\dot{p} = c_{(\hat{L}_z, \hat{Q})(e, \iota)} \frac{dE}{dt} + c_{(E, \hat{Q})(\iota, e)} \frac{d\hat{L}_z}{dt} + c_{(E, \hat{L}_z)(e, \iota)} \frac{d\hat{Q}}{dt}, \quad (3.57a)$$

$$\dot{e} = c_{(\hat{L}_z, \hat{Q})(\iota, p)} \frac{dE}{dt} + c_{(E, \hat{Q})(p, \iota)} \frac{d\hat{L}_z}{dt} + c_{(E, \hat{L}_z)(\iota, p)} \frac{d\hat{Q}}{dt}, \quad (3.57b)$$

$$\dot{\iota} = c_{(\hat{L}_z, \hat{Q})(p, e)} \frac{dE}{dt} + c_{(E, \hat{Q})(e, p)} \frac{d\hat{L}_z}{dt} + c_{(E, \hat{L}_z)(p, e)} \frac{d\hat{Q}}{dt}, \quad (3.57c)$$

where the coefficients are given by

$$c_{(C_1, C_2)(o_1, o_2)} = \frac{\left[\frac{\partial C_1}{\partial o_1} \frac{\partial C_2}{\partial o_2} - \frac{\partial C_1}{\partial o_2} \frac{\partial C_2}{\partial o_1} \right]}{\left[\frac{\partial E}{\partial \iota} \frac{\partial \hat{L}_z}{\partial p} - \frac{\partial E}{\partial p} \frac{\partial \hat{L}_z}{\partial \iota} \right] \frac{\partial \hat{Q}}{\partial e} + \left[\frac{\partial E}{\partial e} \frac{\partial \hat{L}_z}{\partial \iota} - \frac{\partial E}{\partial \iota} \frac{\partial \hat{L}_z}{\partial e} \right] \frac{\partial \hat{Q}}{\partial p} + \left[\frac{\partial E}{\partial p} \frac{\partial \hat{L}_z}{\partial e} - \frac{\partial E}{\partial e} \frac{\partial \hat{L}_z}{\partial p} \right] \frac{\partial \hat{Q}}{\partial \iota}}, \quad (3.57d)$$

where $C = \{E, \hat{L}_z, \hat{Q}\}$ and $o = \{p, e, \iota\}$, and the derivatives can be computed from the expressions in Eqs. (3.18) and (3.23). We do not write the complete expressions here because they can be calculated quite directly but are very long.

Once we have the GW fluxes \dot{E} , $\dot{\hat{L}}_z$, and $\dot{\hat{Q}}$, the orbital evolution can be obtained. We have no plan to introduce the detailed fluxes in the present paper, and thus we do not calculate the specific orbital evolution here. We will leave this task to the next paper on gravitational waveforms. In this work, we just focus on the geometrization of the EOB formalism in extreme-mass-ratio cases.

The final set of EOB equations of motion with radiation reaction are Eqs. (3.57a)–(3.57b) together with the evolution of the phases described by Eqs. (3.55) and (3.57c), and the radius of motion at any arbitrary moment is given by Eq. (3.51a). Now all of the equations of motion are expressed in terms of only the geometric parameters (p, e, ι, ξ, χ) and the effective Kerr parameter a .

IV. CONCLUSIONS AND OUTLOOK

In the present paper, based on the EOB deformed metric and Hamiltonian, we gave the geometrized formalism of the equations of motion for the inclined-eccentric EMRIs with spinning black holes. The solutions were derived with the geometric parameters p , e , and ι instead of the EOB coordinates and momentum. The fundamental properties of the motion due to the mass ratio and black hole spin were discussed. We also gave expressions for three orbital frequencies ω_r , ω_θ , and ω_ϕ . With these formalisms in hand, it is convenient to obtain the motion of a compact object around a supermassive black hole with the orbital parameters p , e , ι .

Our results show that the influence on the orbital motion due to the small compact object's gravitational self-force on

the background of a SMBH cannot be ignored. The analytical formalism in this work makes the inclusion of the mass ratio in the motion much more intuitive. Although we did not give a waveform template in the present work, we believe that our analytical method (not the formalisms themselves) should be a useful way to build waveform models in the future for EMRIs to replace the test-particle approximation, which is used in popular waveform models.

In the present work, a few approximations have been used. As we mentioned before, in the present model we temporarily omitted the effective spin of the small object. In the EOB theory, this spin of the effective test particle is $\sim \mu a/M$ even if the small object does not rotate. The omission of this term will only induce a relative error of the Hamiltonian at least two orders lower than the mass ratio. Furthermore, for decoupling the equations of motion, we used the approximation $F(r, \theta) \approx F(r)$. This usually induces an error at $O(10^{-2})\nu$ order, even at the edge of the LSO, the error still $\lesssim 0.1\nu$. The analysis of these approximations performed here showed that the errors could be ignored for EMRIs due to the very small mass ratio (see Table I). By encoding the mass-ratio correction in the Hamiltonian H_{NS} , our expressions may be an improvement compared to the test-particle approximation.

However, as stated in the previous sections, the EOB formalism's description in the extreme-mass-ratio limit does not get guarantee. Considering that the comparison of the ISCO shifts with the gravitational self-force has obvious deviations for the extreme spin cases, we can only state that our detailed results qualitatively reveal the influences of the mass ratio on the conservative dynamics. Fortunately, our analytical technology presented in this paper can be used for any similar deformed Kerr metrics. Once there is an updated version of the EOB resummation approximation, the EOB corrections in the extreme-mass-ratio limit can be improved or guaranteed, our results can be

easily updated by replacing the EOB potentials, and then we can get an accurate analytical formalism with mass-ratio corrections for the EMRIs.

One of the scientific targets of EMRIs is to detect the spacetime geometry of a SMBH. For this target, an accurate and efficient waveform template is needed. However, this is still a challenge. The analytical orbital solution including the mass ratio, eccentricity, and orbital inclination given in this paper is more accurate than the test-particle model and more convenient than the original EOB equations for inclined-eccentric orbits over a long-term evolution.

Due to the analytical frequencies, a combination with the frequency-domain Teukolsky equation [8] will be more convenient and can generate numerical waveforms. In the future, we will use the formalisms in this work to generate the orbital evolution and waveforms for EMRIs.

ACKNOWLEDGMENTS

This work is supported by NSFC No. 11773059, and we also appreciate the anonymous referee's suggestions about our work.

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