Double null coordinates for Kerr spacetime

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(Received 19 May 2021; accepted 28 June 2021; published 19 July 2021)

We present a definition for a pair of null coordinates that are naturally adapted to the horizons and future null infinity of Kerr spacetime, and that are generated by the center-of-mass sections at future null infinity. They are a smooth, round family of null hypersurfaces which foliate the Kerr spacetime in an outgoing and an ingoing sense, respectively, and they have a regular extension across the horizons. Because of Kerr's peculiar geometry, the construction involves a nonlinear differential equation for a scalar function related to Carter's constant, whose solution cannot be expressed in terms of simple analytic functions. We present the numerical solution of this scalar for a particular choice of the geometrical parameters. In this setting, there naturally appears a two-dimensional spacelike family of round surfaces S_{r_s} that are parametrized by r_s, which are the intersections of both null coordinates, where r_s can be thought of as the tortoise coordinate extension for the Kerr spacetime. The S_{r_s} surfaces are axially symmetric, but they have an (r, θ) dependence in Boyer-Lindquist coordinates. They can also be characterized in a complete geometrical way by their Gaussian and extrinsic curvature scalars, which we were able to compute by the use of Geroch-Held-Penrose formalism. We compare our definition with other previous attempts in the literature, and we show that all of them have divergent behavior at the axis of symmetry. Thus, our construction presents the first double null coordinate system which makes possible computations over all of the Kerr spacetime.

DOI: 10.1103/PhysRevD.104.024049

I. INTRODUCTION

The significance of null coordinates can be traced back to the beginnings of general relativity. The first solution of the Einstein-Hilbert equations was discovered for a vacuum and spherical symmetric spacetime by Schwarzschild very early in 1916 [1]. But it took several decades to understand the true meaning of the coordinate singularity $r = 2m$ in terms of the original coordinate system. The issue was clarified in the works of Kruskal [2] and Szekeres [3], where the use of null coordinates was essential in the process to remove the coordinate singularity. Moreover, such definitions allowed a complete understanding of the causal structure and the most important feature of a black hole spacetime, the event horizon.

Null coordinates also opened a broad spectrum of studies, including Vaydia spacetimes, Hawking radiation, and the stability of black holes. Nevertheless, Schwarzschild spacetimes are very restrictive and cannot model the final state of a generic black hole.

This inevitably leads to the study of other vacuum solutions that could account for a final angular momentum content. This was achieved by the Kerr solution [4]. Although presented more than fifty years ago, it is still the subject of interesting studies and discoveries. At present, the Kerr geometry has acquired renewed relevance, since it is used to model the spacetime associated with the first observed picture of a black hole, which corresponds to a supermassive black hole in M87 [5–10] with a mass of $6.5 \times 10^9\ M_\odot$ and a favored angular momentum parameter of about $a = 0.94M$, although other astrophysical studies [11] set this relation to $a = 0.98M$.

The general understanding [12] is that these metrics are the general final stage for dynamically evolving isolated black holes, so they are the natural candidates to model black holes with angular momentum as they settle down to a stationary state. It is for these reasons that the Kerr spacetimes are continuously studied and new properties are regularly presented in the literature.

The possibility to generalize the previous knowledge gained for the Schwarzschild case to this axis-symmetric spacetime makes the calculation of null coordinates for Kerr geometry an interesting and necessary subject. However, it has also been remarkably elusive.

It is an interesting question, since in its construction one can grasp the details of the geometry encoded in the Kerr metric. It is a necessary and useful construction for the discussion and calculation of characteristic problems in the Kerr geometry, from a theoretical and numerical point of view. It is elusive, since before this work, there were no presentations of complete, round, smooth null coordinates that enfold the horizons in a regular way.

For spacetimes with spherical symmetry, like Schwarzschild or Reissner-Nordström, null coordinates are well known. It is advantageous that in those cases, principal null congruences do not have twist, so it is possible to define null coordinates adapted to such congruences, like Edington-Finkelstein and Kruskal-Skeres for Schwarzschild. But in Kerr geometry, with less symmetry, the ingredients to define null coordinates are not that easy to find. In this case, principal null congruences have twist, so it is not possible to define null coordinates adapted to them.

The use of functions that are null at certain points in Kerr spacetime can be traced to Carter's work [13]. In this way, he was able to study the causal structure of Kerr geometry at the axis of symmetry. In many textbooks like Refs. [14] or [15], inspired by principal null congruences, Carter's construction is reproduced. But it must be noted that such constructions cannot be extended as null coordinates to other spacetime regions, due to the presence of twist in the principal null directions. Moreover, all compactified causal diagrams related to this construction are incomplete, since they are only valid at the axis of symmetry of Kerr spacetime.

In the literature, there are several works which deal with the construction of null coordinates in Kerr spacetime. Some remarkable ones are Refs. [16–19], each of which contributes from different approaches with a characteristic point of view. But as we will discuss at the end of this manuscript, in spite of all these remarkable contributions, such definitions still have serious problems with the presence of divergent behaviors. Therefore, they become useless for application purposes, failing at the main task of any coordinate system.

In this work, we solve this situation by presenting a double null coordinate system for Kerr spacetime. These null functions surround the black hole in a smooth manner and fill the spacetime, and coincide at future null infinity with the *center-of-mass sections* [20,21]. In the interior of the black hole, they are also smooth and can be extended all the way up to the interior horizons, and across them too, up to the region containing Kerr's ring singularity.

This article is organized as follows: In Sec. II, we define the new double null coordinates for Kerr spacetime, starting from the most general null geodesic congruence. We also give the integral expressions of each null coordinate function u and v, together with a plot that compares Kerr's and Schwarzschild's outgoing null functions. In Sec. III, we present the null tetrad adapted to the new double null coordinates. To obtain a deeper geometric perspective, we compute the spin coefficients and the Weyl scalars that give geometric details of the null congruences used in our definition. In Sec. IV, we present the surface family S_{r_s} as the intersection of both null coordinates u and v; where r_s can be interpreted as the Kerr extension of the Schwarzschild's tortoise coordinate. We give a complete geometric description of such two-dimensional spacelike surfaces in terms of their Gaussian and extrinsic curvature scalars. In Sec. V, we express the Kerr metric in terms of the null coordinates u and v. We also define a new angular coordinate and the extended versions of the null coordinates, which we call U and V. These extensions allow the crossing of the past and future event horizon in a regular way. In Sec. VI, we compare our definition and results with other previous attempts found in the literature. We show that in all those works, there is divergent behavior over the axis of symmetry, and we explain why such definitions fail as candidates for coordinate systems and therefore are unsuitable to be used in many studies of interest. Instead, with our definition, one obtains a regular behavior, opening a broad spectrum for possible applications. In Sec. VII, we give final comments summarizing important aspects of our contribution.

II. NULL COORDINATES DEFINITION

A. Basic construction

To make our notation explicit, let us begin by writing the Kerr line element and its inverse using Boyer-Lindquist [22] coordinates:

$$ds^2 = (1-\Phi)dt^2 + 2\Phi a \sin^2(\theta) dt d\phi - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 - \frac{\Upsilon}{\Sigma}\sin^2(\theta)d\phi^2, \tag{1}$$

$$\left(\frac{\partial}{\partial s}\right)^2 = \frac{\Upsilon}{\Sigma\Delta}\left(\frac{\partial}{\partial t}\right)^2 + \frac{4amr}{\Sigma\Delta}\left(\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial \phi}\right) - \frac{\Delta}{\Sigma}\left(\frac{\partial}{\partial r}\right)^2 - \frac{1}{\Sigma}\left(\frac{\partial}{\partial \theta}\right)^2 - \frac{\Delta - a^2\sin^2(\theta)}{\Sigma\Delta\sin^2(\theta)}\left(\frac{\partial}{\partial \phi}\right)^2, \tag{2}$$

with

$$\Sigma = r^2 + a^2\cos^2(\theta), \qquad \Delta = r^2 + a^2 - 2mr,$$
$$\Upsilon = (r^2+a^2)^2 - \Delta a^2 \sin^2(\theta), \qquad \Phi = \frac{2mr}{\Sigma}, \tag{3}$$

where the parameter m denotes the mass, and the angular momentum of the black holes is given by $J = am$.

Since the principal null congruences have twist, they do not help in the search for natural null hypersurfaces. We have to start by considering all possible null geodesics. It is well known [14,23] that the most general null geodesics can be put in terms of first-order derivatives with respect to an affine parameter λ:

$$\frac{dt}{d\lambda} = \dot{t} = \frac{1}{\Sigma\Delta}[E\Upsilon - 2amrL_z], \tag{4}$$

$$\frac{dr}{d\lambda} = \dot{r} = \pm_{oi} \frac{\sqrt{[(r^2+a^2)E - aL_z]^2 - K\Delta}}{\Sigma}, \qquad (5)$$

$$\frac{d\theta}{d\lambda} = \dot{\theta} = \pm\frac{1}{\Sigma}\sqrt{K - \left[Ea\sin(\theta) - \frac{L_z}{\sin(\theta)}\right]^2}, \qquad (6)$$

$$\frac{d\phi}{d\lambda} = \dot{\phi} = \frac{1}{\Sigma\Delta}\left[2amrE + (\Sigma - 2mr)\frac{L_z}{\sin^2(\theta)}\right]. \qquad (7)$$

Then the most general null geodesic congruence for the Kerr spacetime can be expressed in terms of its tangent vector

$$V^a = \dot{t}\left(\frac{\partial}{\partial t}\right)^a + \dot{r}\left(\frac{\partial}{\partial r}\right)^a + \dot{\theta}\left(\frac{\partial}{\partial \theta}\right)^a + \dot{\phi}\left(\frac{\partial}{\partial \phi}\right)^a. \qquad (8)$$

All the steps of our definition will be shorter and simpler if we work with the one form V_a, which in Boyer-Lindquist coordinates is

$$\begin{aligned} V_a &= g_{ab}V^b \\ &= Edt_a - \dot{r}\frac{\Sigma}{\Delta}dr_a - \dot{\theta}\Sigma d\theta_a - L_z d\phi_a \\ &= Edt_a - \frac{\pm_{oi}\sqrt{[(r^2+a^2)E - aL_z]^2 - K\Delta}}{\Delta}dr_a \\ &\quad - \left(\pm\sqrt{K - \left[Ea\sin(\theta) - \frac{L_z}{\sin(\theta)}\right]^2}\right)d\theta_a \\ &\quad - L_z d\phi_a, \end{aligned} \qquad (9)$$

where E, L_z, and K (the Carter constant) are conserved quantities along each geodesic.

The sign $\pm_{oi}$ determines the character of the congruence. We will use ℓ_a (with $\pm_{oi} = +$) to denote the most general outgoing null congruence, and $\mathfrak{n}_a$ (with $\pm_{oi} = -$) to denote the ingoing null congruence. For the purpose of simpler presentation, below we will present our definition considering the outgoing ℓ_a, but all the steps and main results can also be obtained for the ingoing one $\mathfrak{n}_a$.

In what follows, without loss of generality, we will take $E = 1$, in the outer region. At each point of the spacetime, the choice of the constants L_z and K singles out a point in the sphere of directions.

This raises the question of how one can choose the "constants" L_z and K so that *locally* they define a hypersuface orthogonal null congruence with the properties that we want—that is, although they are constant along each null geodesic, we are free to choose them with different values for each geodesic. The guiding idea is that ℓ_a must be an exact differential, but we also demand the congruence to be orthogonal to a sphere at future null infinity, which coincides with the center-of-mass section.

As a first step, let us consider a surface S_r, defined by (t = constant and r = constant), with tangent vectors $(\frac{\partial}{\partial\theta})^b$ and $(\frac{\partial}{\partial\phi})^b$. We are interested in the limiting case, where this surface tends to a sphere at infinity, $S_r \to S_\infty$. In our approach, we select the congruence ℓ^a which is orthogonal to S_∞—that is, that

$$\begin{aligned} \lim_{r\to\infty} g_{ab}\ell^a\left(\frac{\partial}{\partial\theta}\right)^b &= 0, \\ \lim_{r\to\infty} g_{ab}\ell^a\left(\frac{\partial}{\partial\phi}\right)^b &= 0. \end{aligned} \qquad (10)$$

From Eqs. (9) and (10), we obtain for each geodesic $L_z = 0$ and $K = a^2\sin(\theta^*)^2$, where $\theta^* = \theta|_{r=\infty}$. These conditions fix the congruence completely, and since it started orthogonal to a topological 2-sphere, it is hypersurface orthogonal.

It is convenient to mention that the center-of-mass sections which motivate our definition can be obtained as the limit of known coordinates as one approaches future null infinity. The retarded version of the original Kerr coordinate [4,14] can be defined by $d\tilde{u} = dt - \frac{r^2+a^2}{\Delta}dr$, although we note that this is not a null coordinate. Then, one can check that the limit where $\tilde{u}$ = constant, $r \to \infty$ goes to the center-of-mass sections S_∞ at future null infinity [20].

At an interior point of the spacetime with coordinates (r,θ), the quantity K will pick the value from the corresponding null geodesic passing through this point. Then, one can think in terms of the functional relation $K(r,\theta)$. This allows one to change the logic and ask for the condition of $K(r,\theta)$, so that ℓ^a is a hypersurface orthogonal outgoing null congruence—that is, without twist—which also reaches future null infinity with $K = a^2\sin(\theta^*)^2$. Let us note that we are requesting $L_z = 0$ and $\dot{r} > 0$ for this congruence.

The sign for $\dot{\theta}$ is chosen by thinking on the behavior of spheroidal coordinates (close to a sphere) in the limit as one approaches future null infinity following an outgoing null geodesic, so that we take (+) for the northern hemisphere and (−) for the southern hemisphere, and we will express this by $\pm|_h$. Therefore, we have

$$\begin{aligned} \ell_a = dt_a &- \frac{\sqrt{(r^2+a^2)^2 - K(r,\theta)\Delta}}{\Delta}dr_a \\ &- \pm|_h\sqrt{K(r,\theta) - a^2\sin(\theta)^2}\,d\theta_a, \end{aligned} \qquad (11)$$

$$\begin{aligned} \mathfrak{n}_a = dt_a &+ \frac{\sqrt{(r^2+a^2)^2 - K(r,\theta)\Delta}}{\Delta}dr_a \\ &\pm|_h\sqrt{K(r,\theta) - a^2\sin(\theta)^2}\,d\theta_a. \end{aligned} \qquad (12)$$

The condition for ℓ_a to be hypersurface orthogonal (without twist) is equivalent to that for being the differential of a null function u (outgoing)—that is,

$$(du)_a = \ell_a. \tag{13}$$

Therefore, the exterior derivative of ℓ_a must vanish—namely,

$$(d\ell_a)_b \equiv \left[\frac{1}{2\sqrt{(r^2+a^2)^2 - K\Delta}}\frac{\partial K}{\partial \theta} d\theta \wedge dr \pm|_h \frac{1}{2\sqrt{K-(a\sin(\theta))^2}}\frac{\partial K}{\partial r} d\theta \wedge dr\right] = 0, \tag{14}$$

which one can also express as

$$\sqrt{(r^2+a^2)^2 - K\Delta}\frac{\partial K}{\partial r} \pm \bigg|_h \sqrt{K-(a\sin(\theta))^2}\frac{\partial K}{\partial \theta} = 0. \tag{15}$$

Thus, Eq. (15) constitutes the integrability condition for the one form ℓ_a to be the differential of a null function that we call u.

The natural question arises: Are the solutions of Eq. (15) consistent with the property that K must be constant along each null geodesic? To answer this, let us just calculate the derivative of K with respect to each affine parameter—namely,

$$\frac{dK}{d\lambda} = \frac{\partial K}{\partial r}\dot{r} + \frac{\partial K}{\partial \theta}\dot{\theta} = \frac{\sqrt{(r^2+a^2)^2 - K\Delta}}{\Sigma}\frac{\partial K}{\partial r} \pm|_h \frac{\sqrt{K-(a\sin(\theta))^2}}{\Sigma}\frac{\partial K}{\partial \theta} = 0. \tag{16}$$

It can be seen, then, that by imposing Eq. (15), one guarantees that K is constant along each null geodesic of the congruence.

The same result [Eq. (15)] is obtained when one considers the ingoing null congruence $\mathfrak{n}_a$ [Eq. (12)] together with both conditions of our definition: the asymptotic [Eq. (10)] and local [Eq. (14)] ones. In such case, the null function is called v (ingoing), where $(dv)_a = \mathfrak{n}_a$.

B. Solving for the function $K(r,\theta)$

The structure of Eq. (15) suggests that we work with the auxiliary function k defined from

$$K(r,\theta) = a^2\sin(\theta)^2 + k^2(r,\theta) \tag{17}$$

so that the function $K(r,\theta)$ can be expressed in terms of $k(r,\theta)$. We also use the variable

$$\xi = \frac{1}{r} \tag{18}$$

for the computation. One can see then that the boundary condition

$$\lim_{r\to\infty} K = a^2\sin(\theta^*)^2 \tag{19}$$

is equivalent to

$$\lim_{\xi\to 0} k = 0. \tag{20}$$

In terms of these new variables, Eq. (15) becomes

$$-2\xi^2 k\frac{\partial k}{\partial \xi} = -\frac{(\pm|_h\xi^2|k|)(2a^2\sin(\theta)\cos(\theta) + 2k\frac{\partial k}{\partial\theta})}{\sqrt{(1+\xi^2a^2)^2 - \xi^4 K\Delta}}, \tag{21}$$

where $\frac{\partial K}{\partial r} = -2\xi^2 k\frac{\partial k}{\partial \xi}$. Such an equation is invariant under the exchange $k\to -k$ with the boundary condition in Eq. (20).

Note that in the northern hemisphere, $\sin(\theta)\cos(\theta)\geq 0$, so after we start at $\xi = 0$ with $k = 0$, we can elect the sign of k. If we assume $k\geq 0$, we have to solve

$$\frac{\partial k}{\partial \xi} = \frac{a^2\sin(\theta)\cos(\theta) + k\frac{\partial k}{\partial\theta}}{\sqrt{(1+\xi^2a^2)^2 - \xi^4\Delta(a^2\sin(\theta)^2 + k^2)}}. \tag{22}$$

Otherwise, if we assume $k\leq 0$ in the northern hemisphere, we have to use Eq. (22) with opposite sign. Note that the sign of k does not interfere with the sign of $\dot{\theta}$. For simplicity, we assume $k\geq 0$ in the northern hemisphere, in the vicinity of future null infinity. Let us note that with this choice, we simply have

$$\pm|_h\sqrt{K-(a\sin(\theta))^2} = k. \tag{23}$$

It is important to remark that from an analytical point of view, the differential equation (22), together with its boundary condition [Eq. (20)], can be integrated over the entire spacetime, since it is well behaved everywhere for every value of the (r,θ) coordinates where they make sense. This means that one can integrate the equation even at the horizon (where $\Delta = 0$), at the interior horizons, and across them, too, up to the region containing the Kerr ring singularity.

Some important properties of k and K can be obtained from the differential equation (22), together with its boundary condition (20). We know that at infinity ($k_{|_{\xi=0}} = 0$); therefore, from Eq. (22) ($\partial_\xi k_{|_{\xi=0,\theta=0,\pi}} = 0$), we find that k remains zero as one integrates over ξ. Then, at the axis of symmetry, we have

$$k(r,\theta)|_{\theta=0,\pi} = 0, \qquad K(r,\theta)|_{\theta=0,\pi} = 0. \tag{24}$$

In the same way, we can see that at the equator

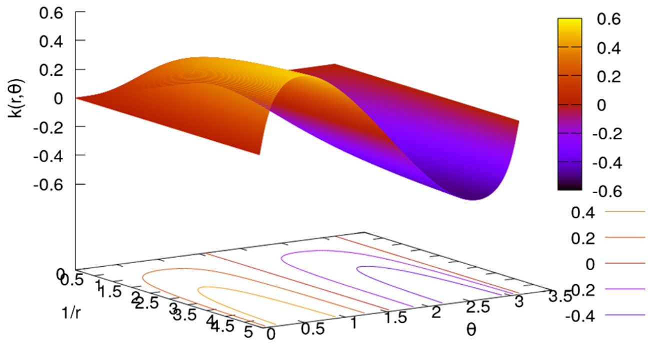

FIG. 1. Numerical solution $k(r,\theta)$, with $\xi \in [0, \frac{2}{r_-}]$. Using parameters $m = 1$, $a = 0.8$.

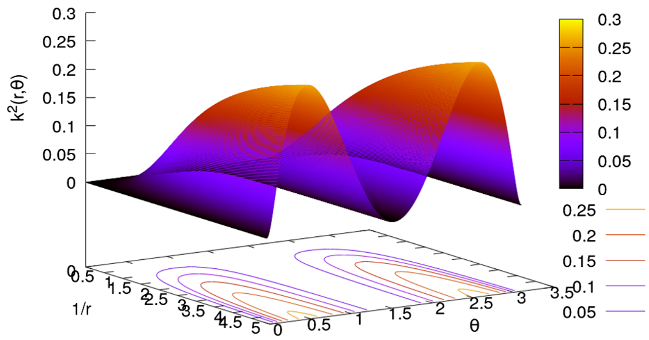

FIG. 2. Numerical solution $k^2(r,\theta)$, with $\xi \in [0, \frac{2}{r_-}]$. Using parameters $m = 1$, $a = 0.8$.

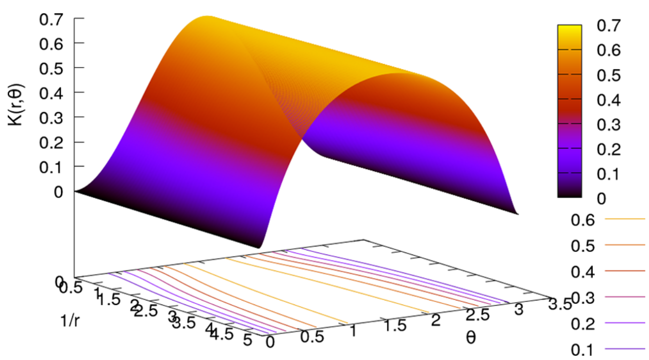

FIG. 3. Numerical solution $K(r,\theta)$, with $\xi \in [0, \frac{2}{r_-}]$. Using parameters $m = 1$, $a = 0.8$.

$$k(r,\theta)|_{\theta=\frac{\pi}{2}} = 0, \qquad K(r,\theta)|_{\theta=\frac{\pi}{2}} = a^2. \tag{25}$$

To obtain a solution, we have to solve a nonlinear differential equation. Unfortunately, the solution to Eq. (22) cannot be expressed in terms of elementary functions, like trigonometric functions mixed with powers of r or ξ. For this reason, we have integrated this equation numerically, which allows us to present the results by means of computed graphs. This also has the advantage of showing explicitly that our construction is well behaved for all

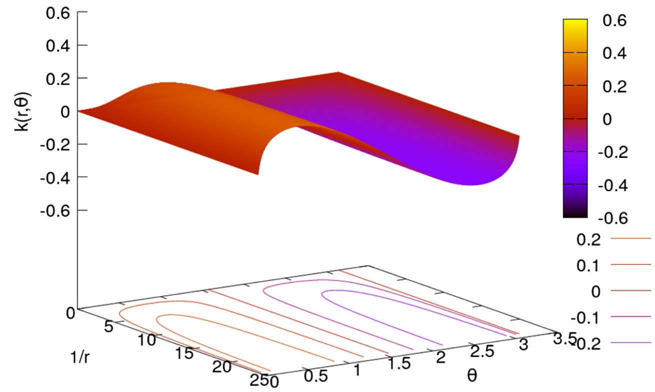

FIG. 4. Numerical solution $k(r,\theta)$, with $\xi \in [0, \frac{2}{r_-}]$. Using parameters $m = 1$, $a = 0.4$.

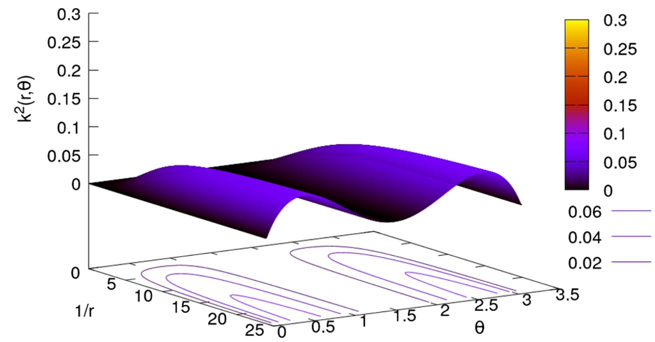

FIG. 5. Numerical solution $k^2(r,\theta)$, with $\xi \in [0, \frac{2}{r_-}]$. Using parameters $m = 1$, $a = 0.4$.

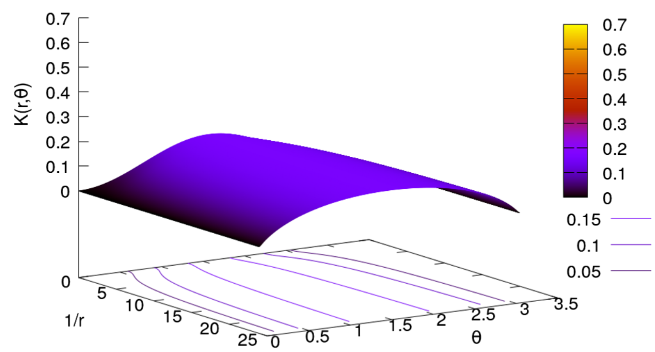

FIG. 6. Numerical solution $K(r,\theta)$, with $\xi \in [0, \frac{2}{r_-}]$. Using parameters $m = 1$, $a = 0.4$.

regions of the spacetime, and all values of the angular coordinate θ.

The numerical scheme was implemented with fourth-order finite difference approximations for angular derivatives, and a fourth-order Runge-Kutta method to integrate along ξ. The residual error was computed with output values of order 1×10^{-14} (close to double-precision rounding error).

The solution can be seen in Figs. 1–6 for different quotient relations (a/m). One can clearly see how the functions k and K tend to zero as one reduces the angular momentum parameter ($a \to 0$).

C. Null functions u and v

From the knowledge of k, we can define the null functions u (outgoing) and v (ingoing) as the hypersurfaces generated by the outgoing ℓ_a and ingoing $\mathfrak{n}_a$ null geodesics congruences, respectively; namely,

$$du = dt - \frac{\sqrt{(r^2+a^2)^2 - K\Delta}}{\Delta} dr - \pm|_h \sqrt{\Theta(r,\theta)} d\theta, \tag{26}$$

$$dv = dt + \frac{\sqrt{(r^2+a^2)^2 - K\Delta}}{\Delta} dr \pm |_h \sqrt{\Theta(r,\theta)} d\theta, \tag{27}$$

where $K = K(r,\theta)$ is a solution of Eq. (15), and

$$\Theta(r,\theta) = K(r,\theta) - a^2 \sin^2(\theta). \tag{28}$$

To obtain integrated expressions of u and v, we need to integrate Eqs. (26) and (27). Because they are very similar, for simplicity we will work with u, but the same process follows for v.

We can integrate du along any curve $\gamma(s')$, which connects an initial point $(t_0, r_0, \theta_0, \phi_0)$ to a final point (t, r, θ, ϕ). Given a curve γ, with $s' \in [s_0, s]$, one can express

$$\begin{aligned} u(t,r,\theta,\phi) - u_0(t_0,r_0,\theta_0,\phi_0) &= t - t_0 \\ &- \int_{s_0}^{s} \frac{\sqrt{(r'^2+a^2)^2 - \Delta K(r',\theta'')}}{\Delta} \frac{dr'}{ds'} ds' \\ &- \int_{s_0}^{s} \pm|_h \sqrt{\Theta(r'',\theta')} \frac{d\theta'}{ds'} ds'. \end{aligned} \tag{29}$$

As we know, the difference $u(t,r,\theta,\phi) - u_0(t_0,r_0,\theta_0,\phi_0)$ only depends on the initial and final points. Nevertheless, to proceed and find a final computable expression, it is useful to consider an integration path. Because both integrands depends only on two coordinates (r,θ), we have two natural paths to connect the initial and final points. One of them is $[(r_0;\theta_0) \to (r;\theta_0) \to (r;\theta)]$; in that case, Eq. (29) becomes

$$\begin{aligned} u(t,r,\theta,\phi) - u_0(t_0,r_0,\theta_0,\phi_0) &= t - t_0 \\ &- \int_{r_0}^{r} \frac{\sqrt{(r'^2+a^2)^2 - \Delta K(r',\theta_0)}}{\Delta} dr' \\ &- \int_{\theta_0}^{\theta} \pm|_h \sqrt{\Theta(r,\theta')} d\theta'. \end{aligned} \tag{30}$$

The other one is $[(r_0;\theta_0) \to (r_0;\theta) \to (r;\theta)]$; in that case, Eq. (29) becomes

$$\begin{aligned} u(t,r,\theta,\phi) - u_0(t_0,r_0,\theta_0,\phi_0) &= t - t_0 \\ &- \int_{r_0}^{r} \frac{\sqrt{(r'^2+a^2)^2 - \Delta K(r',\theta)}}{\Delta} dr' \\ &- \int_{\theta_0}^{\theta} \pm|_h \sqrt{\Theta(r_0,\theta')} d\theta'. \end{aligned} \tag{31}$$

In what follows, we work with both expressions. Because they are very similar, for simplicity we show the complete process only for Eq. (30), but the same final result follows from Eq. (31). We start with the behavior analysis of the second term's integrand at ($\Delta \approx 0$):

$$\begin{aligned} &\frac{\sqrt{(r'^2+a^2)^2 - \Delta K(r',\theta_0)}}{\Delta} \\ &= \frac{(r'^2+a^2)}{\Delta} - \frac{K}{2(r'^2+a^2)} - \frac{K^2\Delta}{8(r'^2+a^2)^3} + \mathcal{O}(\Delta^2). \end{aligned} \tag{32}$$

From Eq. (32), it is clear that Eq. (30) has a divergent term at ($\Delta \approx 0$). We can isolate such behavior in one simpler term, by simply adding and subtracting $(r'^2+a^2)/\Delta$, to obtain

$$\begin{aligned} &u(t,r,\theta,\phi) - u_0(t_0,r_0,\theta_0,\phi_0) \\ &= t - t_0 - \int_{r_0}^{r} \left(\frac{(r'^2+a^2)}{\Delta}\right) dr' \\ &- \int_{r_0}^{r} \left(\frac{\sqrt{(r'^2+a^2)^2 - \Delta K(r',\theta_0)}}{\Delta} - \frac{(r'^2+a^2)}{\Delta}\right) dr' \\ &- \int_{\theta_0}^{\theta} \pm|_h \sqrt{\Theta(r,\theta')} d\theta', \end{aligned} \tag{33}$$

where the divergent term can be integrated analytically:

$$\begin{aligned} \int_{r_0}^{r} \left(\frac{(r'^2+a^2)}{\Delta}\right) dr' = \Bigg[r &+ \frac{r_+^2+a^2}{r_+ - r_-} \ln\left(\frac{r}{r_+} - 1\right) \\ &- \frac{r_-^2+a^2}{r_+ - r_-} \ln\left(\frac{r}{r_-} - 1\right) \Bigg] \Bigg|_{r_0}^{r}, \end{aligned} \tag{34}$$

where r_+ and r_- are the solutions of $\Delta = 0$—namely, $r_+ = m + \sqrt{m^2 - a^2}$ and $r_- = m - \sqrt{m^2 - a^2}$.

Then, we have an equivalent expression of Eq. (30):

$$\begin{aligned} &u(t,r,\theta,\phi) - u_0(t_0,r_0,\theta_0,\phi_0) \\ &= t - r - \left(\frac{r_+^2+a^2}{r_+ - r_-} \ln\left(\frac{r}{r_+} - 1\right) - \frac{r_-^2+a^2}{r_+ - r_-} \ln\left(\frac{r}{r_-} - 1\right)\right) \\ &- \int_{r_0}^{r} \left(\frac{\sqrt{(r'^2+a^2)^2 - \Delta K(r',\theta_0)}}{\Delta} - \frac{(r'^2+a^2)}{\Delta}\right) dr' \\ &- \int_{\theta_0}^{\theta} \pm|_h \sqrt{\Theta(r,\theta')} d\theta' + C_0(t_0, r_0), \end{aligned} \tag{35}$$

where C_0 is a constant. The strategy is to elect

$$u_0(t_0, r_0, \theta_0, \phi_0) = -C_0(t_0, r_0) \tag{36}$$

in such a way that for any initial point $(t_0, r_0, \theta_0, \phi_0)$,

$$u(t,r,\theta,\phi) = t - r - \left(\frac{r_+^2 + a^2}{r_+ - r_-}\ln\left(\frac{r}{r_+} - 1\right) - \frac{r_-^2 + a^2}{r_+ - r_-}\ln\left(\frac{r}{r_-} - 1\right)\right) - \int_{r_0}^{r}\left(\frac{\sqrt{(r'^2 + a^2)^2 - \Delta K(r', \theta_0)}}{\Delta} - \frac{(r'^2 + a^2)}{\Delta}\right)dr' - \int_{\theta_0}^{\theta} \pm|_h \sqrt{\Theta(r, \theta')}d\theta'. \tag{37}$$

The final step in obtaining the coordinate $u(t, r, \theta, \phi)$ is to elect the initial point $(t_0, r_0, \theta_0, \phi_0)$. We are interested in taking its value at future null infinity—that is, at $r_0 \to \infty$, $t_0 \to \infty$, $\theta_0 \to \theta_\infty$, $\phi_0 \to \phi_\infty$, where θ_∞ and ϕ_∞ are finite. Note that in Eq. (37), there is no dependence on ϕ_0, so we are free to choose ϕ_∞ without any change in u. Also, it can be noted that C_0 is finite in this limit. Then, since we can still use Eq. (36), we have

$$u(t,r,\theta,\phi) = t - r - \left(\frac{r_+^2 + a^2}{r_+ - r_-}\ln\left(\frac{r}{r_+} - 1\right) - \frac{r_-^2 + a^2}{r_+ - r_-}\ln\left(\frac{r}{r_-} - 1\right)\right) - \int_{r_\infty}^{r}\left(\frac{\sqrt{(r'^2 + a^2)^2 - \Delta K(r', \theta_\infty)}}{\Delta} - \frac{(r'^2 + a^2)}{\Delta}\right)dr' - \int_{\theta_\infty}^{\theta} \pm|_h \sqrt{K(r, \theta') - a^2\sin(\theta')^2}d\theta'. \tag{38}$$

In the same way, we can start with the second natural integration path [Eq. (31)] and repeat all the steps to obtain

$$u(t,r,\theta,\phi) = t - r - \left(\frac{r_+^2 + a^2}{r_+ - r_-}\ln\left(\frac{r}{r_+} - 1\right) - \frac{r_-^2 + a^2}{r_+ - r_-}\ln\left(\frac{r}{r_-} - 1\right)\right) - \int_{r_\infty}^{r}\left(\frac{\sqrt{(r'^2 + a^2)^2 - \Delta K(r', \theta)}}{\Delta} - \frac{(r'^2 + a^2)}{\Delta}\right)dr' - \int_{\theta_\infty}^{\theta} \pm|_h \sqrt{K(r_\infty, \theta') - a^2\sin(\theta')^2}d\theta'. \tag{39}$$

Note that when $a \to 0$, we have $K \to 0$. Then the last two terms in Eq. (38) and (39) become zero, because each integrand is zero. Also, from (34), when $a \to 0$, we have $r_- \to 0$, so we can simply take $a = 0$ and $r_- = 0$. In this way, we recover Eddington's outgoing null function for Schwarzschild spacetime:

$$\lim_{a\to 0} u(t, r, \theta, \phi) = t - \left(r + 2m\ln\left(\frac{r}{2m} - 1\right)\right). \tag{40}$$

D. Shorter expressions for *u* and *v*

We have already mentioned that there are two natural integration paths [Eqs. (30) and (31)]. In this part, we will show how these paths allow us to find shorter expressions for u and v.

Remember that the first path is $[(r_0; \theta_0) \to (r; \theta_0) \to (r; \theta)]$, and that we locate the initial point at $r_0 = r_\infty$, $\theta_0 = \theta_\infty$. To obtain a shorter expression, we can take advantage of the election freedom and choose $(\theta_0 = \theta_\infty = 0)$. Then, we have to consider the property [Eq. (24)]

$$K(r, \theta = 0) = 0 \tag{41}$$

to obtain the shortest expression for u; by substituting Eq. (41) into Eq. (38) and using the properties $(r_+^2 + a^2 = 2mr_+)$ and $(r_-^2 + a^2 = 2mr_-)$, we find

$$u(t,r,\theta,\phi) = t - r - \left(\frac{2mr_+}{r_+ - r_-}\ln\left(\frac{r}{r_+} - 1\right) - \frac{2mr_-}{r_+ - r_-}\ln\left(\frac{r}{r_-} - 1\right)\right) - \int_0^{\theta} \pm|_h \sqrt{K(r, \theta') - a^2\sin(\theta')^2}d\theta'. \tag{42}$$

The second natural path is $[(r_0;\theta_0) \to (r_0;\theta) \to (r;\theta)]$, with the initial point at $r_0 = r_\infty$, $\theta_0 = \theta_\infty$. To obtain a shorter expression, we can take advantage of the election freedom and choose $(r_0 = r_\infty = \infty)$. Then, we have to consider another property that comes from the boundary condition [Eq. (19)]:

$$K(r = \infty, \theta') = a^2 \sin(\theta')^2 \tag{43}$$

to obtain another short expression. By substituting Eq. (43) into Eq. (39), we find

$$u(t,r,\theta,\phi) = t - r - \left(\frac{2mr_+}{r_+ - r_-}\ln\left(\frac{r}{r_+} - 1\right) - \frac{2mr_-}{r_+ - r_-}\ln\left(\frac{r}{r_-} - 1\right)\right) - \int_\infty^r \left(\frac{\sqrt{(r'^2 + a^2)^2 - \Delta K(r',\theta)}}{\Delta} - \frac{(r'^2 + a^2)}{\Delta}\right) dr'. \tag{44}$$

In an analogous way, we can start with Eq. (27) to obtain the corresponding expressions for $v(t,r,\theta,\phi)$. The expressions are

$$v(t,r,\theta,\phi) = t + r + \left(\frac{2mr_+}{r_+ - r_-}\ln\left(\frac{r}{r_+} - 1\right) - \frac{2mr_-}{r_+ - r_-}\ln\left(\frac{r}{r_-} - 1\right)\right) + \int_0^\theta \pm|_h \sqrt{K(r,\theta') - a^2\sin(\theta')^2}\, d\theta' \tag{45}$$

and

$$v(t,r,\theta,\phi) = t + r + \left(\frac{2mr_+}{r_+ - r_-}\ln\left(\frac{r}{r_+} - 1\right) - \frac{2mr_-}{r_+ - r_-}\ln\left(\frac{r}{r_-} - 1\right)\right) + \int_\infty^r \left(\frac{\sqrt{(r'^2 + a^2)^2 - \Delta K(r',\theta)}}{\Delta} - \frac{(r'^2 + a^2)}{\Delta}\right) dr'. \tag{46}$$

Then, we can compute u with Eqs. (42) or (44) and v with Eqs. (45) or (46) to obtain the same result. This has to do with the path-independent character of the difference $u(t,r,\theta,\phi) - u_0(t_0,r_0,\theta_0,\phi_0)$, which we integrate from the beginning [Eq. (29)]. To see this in more detail, note that the slightly longer expressions for u [Eq. (44)] and v [Eq. (46)] have the same value for any initial angle at the asymptotic two-dimensional sphere $S_{r=\infty}$—that is, $\forall \theta_\infty \in [0,\pi]$, $\forall \phi_\infty \in [0,2\pi]$. Meanwhile, the shortest expressions for u [Eq. (42)] and v [Eq. (45)] are obtained by the election $\theta_0 = \theta_\infty = 0$.

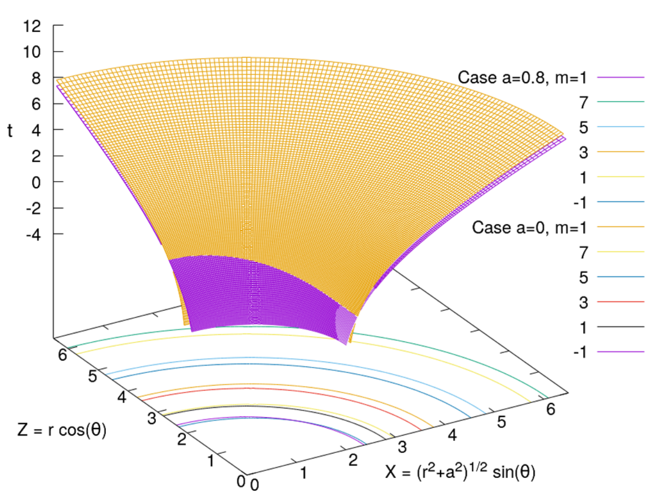

FIG. 7. Center-of-mass null surface ($u = 0$) for Kerr and Schwarzschild spacetimes.

E. Plot of u

Figure 7 shows the Kerr center-of-mass null surface when $u = 0$, with angular parameter values $a = 0.8$ and $a = 0$, and so includes the Schwarzschild case for comparison. Note that this comparison is only about the functional dependence of $u(t,r,\theta)$ with respect to the coordinates (t,r,θ).

We have computed numerically both expressions (42) and (44). We corroborate that they have the same value up to machine rounding error, 1×10^{-15}.

III. ADAPTED NULL TETRAD

A. The tetrad

From Eqs. (26) and (27), we have two null directions $(du)_a = \ell_a$, $(dv)_a = \mathfrak{n}_a$ to build a null tetrad. As usual, we have to use a normalized direction to satisfy Eq. (58):

$$n_a = \frac{1}{2}\frac{\Sigma\Delta}{\Upsilon}\mathfrak{n}_a. \tag{47}$$

Then, we can compute the spin coefficients in the Geroch-Held-Penrose (GHP) [24] notation to make a complete geometric analysis.

In what follows, from Eqs. (23) and (28), we can simplify our notation by taking

$$\pm|_h\sqrt{\Theta} = k. \tag{48}$$

Then, the *center-of-mass null tetrad* in its covariant form is

$$\ell_a = dt_a - \frac{\sqrt{\mathcal{R}}}{\Delta} dr_a - kd\theta_a, \tag{49}$$

$$n_a = \frac{1}{2}\frac{\Sigma\Delta}{\Upsilon}\left(dt_a + \frac{\sqrt{\mathcal{R}}}{\Delta} dr_a + kd\theta_a\right), \tag{50}$$

$$m_a = i\frac{\sqrt{2}amr\sin(\theta)}{\sqrt{\Upsilon\Sigma}} dt_a + k\sqrt{\frac{\Sigma}{2\Upsilon}} dr_a - \sqrt{\frac{\Sigma\mathcal{R}}{2\Upsilon}} d\theta_a - i\sqrt{\frac{\Upsilon}{2\Sigma}}\sin(\theta) d\phi_a, \tag{51}$$

and in its contravariant form, it is

$$\ell^a = \frac{\Upsilon}{\Sigma\Delta}\partial_t^a + \frac{\sqrt{\mathcal{R}}}{\Sigma}\partial_r^a + \frac{k}{\Sigma}\partial_\theta^a + \frac{2amr}{\Delta\Sigma}\partial_\phi^a, \tag{52}$$

$$n^a = \frac{\partial_t^a}{2} - \frac{\sqrt{\mathcal{R}}\Delta}{2\Upsilon}\partial_r^a - \frac{k\Delta}{2\Upsilon}\partial_\theta^a + \frac{2amr}{2\Upsilon}\partial_\phi^a, \tag{53}$$

$$m^a = -\frac{k\Delta}{\sqrt{2\Upsilon\Sigma}}\partial_r^a + \sqrt{\frac{\mathcal{R}}{2\Upsilon\Sigma}}\partial_\theta^a + i\sqrt{\frac{\Sigma}{2\Upsilon\sin^2(\theta)}}\partial_\phi^a, \tag{54}$$

where

$$\mathcal{R} = (r^2 + a^2)^2 - K\Delta, \tag{55}$$

$$\Upsilon = \mathcal{R} + \Theta\Delta = \mathcal{R} + k^2\Delta. \tag{56}$$

It can be verified that the null tetrad relations are satisfied:

$$\ell^a\ell_a = n^a n_a = m^a m_a = \bar{m}^a\bar{m}_a = 0, \tag{57}$$

$$\ell^a n_a = 1, \tag{58}$$

$$m^a\bar{m}_a = -1; \tag{59}$$

where $\bar{m}$ is obtained from m by the exchange of i for $-i$. We have preserved the traditional notation for each null tetrad element, where m_a and $\bar{m}_a$ must not be confused with the spacetime mass parameter m.

B. The spin coefficients

For the spin coefficients, we use the GHP notation of Ref. [24]. We will focus on those spin coefficients which capture the most relevant congruence features. For example, κ_{GHP} indicates if the congruence is geodesic ($\kappa_{\mathrm{GHP}} = 0$), and ρ indicates if it is hypersurface orthogonal ($\rho = \bar{\rho}$). The extra notation of GHP in the scalar function κ_{GHP} is used to clearly distinguish it with respect to a parameter that will be used later.

The spin coefficient κ_{GHP} is given by

$$\kappa_{\mathrm{GHP}} = \frac{\sqrt{2}}{2}\frac{\sqrt{\Upsilon}}{\Sigma^{\frac{3}{2}}\Delta}\left(\frac{\partial}{\partial\theta}\sqrt{\mathcal{R}} - \Delta\frac{\partial k}{\partial r}\right). \tag{60}$$

It can be seen that in our case $\kappa_{\mathrm{GHP}} = 0$, because of Eq. (15). Such a result is consistent with our definition, because we start with a geodesic null congruence.

Then, we have ρ:

$$\rho = -\frac{1}{2\Sigma}\left(\frac{\partial}{\partial r}\sqrt{\mathcal{R}} + \frac{\partial k}{\partial\theta} + \frac{\cos(\theta)}{\sin(\theta)}k\right). \tag{61}$$

Clearly it is a real quantity. It means that the congruence is hypersurface orthogonal, which is in total consistence with our definition; see Eq. (14). Note that in the poles ($\theta = 0, \pi$), the function $k(r,\theta) \to 0$; see Eq. (24). It can be shown that

$$\lim_{\theta\to 0,\pi}\frac{\cos(\theta)}{\sin(\theta)}k = \frac{\partial k}{\partial\theta}, \tag{62}$$

where $\frac{\partial k(r,\theta)}{\partial\theta}$ has a smooth behavior at the poles.

Another important spin coefficient is σ. Remember that the principal null congruences have $\sigma = 0$. But in the case of the *center-of-mass null coordinates*, its associated null tetrad has

$$\begin{aligned}\sigma = &-i\frac{am\sin(\theta)}{\Sigma^2\Upsilon}[\sqrt{\mathcal{R}}a^2r\cos(\theta)\sin(\theta) + k(2r^4 + 2a^2r^2 + (a^2mr - a^2r^2)\sin^2(\theta))]\\ &+ \frac{1}{\Upsilon\Sigma}[(2r^3 + (r+m)a^2 + (r-m)a^2\cos^2(\theta))\sqrt{\mathcal{R}} - k\Delta a^2\cos(\theta)\sin(\theta)]\\ &+ \frac{1}{4\sin(\theta)\Sigma^2 k}(-\sin(\theta)\Sigma\partial_\theta K + (2K\Sigma - 4a^2\sin^2(\theta)K + 4a^4\sin^4(\theta))\cos(\theta))\\ &+ \frac{1}{\Sigma^2\sqrt{\mathcal{R}}}\left[\frac{\Sigma\Delta\partial_\theta K}{4} - r\mathcal{R} + \frac{(r-m)K}{2} + ra^2 - r^3\right].\end{aligned} \tag{63}$$

In further discussions of this work, we will use two other spin coefficients that in GHP notation [24] are

$$\rho' = -\frac{1}{2}\frac{\Sigma\Delta}{\Upsilon}\rho, \tag{64}$$

$$\sigma' = -\frac{1}{2}\frac{\Sigma\Delta}{\Upsilon}\sigma. \tag{65}$$

All spin coefficients were computed with the tensorial manipulation software GRTensorII.

C. Weyl scalars

In this section, we will obtain the expressions of all the Weyl scalars $\{\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4\}$, with particular focus on Ψ_2, which will be useful in the next section. To accomplish such a task, we will make a tetrad rotation from the principal null tetrad to the center-of-mass null tetrad [Eqs. (52), (53), (54)].

For the principal null congruence, it is well known that the only nonzero Weyl scalar is

$$\Psi_{2p} = -\frac{m}{(r + ia\cos(\theta))^3}, \tag{66}$$

which is written in Boyer-Lindquist coordinates. Then, if we make the mentioned rotation, from general change rules under rotations, we can obtain the expressions of all of Weyl's scalars in terms of Ψ_{2p}.

1. Tetrad rotation

We start with the principal null congruence and then rotate it to obtain the center-of-mass null tetrad. The rotation is made up of a sequence of three different types of rotations, called types I, II, and III; see the Appendix of Ref. [25]. The sequence we follow is type II, type III, and then type I:

$$\hat{n}^a_{\mathrm{II}} = \mathsf{n}^a, \tag{67}$$

$$\hat{m}^a_{\mathrm{II}} = \mathsf{m}^a + \Lambda\mathsf{n}^a, \tag{68}$$

$$\hat{\ell}^a_{\mathrm{II}} = \mathsf{l}^a + \Lambda\bar{\mathsf{m}}^a + \bar{\Lambda}\mathsf{m}^a + \Lambda\bar{\Lambda}\mathsf{n}^a; \tag{69}$$

$$\hat{n}^a_{\mathrm{II.III}} = Z^{-1}\hat{n}^a_{\mathrm{II}}, \tag{70}$$

$$\hat{m}^a_{\mathrm{II.III}} = e^{is}\hat{m}^a_{\mathrm{II}}, \tag{71}$$

$$\hat{\ell}^a_{\mathrm{II.III}} = Z\hat{\ell}^a_{\mathrm{II}}; \tag{72}$$

$$\hat{n}^a_{\mathrm{I.II.III}} = \hat{n}^a_{\mathrm{II.III}} + \Gamma\bar{\hat{m}}^a_{\mathrm{II.III}} + \bar{\Gamma}\hat{m}^a_{\mathrm{II.III}} + \Gamma\bar{\Gamma}\hat{\ell}_{\mathrm{II.III}}, \tag{73}$$

$$\hat{m}^a_{\mathrm{I.II.III}} = \hat{m}^a_{\mathrm{II.III}} + \Gamma\hat{\ell}^a_{\mathrm{II.III}}, \tag{74}$$

$$\hat{\ell}^a_{\mathrm{I.II.III}} = \hat{\ell}_{\mathrm{II.III}}. \tag{75}$$

All involved coefficients are determined by the condition that after the last rotation, the center-of-mass null tetrad is obtained. If we write down everything in terms of principal null tetrad elements, following each conversion, we have

$$\ell^a = Z\mathsf{l}^a + Z\Lambda\bar{\mathsf{m}}^a + Z\bar{\Lambda}\mathsf{m}^a + Z\Lambda\bar{\Lambda}\mathsf{n}^a, \tag{76}$$

$$m^a = \Gamma Z\mathsf{l}^a + \Gamma Z\Lambda\bar{\mathsf{m}}^a + (\Gamma Z\bar{\Lambda} + e^{is})\mathsf{m}^a + \Lambda(\Gamma Z\bar{\Gamma} + e^{is})\mathsf{n}^a, \tag{77}$$

$$n^a = \Gamma\bar{\Gamma}Z\mathsf{l}^a + (\bar{\Gamma}e^{-is} + \bar{\Gamma}\Gamma Z\Lambda)\bar{\mathsf{m}}^a + \left(\frac{1}{Z} + \Gamma\bar{\Lambda}e^{-is} + \bar{\Gamma}\Lambda e^{is} + \Gamma\bar{\Gamma}Z\Lambda\bar{\Lambda}\right)\mathsf{n}^a. \tag{78}$$

Finally, each coefficient Z, Λ, Γ, e^{is} is calculated using the contraction properties of the principal null congruence elements (the only nonzero ones are $\mathsf{l}^a\mathsf{n}_a = 1$ and $\mathsf{m}^a\bar{\mathsf{m}}_a = -1$). In this way, we obtain

$$Z = \mathsf{n}_a\ell^a = \frac{\sqrt{\mathcal{R}} + r^2 + a^2}{2\Sigma}, \tag{79}$$

$$\Lambda = -\frac{\mathsf{m}_a\ell^a}{Z} = \frac{1}{Z}\frac{(k - ia\sin(\theta))}{\sqrt{2}(r + ia\cos(\theta))}, \tag{80}$$

$$\Gamma = \frac{\mathsf{n}_a m^a}{Z} = -\frac{1}{Z}\frac{\sqrt{2}}{4}\frac{(ia\sin(\theta) + k)\Delta}{\sqrt{\Upsilon\Sigma}}, \tag{81}$$

$$\begin{aligned} e^{is} &= \bar{\mathsf{m}}_a m^a - \mathsf{n}_a m^a\bar{\Lambda} \\ &= \frac{a\Delta[k\sin(\theta)(ir - a\cos(\theta))]}{\sqrt{\Upsilon\Sigma}(\sqrt{\mathcal{R}} + r^2 + a^2)} \\ &\quad + \frac{a\Delta[-ar\sin^2(\theta) + ik^2\cos(\theta)]}{\sqrt{\Upsilon\Sigma}(\sqrt{\mathcal{R}} + r^2 + a^2)}. \end{aligned} \tag{82}$$

In the work of Ref. [26] similar computations were performed to obtain a rotated arbitrary null tetrad, but it differs from our results, because in our work rotations are taken to reach the center-of-mass null tetrad.

2. Final expressions of Weyl scalars

From previous results, we can compute all Weyl scalars related to the center-of-mass null tetrad. Under such a sequence of rotations, we have

$$
\begin{aligned}
\Psi_0 &= 6Z^2 e^{2is}\Lambda^2\Psi_{2p},\\
\Psi_1 &= [3Ze^{is}\Lambda + 6\bar{\Gamma}Z^2e^{2is}\Lambda^2]\Psi_{2p},\\
\Psi_2 &= [1+6(\bar{\Gamma}Ze^{is}\Lambda + \bar{\Gamma}^2Z^2e^{2is}\Lambda^2)]\Psi_{2p},\\
\Psi_3 &= [3\bar{\Gamma} + 9\bar{\Gamma}^2Ze^{is}\Lambda + 6\bar{\Gamma}^3Z^2e^{2is}\Lambda^2]\Psi_{2p},\\
\Psi_4 &= [6\bar{\Gamma}^2 + 12\bar{\Gamma}^3Ze^{is}\Lambda + 6\bar{\Gamma}^4Z^2e^{2is}\Lambda^2]\Psi_{2p}.
\end{aligned} \tag{83}
$$

For our main purpose, we focus on

$$\Psi_2 = [1+6(\bar{\Gamma}Ze^{is}\Lambda + \bar{\Gamma}^2Z^2e^{2is}\Lambda^2)]\Psi_{2p}. \tag{84}$$

We can work out and conveniently express

$$
\begin{aligned}
\Psi_{2p} &= (\Psi_{2p})_{\mathrm{Re}} + i(\Psi_{2p})_{\mathrm{Im}}\\
&= \frac{3mra^2\cos^2(\theta) - mr^3}{\Sigma^3} + i\,\frac{ma\cos(\theta)(3r^2 - a^2\cos^2(\theta))}{\Sigma^3}
\end{aligned} \tag{85}
$$

and

$$1+6(\bar{\Gamma}Ze^{is}\Lambda + \bar{\Gamma}^2Z^2e^{2is}\Lambda^2) = 1+6(\alpha+\alpha^2), \tag{86}$$

where the real and imaginary parts of α are

$$\alpha_{\mathrm{Re}} = \frac{-Ka^2\Delta^2(k\cos(\theta) + r\sin(\theta))^2}{2\Upsilon\Sigma(\sqrt{R}+r^2+a^2)^2}, \tag{87}$$

$$\alpha_{\mathrm{Im}} = \frac{-Ka^2\Delta^2(k^2r\cos(\theta) + k\sin(\theta)(r^2 - a^2\cos(\theta)^2))}{2\Upsilon\Sigma(\sqrt{R}+r^2+a^2)^2} - \frac{Ka^2\Delta^2(a^2r\cos(\theta)^3 - a^2r\cos(\theta))}{2\Upsilon\Sigma(\sqrt{R}+r^2+a^2)^2}. \tag{88}$$

Finally, we can write a short version for each component of Ψ_2. In terms of the real and imaginary parts of Ψ_{2p} and α,

$$(\Psi_2)_{\mathrm{Re}} = (1+6\alpha_{\mathrm{Re}} + 6\alpha_{\mathrm{Re}}^2 - 6\alpha_{\mathrm{Im}}^2)(\Psi_{2p})_{\mathrm{Re}} - 12(\Psi_{2p})_{\mathrm{Im}}\alpha_{\mathrm{Im}}\left(\frac{1}{2} + \alpha_{\mathrm{Re}}\right), \tag{89}$$

$$(\Psi_2)_{\mathrm{Im}} = (1+6\alpha_{\mathrm{Re}} + 6\alpha_{\mathrm{Re}}^2 - 6\alpha_{\mathrm{Im}}^2)(\Psi_{2p})_{\mathrm{Im}} + 12(\Psi_{2p})_{\mathrm{Re}}\alpha_{\mathrm{Im}}\left(\frac{1}{2} + \alpha_{\mathrm{Re}}\right). \tag{90}$$

IV. SURFACE FAMILY S_{r_s}

Note that Eqs. (26) and (27) can be written as

$$du = dt - dr_s, \tag{91}$$

$$dv = dt + dr_s, \tag{92}$$

where

$$dr_s \equiv \frac{\sqrt{\mathcal{R}}}{\Delta}dr + k d\theta. \tag{93}$$

The intersection of both null coordinate families u and v—that is, $(du = dv = 0)$—define a family of two-dimensional spacelike surfaces S_{r_s}, where the function $r_s = (v-u)/2$ is constant. In terms of Boyer-Lindquist coordinates, such a surface $(dr_s = 0)$ is given by $r(\theta)$, which satisfies

$$\frac{dr}{d\theta} = -\frac{k\Delta}{\sqrt{\mathcal{R}}}. \tag{94}$$

Note that r_s can be interpreted as the Kerr extension of the Schwarzschild's tortoise coordinate. In particular, we can express the function r_s in the exterior region as

$$r_s(r,\theta) = r + \frac{2mr_+}{r_+ - r_-}\ln\left(\frac{r}{r_+} - 1\right) - \frac{2mr_-}{r_+ - r_-}\ln\left(\frac{r}{r_-} - 1\right) + \int_0^\theta k(r,\theta')d\theta'. \quad (95)$$

It is easy to show that the induced metric over the two-dimensional spacelike surface family S_{r_s} is given by

$$ds^2 = -\frac{\Upsilon\Sigma}{\mathcal{R}}d\theta^2 - \frac{\Upsilon}{\Sigma}\sin^2(\theta)d\phi^2, \quad (96)$$

which can be obtained directly from Eq. (104) below, by simply taking ($du = dv = 0$). Note that in Boyer-Lindquist coordinates at each surface S_{r_s}, Eq. (94) is satisfied, which also implies that at the horizons the surface S_{r_s} is given by $r = r_+, r_-$, as one could expect. But it should be remarked that Eq. (94) still makes sense even at the horizons where r_s diverges—that is, the family S_{r_s} has a smooth extension at the horizons, even though the behavior of r_s.

A convenient geometric description of such a surface family can be made in terms of their Gaussian and extrinsic curvatures, as described in the GHP [24] formalism. Then, one has

$$C_{\text{Gaussian}} = (\bar{Q}_{\text{GHP}} + Q_{\text{GHP}}), \quad (97)$$

$$C_{\text{Extrinsic}} = i(\bar{Q}_{\text{GHP}} - Q_{\text{GHP}}), \quad (98)$$

which are given in terms of the complex curvature scalar Q_{GHP}, given by

$$Q_{\text{GHP}} = \sigma\sigma' - \rho\rho' - \Psi_2 + \Lambda + \Phi_{11}, \quad (99)$$

where $\sigma, \sigma', \rho, \rho'$ are the spin coefficients in the GHP formalism [see Eqs. (61), (63), (64), and (65)], and in the Kerr spacetime one has $\Lambda = \Phi_{11} = 0$.

Figures 8 and 9 show the numerical calculation of these quantities, where their smooth nature can be inferred.

It is important to mention that in the limit case where $a = 0$ (Schwarzschild), one has

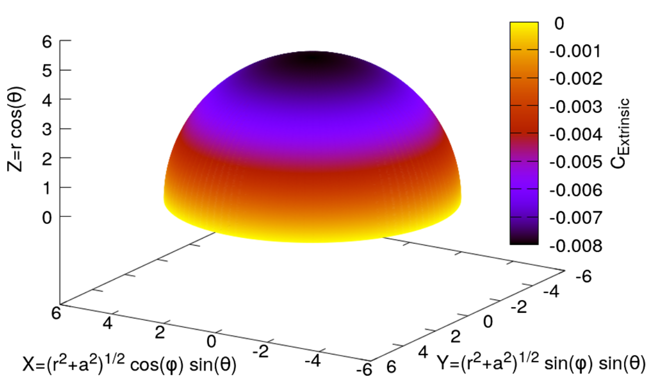

FIG. 8. Extrinsic curvature of S_{r_s}, using parameters $a = 0.8$, $m = 1.0$. Case $r(\theta) \approx 3r_+$.

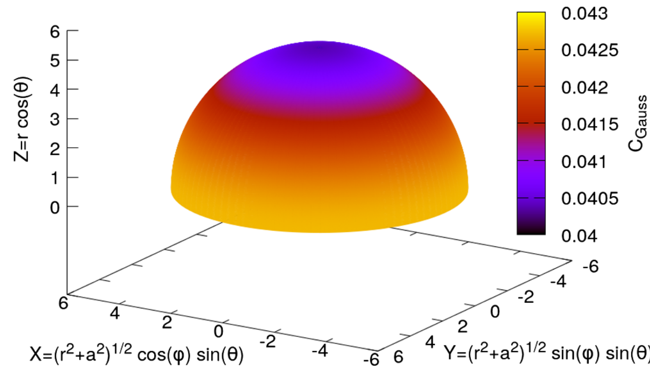

FIG. 9. Gaussian curvature of S_{r_s}, using parameters $a = 0.8$, $m = 1.0$. Case $r(\theta) \approx 3r_+$.

$$\lim_{a\to 0} C_{\text{Gauss}} = \frac{1}{r^2}, \quad (100)$$

$$\lim_{a\to 0} C_{\text{Extrinsic}} = 0, \quad (101)$$

where the two-dimensional spacelike surfaces ($dr_s = 0$) in Schwarzschild's case are given by ($dr = 0$). But in Kerr, without spherical symmetry, these curvatures are not constant (see Figs. 8 and 9). In particular, Gaussian curvature gives information about the intrinsic geometry of S_{r_s}.

It is even more interesting what happens in the interior regions, close to the horizon and the ring's singularity. In Figs. 10, 11, and 12, we plot the Gaussian curvature of S_{r_s} in those regions.

The intrinsic character of Gaussian curvature allows us to establish an analogy with two-dimensional spacelike surfaces embedded in $\mathbb{R}^3$. Far away from the horizon, $r \gg r_+$, the Gaussian curvature is almost constant (see Fig. 9), so the analogous embedded surface is almost a sphere. As one goes closer to the horizon, Gaussian curvature has lower values at the poles ($\theta = 0, \pi$) and bigger ones at the equator ($\theta = \pi/2$), so the analogous embedded surface is an oblate spheroid (see Fig. 10). In Fig. 11, we have negative values of Gaussian curvature at the poles, so the analogous

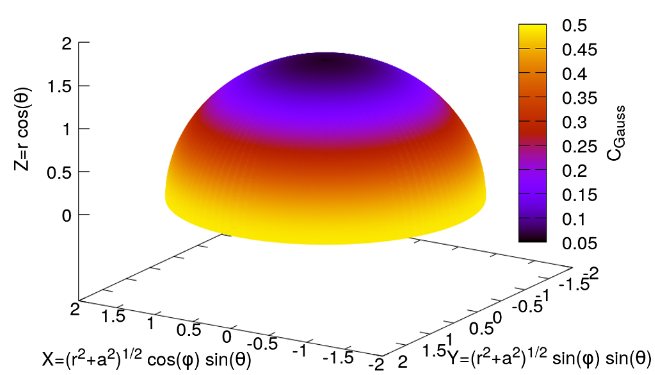

FIG. 10. Gaussian curvature of S_{r_s}, using parameters $a = 0.8$, $m = 1.0$. Case $r(\theta) \approx r_+$.

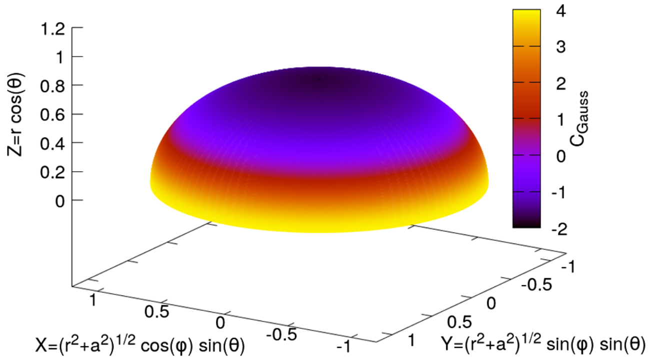

FIG. 11. Gaussian curvature of $r_s = \text{const.}$, $\rightarrow r(\theta)$, using parameters $a = 0.8$, $m = 1.0$. Case $r(\theta) \approx 0.5 r_+$.

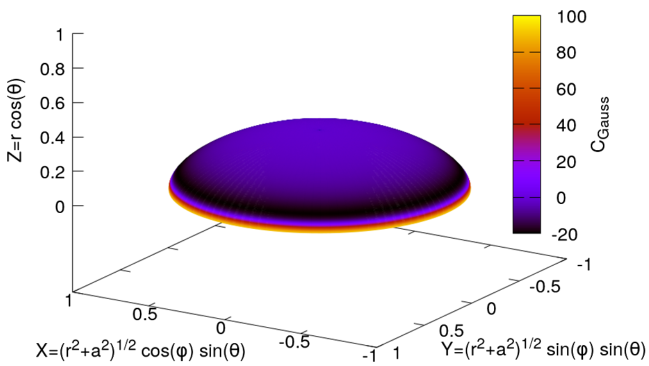

FIG. 12. Gaussian curvature of $r_s = \text{const.}$, $\rightarrow r(\theta)$, using parameters $a = 0.8$, $m = 1.0$. Case $r(\theta) \approx 0.85 r_-$.

embedded surface should have a pothole at the poles. Finally, in Fig. 12 (the closer region to the ring's singularity), the Gaussian curvature is very close to zero at the poles, negative as one approaches the equator, and positive with high values at the equator, so the analogous embedded surface in this case can be constructed as a torus with a disk at the center instead of a hole.

Let us note that for each surface S_{r_s}, we have computed the Gaussian and extrinsic curvature scalars at different spacetime regions using the presented expressions, although in the case of interior regions ($r < r_+$), one has to adapt the null coordinates definitions in a similar way to what we have done in Sec. II. The complete and detailed description of interior regions in terms of double null coordinates will be published elsewhere.

V. KERR METRIC IN DOUBLE NULL COORDINATES

In what follows, we will start from Kerr's metric in Boyer-Lindquist coordinates [Eq. (1)], and then we will make a coordinate transformation to the center-of-mass null coordinates.

A. Coordinates $\{u, v, \theta, \phi\}$

From Eqs. (26) and (27), we can write

$$dt = \frac{dv + du}{2}, \tag{102}$$

$$dr = \left[\frac{(dv - du)}{2} - k d\theta\right] \frac{\Delta}{\sqrt{\mathcal{R}}}. \tag{103}$$

Then, we can substitute Eqs. (102) and (130) into Eq. (1) to obtain

$$\begin{aligned} ds^2 = & \frac{1}{4}\left(1 - \frac{2mr}{\Sigma} - \frac{\Sigma\Delta}{\mathcal{R}}\right)(du^2 + dv^2) \\ & + \frac{1}{2}\left(1 - \frac{2mr}{\Sigma} + \frac{\Sigma\Delta}{\mathcal{R}}\right) du dv \\ & + dv\left(\frac{2amr\sin^2(\theta)}{\Sigma} d\phi + \frac{\Sigma\Delta}{\mathcal{R}} k d\theta\right) \\ & + du\left(\frac{2amr\sin^2(\theta)}{\Sigma} d\phi - \frac{\Sigma\Delta}{\mathcal{R}} k d\theta\right) \\ & - \frac{\Upsilon\Sigma}{\mathcal{R}} d\theta^2 - \frac{\Upsilon}{\Sigma}\sin^2(\theta) d\phi^2. \end{aligned} \tag{104}$$

Note that in the asymptotic limit $r \rightarrow \infty$, we recover Minkowski geometry, and when $a \rightarrow 0$, one obtains the Schwarzschild metric. In both limiting cases, the quadratic terms in du and dv go to zero, as one could expect:

$$\lim_{r\rightarrow\infty}\left[\left(1 - \frac{2mr}{\Sigma}\right) - \frac{\Sigma\Delta}{\mathcal{R}}\right] = 0, \tag{105}$$

$$\lim_{a\rightarrow 0}\left[\left(1 - \frac{2mr}{\Sigma}\right) - \frac{\Sigma\Delta}{\mathcal{R}}\right] = 0. \tag{106}$$

We can also obtain the contravariant expression of Eq. (104) if we start with the inverse metric line element in Boyer-Lindquist coordinates [Eq. (2)]. Formally, we will use a notation to distinguish old and new coordinates. We will start from $\{t, r, \theta, \phi\}$, and then transform to $\{u, v, \tilde{\theta}, \tilde{\phi}\}$. Note that the angular coordinates θ, ϕ are still unaltered, and we only introduce the new coordinates u, v. To start, it will be useful to consider the differentials of the new coordinates in terms of the old ones:

$$du = dt - \frac{\sqrt{\mathcal{R}}}{\Delta} dr - k d\theta, \tag{107}$$

$$dv = dt + \frac{\sqrt{\mathcal{R}}}{\Delta} dr + k d\theta, \tag{108}$$

$$d\tilde{\theta} = d\theta, \tag{109}$$

$$d\tilde{\phi} = d\phi. \tag{110}$$

Then, each coordinate vector is

$$\frac{\partial}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial}{\partial v} + \frac{\partial \tilde{\theta}}{\partial t}\frac{\partial}{\partial \tilde{\theta}} + \frac{\partial \tilde{\phi}}{\partial t}\frac{\partial}{\partial \tilde{\phi}} = \frac{\partial}{\partial v} + \frac{\partial}{\partial u}, \tag{111}$$

$$\frac{\partial}{\partial r} = \frac{\partial u}{\partial r}\frac{\partial}{\partial u} + \frac{\partial v}{\partial r}\frac{\partial}{\partial v} + \frac{\partial \tilde{\theta}}{\partial r}\frac{\partial}{\partial \tilde{\theta}} + \frac{\partial \tilde{\phi}}{\partial r}\frac{\partial}{\partial \tilde{\phi}} = \frac{\sqrt{\mathcal{R}}}{\Delta}\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right), \tag{112}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial u}{\partial \theta}\frac{\partial}{\partial u} + \frac{\partial v}{\partial \theta}\frac{\partial}{\partial v} + \frac{\partial \tilde{\theta}}{\partial \theta}\frac{\partial}{\partial \tilde{\theta}} + \frac{\partial \tilde{\phi}}{\partial \theta}\frac{\partial}{\partial \tilde{\phi}} = k\left(\frac{\partial}{\partial v} - \frac{\partial}{\partial u}\right) + \frac{\partial}{\partial \tilde{\theta}}, \tag{113}$$

$$\frac{\partial}{\partial \phi} = \frac{\partial u}{\partial \phi}\frac{\partial}{\partial u} + \frac{\partial v}{\partial \phi}\frac{\partial}{\partial v} + \frac{\partial \tilde{\theta}}{\partial \phi}\frac{\partial}{\partial \tilde{\theta}} + \frac{\partial \tilde{\phi}}{\partial \phi}\frac{\partial}{\partial \tilde{\phi}} = \frac{\partial}{\partial \tilde{\phi}}. \tag{114}$$

Finally, if we replace everything in Eq. (2) and we take out accent marks on $\tilde{\theta}, \tilde{\varphi}$, we have

$$\left(\frac{\partial}{\partial s}\right)^2 = 4\frac{\Upsilon}{\Sigma\Delta}\left(\frac{\partial}{\partial u}\right)\left(\frac{\partial}{\partial v}\right) - \frac{1}{\Sigma}\left(\frac{\partial}{\partial \theta}\right)^2 + 2\left(\frac{\partial}{\partial u}\right)\left[\frac{2amr}{\Sigma\Delta}\left(\frac{\partial}{\partial \phi}\right) + \frac{k}{\Sigma}\left(\frac{\partial}{\partial \theta}\right)\right] + 2\left(\frac{\partial}{\partial v}\right)\left[\frac{2amr}{\Sigma\Delta}\left(\frac{\partial}{\partial \phi}\right) - \frac{k}{\Sigma}\left(\frac{\partial}{\partial \theta}\right)\right] - \frac{\Delta - a^2\sin^2(\theta)}{\Sigma\Delta\sin^2(\theta)}\left(\frac{\partial}{\partial \phi}\right)^2. \tag{115}$$

It is easy to verify that $\ell_a = (du)_a$ and $\mathfrak{n}_a = (dv)_a$ are null covectors, which is consistent with our definition.

B. New angular coordinate φ

Let us start analyzing the most general null geodesic, where the angular coordinate ϕ changes as in Eq. (7). It is clear that

$$\lim_{\Delta\to 0}\frac{d\phi}{d\lambda} \to \infty, \tag{116}$$

where $\Delta = (r - r_+)(r - r_-)$—that is, $\Delta \to 0$, when $r \to r_+$ or $r \to r_-$.

We want to define a new angular coordinate without that bad behavior at $\Delta = 0$. We will consider two ways to accomplish the task.

One way is to define a coordinate $\tilde{\varphi}$ which remains constant along the *center-of-mass null congruence*, for which we have (with $\delta = 0$, $L = 0$, $E = 1$)

$$\frac{dr}{d\lambda} = -\pm_{oi}\frac{\sqrt{(r^2 + a^2)^2 - K(r,\theta)\Delta}}{\Sigma}, \tag{117}$$

$$\frac{d\phi}{d\lambda} = \frac{2amr}{\Sigma\Delta}. \tag{118}$$

This suggests that we define

$$\tilde{\varphi}_{\pm_{oi}} = \phi - \pm_{oi}\int\frac{2amr'}{\Delta\sqrt{(r'^2 + a^2)^2 - K\Delta}}dr \tag{119}$$

such that with the correct election of $\pm_{oi}$, for both outgoing and ingoing center-of-mass null congruences we have

$$\frac{d\tilde{\varphi}_{\pm_{oi}}}{d\lambda} = 0. \tag{120}$$

Depending on the election of $\pm_{oi}$, is well behaved for the outgoing null congruence ($\pm_{oi} = +$) and for the ingoing one ($\pm_{oi} = -$). Such a coordinate also depends on θ, where $\frac{\partial\tilde{\varphi}_{\pm_{oi}}}{\partial\theta} \neq 0$. This possible new definition also has been suggested in Ref. [19]; nevertheless, we have decided not to use it, because the metric expressions get more complicated.

The other way, the one we elected in this work, is the usual definition [15]

$$d\varphi_{\pm_{oi}} = d\phi - \pm_{oi}\frac{a}{\Delta}dr, \tag{121}$$

which has an integral expression given by

$$\varphi_{\pm_{oi}} = \phi - \pm_{oi}\frac{a}{2\sqrt{m^2 - a^2}}\ln\left|\frac{r - r_+}{r - r_-}\right|. \tag{122}$$

Note that this construction gives two possible angular coordinates which are well behaved across both horizons r_+, r_-. We have φ_+, which is well behaved as one approaches the past horizon following an outgoing null congruence ($\pm_{oi} = +$), and φ_-, which is well behaved as one approaches the future horizon following an ingoing one ($\pm_{oi} = -$). In this case,

$$\frac{d\varphi_{\pm_{oi}}}{d\lambda} \neq 0, \tag{123}$$

which indicates that the coordinate φ changes along the center-of-mass null congruences.

In the next section, it will be useful to express the differential of Eq. (121) in terms of center-of-mass null coordinates u, v; which from Eq. (103) can be expressed as

$$d\varphi_{\pm_{oi}} = d\phi - \pm_{oi}\frac{a}{\sqrt{\mathcal{R}}}\left[\frac{(dv-du)}{2} - kd\theta\right]. \tag{124}$$

C. Extended null functions U, V

A natural geometric way to study the behavior of a function (U) across a hypersurface is to study its behavior along geodesics that cross this hypersurface, since its dependence on the affine parameters is geometric and independent of coordinate choices. Therefore, we begin by studying the asymptotic behavior of the null coordinate u along null geodesics that reach the future horizon H_f. Since it is immaterial which geodesics they are, we choose null geodesics that are contained in the congruence $v = \text{const.}$—that is, ingoing geodesics for which one has to take ($\pm_{oi} = -$) and ($\dot{\theta} = -\pm_h\sqrt{\Theta} = -k$); see Eq. (12). From the first-order expressions of the geodesic equations and $dv = 0$, which allows to express $\dot{t} = -\frac{\sqrt{\mathcal{R}}}{\Delta}\dot{r} - k\dot{\theta}$, one obtains

$$\dot{u}|_{v=\text{const.}} = -2\frac{\sqrt{\mathcal{R}}}{\Delta}\dot{r} - 2k\dot{\theta} = 2\left[\frac{\mathcal{R}}{\Sigma\Delta} + \frac{k^2}{\Sigma}\right]. \tag{125}$$

Defining the null function U by

$$U = -\exp(-\kappa u), \tag{126}$$

one finds the behavior

$$\dot{U} = -\kappa U\dot{u} = -2\kappa U\left[\frac{\mathcal{R}}{\Sigma\Delta} + \frac{k^2}{\Sigma}\right], \tag{127}$$

which indicates a divergent behavior of the first term. Let us study this in more detail. To see the behavior of the first term as a function of the affine parameter λ, let us recall from Eq. (5) that at the outer horizon one has

$$-\frac{\Sigma(r_+,\theta)dr}{\sqrt{\mathcal{R}(r_+)}} = d\lambda, \tag{128}$$

so that to first order, one has

$$\begin{aligned}\Delta &= (r-r_+)(r-r_-)\\ &= -\frac{\sqrt{\mathcal{R}(r_+)}}{\Sigma(r_+,\theta)}(\lambda-\lambda_+)(r_+-r_-) + \mathcal{O}((\lambda-\lambda_+)^2),\end{aligned} \tag{129}$$

where λ_+ is the value of the affine parameter at the horizon. Then, the divergent behavior in Eq. (127) can be expressed as

$$\frac{dU}{U} = 2\kappa\frac{\sqrt{\mathcal{R}(r_+)}}{(\lambda-\lambda_+)(r_+-r_-)}d\lambda + \mathcal{O}((\lambda-\lambda_+)^0)d\lambda, \tag{130}$$

and noting that

$$\frac{(r_+-r_-)}{2\sqrt{\mathcal{R}(r_+)}} = \frac{(r_+-r_-)}{2(r_+^2+a^2)} = \frac{\sqrt{m^2-a^2}}{2mr_+} = \kappa_+, \tag{131}$$

where κ_+ is customarily referred to as the surface gravity of the black hole, one has the leading behavior

$$\frac{dU}{U} = \frac{\kappa}{\kappa_+}\frac{d\lambda}{(\lambda-\lambda_+)} \tag{132}$$

so that one must take $\kappa = \kappa_+$ in order to have a smooth behavior of U as a function of the affine parameter λ.

Note that we have just shown that at the future horizon, one has

$$U \propto (\lambda-\lambda_+) \propto \Delta, \tag{133}$$

where the proportionality factors are smooth functions on the horizon. In order to have a double null system that is smooth across the outer past event horizon, we also define the null function V in a similar way, so that we have

$$U = -\exp(-\kappa_+ u) \tag{134}$$

and

$$V = \exp(\kappa_+ v). \tag{135}$$

Thus, using this general geometric approach, we have determined the correct functional form of the new null function U to be regular at both sides of the future event horizon, and that of the null function V to be regular at both sides of the past event horizon.

Let us remark that we have been studying the asymptotic behavior approaching the horizon from the outside region where $\lambda < \lambda_+$. In the inner region, $U > 0$, and one would use the relation

$$U = \exp(\kappa_+ u_{\text{inner}}), \tag{136}$$

where u_{inner} is the analogous inner version of the null coordinate u in the outer region. The complete and detailed description of interior regions in terms of double null coordinates, will be published elsewhere.

D. Coordinates $\{U, V, \theta, \varphi\}$

From Eqs. (134) and (135), we have

$$du = -\frac{1}{\kappa_+}\frac{dU}{U}, \tag{137}$$

$$dv = \frac{1}{\kappa_+}\frac{dV}{V}. \tag{138}$$

Then, we substitute Eqs. (124), (137), and (138) into Eq. (104) to obtain

$$\begin{aligned} ds^2 = &\frac{1}{4}\left(1-\frac{2mr}{\Sigma}-\frac{\Upsilon a^2\sin^2(\theta)+\Sigma^2\Delta}{\Sigma\mathcal{R}}-\frac{\pm_{oi}4mra^2\sin^2(\theta)}{\Sigma\sqrt{\mathcal{R}}}\right)\frac{1}{\kappa^2U^2}dU^2 \\ &+\frac{1}{4}\left(1-\frac{2mr}{\Sigma}-\frac{\Upsilon a^2\sin^2(\theta)+\Sigma^2\Delta}{\Sigma\mathcal{R}}\pm_{oi}\frac{4mra^2\sin^2(\theta)}{\Sigma\sqrt{\mathcal{R}}}\right)\frac{1}{\kappa^2V^2}dV^2-\frac{1}{2}\left(1-\frac{2mr}{\Sigma}+\frac{\Upsilon a^2\sin^2(\theta)+\Sigma^2\Delta}{\Sigma\mathcal{R}}\right)\frac{1}{\kappa^2UV}dUdV \\ &+\left[\frac{(\Upsilon a^2\sin^2(\theta)+\Delta\Sigma^2)}{\Sigma\mathcal{R}}\pm_{oi}\frac{2mra^2\sin^2(\theta)}{\Sigma\sqrt{\mathcal{R}}}\right]\frac{k}{\kappa U}dUd\theta+\left[\frac{(\Upsilon a^2\sin^2(\theta)+\Delta\Sigma^2)}{\Sigma\mathcal{R}}-\frac{\pm_{oi}2mra^2\sin^2(\theta)}{\Sigma\sqrt{\mathcal{R}}}\right]\frac{k}{\kappa V}dVd\theta \\ &-\left(\frac{2amr\sin^2(\theta)}{\Sigma}\pm_{oi}\frac{\Upsilon a\sin^2(\theta)}{\Sigma\sqrt{\mathcal{R}}}\right)\frac{1}{\kappa U}dUd\varphi+\left(\frac{2amr\sin^2(\theta)}{\Sigma}-\pm_{oi}\frac{\Upsilon a\sin^2(\theta)}{\Sigma\sqrt{\mathcal{R}}}\right)\frac{1}{\kappa V}dVd\varphi \\ &-\left[\Sigma+\frac{k^2(\Upsilon a^2\sin^2(\theta)+\Sigma^2\Delta)}{\Sigma\mathcal{R}}\right]d\theta^2\pm_{oi}\frac{2\Upsilon a\sin^2(\theta)}{\Sigma\sqrt{\mathcal{R}}}kd\theta d\varphi-\frac{\Upsilon}{\Sigma}\sin^2(\theta)d\varphi^2, \end{aligned} \tag{139}$$

where one has to consider $\kappa = \kappa_+$. Alternatively, using the explicit appearance of Φ, one can express each of the metric components for the ingoing case as follows:

$$g_{UU} = \frac{1}{4\kappa_+^2U^2}\left(1-\Phi-\frac{\Delta\Sigma}{\mathcal{R}}-\frac{a^2\sin(\theta)^2}{\mathcal{R}}(r^2+a^2+\Phi a^2\sin(\theta)^2-2\Phi\sqrt{\mathcal{R}})\right), \tag{140}$$

$$g_{\mathrm{UV}} = -\frac{1}{4\kappa_+^2UV}\left(1-\Phi+\frac{\Delta\Sigma}{\mathcal{R}}+\frac{a^2\sin(\theta)^2}{\mathcal{R}}(r^2+a^2+\Phi a^2\sin(\theta)^2)\right), \tag{141}$$

$$g_{U\theta} = \frac{k}{2\mathcal{R}\kappa_+U}(\Delta\Sigma+a^2\sin(\theta)^2(r^2+a^2+\Phi a^2\sin(\theta)^2-\Phi\sqrt{\mathcal{R}})), \tag{142}$$

$$g_{U\varphi} = -\frac{a\sin(\theta)^2}{2\kappa_+U}\left(\Phi-\frac{r^2+a^2+\Phi a^2\sin(\theta)^2}{\sqrt{\mathcal{R}}}\right), \tag{143}$$

$$g_{VV} = \frac{1}{4\kappa_+^2V^2}\left(1-\Phi-\frac{\Delta\Sigma}{\mathcal{R}}-\frac{a^2\sin(\theta)^2}{\mathcal{R}}(r^2+a^2+\Phi a^2\sin(\theta)^2+2\Phi\sqrt{\mathcal{R}})\right), \tag{144}$$

$$g_{V\theta} = \frac{k}{2\mathcal{R}\kappa_+V}(\Delta\Sigma+a^2\sin(\theta)^2(r^2+a^2+\Phi a^2\sin(\theta)^2+\Phi\sqrt{\mathcal{R}})), \tag{145}$$

$$g_{V\varphi} = \frac{a\sin(\theta)^2}{2\kappa_+V}\left(\Phi+\frac{r^2+a^2+\Phi a^2\sin(\theta)^2}{\sqrt{\mathcal{R}}}\right), \tag{146}$$

$$g_{\theta\theta} = -\Sigma-\frac{k^2}{\mathcal{R}}(\Delta\Sigma+a^2\sin(\theta)^2(r^2+a^2+\Phi a^2\sin(\theta)^2)), \tag{147}$$

$$g_{\theta\varphi} = -ka\sin(\theta)^2\left(\frac{r^2+a^2+\Phi a^2\sin(\theta)^2}{\sqrt{\mathcal{R}}}\right), \tag{148}$$

$$g_{\varphi\varphi} = -\sin(\theta)^2(r^2+a^2+\Phi a^2\sin(\theta)^2). \tag{149}$$

To compute the contravariant expression, we can start from Eq. (115) to make a coordinate transformation. Formally, we will use a notation to distinguish old and new coordinates. We will start from $\{u, v, \theta, \phi\}$, and then transform to $\{U, V, \tilde{\theta}, \varphi\}$. Note that the angular coordinate θ is still unaltered, and we only introduce the new coordinates U, V, φ. It is useful to consider the differentials of the new coordinates in terms of the old ones:

$$dU = -\kappa U du, \tag{150}$$

$$dV = \kappa V dv, \tag{151}$$

$$d\tilde{\theta} = d\theta, \tag{152}$$

$$d\varphi = d\phi - \frac{\pm_{oi} a}{\sqrt{\mathcal{R}}}\left[\frac{(dv - du)}{2} - k d\theta\right]. \tag{153}$$

Then, each coordinate vector is

$$\frac{\partial}{\partial u} = \frac{\partial U}{\partial u}\frac{\partial}{\partial U} + \frac{\partial V}{\partial u}\frac{\partial}{\partial V} + \frac{\partial \tilde{\theta}}{\partial u}\frac{\partial}{\partial \tilde{\theta}} + \frac{\partial \varphi}{\partial u}\frac{\partial}{\partial \varphi} = -\kappa U \frac{\partial}{\partial U} \pm_{oi} \frac{a}{2\sqrt{\mathcal{R}}}\frac{\partial}{\partial \varphi}, \tag{154}$$

$$\frac{\partial}{\partial v} = \frac{\partial U}{\partial v}\frac{\partial}{\partial U} + \frac{\partial V}{\partial v}\frac{\partial}{\partial V} + \frac{\partial \tilde{\theta}}{\partial v}\frac{\partial}{\partial \tilde{\theta}} + \frac{\partial \varphi}{\partial v}\frac{\partial}{\partial \varphi} = \kappa V \frac{\partial}{\partial V} - \pm_{oi} \frac{a}{2\sqrt{\mathcal{R}}}\frac{\partial}{\partial \varphi}, \tag{155}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial U}{\partial \theta}\frac{\partial}{\partial U} + \frac{\partial V}{\partial \theta}\frac{\partial}{\partial V} + \frac{\partial \tilde{\theta}}{\partial \theta}\frac{\partial}{\partial \tilde{\theta}} + \frac{\partial \varphi}{\partial \theta}\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial \tilde{\theta}} \pm_{oi} \frac{ak}{\sqrt{\mathcal{R}}}\frac{\partial}{\partial \varphi}, \tag{156}$$

$$\frac{\partial}{\partial \phi} = \frac{\partial U}{\partial \phi}\frac{\partial}{\partial U} + \frac{\partial V}{\partial \phi}\frac{\partial}{\partial V} + \frac{\partial \tilde{\theta}}{\partial \phi}\frac{\partial}{\partial \tilde{\theta}} + \frac{\partial \varphi}{\partial \phi}\frac{\partial}{\partial \varphi} = \frac{\partial}{\partial \varphi}. \tag{157}$$

Finally, if we replace everything in Eq. (115) and we take out tilde mark on $\tilde{\theta}$, we have

$$\begin{aligned}\left(\frac{\partial}{\partial s}\right)^2 = {} & -4\kappa^2 \frac{\Upsilon}{\Sigma\Delta} UV \left(\frac{\partial}{\partial U}\right)\left(\frac{\partial}{\partial V}\right) \\ & - \frac{2\kappa k}{\Sigma}\left[U\left(\frac{\partial}{\partial U}\right) + V\left(\frac{\partial}{\partial V}\right)\right]\left(\frac{\partial}{\partial \theta}\right) \\ & - \frac{2\kappa a U}{\Sigma\Delta}(2mr - \pm_{oi}\sqrt{R})\left(\frac{\partial}{\partial U}\right)\left(\frac{\partial}{\partial \varphi}\right) \\ & + \frac{2\kappa a V}{\Sigma\Delta}(2mr \pm_{oi} \sqrt{R})\left(\frac{\partial}{\partial V}\right)\left(\frac{\partial}{\partial \varphi}\right) \\ & - \frac{1}{\Sigma}\left(\frac{\partial}{\partial \theta}\right)^2 - \frac{1}{\Sigma \sin^2(\theta)}\left(\frac{\partial}{\partial \varphi}\right)^2, \end{aligned} \tag{158}$$

where one has to consider $\kappa = \kappa_+$.

We have already checked that Eqs. (139) and (158) satisfy $g_{ab}g^{ab} = \mathbb{I}$. Moreover, its determinant is given by

$$g = -\frac{\Delta^2 \Sigma^2 \sin(\theta)^2}{4\kappa^4 \mathcal{R} U^2 V^2}. \tag{159}$$

E. Metric near the outer horizons

Since the metric components in the new double null coordinate system are rather complicated, it is interesting to see explicitly that each of the components is a regular function in a neighborhood of the outer horizon, although we know that this should be so, since the coordinates are well behaved at the event horizon.

Therefore, here we will present the expansion of each metric component around $(\Delta = 0)$. These calculations were performed with algebraic manipulation programs:

$$(g_{UU}\kappa^2 U^2)|_{(\pm_{oi}=-)} \approx \frac{K[(K - 8mr)a^2 \sin^2(\theta) - 16m^2 r^2]}{256 m^4 r^4 (2mr + a^2 \sin^2(\theta))}\Delta^2 + \mathcal{O}(\Delta^3), \tag{160}$$

$$(g_{VV}\kappa^2 V^2)|_{(\pm_{oi}=+)} \approx \frac{K[(K - 8mr)a^2 \sin^2(\theta) - 16m^2 r^2]}{256 m^4 r^4 (2mr + a^2 \sin^2(\theta))}\Delta^2 + \mathcal{O}(\Delta^3), \tag{161}$$

$$(g_{UU}\kappa^2 U^2)|_{(\pm_{oi}=+)} \approx \frac{-a^2 \sin^2(\theta)}{2mr + a^2 \sin^2(\theta)} + \mathcal{O}(\Delta), \tag{162}$$

$$(g_{VV}\kappa^2 V^2)|_{(\pm_{oi}=-)} \approx \frac{-a^2 \sin^2(\theta)}{2mr + a^2 \sin^2(\theta)} + \mathcal{O}(\Delta), \tag{163}$$

$$(g_{\mathrm{UV}}\kappa^2 UV) = (g_{VU}\kappa^2 UV) \approx \frac{(4mr - K)a^2 \sin^2(\theta) + 8m^2 r^2}{16 m^2 r^2 (2mr + a^2 \sin^2(\theta))}\Delta + \mathcal{O}(\Delta^2). \tag{164}$$

The null-angular components are

$$(g_{U\theta}\kappa U)|_{(\pm_{oi}=-)} \approx -\frac{1}{8}\frac{k((4mr-K)a^2\sin^2(\theta)-8m^2r^2)}{r^2m^2(2mr-a^2\sin^2(\theta))}\Delta+\mathcal{O}(\Delta^2), \tag{165}$$

$$(g_{V\theta}\kappa V)|_{(\pm_{oi}=+)} \approx -\frac{1}{8}\frac{k((4mr-K)a^2\sin^2(\theta)-8m^2r^2)}{r^2m^2(2mr-a^2\sin^2(\theta))}\Delta+\mathcal{O}(\Delta^2), \tag{166}$$

$$(g_{U\varphi}\kappa U)|_{(\pm_{oi}=-)} \approx \frac{1}{4}\frac{(4mr+K-2a^2\sin^2(\theta))a^2\sin^2(\theta)}{mr(2mr-a^2\sin^2(\theta))}\Delta+\mathcal{O}(\Delta^2), \tag{167}$$

$$(g_{V\varphi}\kappa V)|_{(\pm_{oi}=+)} \approx -\frac{1}{4}\frac{(4mr+K-2a^2\sin^2(\theta))a^2\sin^2(\theta)}{mr(2mr-a^2\sin^2(\theta))}\Delta+\mathcal{O}(\Delta^2), \tag{168}$$

$$(g_{U\theta}\kappa U)|_{(\pm_{oi}=+)} \approx \frac{2ka^2\sin^2(\theta)}{2mr-a^2\sin^2(\theta)}+\mathcal{O}(\Delta), \tag{169}$$

$$(g_{V\theta}\kappa V)|_{(\pm_{oi}=-)} \approx \frac{2ka^2\sin^2(\theta)}{2mr-a^2\sin^2(\theta)}+\mathcal{O}(\Delta), \tag{170}$$

$$(g_{U\varphi}\kappa U)|_{(\pm_{oi}=+)} \approx -\frac{4amr\sin^2(\theta)}{2mr-a^2\sin^2(\theta)}+\mathcal{O}(\Delta), \tag{171}$$

$$(g_{V\varphi}\kappa V)|_{(\pm_{oi}=-)} \approx \frac{4amr\sin^2(\theta)}{2mr-a^2\sin^2(\theta)}+\mathcal{O}(\Delta), \tag{172}$$

and the angular-angular components are

$$g_{\theta\varphi} \approx \pm_{oi}\frac{4m^2r^2a\sin^2(\theta)k}{mr(2mr-a^2\sin^2(\theta))}+\mathcal{O}(\Delta), \tag{173}$$

$$g_{\theta\theta} \approx -\frac{4m^2r^2-(4mr-K)}{2mr-a^2\sin^2(\theta)}+\mathcal{O}(\Delta), \tag{174}$$

$$g_{\varphi\varphi} \approx \frac{4m^2r^2\sin(\theta)^2}{2mr-a^2\sin(\theta)^2}+\mathcal{O}(\Delta). \tag{175}$$

From the discussions in Sec. V, V C, it is deduced that all the components of the metric are smooth across the future event horizon, and it can easily be seen that the same observation is true for the past outer horizon.

With respect to the contravariant metric components [Eq. (158)], the Δ dependence is explicit, so there is no need to take any expansion around ($\Delta = 0$).

F. Other algebraic study of metric components through the outer horizon H

Since the expressions involving the coordinates U and V are rather complicated, and the subject of the smoothness of the metric across the horizon is a delicate one, we will next show another algebraic study of the well-behaved-ness of metric components at the outer horizon.

One can distinguish two exterior horizons where $r = r_+$: the future exterior horizon, H_f, and the past one, H_p. To cross both of them, it has to be taken that $\kappa = \kappa_+$ as given by Eq. (131). To show that the Kerr metric in extended center-of-mass null coordinates is well behaved through r_+, one can choose a path to cross H_f.

The first step is to note that the product between U [Eq. (134)] and V [Eq. (135)] is

$$UV|_{\kappa=\kappa_+} = \frac{-e^{2\kappa_+(r+\int_0^\theta k(r,\theta')d\theta')}(r_-)^{\frac{r_-}{r_+}}}{(r_+)(r-r_-)^{\frac{r_++r_-}{r_+}}}\Delta. \tag{176}$$

We will study its behavior as one crosses H_f and H_p.

1. Crossing future horizon H_f

The next step is to choose the path to cross H_f. For example, we can approach H_f maintaining V as constant—that is, $V = V_0$ (following a future-directed ingoing *center-of-mass null geodesic*). Then we have

$$U|_{\kappa_+, V=V_0} = \frac{-e^{2\kappa_+(r+\int_0^\theta k(r,\theta')d\theta')}(r_-)^{\frac{r_-}{r_+}}}{(r_+)(r-r_-)^{\frac{r_++r_-}{r_+}} V_0} \Delta. \quad (177)$$

We can start with the covariant metric expression (139). If one follows such a path where $dV = 0$, the only covariant metric components to analyze are g_{UU}, $g_{U\theta}$, and $g_{U\varphi}$. Because such a path follows an ingoing center-of-mass null geodesic, the correct angular coordinate $\varphi = \varphi_-$ has to be considered—that is, $\pm_{oi} = -$. Then, from Eqs. (160), (165), and (170), it is straightforward to show that the metric components g_{UU}, $g_{U\theta}$, and $g_{U\varphi}$ are well behaved and different from zero at $r = r_+$ and nearby regions. One can see that every factor which involves Δ gets simplified. The remaining factors, including $(r - r_-)^{\frac{r_++r_-}{r_+}}$, are well behaved at $r = r_+$.

To analyze the contravariant metric expression (158), we only have to look at the components g^{UV}, $g^{U\varphi}$, and $g^{V\varphi}$, which seems to be problematic because of Δ in the denominator. From Eqs. (176), (177), and this path election where $\pm_{oi} = -$, it is clear that g^{UV} and $g^{U\varphi}$ are well behaved and different from zero at $r = r_+$. In the case of $g^{V\varphi}$, we have that $(2mr - \sqrt{\mathcal{R}})|_{r\approx r_+} \approx -\Delta$, so the problematic factor gets simplified while $V = V_0$; therefore such a metric component is also well behaved and different from zero at $r = r_+$.

If one considers the behavior of the metric determinant (159) at H_f and H_p, it is easy to show from Eq. (176) that the factor Δ^2 gets simplified, so that the determinant is well behaved and different from zero at $r = r_+$.

Therefore, we have shown that the metric components, in both covariant [Eq. (139)] and contravariant [Eq. (158)] expressions, are well behaved across the future horizon H_f.

2. Crossing past horizon H_p

We have to choose the path to cross H_p. For example, we can approach H_p maintaining U as constant—that is, $U = U_0$ (following a past-directed center-of-mass null geodesic). Then we have

$$V|_{\kappa_+, U=U_0} = \frac{-e^{2\kappa_+(r+\int_0^\theta k(r,\theta')d\theta')}(r_-)^{\frac{r_-}{r_+}}}{(r_+)(r-r_-)^{\frac{r_++r_-}{r_+}} U_0} \Delta. \quad (178)$$

We can start with the covariant metric expression (139). If one follows such a path where $dU = 0$, the only covariant metric components to analyze are g_{VV}, $g_{V\theta}$, and $g_{V\varphi}$. Because such a path follows an outgoing center-of-mass null geodesic, the correct angular coordinate $\varphi = \varphi_+$ has to be considered—that is, $\pm_{oi} = +$. Then, from Eqs. (161), (166), and (168), it is straightforward to show that the metric components g_{VV}, $g_{V\theta}$, and $g_{V\varphi}$ are well behaved and different from zero at $r = r_+$. One can see how every factor that involves Δ gets simplified. The remaining factors, including $(r - r_-)^{\frac{r_++r_-}{r_+}}$, are well behaved at $r = r_+$.

To analyze the contravariant metric expression (158), we only have to look at the components g^{UV}, $g^{U\varphi}$, and $g^{V\varphi}$, which seems to be problematic because of Δ in the denominator. From Eqs. (176), (178), and this path election where $\pm_{oi} = +$, it is clear that g^{UV} and $g^{V\varphi}$ are well behaved and different from zero at $r = r_+$. In the case of $g^{U\varphi}$, we have that $(2mr - \sqrt{\mathcal{R}})|_{r\approx r_+} \approx -\Delta$, so the problematic factor gets simplified while $U = U_0$; therefore such a metric component is also well behaved and different from zero at $r = r_+$.

In this way, we have shown that the metric components, in both covariant [Eq. (139)] and contravariant [Eq. (158)] expressions, are well behaved across both the future and past horizons H_f and H_p $(r = r_+)$.

VI. COMPARISON WITH RELATED WORKS

Our work improves upon several attempts that can be found in the literature. We compare previous works in detail in the rest of this section.

A remarkable one is developed in Ref. [16], where the null hypersurfaces they construct unfortunately have a conical singularity all along the axis of symmetry and do not include the null geodesic along the axis of symmetry. In order to compare it with our coordinates, we consider from Ref. [16] the null function

$$u^* = t^* - r^* = t - a\sin(\theta) - r^*(r), \quad (179)$$

which they call ξ^-, and where the analog to our natural spheres is the intersection of u^* with the Boyer-Lindquist coordinate t, which can be parametrized by r_{sH}, and that we call $S_{r_{sH}}$. Just by looking at Eq. (179), one can forecast some kind of problem, since the derivative of the null coordinate is direction dependent at the axis of symmetry, but one can also see this graphically. In Fig. 13, it can be seen that their surface, $S_{r_{sH}}$, has a discontinuity in the derivatives at $(\theta = 0, \pi)$, while our S_{r_s} is clearly smooth. Such discontinuity is not a scale effect. The Gaussian curvature of $S_{r_{sH}}$ diverges at the poles, while it is well behaved for S_{r_s}. The intrinsic character of Gaussian curvature clears any doubt about the presence of such a problematic cusp in $S_{r_{sH}}$.

As a test case for the use of coordinate systems, one can consider the massless scalar field equation for Kerr spacetime. To study the scalar wave equation, one can start from the expressions of Ref. [27] and make the appropriate

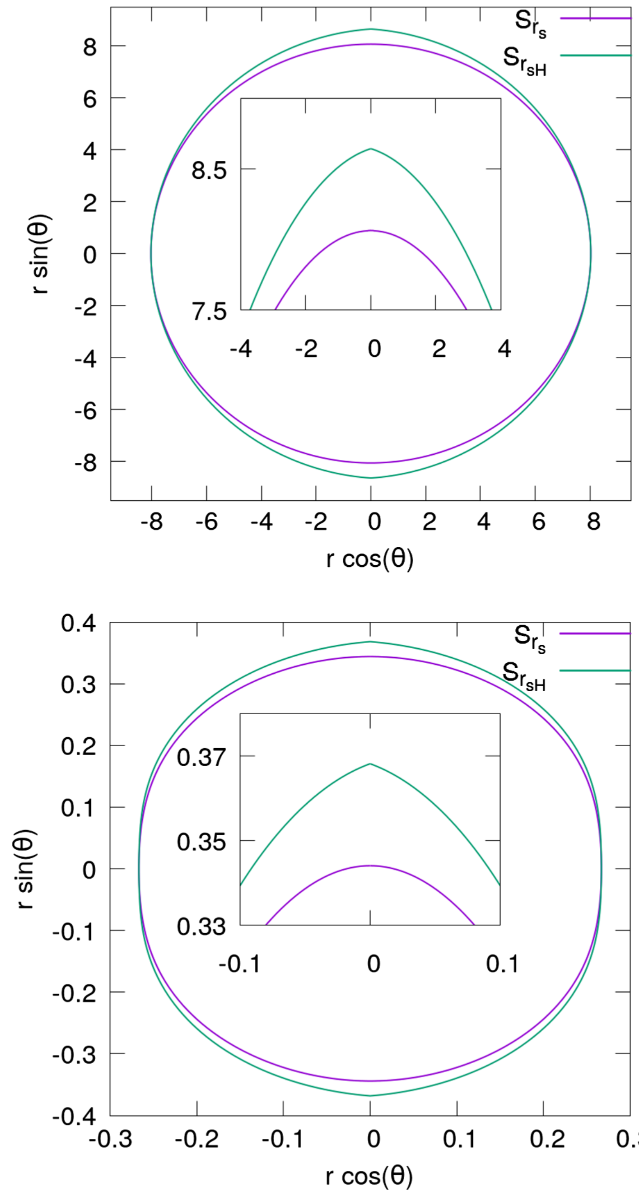

FIG. 13. Comparison between two-dimensional spacelike surfaces S_{r_s} (of center-of-mass null coordinates) and $S_{r_{sH}}$ (of Hayward's definition [16]). It can be seen that S_{r_s} is completely smooth, while $S_{r_{sH}}$ has a cusp or peak with a discontinuity at the poles $(\theta = 0, \pi)$, which is not a scale effect. The Gaussian curvature and spin coefficient ρ_{Hayward} have divergent behavior on the surface $S_{r_{sH}}$, indicating the presence of caustics along the axis of symmetry. Two cases are shown, one at $(r \approx 3r_+)$ and the other at the interior $(r \approx 0.85r_-)$, using the parameters $m = 1,\ a = 0.8$.

coordinate transformations. If one does so, in such an equation, the expansion spin coefficient ρ of the corresponding null geodesic congruence appears explicitly contained in the null hypersurfaces, which for the center-of-mass null coordinates is regular and well behaved. But if one uses Hayward's definition [16], which can be obtained from our expressions by simply taking $K = a^2$, one will get a divergent coefficient at the poles. Let us look at the third term of ρ [see Eq. (61)], namely

$$\frac{\cos(\theta)}{\sin(\theta)}\sqrt{\Theta} = \frac{\cos(\theta)}{\sin(\theta)}\sqrt{K(r,\theta) - a^2\sin^2(\theta)}. \tag{180}$$

Note that in Hayward's case, it has a divergent behavior at the poles, since

$$\lim_{\theta\to 0,\pi}\frac{\cos(\theta)}{\sin(\theta)}\sqrt{\Theta}_{|\text{Hayward}} = \lim_{\theta\to 0,\pi}\frac{a\cos^2(\theta)}{\sin(\theta)} = \infty. \tag{181}$$

Therefore, the cusp that one can see in Fig. 13 points out a serious geometric problem in Hayward's definition. The divergent behavior in both geometric quantities, Gaussian curvature and the expansion spin coefficient ρ_{Hayward}, indicates that the null coordinates they have constructed are adapted to null congruences which have *caustics* along the axis of symmetry. Therefore, it is impossible to solve the wave equation by any analytic or numeric treatment if one uses Hayward's coordinates. It is intriguing to note that the Kerr metric in Hayward's null coordinates has no divergences or bad behavior in its components (with the exception of typical problems of spherical coordinates). The problem instead is related to its derivatives. The spin coefficient ρ_{Hayward} is one of the connection components, which involves metric derivatives. Since the curvature is determined from its connection, we can foretell the presence of divergent behaviors in many other studies of interest. We have already studied the scalar wave equation with our setting, and our results will be published elsewhere.

Another approach is the work of Ref. [17], where they also consider the possibility of a constant $K = a^2E^2$. They call this condition $X^2 = 1$, where $X^2 = \frac{K}{a^2E^2}$. Then, such a definition also has the same pathology as Ref. [16].

Our approach is more related to the work of Ref. [19], although their treatment only covers the northern hemisphere, but their expressions also fail to deal with the north pole and are very difficult to compute, even numerically [18]. More concretely, for example, it will be impossible to treat the scalar wave equation numerically with the coordinate system of Ref. [19], since their expressions are divergent at the poles.

Finally, after all these comparisons with related works, we can conclude that the center-of-mass null coordinates form the first complete double null coordinate system in the sense that they allow us to treat and solve fundamental applications problems where previous definitions have failed.

VII. FINAL COMMENTS

The condition for the construction of our null coordinate system can be readily recapitulated in the following main points: (i) We base our construction on the existence of a particular null geodesic congruence. (ii) The condition for the congruence to be hypersurface orthogonal is equivalent to considering a space-dependent Carter "constant" K. (iii) The congruence is fixed by conditions at future null infinity, which is equivalent to a boundary condition for the differential equation for K. (iv) This equation cannot be solved in terms of elementary functions, and therefore is solved numerically. (v) As a consequence of the previous requirements, our congruence does not have caustics.

The construction of one such null congruence guarantees the existence of its dual congruence, since for each outgoing one there exists the corresponding incoming one.

It is worthwhile to notice that the geometrical construction of the null coordinate u, based on the center-of-mass sections, can be extended to nonstationary, radiating, asymptotically flat spacetimes at future null infinity [28].

As one could expect, the new double null coordinates for Kerr spacetime have an explicit dependence with the function $K(r,\theta)$. Nevertheless, in this construction, one can proceed analytically to compute any geometric quantity of interest, as we did when we computed the spin coefficients and Weyl scalars. For other cases, where the complete solution $K(r,\theta)$ is needed, it can be easily obtained from the presented numeric scheme. We have used it to compute all the plots of this manuscript.

We have also studied the surface family S_{r_s}, defined by the intersection of both null coordinates u and v, where r_s can be interpreted as the Kerr extension of Schwarzschild's tortoise coordinate. We give a complete geometric description of such two-dimensional spacelike surfaces in terms of their Gaussian and extrinsic curvature scalars. One can clearly appreciate their roundness far away from the event horizon, and how their geometry changes as one approaches the interior regions ($r < r_+$).

Since our family of null surfaces surround the black hole in a smooth manner, we were also able to define Kruskal-like extensions U and V for the outgoing u and ingoing v null functions. Those extensions, together with the definition of a new axial angular coordinate, allow us to overcome the Boyer-Lindquist coordinate singularities. This makes it possible to extend the metric across the event horizon in a regular way. A detailed discussion for the construction of Kruskal-like coordinates in the vicinity of the interior horizons will be carried out in a separate article.

We have also compared our definition and results with other previous attempts in literature. We show that in all those works, there is divergent behavior over the axis of symmetry; with our definition, one instead obtains regular behavior. We have considered all those definitions with an application problem: the computation of a massless scalar field in Kerr spacetime. The definitions of Refs. [16,17] have divergent behavior in one of the terms of the wave equation and in the Gaussian curvature over their related two-dimensional spacelike surface family. In the case of Ref. [19], their expressions for null coordinates are directly divergent at the axis of symmetry. For these reasons, all the previous definitions cannot be taken as candidates for coordinate systems and are therefore unsuitable to be used in many problems of interest.

We expect that this double null foliation that we are presenting will be useful for several studies of the Kerr geometry, including electromagnetic radiation powered by rotating black holes or black hole evaporation. Many lines of studies have been affected due to the lack of these null coordinates—for example, the Vaydia extension for spacetimes without spherical symmetry, where previous attempts [29] did not have a complete null coordinate system to work with. Also, the stability studies of the interior of Kerr spacetime have a huge dependence on the null coordinate system [30].

We plan to study perturbations of the Kerr geometry calculated from data on these characteristic surfaces in the near future.

ACKNOWLEDGMENTS

We have benefited from comments by F. Pretorius on his work and ours. We acknowledge support from CONICET, SeCyT-UNC, and FONCYT.

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