Gravitational decoherence: A nonrelativistic spin 1/2 fermionic model

L. Asprea[1,2,*] and G. Gasbarri[3,4,†]

[1]*Department of Physics, University of Trieste, Strada costiera 11, 34151 Trieste, Italy*
[2]*Istituto Nazionale di Fisica Nucleare, Trieste Section, Via Valerio 2, 34127 Trieste, Italy*
[3]*Física Teòrica: Informació i Fenòmens Quàntics, Department de Física, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain*
[4]*Department of Physics and Astronomy, University of Southampton Highfield Campus, Southampton SO17 1BJ, United Kingdom*

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In a previous work [L. Asprea, G. Gasbarri, and A. Bassi, Gravitational decoherence: A general nonrelativistic model, Phys. Rev. D **103**, 104041 (2021).] we derived a quantum master equation for the dynamics of a scalar bosonic particle interacting with a weak, stochastic and classical gravitational field. As standard matter is made of fermions, such an equation should be suitably extended to describe more relevant experimental situations. Here we derive a nonrelativistic model for the gravitational decoherence of spin 1/2 particles. We enrich the treatment by also considering a coupling with an external classical electromagnetic field. We comment on the differences with the scalar bosonic model and we describe the regimes in which they become negligible.

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I. INTRODUCTION

The recent exciting first detections of gravitational waves [1,2], which marked a new era in astrophysics and cosmology, have pushed the scientific community towards the construction of ever more sophisticated ground and space based detectors [3–7] to observe waves in a variety of ranges in order to sketch a first map of the stochastic gravitational background. Within the framework of quantum theory, a stochastic gravitational background affects the dynamics of matter propagation [8,9] and, when the quantum state is in a superposition, it leads to decoherence effects, as typical of noisy environments. Different models for the description of this phenomenon have been proposed [10–16]. However, they do not agree on the decoherence mechanism (the preferred basis and rates) at which it takes place. In order to solve such apparent contradictory results, we derived in [17] a novel model for the decoherence effect induced by a stochastic gravitational perturbation on nonrelativistic scalar bosonic particles. Our model has so far proven to be able to describe more general scenarios than those present in the literature, as it is able to qualitatively recover them [10–16] as appropriate limiting cases, thus solving the decoherence basis puzzle. However, it might not be general enough to describe the outcome of a real experiment. The particles commonly employed in experiments (atoms, neutrons, electrons...) in fact have a charge, a spin, and could be coupled to other external fields, like the Maxwell one for instance. For the above reasons, in this paper we will derive an analogous model, this time for spin 1/2 fermions interacting with both a gravitational perturbation and an external electromagnetic field.

The paper is organized as follows. In Sec. II we derive the equations of motion in Hamiltonian form for a spin 1/2 fermionic field minimally coupled to a weakly perturbed flat metric. We then specialize such equation to the nonrelativistic regime in Sec. III and proceed with the canonical quantization of the bosonic field in the single particle sector, obtaining a Schrödinger like equation for a test particle interacting with a weakly perturbed gravitational field.

In Sec. IV we compare the fermionic model derived here with the bosonic one derived in [17].

In Sec. V we specialize to the case of a stochastic gravitational perturbation and derive the corresponding master equation. We discuss the decoherence effect with explicit reference to the preferred eigenbasis and characteristic decoherence time. We also show under which assumptions our master equation is able to reproduce decoehrence in the position or momenutm eigenbasis only thus recovering the results of the literature [11–15,17].

II. EQUATIONS OF MOTION

We first derive the equations of motion (EOM) for a spin 1/2 fermionic field minimally coupled to linearized gravity. We start from the action for the Dirac field in curved spacetime [18]

*lorenzo.asprea@phd.units.it
†giulio.gasbarri@uab.cat

$$S = \int d^4x \sqrt{-g}\mathcal{L} \tag{1}$$

with the Lagrangian density

$$\mathcal{L} = \frac{i\hbar c}{2}[\bar{\psi}\gamma^\mu e^A{}_\mu \mathcal{D}_A\psi - e^A{}_\mu \mathcal{D}_A\bar{\psi}\gamma^\mu\psi] - mc^2\bar{\psi}\psi, \tag{2}$$

where $e^A{}_\mu(x)$ is the so called vierbein field [19], an auxiliary field used in order to extend the definition of fermions as irreducible spin 1/2 representations of the Poincaré group to curved spacetimes (see Appendix A), and

$$\mathcal{D}_\mu\psi = \partial_\mu\psi + \frac{1}{8}[\gamma_a, \gamma_b]\omega_\mu{}^{ab}\psi + \frac{ie}{\hbar c}A_\mu\psi \tag{3}$$

is the covariant derivative with respect to both the spin (ω^{ab}_μ) and the electromagnetic (A_μ) connections. In this framework the metric tensor g_{AB} and the affine connection Γ^{BC}_A are described by the following pair of equations:

$$\begin{cases} e_A{}^\mu \eta_{\mu\nu} e_B{}^\nu = g_{AB} \\ \omega_A{}^{\mu\nu} = e_B{}^\mu \eta^{\nu\rho}\partial_A e^B{}_\rho + e_B{}^\mu\eta^{\nu\rho}e^C{}_\rho \Gamma^B{}_{AC}. \end{cases} \tag{4}$$

Note that Eq. (4) holds only for a torsion free, metric compatible connection [19].

In order to describe a weak perturbation of the metric, we now write the metric as the sum of a flat background $\eta_{\mu\nu} = \mathrm{diag}(+---)$, and a perturbation $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{5}$$

and since we are interested in studying the dynamics of the Dirac field interacting with a weak gravitational perturbation, perform a Taylor expansion of the fermionic action around the flat background metric and truncate the series at the first perturbative order (See Appendix B for the explicit calculation). Thus, we obtain the effective Lagrangian $\mathcal{L}_{\mathrm{eff}}$ acting on flat spacetime

$$\begin{aligned} S = &\int d^4x \left(\frac{i\hbar c}{2}[\bar{\psi}\gamma^\mu\nabla_\mu\psi - \nabla_\mu(\bar{\psi})\gamma^\mu\psi]\left(1 + \frac{tr(h)}{2}\right) \right. \\ &\left. - \left(1 + \frac{tr(h)}{2}\right) mc^2\bar{\psi}\psi - \frac{i\hbar c}{4}h_{\mu\nu}[\bar{\psi}\gamma^\mu\nabla^\nu\psi - \nabla^\nu(\bar{\psi})\gamma^\mu\psi] \right) \\ &+ O(h^2) \\ \equiv &\int d^4x \mathcal{L}_{\mathrm{eff}} + O(h^2), \end{aligned} \tag{6}$$

where ∇_α is the flat covariant derivative with respect to the electromagnetic connection. The EOM for the matter field are obtained (at first order in the perturbation $h_{\mu\nu}$) from the Euler Lagrange equations

$$\frac{\partial\mathcal{L}_{\mathrm{eff}}}{\partial\bar{\psi}} - \nabla_\alpha \frac{\partial\mathcal{L}_{\mathrm{eff}}}{\partial\nabla_\alpha\bar{\psi}} = 0 \tag{7}$$

and in the harmonic gauge they read

$$\begin{aligned} i\hbar\partial_t\psi = &\, eA_0\psi + mc^2\left(1 + \frac{h_{00}}{2}\right)\gamma^0\psi - \frac{mc^2}{2}h_{0j}\gamma^j\psi \\ &- i\hbar c\left(1 + \frac{h_{00}}{2}\right)\gamma^0\gamma^i\left(\partial_i + \frac{ie}{\hbar c}A_i\right)\psi \\ &+ \frac{i\hbar c}{2}h_{0i}\left(\partial^i + \frac{ie}{\hbar c}A^i\right)\psi \\ &+ \frac{i\hbar c}{2}h_{ij}\gamma^0\gamma^i\left(\partial^j + \frac{ie}{\hbar c}A^j\right)\psi \\ &+ \frac{i\hbar c}{2}h_{0i}\gamma^i\gamma^j\left(\partial_j + \frac{ie}{\hbar c}A_j\right)\psi \\ &- \frac{i\hbar c}{8}\partial_\alpha(tr(h))\gamma^0\gamma^\alpha\psi + O(h^2)\psi \\ = &: \mathcal{H}\psi + O(h^2)\psi. \end{aligned} \tag{8}$$

As in the case of a scalar field studied in [17], we are not allowed to give a probabilistic interpretation to the field ψ, because the conserved charged Q associated to the internal $U(1)$ symmetry ($\psi \to e^{ie}\psi$; $\bar{\psi} \to e^{-ie}\bar{\psi}$) via Noether's Theorem reads

$$\begin{aligned} Q \equiv &-ie\int d^3x\left(\frac{\partial\mathcal{L}_{\mathrm{eff}}}{\partial(\nabla_0\psi)}\psi - \psi^\dagger\frac{\partial\mathcal{L}_{\mathrm{eff}}}{\partial(\nabla_0\psi^\dagger)}\right) \\ = &\,\hbar ec\int d^3x\left(\psi^\dagger\left(1 - tr(h) - \frac{h_{00}}{2}\right)\psi - \psi^\dagger\frac{h_{0i}}{2}\gamma^0\gamma^i\psi\right), \end{aligned} \tag{9}$$

instead of the required

$$\rho = \int d^3x\psi^\dagger\psi. \tag{10}$$

We therefore apply the transformation

$$\begin{cases} T &= \left(1 - \frac{tr(h)}{2} - \frac{h_{00}}{4} - \frac{h_{0i}}{4}\gamma^0\gamma^i\right) \\ \psi &\to T\psi \\ \mathcal{H} &\to \mathfrak{H} \coloneqq T\mathcal{H}T^{-1} + i\hbar T\partial_t(T^{-1}) \end{cases} \tag{11}$$

so that, in the new representation,the conserved charge can be expressed by the standard form in Eq. (10). After some algebra the EOM (8) reads

$$i\hbar\partial_t\psi = [mc^2\gamma^0 + \mathfrak{E} + \mathcal{O}]\psi, \tag{12}$$

where

$$\mathfrak{E} = eA_0 + \frac{mc^2}{2}h_{00}\gamma^0 + i\hbar c h_{0i}\left(\partial^i - \frac{ie}{\hbar c}A^i\right) + \frac{i\hbar c}{4}\partial_i(h^i_0) + \frac{\hbar c}{4}\epsilon^{ijk}\partial_i(h_{0j})\Sigma_k - \frac{3i\hbar}{8}\partial_t(tr(h)) + \frac{i\hbar}{4}\partial_t(h_{00}), \tag{13}$$

$$\mathcal{O} = -i\hbar c\left(1+\frac{h_{00}}{2}\right)\left(\partial_j - \frac{ie}{\hbar c}A_j\right)\alpha^j + \frac{i\hbar}{4}\partial_t(h_{0i})\alpha^i + \frac{i\hbar c}{2}h_{ij}\left(\partial^j - \frac{ie}{\hbar c}A^j\right)\alpha^i + \frac{i\hbar c}{4}\partial_i\left(\frac{tr(h)}{2} - h_{00}\right)\alpha^i, \tag{14}$$

are respectively the even (diagonal) and odd (off diagonal) parts of the Hamiltonian $\mathfrak{H}$, with $\alpha^\mu = \gamma^0\gamma^\mu$ and $\Sigma^i = \mathrm{diag}(\sigma^i, \sigma^i)$.

We are interested in the description of the dynamics of a positive energy particle system in the nonrelativistic limit. In such a limit, the particle and antiparticle sectors are noninteracting with one another, that is to say, the EOM (8) can be recast to a system of two decoupled equations respectively for the large (ψ_L) and the small (ψ_s) component of the bispinor $\psi = (\psi_L \psi_s)$. While this is evident and straightforward for the free case [20], for an interacting theory decoupling the two components is a very complicated task that can only be achieved perturbatively.

In the next section we will provide a standard prescription for the diagonalization of the EOM in the nonrelativistic limit.

III. NONRELATIVISTIC LIMIT AND CANONICAL QUANTIZATION

We aim to find a representation of the bispinor field ψ in which the EOM (12) are diagonal. This representation can be found in non relativistic limit following the Foldy-Wouthuysen Method [21], which allows one to write perturbatively (at any order in $\frac{v}{c}$) two decoupled equations, one for each component of the field. The method is operatively characterized by the application of an appropriate unitary transformation U,

$$\psi \to \psi' = U\psi \tag{15}$$

$$\mathfrak{H} \to \mathfrak{H}' = U(\mathfrak{H} - i\hbar\partial_t)U^{-1} = mc^2\gamma^0 + \mathfrak{E}' + \mathcal{O}' + O(h^2), \tag{16}$$

such that, in the new representation, the antidiagonal part $\mathcal{O}'$ is of higher order in $\frac{v}{c}$ than the diagonal $\mathfrak{E}'$. By neglecting $\mathcal{O}'$ one recovers two decoupled equations. By performing iteratively the transformation, one can always find a representation of the bispinor field for which the EOM are diagonal at any desired order in $\frac{v}{c}$.

In our case, the task is easily achieved by applying the subsequent transformations

$$\begin{cases} U = & e^{-i\gamma^0\mathcal{O}/(2mc^2)} \\ U' = & e^{-i\gamma^0\mathcal{O}'/(2mc^2)} \\ U'' = & e^{-i\gamma^0\mathcal{O}''/(2mc^2)} \end{cases} \tag{17}$$

after which, with some algebra (see Appendix C) and by neglecting the terms containing the derivatives of the gravitational perturbation of order $\frac{v^3}{c^3}$ or higher[22], the Hamiltonian density to order $\frac{v^4}{c^4}$ reads

$$\begin{aligned} H = {} & eA_0 + \gamma^0\Bigg[mc^2\left(1+\frac{h_{00}}{2}\right) - \frac{\hbar^2}{2m}\left(1+\frac{h_{00}}{2}\right)\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2 - \frac{\hbar e}{2mc}\left(1+\frac{h_{00}}{2}\right)B^k\Sigma_k - \frac{\hbar^2}{2m}h_{ij}\left(\partial^i - \frac{ie}{\hbar c}A^i\right)\left(\partial^j - \frac{ie}{\hbar c}A^j\right) \\ & + \frac{\hbar e}{4mc}\epsilon^{ijl}h_{jk}F_i{}^k\Sigma_l\Bigg] + \frac{i\hbar^2 e}{4m^2c^2}\left(1+\frac{h_{00}}{2}\right)\left(\frac{\nabla}{2}\times\mathbf{E} - \mathbf{E}\times\nabla\right)\cdot\mathbf{\Sigma} - (1+h_{00})\frac{\hbar^2 e}{8m^2c^2}\nabla\cdot\mathbf{E} \\ & - \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ikl}h_{ij}\partial^j(E_k)\Sigma_l - \frac{i\hbar^2 e}{8m^2c^2}\epsilon^{ikl}h_{ij}E_k\left(\partial^j - \frac{ie}{\hbar c}A^j\right)\Sigma_l + \frac{i\hbar^2 e}{4m^2c^2}\epsilon^{ijl}h_{0k}F_j{}^k\left(\partial_i - \frac{ie}{\hbar c}A_i\right)\Sigma_l \\ & - \frac{\hbar^2 e}{8m^2c^2}h_{0j}\partial_i(F^{ij}) + \frac{i\hbar^2 e}{8m^2c^2}\epsilon^{ijl}h_{0k}\partial_i(F_j{}^k)\Sigma_l - \frac{\gamma^0}{8m^3c^6}\Bigg[\hbar^4c^4(1+2h_{00})\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^4 + \hbar^2ec^2(1+2h_{00})B^2 \\ & + 2\hbar^4c^4h_{ij}\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2\left(\partial^i - \frac{ie}{\hbar c}A^i\right)\left(\partial^j - \frac{ie}{\hbar c}A^j\right) - \frac{\hbar^3ec^3}{2}\epsilon^{ijl}h_{jm}F_i{}^mB^k\{\Sigma_k, \Sigma_l\} \\ & + \frac{\hbar^3ec^3}{2}\epsilon^{ijl}\left\{\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2, h_{jk}F_i{}^k\right\}\Sigma_l - \hbar^3ec^3(1+2h_{00})\left\{\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2, B^k\right\}\Sigma_k\Bigg] \\ & + H_d + O(h^2) + O(\partial h) + O\left(\frac{v^5}{c^5}\right), \end{aligned} \tag{18}$$

where B and E are the magnetic and electric fields, and in terms of the four-potential they read:

$$\begin{cases} \mathbf{E} = & -\nabla A_0 - \frac{1}{c}\dot{\mathbf{A}} \\ \mathbf{B} = & \nabla \times \mathbf{A} \\ B^k = & -\frac{1}{2}\epsilon^{ijk}F_{ij} \\ F_{ij} = & -\epsilon_{ijk}B^k. \end{cases} \tag{19}$$

ϵ_{ijk} represent the Levi-Civita symbol, and

$$\begin{aligned} H_d = &-\frac{\hbar^2}{8m}\partial_i(h_{00})\left(\partial^i - \frac{ie}{\hbar c}A^i\right)\gamma^0 - \frac{\hbar^2}{16m}\partial^i\partial_i(h_{00})\gamma^0 \\ &+ \frac{i\hbar c}{4}\partial_i(h_0^i) + \frac{\hbar c}{4}\epsilon^{ijk}\partial_i(h_{0j})\Sigma_k - \frac{3i\hbar}{8}\partial_t(tr(h)) \\ &+ \frac{i\hbar}{4}\partial_t(h_{00}) + \gamma^0\left[\frac{\hbar^2}{2m}\partial^i(h_{00})\nabla_i - \frac{\hbar^2}{4m}\partial^i(h_{ij})\nabla^j \right. \\ &- \frac{\hbar^2}{2m}\partial_i\left(\frac{tr(h)}{2} - h_{00}\right)\nabla^i - \frac{\hbar^2}{4m}\partial^i\partial_i\left(\frac{tr(h)}{2} - h_{00}\right) \\ &\left. - \frac{i\hbar^2}{4m}\epsilon^{ijk}(\partial_i(h_{00})\nabla_j - \partial_i(h_{jl})\nabla^l)\Sigma_k\right]. \end{aligned} \tag{20}$$

Note that as the transformations (17) are unitary [23], they preserve the conserved charge in (9) i.e., the probability density in the nonrelativistic limit.

In the nonrelativistic limit the EOM (18) does not mix the two components ψ_L and ψ_s of the field (up to a very small correction). As we are only interested in the dynamics of particles, we restrict the analysis to the first field component ψ_L, that we rename as ψ in what follows.

Since the dynamics preserves the probability density, we are allowed to apply the canonical quantization prescription and impose the equal time commutation relations

$$\begin{aligned} &[\hat{\psi}(t,\mathbf{x}), \hat{\psi}(t,\mathbf{x}')] = [\hat{\psi}^\dagger(t,\mathbf{x}), \hat{\psi}^\dagger(t,\mathbf{x}')] = 0 \\ &[\hat{\psi}(t,\mathbf{x}), \hat{\psi}^\dagger(t,\mathbf{x}')] = \delta^3(\mathbf{x}-\mathbf{x}'), \end{aligned} \tag{21}$$

to obtain the EOM for the quantum field. The equation thus obtained does not allow for the creation or annihilation of particles. We can thus safely project it onto a single particle sector to obtain the single particle Schrödinger-like equation

$$i\hbar\partial_t|\phi(t)\rangle = (\hat{H}_0 + \hat{H}_r + \hat{H}_p + \hat{H}_{rp} + \hat{H}_d)|\phi(t)\rangle \tag{22}$$

with

$$\hat{H}_0 = mc^2 + \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}(\hat{\mathbf{x}})\right)^2 + eA_0(\hat{\mathbf{x}}) - \frac{\hbar e}{2mc}\mathbf{B}(\hat{\mathbf{x}})\cdot\boldsymbol{\sigma} \tag{23}$$

$$\begin{aligned} \hat{H}_r = &\frac{\hbar e}{4m^2c^2}\left(\frac{\hat{\mathbf{p}}}{2}\times\mathbf{E}(\hat{\mathbf{x}}) - \mathbf{E}(\hat{\mathbf{x}})\times\hat{\mathbf{p}}\right)\cdot\boldsymbol{\sigma} \\ &- \frac{\hbar^2 e}{8m^2c^2}\nabla\cdot\mathbf{E}(\hat{\mathbf{x}}) - \frac{\gamma^0}{8m^3c^6}\left[c^4\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}(\hat{\mathbf{x}})\right)^4 \right. \\ &\left. + \hbar^2 ec^2B^2(\hat{\mathbf{x}}) - \hbar ec^3\left\{\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}(\hat{\mathbf{x}})\right)^2, B^k(\hat{\mathbf{x}})\right\}\sigma_k\right] \end{aligned} \tag{24}$$

$$\begin{aligned} \hat{H}_p \\ = &+\frac{mc^2}{2}h^{00}(t,\hat{\mathbf{x}}) - \frac{1}{8m}\left\{h^{00}(t,\hat{\mathbf{x}}), \left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}(\hat{\mathbf{x}})\right)^2\right\} \\ &+ \frac{c}{2}\{h_{0i}(t,\hat{\mathbf{x}}), \hat{p}^i\} - \frac{\hbar e}{4mc}\epsilon^{ikl}h_{ij}(\hat{\mathbf{x}},t)F^j{}_k(\hat{\mathbf{x}})\sigma_l \\ &- \frac{1}{4m}\left\{h^{ij}(t,\hat{\mathbf{x}}), \left(\hat{p}_i - \frac{e}{c}A_i(t,\hat{\mathbf{x}})\right)\left(\hat{p}_j - \frac{e}{c}A_j(t,\hat{\mathbf{x}})\right)\right\} \\ &- \frac{\hbar e}{4mc}h_{00}(\hat{\mathbf{x}},t)\mathbf{B}(\hat{\mathbf{x}})\cdot\boldsymbol{\sigma}, \end{aligned} \tag{25}$$

$$\begin{aligned} \hat{H}_{rp} = &\frac{\hbar e}{16m^2c^2}\left\{h_{00}(\hat{\mathbf{x}},t), \left(\frac{\hat{\mathbf{p}}}{2}\times\mathbf{E}(\hat{\mathbf{x}}) - \mathbf{E}(\hat{\mathbf{x}})\times\hat{\mathbf{p}}\right)\cdot\boldsymbol{\sigma}\right\} - \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ikl}h_{ij}(\hat{\mathbf{x}})\partial^j(E_k(\hat{\mathbf{x}}))\Sigma_l - \frac{\hbar^2 e}{8m^2c^2}h_{00}(\hat{\mathbf{x}})(\hat{\mathbf{x}},t)\nabla\cdot\mathbf{E}(\hat{\mathbf{x}}) \\ &- \frac{\hbar e}{16m^2c^2}\epsilon^{ikl}\left\{h_{ij}(\hat{\mathbf{x}},t), E_k(\hat{\mathbf{x}})\left(\hat{p}^j - \frac{e}{c}A^j(\hat{\mathbf{x}})\right)\right\}\Sigma_l + \frac{\hbar e}{8m^2c^2}\epsilon^{ijl}\left\{h_{0k}(\hat{\mathbf{x}}), F_j{}^k\left(\hat{p}_i - \frac{e}{c}A_i\right)\right\}\Sigma_l \\ &- \frac{\hbar^2 e}{8m^2c^2}h_{0j}(\hat{\mathbf{x}})\partial_i(F^{ij}(\hat{\mathbf{x}})) + \frac{i\hbar^2 e}{8m^2c^2}\epsilon^{ijl}h_{0k}(\hat{\mathbf{x}})\partial_i(F_j{}^k(\hat{\mathbf{x}}))\Sigma_l - \frac{\gamma^0}{8m^3c^6}\left[c^4\left\{h_{00}(\hat{\mathbf{x}}), \left(\nabla - \frac{e}{c}\mathbf{A}(\hat{\mathbf{x}})\right)^4\right.\right. \\ &+ c^4\left\{h_{ij}(\hat{\mathbf{x}}), \left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}(\hat{\mathbf{x}})\right)^2\left(\hat{p}^i - \frac{e}{c}A^i(\hat{\mathbf{x}})\right)\left(\hat{p}^j - \frac{e}{c}A^j(\hat{\mathbf{x}})\right)\right\} - \frac{\hbar^3 ec^3}{2}\epsilon^{ijl}h_{jm}(\hat{\mathbf{x}})F_i{}^m(\hat{\mathbf{x}})B^k(\hat{\mathbf{x}})\{\Sigma_k, \Sigma_l\} \\ &\left. + 2\hbar^2 ec^2h_{00}(\hat{\mathbf{x}})B^2(\hat{\mathbf{x}}) + \frac{\hbar ec^3}{2}\epsilon^{ijl}\left\{\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2, h_{jk}(\hat{\mathbf{x}})F_i{}^k(\hat{\mathbf{x}})\right\}\Sigma_l - \hbar ec^3\left\{h_{00}(\hat{\mathbf{x}})\left\{\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2, B^k\right\}\right\}\Sigma_k\right], \end{aligned} \tag{26}$$

$$\hat{H}_d = -\frac{\hbar}{16m}\left\{\partial_i(h_{00}(\hat{\mathbf{x}})),\left(\hat{\mathbf{p}}^i-\frac{e}{c}A^i(\hat{\mathbf{x}})\right)\right\}\gamma^0-\frac{\hbar^2}{16m}\partial^i\partial_i(h_{00}(\hat{\mathbf{x}}))\gamma^0+\frac{i\hbar c}{4}\partial_i(h_0^i(\hat{\mathbf{x}}))+\frac{\hbar c}{4}\epsilon^{ijk}\partial_i(h_{0j}(\hat{\mathbf{x}}))\sigma_k$$
$$-\frac{3i\hbar}{8}\partial_t(tr(h(\hat{\mathbf{x}})))+\frac{i\hbar}{4}\partial_t(h_{00}(\hat{\mathbf{x}}))+\gamma^0\left[\frac{\hbar^2}{4m}\left\{\partial^i(h_{00}(\hat{\mathbf{x}})),\left(\hat{\mathbf{p}}_i-\frac{e}{c}A_i(\hat{\mathbf{x}})\right)\right\}-\frac{\hbar^2}{8m}\left\{\partial^i(h_{ij}(\hat{\mathbf{x}})),\left(\hat{\mathbf{p}}^j-\frac{e}{c}A^j(\hat{\mathbf{x}})\right)\right\}\right.$$
$$-\frac{\hbar^2}{4m}\left\{\partial_i\left(\frac{tr(h(\hat{\mathbf{x}}))}{2}-h_{00}(\hat{\mathbf{x}})\right),\left(\hat{\mathbf{p}}^i-\frac{e}{c}A^i(\hat{\mathbf{x}})\right)\right\}-\frac{\hbar^2}{4m}\partial^i\partial_i\left(\frac{tr(h(\hat{\mathbf{x}}))}{2}-h_{00}(\hat{\mathbf{x}})\right)-\frac{e}{c}A^l(\hat{\mathbf{x}})\Big)\Big\}\Big)\sigma_k$$
$$\left.-\frac{i\hbar^2}{8m}\epsilon^{ijk}\left(\left\{\partial_i(h_{00}(\hat{\mathbf{x}})),\left(\hat{\mathbf{p}}_j-\frac{e}{c}A_j(\hat{\mathbf{x}})\right)\right\}-\left\{\partial_i(h_{jl}(\hat{\mathbf{x}})),\hat{\mathbf{p}}^l-\frac{e}{c}A^l(\hat{\mathbf{x}})\right\}\right]\right., \tag{27}$$

where $\hat{\mathbf{x}}, \hat{\mathbf{p}}$ are respectively the single particle position and the momentum operator. The term $\hat{H}_0$ is the usual Pauli Hamiltonian [20] plus an irrelevant global phase mc^2 that can be reabsorbed with the transformation

$$|\phi(t)\rangle \to e^{imc^2t/\hbar}|\phi(t)\rangle. \tag{28}$$

The term $\hat{H}_r$ encodes the standard relativistic corrections [23] due to the presence of the electromagnetic field up to order $\frac{v^4}{c^4}$. Finally, $\hat{H}_p$, $\hat{H}_d$ and $\hat{H}_{rp}$ respectively account for the corrections due to the presence of the weak gravitational field of $\hat{H}_0$ and $\hat{H}_r$.

IV. DIFFERENCES WITH THE BOSONIC MODEL

Equation (18) is rather instructive as it extends the nonrelativistic Hamiltonian obtained in [17] for the scalar field which we recall below [24]

$$\begin{aligned}\hat{H}_0{}^{(B)} &= mc^2+\frac{\hat{\mathbf{p}}^2}{2m}\\ \hat{H}_p{}^{(B)} &= \frac{mc^2}{2}h^{00}(t,\hat{\mathbf{x}})-\frac{\hbar^2}{8m}\{h^{00}(t,\hat{\mathbf{x}}),\hat{\mathbf{p}}^2\}+\frac{c}{2}\{h^{0i}(t,\hat{\mathbf{x}}),\hat{p}_i\}-\frac{1}{4m}\{h^{ij}(t,\hat{\mathbf{x}}),\hat{p}_i\hat{p}_j\}\\ \hat{H}_d{}^{(B)} &= \frac{\hbar^2}{8m}\nabla^2(tr[h^{\mu\nu}(t,\hat{\mathbf{x}})])+\frac{i\hbar}{2}\partial_t(h^{00}(t,\hat{\mathbf{x}}))-\frac{i\hbar}{4}\partial_t(tr[h^{\mu\nu}(t,\hat{\mathbf{x}})]),\end{aligned} \tag{29}$$

to describe the dynamics of a charged quantum particle (with spin 1/2) subject to a gravitational perturbation and to an external electromagnetic field. It is however as instructive to consider the electromagnetic free case i.e., the limit $A(t,\hat{\mathbf{x}})\to 0$, in order to directly compare the fermionic and bosonic Hamiltonian. By taking the limit $A(t,\hat{\mathbf{x}})\to 0$ of Eq. (18), we obtain

$$\begin{aligned}\hat{H}_0{}^{(F)} &= mc^2+\frac{\hat{\mathbf{p}}^2}{2m}\\ \hat{H}_p{}^{(F)} &= \frac{mc^2}{2}h^{00}(t,\hat{\mathbf{x}})-\frac{\hbar^2}{8m}\{h^{00}(t,\hat{\mathbf{x}}),\hat{\mathbf{p}}^2\}+\frac{c}{2}\{h^{0i}(t,\hat{\mathbf{x}}),\hat{p}_i\}-\frac{1}{4m}\{h^{ij}(t,\hat{\mathbf{x}}),\hat{p}_i\hat{p}_j\}\end{aligned} \tag{30}$$

$$\hat{H}_d{}^{(F)} = -\frac{\hbar}{16m}\left\{\partial_i(h_{00}(\hat{\mathbf{x}})),\left(\hat{\mathbf{p}}^i-\frac{e}{c}A^i(\hat{\mathbf{x}})\right)\right\}\gamma^0-\frac{\hbar^2}{16m}\partial^i\partial_i(h_{00}(\hat{\mathbf{x}}))\gamma^0+\frac{i\hbar c}{4}\partial_i(h_0^i(\hat{\mathbf{x}}))-\frac{3i\hbar}{8}\partial_t(tr(h(\hat{\mathbf{x}})))$$
$$+\frac{\hbar c}{4}\epsilon^{ijk}\partial_i(h_{0j}(\hat{\mathbf{x}}))\sigma_k+\frac{i\hbar}{4}\partial_t(h_{00}(\hat{\mathbf{x}}))+\gamma^0\left[\frac{\hbar^2}{4m}\left\{\partial^i(h_{00}(\hat{\mathbf{x}})),\left(\hat{\mathbf{p}}_i-\frac{e}{c}A_i(\hat{\mathbf{x}})\right)\right\}\right.$$
$$-\frac{\hbar^2}{8m}\left\{\partial^i(h_{ij}(\hat{\mathbf{x}})),\left(\hat{\mathbf{p}}^j-\frac{e}{c}A^j(\hat{\mathbf{x}})\right)\right\}-\frac{\hbar^2}{4m}\left\{\partial_i\left(\frac{tr(h(\hat{\mathbf{x}}))}{2}-h_{00}(\hat{\mathbf{x}})\right),\left(\hat{\mathbf{p}}^i-\frac{e}{c}A^i(\hat{\mathbf{x}})\right)\right\}$$
$$-\frac{\hbar^2}{4m}\partial^i\partial_i\left(\frac{tr(h(\hat{\mathbf{x}}))}{2}-h_{00}(\hat{\mathbf{x}})\right)-\frac{i\hbar^2}{8m}\epsilon^{ijk}\left(\left\{\partial_i(h_{00}(\hat{\mathbf{x}})),\left(\hat{\mathbf{p}}_j-\frac{e}{c}A_j(\hat{\mathbf{x}})\right)\right\}\right.$$
$$\left.\left.-\left\{\partial_i(h_{jl}(\hat{\mathbf{x}})),\left(\hat{\mathbf{p}}^l-\frac{e}{c}A^l(\hat{\mathbf{x}})\right)\right\}\right)\sigma_k\right]. \tag{31}$$

As expected, the bosonic and the fermionc description match for the gravity free case ($\hat{H}_0{}^{(B)} = \hat{H}_0{}^{(F)}$). They also match for the terms proportional to the gravitational perturbation $h_{\mu\nu}$. This is also to be expected—suppose in fact that there was actually a difference in the terms $h_{00}\hat{\mathbf{p}}^2/2m$ or $mc^2 h_{00}$. This would imply that e.g., a simple change form Cartesian to Rindler [25] coordinates would predict that a boson and a fermion would fall with the same acceleration in the first (Cartesian) but not the second (Rindler) reference frame, which would violate the weak equivalence principle. The same line of reasoning can be applied to the other terms containing h_{ij} and h_{0i}. It is interesting however to notice that some differences arise for the terms containing the derivatives of the gravitational perturbation $\partial h_{\mu\nu}$. Such differences originated when we required the matter field to allow for a probabilistic interpretation in order to canonically quantize the system (see Eq. (11) and Eq. (x) of [17]).

V. MASTER EQUATION WITH ELECTROMAGNETIC FIELD

In this section we derive a master equation to describe the decoherence effect induced by a weak stochastic gravitational perturbation on a spin 1/2 fermionic particle. For the sake of simplicity and compactness of the result, we will restrict our analysis to the Pauli Hamiltonian $\hat{H}_0$ and its gravitational corrections $\hat{H}_p$, as the terms $\hat{H}_r$ and $\hat{H}_{rp}$ are of higher order in the nonrelativistic expansion [26], and the term $\hat{H}_d$ contains derivatives of the gravitational perturbations (as in typical experimental situations [1,3–6] they are negligible and would not add any further informative content to the analysis in any case).

This means that we approximate Eq. (22) to

$$i\hbar\partial_t|\phi(t)\rangle = (\hat{H}_0 + \hat{H}_p)|\phi(t)\rangle. \tag{32}$$

If the metric is random, Eq. (32) becomes a stochastic differential equation. As a consequence the predictions are given by taking the stochastic average over the random gravitational field. We then need to specify its stochastic properties.

As done for the bosonic particle case, we assume the noise to be Gaussian with zero mean. For the sake of simplicity, we also assume the different components of the metric fluctuation to be uncorrelated. This means that the noise is fully characterized by

$$\begin{aligned} \mathbb{E}[h_{\mu\nu}(\mathbf{x},t)] &= 0 \\ \mathbb{E}[h_{\mu\nu}(\mathbf{x},t)h_{\mu\nu}(\mathbf{y},s)] &= \alpha^2 f_{\mu\nu}(\mathbf{x},\mathbf{y};t,s), \end{aligned} \tag{33}$$

where we recall that $\mathbb{E}[\cdot]$ denotes the stochastic average, α represents the strength of the gravitational fluctuations, and $f(\mathbf{x},\mathbf{y};t,s)$ is the two-point correlation function.

We move to the density operator formalism, and write the von Neumann equation for the averaged density matrix

$$\begin{aligned} \partial_t\hat{\rho}(t) &= -\frac{i}{\hbar}[\hat{H}_0(t),\hat{\rho}(t)] - \frac{i}{\hbar}\mathbb{E}[[\hat{H}_p(t),\hat{\Omega}(t)]] \\ &\equiv \mathbb{E}[\mathfrak{L}[\hat{\Omega(t)}]], \end{aligned} \tag{34}$$

where $\hat{\rho}(t) = \mathbb{E}[\Omega(t)]$. We solve the above equation perturbatively exploiting the cumulant expansion [27] (see Appendix C of [17]). With the further help of the Gaussianity, zero mean, and noncorrelation of different components, we can rewrite Eq. (34) in Fourier space [28] as

$$\begin{aligned} \partial_t\hat{\rho} = &-\frac{i}{\hbar}[\hat{H}_0,\hat{\rho}(t)] + -\frac{\alpha^2}{\hbar^8}\int\frac{d^3q d^3q'}{(2\pi)^3}\int_0^t dt_1 \frac{\tilde{f}^{00}(\mathbf{q},\mathbf{q}';t,t_1)}{4}[\{e^{i\mathbf{q}\cdot\hat{\mathbf{x}}/\hbar},\Xi_{00}(\hat{\mathbf{x}},\hat{\mathbf{p}})\},[\{e^{i\mathbf{q}'\cdot\hat{\mathbf{x}}_{t_1}/\hbar},\Xi_{00}(\hat{\mathbf{x}}_{t_1},\hat{\mathbf{p}})\},\hat{\rho}(t)]] \\ &-\frac{\alpha^2c^2}{\hbar^8}\int\frac{d^3q d^3q'}{(2\pi)^3}\int_0^t dt_1 \frac{\tilde{f}^{0i}(\mathbf{q},\mathbf{q}';t,t_1)}{4}[\{e^{i\mathbf{q}\cdot\hat{\mathbf{x}}/\hbar},\hat{p}_i\},[\{e^{i\mathbf{q}'\cdot\hat{\mathbf{x}}_{t_1}/\hbar},\hat{p}_i\},\hat{\rho}(t)]] \\ &-\frac{\alpha^2}{\hbar^8}\int\frac{d^3q d^3q'}{(2\pi)^3}\int_0^t dt_1 \frac{\tilde{f}^{ij}(\mathbf{q},\mathbf{q}';t,t_1)}{4}[\{e^{i\mathbf{q}\cdot\hat{\mathbf{x}}/\hbar},\Xi_{ij}(\hat{\mathbf{x}},\hat{\mathbf{p}})\},[\{e^{i\mathbf{q}'\cdot\hat{\mathbf{x}}_{t_1}/\hbar},\Xi_{ij}(\hat{\mathbf{x}}_{t_1},\hat{\mathbf{p}})\},\hat{\rho}(t)]] + O(t\alpha^3\tau_c^2), \end{aligned} \tag{35}$$

where we have introduced

$$\begin{aligned} \Xi_{00}(\hat{\mathbf{x}},\hat{\mathbf{p}}) &= \frac{(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}(t,\hat{\mathbf{x}}))^2}{4m} + \frac{mc^2}{2} - \frac{\hbar e}{2mc}\mathbf{B}(t,\hat{\mathbf{x}})\cdot\boldsymbol{\sigma} \\ \Xi_{ij}(\hat{\mathbf{x}},\hat{\mathbf{p}}) &= \frac{(\hat{p}_i - \frac{e}{c}A_i(t,\hat{\mathbf{x}}))(\hat{p}_j - \frac{e}{c}A_j(t,\hat{\mathbf{x}}))}{4m} + \frac{mc^2}{2} \\ &\quad + \frac{\hbar e}{2mc}\epsilon_{kil}F^k{}_j(t,\hat{\mathbf{x}})\sigma^l, \end{aligned} \tag{36}$$

for the sake of compactness, and $\hat{x}_{t_1} = e^{i\hat{H}_0 t_1}\hat{x}e^{-i\hat{H}_0 t_1}$.

The above equation describes the dynamics of a pointlike spin 1/2 fermionic particle in presence of an external weak, stochastic gravitational field (with the further assumptions made in this section) and an external electromagnetic field.

We specialize Eq. (35) to the Markovian limit, i.e., we assume the noise to be delta correlated in time, with the further assumptions of isotropy and homogeneity of the noise, so that its correlation function reads

$$f^{\mu\nu}(\mathbf{x},\mathbf{y};t,s) = \lambda u^{\mu\nu}(\mathbf{x}-\mathbf{y})\delta(t-s), \tag{37}$$

where the factor λ is, in principle, a generic coefficient with the dimension of a time. Note that the white noise assumption makes physical sense only if the correlation time (τ_c) of the gravitational fluctuations is much smaller than the free dynamics' characteristic time (τ_{free}), or in the case where the contribution to the dynamics due to the gravitational perturbation is not affected by the free evolution dynamics, i.e., the operators describing the perturbation commute with the free dynamics operator $\hat{H}_0$. In such cases, as a first approximation, we can take λ to be

$$\lambda = \min(\tau_c, t). \tag{38}$$

Note that this choice does not affect the generality of the analysis as we leave $u^{\mu\nu}(\mathbf{x}-\mathbf{y})$ unspecified.

In such a regime Eq. (35) is exact and it is easy to show that it reduces to

$$\begin{aligned}\partial_t\hat{\rho} = &-\frac{i}{\hbar}[\hat{H},\hat{\rho}(t)]\\ &-\frac{\alpha^2\lambda}{(2\pi)^{3/2}\hbar^5}\int d^3q\tilde{u}^{00}(\mathbf{q})[\{e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\Xi_{00}(\hat{\mathbf{x}},\hat{\mathbf{p}})\},[\{e^{-i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\Xi_{00}(\hat{\mathbf{x}},\hat{\mathbf{p}})\},\hat{\rho}(t)]]\\ &-\frac{\alpha^2\lambda c^2}{(2\pi)^{3/2}\hbar^5}\int d^3q\tilde{u}^{0i}(\mathbf{q})[\{e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\hat{P}_i\},[\{e^{-i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\hat{P}_i\},\hat{\rho}(t)]]\\ &-\frac{\alpha^2\lambda}{(2\pi)^{3/2}\hbar^5}\int d^3q\tilde{u}^{ij}(\mathbf{q})[\{e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\Xi_{ij}(\hat{\mathbf{x}},\hat{\mathbf{p}})\},[\{e^{-i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\Xi_{ij}(\hat{\mathbf{x}},\hat{\mathbf{p}})\},\hat{\rho}(t)]].\end{aligned} \tag{39}$$

Eq. (39) describes decoherence in a complex combination of position momentum and energy bases, as it contains double commutators of functions of the position, momentum and free kinetic energy operators with the averaged density matrix.

In what follows we will specialize Eq. (39) to determine under which approximations it recovers decoherence in the position or momentum eigenbasis only.

As for the bosonic case [17], the conditions

$$\begin{cases} h^{00}\gtrsim h^{0i}\\ h^{00}\gtrsim h^{ij}\\ \Delta E\ll Mc^2(1-u^{00}(\Delta\mathbf{x}))\end{cases} \tag{40}$$

are sufficient for our master equation to describe decoherence in the position eigenbasis only, where in this case the energy coherence needs to be modified to take into account the presence of the electromagnetic field, as $E=\frac{(\mathbf{p}-\frac{e}{c}\mathbf{A})^2}{2m}$. Under the above assumptions, Eq. (39) reads

$$\partial_t\hat{\rho} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}(t)]-\frac{\alpha^2\lambda}{(2\pi)^{3/2}\hbar^5}\int d^3q\tilde{u}^{00}(\mathbf{q})\left[e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}\left(\frac{mc^2}{2}-\frac{\hbar e\boldsymbol{\sigma}}{2mc}\cdot\mathbf{B}(t,\hat{\mathbf{x}})\right),\left[\left\{e^{-i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}\left(\frac{mc^2}{2}-\frac{\hbar e\boldsymbol{\sigma}}{2mc}\cdot\mathbf{B}(t,\hat{\mathbf{x}})\right),\hat{\rho}(t)\right]\right]. \tag{41}$$

Contrary to the bosonic case, the condition of low momentum transfer

$$e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}\sim\hat{\mathbb{1}} \tag{42}$$

is necessary, but not sufficient, to recover deocherence in the momentum or energy eigenbasis starting from Eq. (39). One in fact needs the further condition

$$\begin{cases}|\mathbf{p}|\gg|\frac{e}{c}\mathbf{A}|\\ \frac{p^2}{2m}\gg|\frac{\hbar e\boldsymbol{\sigma}}{2mc}\cdot\mathbf{B}|.\end{cases} \tag{43}$$

In this regime, Eq. (39) can be approximated as

$$\begin{aligned}\partial_t\hat{\rho} = &-\frac{i}{\hbar}[\hat{H},\hat{\rho}(t)]\\ &-\frac{\alpha^2\lambda}{(2\pi)^{3/2}\hbar^5}\int d^3q\tilde{u}^{00}(\mathbf{q})\left[\frac{\hat{\mathbf{p}}^2}{2m},\left[\frac{\hat{\mathbf{p}}^2}{2m},\hat{\rho}(t)\right]\right]\\ &-\frac{\alpha^2\lambda c^2}{(2\pi)^{3/2}\hbar^5}\int d^3q\tilde{u}^{0i}(\mathbf{q})[\hat{p}_i,[\hat{p}_i,\hat{\rho}(t)]]\\ &-\frac{\alpha^2\lambda}{(2\pi)^{3/2}\hbar^5}\int d^3q\tilde{u}^{ij}(\mathbf{q})\left[\frac{\hat{p}_i\hat{p}_j}{2m},\left[\frac{\hat{p}_i\hat{p}_j}{2m},\hat{\rho}(t)\right]\right],\end{aligned} \tag{44}$$

which indeed describes decoherence in the momentum eigenbasis.

VI. CONCLUSIONS

In this paper we have extended the results of our previous paper [17] to the ferminonic case. We have derived a model of decoherence for nonrelativistic spin 1/2 particles interacting with both a weak stochastic gravitational perturbation and an external electromagnetic field. The resulting Hamiltonian and master equation correctly reproduce the results of the bosonic model in [17] up to very small corrections. Such corrections account for relativistic effects (different spin of the two kind of particles) and the different quantization scheme employed.

The dynamics predicts also in this case decoherence in the position, momentum and energy eigenbasis, though under different limiting cases than those described in [17].

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APPENDIX A: VIERBEIN (OR TETRAD) FORMULATION OF GRAVITY

We illustrate the basic ingredients of the tetrad formalism of the General Relativity theory. For a more complete treatment we address the reader to [19,29].

The standard geometrical interpretation of the gravitational interaction is based on the notion of the Riemannian metric (g) and the Christoffel connection (Γ). The spacetime curvature, its dynamical evolution, and the interaction with matter sources are described through differential equations involving g and Γ.

It is possible though to equivalently describe the geometry of a Riemannian manifold (M) using the notion of vierbein and local connection. Such a formalism is particularly convenient when one wants to formulate a theory of gravity as a gauge theory, and wants to accommodate the notion of particles as irreducible representations of the Poincaré group in curved spacetimes [30–32].

We know that locally the laws of special relativity are valid. This translates into the consideration that we can attach at each and every point p of the Riemannian manifold M a flat tangent manifold equipped with the flat Minkowski metric.

There is a natural choice for the basis of such a tangent space (T_pM), the coordinate (or differential) basis

$$\hat{e}_{(\mu)} = \partial_{(\mu)} \tag{A1}$$

given by the partial derivatives of the coordinates. It follows that a given four-vector $A \in T_pM$ has components

$$A = A^\mu \hat{e}_{(\mu)} = A^\mu \partial_{(\mu)}. \tag{A2}$$

The dual basis

$$\hat{e}^{(\mu)} = dx^{(\mu)} \tag{A3}$$

spans the cotangent space, and it is given by the differential of the coordinates. A dual vector $B \in T_pM$ then has components

$$B = B_\mu \hat{e}^{(\mu)} = B_\mu dx^{(\mu)}. \tag{A4}$$

As T_pM is a vector space, we are in principle free to choose any orthonormal basis to span it, as long as T_pM preserves the appropriate signature of the manifold. We therefore introduce a set of basis vectors $\hat{e}_a$, which we choose as noncoordinate unit vectors, and we denote this choice by using small Latin letters for the indices of the noncoordinate frame. Such a noncoordinate basis is called a tetrad basis. The condition for preserving the signature of the metric therefore reads

$$g(\hat{e}_a, \hat{e}_b) = \eta_{ab} = \mathrm{diag}(+, - - -). \tag{A5}$$

With this choice, we can clearly find a fixed orthonormal basis that is independent of position. Then, from a local perspective, any vector can be expressed as a linear combination of the fixed tetrad basis vectors at the point in the following way

$$\hat{e}_\mu(x) = e_\mu{}^a(x)\hat{e}_{(a)} \tag{A6}$$

$$V^a = e_\mu{}^a V^\mu. \tag{A7}$$

The 4×4 invertible matrix $e_\mu{}^a(x)$ is called a vierbein field (or tetrad), and it is the transformation matrix that maps the tangent space T_xM into Minkowski space preserving the inner product.

The inverse vierbein field (or tetrad) has components $e^\mu{}_a(x)$, and satisfies the orthonormality condition

$$e^\mu{}_a e_\nu{}^a = \delta^\mu_\nu$$
$$e_\mu{}^a e^\mu{}_b = \delta^a_b, \tag{A8}$$

which come from the preservation of the inner product.

The vierbein fields are mixed indices objects, in the sense that they carry one Minkowski spacetime index (a), and one Riemannian index (μ). Accordingly, they transform under coordinate and Lorentz transformations respectively as

$$e_\mu{}^a \underset{\longrightarrow}{\text{coord}} e'^{\,a}_\mu = \frac{\partial x^\nu}{\partial x'^\mu} e_\nu{}^a \tag{A9}$$

$$e_\mu{}^a(x) \underset{\longrightarrow}{\text{Lorentz}} e'^{\,a}_\mu(x) = \Lambda^a{}_b e_\mu{}^b. \tag{A10}$$

We now consider the covariant derivative ∇X of a vector (X) in the Minkowski frame. It will be given by the standard derivative (∂X) plus a correction given by the affine connection of the Minkowski frame

$$(\nabla_\mu X^a) dx^\mu \otimes \hat{e}_{(a)} = (\partial_\mu X^a + \omega_\mu{}^a{}_b X^b) dx^\mu \otimes \hat{e}_{(a)}. \tag{A11}$$

The expression for the covariant derivative in the coordinate basis instead reads

$$\begin{aligned} \nabla X &= (\nabla_\mu x^\nu) dx^\mu \otimes \partial_\nu \\ &= (\partial_\mu X^\nu + \Gamma^\nu_{\mu\alpha} X^\alpha) dx^\mu \otimes \partial_\nu \\ &= (\partial_\mu X^\nu + \Gamma^\nu_{\mu\alpha} X^\alpha) dx^\mu \otimes e_\nu{}^a(x) \hat{e}_{(a)} \\ &= e_\nu{}^a(x)(\partial_\mu (e^\nu{}_b(x) X^b) + \Gamma^\nu_{\mu\alpha} e^\alpha{}_b(x) X^b) dx^\mu \otimes \hat{e}_{(a)}. \end{aligned} \tag{A12}$$

Upon comparing Eq. (A11) with Eq. (A12), we can express the Minkowski frame or local affine connection in terms of the tetrads and the usual affine connection as

$$\omega_\mu{}^a{}_b X^b = e_\nu{}^a(x) \partial_\mu e^\nu{}_b(x) X^b + e_\nu{}^a(x) e^\alpha{}_b(x) \Gamma^\nu_{\mu\alpha} X^b. \tag{A13}$$

Note that the above relation implies the metric compatibility condition

$$\begin{aligned} \nabla_\mu e^\nu{}_b(x) &= 0, \\ \nabla_\mu g_{\alpha\beta} &= 0. \end{aligned} \tag{A14}$$

Observing that $\nabla_\mu X^a$ must transform under a Lorentz boost as X^a, it follows:

$$\begin{aligned} \nabla_\mu(\Lambda^a{}_b) &= 0 \\ &= \partial_\mu(\Lambda^a{}_b) + \omega_\mu{}^a{}_c \Lambda^c{}_b - \omega_\mu{}^c{}_b \Lambda^a{}_c. \end{aligned} \tag{A15}$$

Upon multiplying the last line of Eq. (A15) by $\Lambda^b{}_d$ on the left, we obtain the following relation:

$$\omega_\mu{}^a{}_d = \Lambda^b{}_d \Lambda^a{}_c \omega_\mu{}^c{}_b - \Lambda^b{}_d \partial_\mu(\Lambda^a{}_b), \tag{A16}$$

which tells us that the affine connection transforms inhomogeneously under Lorentz transformations.

One can construct the usual geometric objects from (e,ω), as it is typically done from (g,Γ), such as the curvature tensor

$$R^{ab}{}_{\mu\nu} = \partial_\mu \omega_\nu{}^a{}_b - \partial_\nu \omega_\mu{}^a{}_b + \omega_\mu{}^a{}_c \omega_\nu{}^c{}_b - \omega_\nu{}^a{}_c \omega_\mu{}^c{}_b \tag{A17}$$

and the torsion

$$\mathfrak{T}^a_{\mu\nu} = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^a{}_b e_\nu{}^b - \omega_\nu{}^a{}_b e_\mu{}^b. \tag{A18}$$

The field equations for the vierbein field can be derived from a variational principle in the same fashion it is typically done for the metric. In order to show it, let us recall the inner product-signature preservation condition Eq. (A5), which can be equivalently recast into

$$g_{\mu\nu} = e_\mu{}^a \eta_{ab} e_\nu{}^b. \tag{A19}$$

It then follows that the variation of the metric can be expressed in terms of the variation of the vierbein filed as

$$\delta g_\mu = e_{\nu a} \delta e_\mu{}^a + e_{\mu a} \delta e_\nu{}^a = -(g_{\mu\lambda} e_\nu{}^a + g_{\nu\lambda} e_\mu{}^a) \delta e_\lambda{}^a. \tag{A20}$$

The variation of the Einstein-Hilbert action ($S = \frac{1}{8\pi G} \times \int d^4x \sqrt{-g} R$ [25]) then reads

$$\begin{aligned} \partial_g S &= \frac{1}{8\pi G} \int d^4x \sqrt{-g} \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \delta g_{\mu\nu} \\ &= \frac{1}{8\pi G} \int d^4x e \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \frac{\partial g_{\mu\nu}}{\partial e^\lambda{}_a} \delta e^\lambda{}_a \\ &= \frac{1}{8\pi G} \int d^4x e \left(R_\lambda{}^\nu e_\nu{}^a - \frac{1}{2} R \delta^\nu_\lambda e_\nu{}^a + R^\mu{}_\lambda e_\mu{}^a \right. \\ &\quad \left. - \frac{1}{2} R \delta^\mu_\lambda e_\mu{}^a \right) \delta e^\lambda{}_a \\ &= \frac{1}{8\pi G} \int d^4x e \left(R^\mu{}_\nu - \frac{1}{2} \delta^\mu_\lambda R \right) e_\mu{}^a \delta e^\lambda{}_a. \end{aligned} \tag{A21}$$

Recalling the expression for the Einstein tensor ($G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$), the above equation yields

$$G^{\mu\nu} e_\mu{}^a = 0, \tag{A22}$$

which must be interpreted as the Einstein's equations for the vierbein field e. Note that in order to switch back to the usual metric formulation it is sufficient to multiply the above equation by $e^\lambda{}_a$. We have thus shown that the vierbein formulation of general relativity is equivalent to the standard metric one.

APPENDIX B: EOM FOR THE SPIN 1/2 FERMIONIC FIELD

We present the explicit steps for the derivation of the effective action of the fermionic action coupled to a weak gravitational perturbation.

Consider the action for the Dirac field in curved spacetime,

$$S = \int d^4x\sqrt{-g}\mathcal{L}_D \tag{B1}$$

with the Lagrangian density

$$\mathcal{L}_D = \frac{i\hbar c}{2}[\bar{\psi}\gamma^\mu e^A{}_\mu \mathcal{D}_A\psi - e^A{}_\mu \mathcal{D}_A\bar{\psi}\gamma^\mu\psi] - mc^2\bar{\psi}\psi, \tag{B2}$$

where $e^A{}_\mu(x)$ is the so called vierbein field, the field that maps the tangent space to the manifold M at point x T_xM (coordinate basis ∂_A) into Minkowski space (noncoordinate basis $\mathbf{e}_\mu$), and

$$\mathcal{D}_\mu\psi = \partial_\mu\psi + \frac{1}{8}[\gamma_a,\gamma_b]\omega_\mu{}^{ab}\psi + \frac{ie}{\hbar c}A_\mu\psi \tag{B3}$$

is the covariant derivative with respect to both the spin and the electromagnetic connections. The pair $(e_A{}^\mu, \omega_A{}^{\mu\nu})$ allows for an equivalent geometrization of the gravitational interaction to the standard one given in terms of the metric and the affine connection $(g_{AB}, \Gamma^A{}_{BC})$; the relation between the two frameworks is given by

$$\begin{cases} e_A{}^\mu \eta_{\mu\nu} e_B{}^\nu = g_{AB} \\ \omega_A{}^{\mu\nu} = e_B{}^\mu \eta^{\nu\rho}\partial_A e^B{}_\rho + e_B{}^\mu\eta^{\nu\rho}e^C{}_\rho\Gamma^B{}_{AC}. \end{cases} \tag{B4}$$

Note that (B4) holds only for a torsion free, metric compatible connection.

We write the metric as the sum of a flat background $\eta_{\mu\nu} = \mathrm{diag}(+---)$, and a perturbation $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{B5}$$

We are interested in studying the dynamics of the Dirac field in presence of a weak gravitational perturbation. We therefore perform a Taylor expansion of the action around the flat background metric and truncate the series at the first perturbative order

$$S \approx \int d^4x(\sqrt{-g}\mathcal{L})|_{g=\eta} - h^{\mu\nu}\left(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}}\right)|_{g=\eta} + O(h^2). \tag{B6}$$

In order to work out the explicit expression for $\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}}$, we look at the variation of the action with respect to the metric tensor

$$\begin{aligned}\delta_g S &= -\frac{1}{2}\int d^4x\sqrt{-g}T^{AB}\delta g_{AB} \\ &= \int d^4x \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{AB}}\delta g_{AB}.\end{aligned} \tag{B7}$$

Notice that the above expression can be equivalently rewritten for a torsion free, metric compatible connection as

$$\begin{aligned}\delta_g S &= \int d^4x \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial e^C{}_\alpha}\delta e^C{}_\alpha + \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial\omega_{A\mu\nu}}\delta\omega_{A\mu\nu} \\ &= \int d^4x\sqrt{-g}\frac{\partial\mathcal{L}}{\partial e^C{}_\alpha}\delta e^C{}_\alpha + \sqrt{-g}\frac{\partial\mathcal{L}}{\partial\omega_{A\mu\nu}}\delta\omega_{A\mu\nu} \\ &\quad + \mathcal{L}\left(\frac{\partial\sqrt{-g}}{\partial e^C{}_\alpha}\delta e^C{}_\alpha + \frac{\partial\sqrt{-g}}{\partial\omega_{A\mu\nu}}\delta\omega_{A\mu\nu}\right).\end{aligned} \tag{B8}$$

By noticing that $\frac{\partial\sqrt{-g}}{\partial\omega_{A\mu\nu}} = 0$, and defining $\frac{\partial\mathcal{L}}{\partial e^C{}_\alpha} =: \mathcal{T}_C{}^\alpha$ and $\frac{\partial\mathcal{L}}{\partial\omega_{A\mu\nu}} =: \mathcal{S}^{A\mu\nu}$, we rewrite the above equation as

$$\begin{aligned}\delta S &= \int d^4x\sqrt{-g}[\mathcal{T}_C{}^\alpha\delta e^C{}_\alpha + \mathcal{S}^{A\mu\nu}\delta\omega_{A\mu\nu} + 2e_C{}^\alpha\mathcal{L}_D\delta e^C{}_\alpha] \\ &= \int d^4x\sqrt{-g}[(\mathcal{T}_C{}^\alpha + 2e_C{}^\alpha\mathcal{L}_D - \mathcal{D}_A[\mathcal{S}^A{}_C{}^\alpha - \mathcal{S}^{A\alpha}{}_C \\ &\quad + \mathcal{S}_C{}^{A\alpha} + \mathcal{S}^{\alpha A}{}_C - \mathcal{S}_C{}^{\alpha A} - \mathcal{S}^\alpha{}_C{}^A])\delta e^C{}_\alpha] \\ &= \int d^4x\sqrt{-g}(B_C{}^\alpha + 2e_C{}^\alpha\mathcal{L}_D)\delta e^C{}_\alpha \\ &=: \int d^4x\sqrt{-g}\Theta_C{}^\alpha\delta e^C{}_\alpha,\end{aligned} \tag{B9}$$

where $B_C{}^\alpha$ is the Belinfante stress energy tensor [33]. In the case of a fermionic field it reads [34]

$$\begin{aligned}B_C{}^\alpha &= \frac{i\hbar c}{4}[\bar{\psi}\gamma^\alpha\mathcal{D}_C\psi - \mathcal{D}_C\bar{\psi}\gamma^\alpha\psi + \bar{\psi}\gamma_C\mathcal{D}^\alpha\psi - \mathcal{D}^\alpha\bar{\psi}\gamma_C\psi] \\ &= \frac{1}{2}(\mathcal{T}_C{}^\alpha + \mathcal{T}^\alpha{}_C).\end{aligned} \tag{B10}$$

Comparing Eq. (B7) and Eq. (B9), we notice

$$\begin{aligned}\Theta_C{}^\alpha\delta e^C{}_\alpha &= -\frac{1}{2}T^{AB}\delta g_{AB} \\ &= \frac{1}{2}T^{AB}(g_{AC}e_B{}^\alpha + g_{BC}e_A{}^\alpha)\delta e^C{}_\alpha \\ &= T_C{}^\alpha\delta e^C{}_\alpha.\end{aligned} \tag{B11}$$

Thus we can write Eq. (B6) as

$$
\begin{aligned}
S &\approx \int d^4x \left[(\sqrt{-g}\mathcal{L}_D)\Big|_{g=\eta} + \frac{\partial(\mathcal{L}_D\sqrt{-g})}{\partial g_{AB}}\Big|_{g=\eta} h_{AB} + O(h^2) \right] \\
&= \int d^4x \left[(\sqrt{-g}\mathcal{L}_D)\Big|_{g=\eta} - \frac{1}{2}(\Theta^{A\alpha} e^{B}{}_{\alpha})\Big|_{g=\eta} h_{AB} + O(h^2) \right] \\
&= \int d^4x \Bigg(\frac{i\hbar c}{2}[\bar{\psi}\gamma^\mu\nabla_\mu\psi - \nabla_\mu(\bar{\psi})\gamma^\mu\psi]\left(1+\frac{tr(h)}{2}\right) \\
&\quad - \left(1+\frac{tr(h)}{2}\right) mc^2\bar{\psi}\psi - \frac{i\hbar c}{4} h_{\mu\nu}[\bar{\psi}\gamma^\mu\nabla^\nu\psi - \nabla^\nu(\bar{\psi})\gamma^\mu\psi] \Bigg) \\
&\quad + O(h^2) \\
&=: \int d^4x \mathcal{L}_{\rm eff} + O(h^2),
\end{aligned} \tag{B12}
$$

and recover Eq. (6) of the main text.

APPENDIX C: FOLDY WOUTHUYSEN METHOD—FERMIONIC MODEL

Here we illustrate the Fouldy Wouthuysen method applied to Eq. (12). Let us consider the transformations

$$\mathfrak{H} \to \mathfrak{H}' = U(\mathfrak{H} - i\hbar\partial_t)U^{-1}, \tag{C1}$$

and specialize U to Eq. (17) i.e.,

$$U = e^{-i\gamma^0\mathcal{O}/(2mc^2)} =: e^{iS}. \tag{C2}$$

With the help of the Baker-Campbell-Hausdorff identity:

$$
\begin{aligned}
\mathfrak{H}' &= e^{iS}(\mathfrak{H} - i\hbar\partial_t)e^{-iS} = \mathfrak{H} + i[S,\mathfrak{H}] + \frac{i^2}{2!}[S[S,\mathfrak{H}]] \\
&\quad + \frac{i^3}{3!}[S[S[S,\mathfrak{H}]]] + \cdots \\
&\quad + \hbar\left(-\dot{S} - \frac{i}{2}[S,\dot{S}] + \frac{1}{6}[S,[S,\dot{S}]] + \cdots\right)
\end{aligned} \tag{C3}
$$

Recalling that:

$$\mathfrak{H} = mc^2\gamma^0 + \mathfrak{E} + \mathcal{O} \tag{C4}$$

and noticing that

$$[\gamma^0, \mathfrak{E}] = 0 \tag{C5}$$

$$\{\gamma^0, \mathcal{O}\} = 0 \tag{C6}$$

$$[\gamma^0\mathcal{O}, \gamma^0] = -2\mathcal{O} \tag{C7}$$

$$[\gamma^0\mathcal{O}, \mathfrak{E}] = \gamma^0[\mathcal{O}, \mathfrak{E}] \tag{C8}$$

$$[\gamma^0\mathcal{O}, \mathcal{O}] = 2\gamma^0\mathcal{O}^2 \tag{C9}$$

we get

$$\mathfrak{H}' = mc^2\gamma^0 + \mathfrak{E}' + \mathcal{O}', \tag{C10}$$

where

$$
\begin{aligned}
\mathfrak{E}' &= \mathfrak{E} + \gamma^0\left(\frac{\mathcal{O}^2}{2mc^2} - \frac{\mathcal{O}^4}{8m^3c^6}\right) - \frac{1}{8m^2c^4}[\mathcal{O},[\mathcal{O},\mathfrak{E}] \\
&\quad + i\hbar\dot{\mathcal{O}}] + \cdots
\end{aligned} \tag{C11}
$$

$$\mathcal{O}' = \frac{1}{2mc^2}\gamma^0[\mathcal{O},\mathfrak{E}] - \frac{\mathcal{O}^3}{3m^2c^4} + \frac{i}{2mc^2}\gamma^0\dot{\mathcal{O}} + \cdots \tag{C12}$$

We note that $\mathcal{O}'$ is of order c^{-1}, meaning that we need to perform a further transformation if we want a nontrivial diagonal EOM. The transformation that we perform is

$$U' = e^{-i\gamma^0\mathcal{O}'/(2mc^2)} \tag{C13}$$

after which the Hamiltonian reads

$$\mathfrak{H}'' = mc^2\gamma^0 + \mathfrak{E}' + \mathcal{O}'' + \cdots \tag{C14}$$

with

$$\mathcal{O}'' = \frac{\gamma^0}{2mc^2}[\mathcal{O}',\mathfrak{E}'] + \frac{i}{2mc^2}\gamma^0\dot{\mathcal{O}}' + \cdots. \tag{C15}$$

As $\mathcal{O}'' \sim O(\frac{v^3}{c^3})$ we need to perform a final transformation

$$U'' = e^{-i\gamma^0\mathcal{O}''/(2mc^2)}. \tag{C16}$$

Finally the Hamiltonian reads

$$H \coloneqq \mathfrak{H}''' = mc^2\gamma^0 + \mathfrak{E}' + O\left(\frac{v^5}{c^5}\right). \tag{C17}$$

In order to calculate the explicit expression of the Hamiltonian in Eq. (C17), we pick the Pauli representation for the Dirac gamma matrices,

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & -\mathbb{1} \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} \mathbb{0} & \sigma^i \\ -\sigma^i & \mathbb{0} \end{pmatrix} \tag{C18}$$

$$\alpha^i \equiv \gamma^0\gamma^i = \begin{pmatrix} \mathbb{0} & \sigma^i \\ \sigma^i & \mathbb{0} \end{pmatrix}, \qquad \Sigma_i = \begin{pmatrix} \sigma_i & \mathbb{0} \\ \mathbb{0} & \sigma_i \end{pmatrix}. \tag{C19}$$

By exploiting the identities

$$
\begin{aligned}
\left[\left(\partial^j - \frac{ie}{\hbar c}A^j\right), \left(\partial_k - \frac{ie}{\hbar c}A_k\right)\right] &= -\frac{ie}{\hbar c}F^j{}_k \\
\alpha^i\alpha^j &= -\eta^{ij} + \epsilon^{ijk}\Sigma_k \\
\{\alpha^i, \alpha^j\} &= -2\eta^{ij} \\
\eta^{ij} &= -\delta^{ij}
\end{aligned} \tag{C20}
$$

it only takes a bit of algebra to show that

$$\frac{\gamma^0}{2mc^2}\mathcal{O}^2 = \frac{\gamma^0}{2mc^2}\left(-i\hbar c\left(1+\frac{h_{00}}{2}\right)\left(\partial_j - \frac{ie}{\hbar c}A_j\right)\gamma^0\gamma^j + \frac{i\hbar c}{2}h_{ij}\left(\partial^j - \frac{ie}{\hbar c}A^j\right)\gamma^0\gamma^i - \frac{i\hbar c}{4}\partial_i\left(\frac{tr(h)}{2} - h_{00}\right)\gamma^0\gamma^i \right.$$
$$\left. + \frac{i\hbar}{4}\partial_t(h_{0i})\gamma^0\gamma^i\right)^2$$
$$= \gamma^0\left[-\frac{\hbar^2}{2m}(1+h_{00})\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2 - \frac{\hbar e}{2mc}(1+h_{00})B^k\Sigma_k - \frac{\hbar^2}{2m}h_{ij}\left(\partial^i - \frac{ie}{\hbar c}A^i\right)\left(\partial^j - \frac{ie}{\hbar c}A^j\right)\right.$$
$$\left. + \frac{\hbar e}{4mc}\epsilon^{ijl}h_{jk}F_i{}^k\Sigma_l\right] + \gamma^0\left[\frac{\hbar^2}{2m}\partial^i(h_{00})\nabla_i - \frac{\hbar^2}{4m}\partial^i(h_{ij})\nabla^j - \frac{\hbar^2}{2m}\partial_i\left(\frac{tr(h)}{2} - h_{00}\right)\nabla^i\right.$$
$$\left. - \frac{i\hbar^2}{4m}\epsilon^{ijk}(\partial_i(h_{00})\nabla_j - \partial_i(h_{jl})\nabla^l)\Sigma_k - \frac{\hbar^2}{4m}\partial^i\partial_i\left(\frac{tr(h)}{2} - h_{00}\right)\right] + O(h^2). \tag{C21}$$

As the above term is of order $\gamma^0\frac{\mathcal{O}^2}{2mc^2} \sim O(\frac{v^2}{c^2})$, it follows that the next term in Eq. (C11) is of order $\gamma^0\frac{\mathcal{O}^4}{8m^3c^6} \sim O(\frac{v^4}{c^4})$. After some algebra it reads

$$\mathcal{O}^4 = \left(\hbar^2c^2(1+h_{00})\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2 - \hbar ec(1+h_{00})B^k\Sigma_k - \frac{i\hbar^2c^2}{2}\epsilon^{ijk}\partial_i(h_{00})\left(\partial_j - \frac{ie}{\hbar c}A_j\right)\Sigma_k\right.$$
$$+ \frac{\hbar^2c^2}{2}\partial^i(h_{00})\left(\partial_i - \frac{ie}{\hbar c}A_i\right) - \hbar^2c^2h_{ij}\left(\partial^i - \frac{ie}{\hbar c}A^i\right)\left(\partial^j - \frac{ie}{\hbar c}A^j\right) + \frac{\hbar ec}{2}\epsilon^{ijl}h_{jl}F_i{}^k\Sigma_l$$
$$- \frac{\hbar^2}{4m}\partial^i(h_{ij})\left(\partial^j - \frac{ie}{\hbar c}A^j\right) + \frac{i\hbar^2c^2}{2}\epsilon^{ijl}\partial_i(h_{jk})\left(\partial^k - \frac{ie}{\hbar c}A^k\right)\Sigma_l$$
$$\left. - \hbar^2c^2\partial_i\left(\frac{tr(h)}{2} - h_{00}\right)\partial^i - \frac{\hbar^2c^2}{2}\partial^i\partial_i\left(\frac{tr(h)}{2} - h_{00}\right)\right)^2$$
$$\coloneqq \hbar^4c^4(1+2h_{00})\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^4 + \hbar^2ec^2(1+2h_{00})B^2 - \hbar^3ec^3(1+2h_{00})\left\{\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2, B^k\right\}\Sigma_k$$
$$+ \frac{\hbar^3ec^3}{2}\epsilon^{ijl}\left\{\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2, h_{jk}F_i{}^k\right\}\Sigma_l + 2\hbar^4c^4h_{ij}\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2\left(\partial^i - \frac{ie}{\hbar c}A^i\right)\left(\partial^j - \frac{ie}{\hbar c}A^j\right)$$
$$- \frac{\hbar^3ec^3}{2}\epsilon^{ijl}h_{jm}F_i{}^mB^k\{\Sigma_k, \Sigma_l\} + \frac{i\hbar^4c^4}{2}\epsilon^{ijk}\left\{\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2, \partial_i(h_{00})\left(\partial_j - \frac{ie}{\hbar c}A_j\right)\right\}\Sigma_k$$
$$- \frac{i\hbar^4c^4}{2}\left\{\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2, \partial^i(h_{00})\left(\partial_i - \frac{ie}{\hbar c}A_i\right)\right\}$$
$$+ \frac{\hbar^4c^4}{2}\left\{\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2, \partial_i(h_{ij})\left(\partial^j - \frac{ie}{\hbar c}A^j\right)\right\} - \frac{\hbar^4c^4}{2}\epsilon^{ijl}\left\{\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2, \partial_i(h_{jk})\left(\partial^k - \frac{ie}{\hbar c}A^k\right)\right\}\Sigma_l$$
$$+ \hbar^4c^4\left\{\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2, \partial_i\left(\frac{tr(h)}{2} - h_{00}\right)\partial_i\right\} + \frac{\hbar^4c^4}{2}\left\{\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2, \partial^i\partial_i\left(\frac{tr(h)}{2} - h_{00}\right)\right\}$$
$$+ \frac{\hbar^3ec^3}{2}\left\{B^k, \partial^i(h_{ij})\left(\partial^j - \frac{ie}{\hbar c}A^j\right)\right\}\Sigma_k + \frac{i\hbar^3ec^3}{2}\epsilon^{ijl}\left\{B^k\Sigma_k, \partial_i(h_{00})\left(\partial_j - \frac{ie}{\hbar c}A_j\right)\Sigma_l\right\}$$
$$- \frac{i\hbar^3ec^3}{e}\left\{B^k, \partial^i(h_{00})\left(\partial_i - \frac{ie}{\hbar c}A_i\right)\right\}\Sigma_k + \frac{i\hbar^3ec^3}{2}\epsilon^{ijl}\left\{B^k\Sigma_k, \partial_i(h_{jm})\left(\partial^m - \frac{ie}{\hbar c}A^m\right)\Sigma_l\right\}$$
$$+ \hbar^3ec^3\left\{B^k, \partial_i\left(\frac{tr(h)}{2} - h_{00}\right)\partial^i\right\}\Sigma_k + \frac{\hbar^3ec^3}{2}\left\{B^k, \partial_i\partial^i\left(\frac{tr(h)}{2} - h_{00}\right)\right\}\Sigma_k. \tag{C22}$$

The last term in Eq. (C11) requires lengthy intermediate calculations in order to get to the final result. We start by considering the expressions $[\mathcal{O}, \mathfrak{E}]$ and $\dot{\mathcal{O}}$ separately. With the help of Eq. (C20) and some algebra

$$[\mathcal{O},\mathfrak{E}] = \left[-i\hbar c\left(1+\frac{h_{00}}{2}\right)\left(\partial_j - \frac{ie}{\hbar c}A_j\right)\gamma^0\gamma^j + \frac{i\hbar c}{2}h_{ij}\left(\partial^j - \frac{ie}{\hbar c}A^j\right)\gamma^0\gamma^i + \frac{i\hbar c}{4}\partial_t(h_{0i})\gamma^0\gamma^i + \frac{i\hbar c}{4}\partial_i\left(\frac{tr(h)}{2} - h_{00}\right)\gamma^0\gamma^i,\right.$$
$$\left. eA_0 + \frac{mc^2}{2}h_{00}\gamma^0 + i\hbar c h_{0i}\left(\partial^i - \frac{ie}{\hbar c}A^i\right) + \frac{i\hbar c}{4}\partial_i(h^i_0) + \frac{\hbar c}{4}\epsilon^{ijk}\partial_i(h_{0j})\Sigma_k - \frac{3i\hbar}{8}\partial_t(tr(h)) + \frac{i\hbar}{4}\partial_t(h_{00})\right]$$
$$= i\hbar mc^3 h_{00}\nabla_i\gamma^i + \frac{i\hbar mc^3}{2}\partial_i(h_{00})\gamma^i - i\hbar ec h_{0j}F_i{}^j\alpha^i + \hbar^2c^2\partial_i(h_{0j})\nabla^j\alpha^i - i\hbar ec\left(1+\frac{h_{00}}{2}\right)\partial_i(A_0)\alpha^i$$
$$+ \frac{i\hbar ec}{2}h_{ij}\partial^j(A_0)\alpha^i + \frac{\hbar^2c^2}{2}(\partial_j(h_{0i})\nabla^i\alpha^j - \partial_j(h_{0i}\nabla^j\alpha^i)) - \frac{i\hbar^2c^2}{4}\epsilon^{jkl}\partial_i\partial_j(h_{0k})\alpha^i\Sigma_l$$
$$- \frac{3\hbar^2 c}{8}\partial_i\partial_t(tr(h))\alpha^i + \frac{\hbar^2 c}{4}\partial_i\partial_t(h_{00})\alpha^i \tag{C23}$$

$$\dot{\mathcal{O}} = -\frac{i\hbar c}{2}\partial_t(h_{00})\nabla_i\alpha^i - \frac{e}{2}h_{00}\partial_t(A_i)\alpha^i + \frac{i\hbar c}{2}\partial_t(h_{ij})\nabla^j\alpha^i + \frac{e}{2}h_{ij}\partial_t(A^j)\alpha^i$$
$$+ \frac{i\hbar c}{4}\partial_t\partial_i\left(\frac{tr(h)}{2} - h_{00}\right)\alpha^i. \tag{C24}$$

Upon plugging the Eqs. (C23) and (C24) into the last term in Eq. (C11), exploiting again the identities in Eq. (C20), and with a lot of algebra, we arrive at the final expression

$$-\frac{[\mathcal{O},[\mathcal{O},\mathfrak{E}] + i\hbar\dot{\mathcal{O}}]}{8m^2c^4} = +\frac{\hbar^2}{4m}h_{00}\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2\gamma^0 + \frac{e\hbar}{4mc}h_{00}\mathbf{B}\cdot\gamma^0\mathbf{\Sigma} + \frac{i\hbar^2 e}{4m^2c^2}\left(1+\frac{h_{00}}{2}\right)\left(\frac{\nabla}{2}\times\mathbf{E} - \mathbf{E}\times\nabla\right)\cdot\mathbf{\Sigma}$$
$$-(1+h_{00})\frac{\hbar^2 e}{8m^2c^2}\nabla\cdot\mathbf{E} - \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ikl}h_{ij}\partial^j(E_k)\Sigma_l - \frac{i\hbar^2 e}{8m^2c^2}\epsilon^{ikl}h_{ij}E_k\left(\partial^j - \frac{ie}{\hbar c}A^j\right)\Sigma_l$$
$$-\frac{\hbar^2 e}{8m^2c^2}h_{0j}\partial_i(F^{ij}) + \frac{i\hbar^2 e}{4m^2c^2}\epsilon^{ijl}h_{0k}F_j{}^k\left(\partial_i - \frac{ie}{\hbar c}A_i\right)\Sigma_l + \frac{i\hbar^2 e}{8m^2c^2}\epsilon^{ijl}h_{0k}\partial_i(F_j{}^k)\Sigma_l$$
$$-\frac{\hbar^2 e}{16m^2c^2}\partial_i(h_{00})E^i - \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ijk}\partial_i(h_{00})E_j\Sigma_k - \frac{\hbar^2 e}{16m^2c^2}\partial^i(h_{ij})E^j + \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ijl}\partial_i(h_{jk})E^k\Sigma_l$$
$$+\frac{\hbar^2}{8m}\partial_i(h_{00})\left(\partial^i - \frac{ie}{\hbar c}A^i\right) - \frac{i\hbar^2}{8m}\epsilon^{ijk}\partial_i(h_{00})\left(\partial_j - \frac{ie}{\hbar c}A_j\right)\Sigma_k$$
$$-\frac{\hbar^2 e}{8m^2c^2}\partial^i(h_{0j})F_i{}^j + \frac{i\hbar^2 e}{8m^2c^2}\epsilon^{ijl}\partial_i(h_{0k})F_j{}^k\Sigma_l - \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ijk}\partial_i\left(\frac{tr(h)}{2} - h_{00}\right)E_j\Sigma_k$$
$$-\frac{i\hbar^3}{16m^2c}\epsilon^{jkl}\epsilon^{ima}\partial_m(h_{0j})\left\{\left(\partial_i - \frac{ie}{\hbar c}A_i\right),\left(\partial_k - \frac{ie}{\hbar c}A_k\right)\right\}\Sigma_a\Sigma_l$$
$$-\frac{\hbar^3}{8m^2c}\epsilon^{jkl}\partial^i(h_{0j})\left(\partial_k - \frac{ie}{\hbar c}A_k\right)\left(\partial_i - \frac{ie}{\hbar c}A_i\right)\Sigma_l$$
$$-\frac{i\hbar^3}{8m^2c}\epsilon^{jkl}\epsilon_l{}^{im}\partial_m(h_{0j})\left(\partial_k - \frac{ie}{\hbar c}A_k\right)\left(\partial_k - \frac{ie}{\hbar c}A_i\right)\Sigma_a\Sigma^a + \frac{i\hbar^2 e}{8m^2c^2}\epsilon^{jkl}\partial^i(h_{0j})F_{ki}\Sigma_l$$
$$+\frac{\hbar^3}{16m^2c}\epsilon^{jkl}\partial^i\partial_i(h_{0j})\left(\partial_k - \frac{ie}{\hbar c}A_k\right)\Sigma_l - \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ikl}\partial_i(h_{0j})F^j{}_k\Sigma_l$$
$$-\frac{i\hbar^3}{8m^2c}\partial^i\partial_i(h_{0j})\left(\partial^j - \frac{ie}{\hbar c}A^j\right) + \frac{\hbar^3}{8m^2c}\epsilon^{jki}\partial_i(h_{0j})\left(\partial_k - \frac{ie}{\hbar c}A_k\right)\left(\partial^l - \frac{ie}{\hbar c}A^l\right)\Sigma_l$$

$$
\begin{aligned}
&+\frac{i\hbar^3}{32m^2c^2}\epsilon^{ijl}\partial_i\partial_t(h_{jk})\left(\partial^k-\frac{ie}{\hbar c}A^k\right)\Sigma_l-\frac{\hbar^3}{16m^2c}\epsilon^{kjl}\partial_k\partial_i(h_{0j})\left(\partial^i-\frac{ie}{\hbar c}A^i\right)\Sigma_l\\
&+\frac{\hbar^3}{16m^2c}\epsilon^{jkl}\partial^i\partial_j(h_{0k})\left(\partial_i-\frac{ie}{\hbar c}A_i\right)\Sigma_l+\frac{i\hbar^3}{16m^2c}\epsilon^{jkl}\epsilon^{ima}\partial_m\partial_j(h_{0k})\bigg(\partial_i\\
&-\frac{ie}{\hbar c}A_i\bigg)\Sigma_l\Sigma_a-\frac{\hbar^3}{16m^2c}\epsilon^{jkl}\partial^i\partial_i\partial_j(h_{0k})\Sigma_l+\frac{i\hbar^3}{32m^2c^2}\partial^i\partial_i\partial_t(tr(h)-h_{00})\\
&-\frac{\hbar^2}{8m}\partial_i(h_{00})\left(\partial^i-\frac{ie}{\hbar c}A^i\right)\gamma^0-\frac{\hbar^2}{16m}\partial^i\partial_i(h_{00})\gamma^0-\frac{i\hbar^3}{16m^2c^2}\partial^i\partial_t(h_{00})\left(\partial^i-\frac{ie}{\hbar c}A^i\right)\\
&-\frac{\hbar^3}{16m^2c^2}\epsilon^{ijk}\partial_j\partial_t(h_{00})\left(\partial_i-\frac{ie}{\hbar c}A_i\right)\Sigma_k-\frac{i\hbar^2e}{8m^2c^3}\epsilon^{ijl}\partial_t(h_{jk})F_i{}^k\Sigma_l+\frac{i\hbar^3}{32m^2c^2}\partial^i\partial_t(h_{ij})\left(\partial^j-\frac{ie}{\hbar c}A^j\right)\\
&+\frac{i\hbar^3}{16m^2c^2}\epsilon^{ijk}\partial_t\partial_j(tr(h)-h_{00})\left(\partial_i-\frac{ie}{\hbar c}A_i\right)\Sigma_k+\frac{i\hbar^3}{32m^2c^2}\partial_t\partial^i(tr(h)-h_{00})\left(\partial_i-\frac{ie}{\hbar c}A_i\right).
\end{aligned}
\tag{C25}
$$

So that the total Hamiltonian reads

$$
\begin{aligned}
H={}&eA_0+\gamma^0\bigg[mc^2\left(1+\frac{h_{00}}{2}\right)-\frac{\hbar^2}{2m}\left(1+\frac{h_{00}}{2}\right)\left(\nabla-\frac{ie}{\hbar c}\mathbf{A}\right)^2\\
&-\frac{\hbar e}{2mc}\left(1+\frac{h_{00}}{2}\right)B^k\Sigma_k-\frac{\hbar^2}{2m}h_{ij}\left(\partial^i-\frac{ie}{\hbar c}A^i\right)\left(\partial^j-\frac{ie}{\hbar c}A^j\right)\\
&+\frac{\hbar e}{4mc}\epsilon^{ijl}h_{jk}F_i{}^k\Sigma_l\bigg]+\frac{i\hbar^2e}{4m^2c^2}\left(1+\frac{h_{00}}{2}\right)\left(\frac{\nabla}{2}\times\mathbf{E}-\mathbf{E}\times\nabla\right)\cdot\mathbf{\Sigma}-(1+h_{00})\frac{\hbar^2e}{8m^2c^2}\nabla\cdot\mathbf{E}\\
&-\frac{i\hbar^2e}{16m^2c^2}\epsilon^{ikl}h_{ij}\partial^j(E_k)\Sigma_l-\frac{i\hbar^2e}{8m^2c^2}\epsilon^{ikl}h_{ij}E_k\left(\partial^j-\frac{ie}{\hbar c}A^j\right)\Sigma_l\\
&+\frac{i\hbar^2e}{4m^2c^2}\epsilon^{ijl}h_{0k}F_j{}^k\left(\partial_i-\frac{ie}{\hbar c}A_i\right)\Sigma_l-\frac{\hbar^2e}{8m^2c^2}h_{0j}\partial_i(F^{ij})+\frac{i\hbar^2e}{8m^2c^2}\epsilon^{ijl}h_{0k}\partial_i(F_j{}^k)\Sigma_l\\
&-\frac{\gamma^0}{8m^3c^6}\bigg[\hbar^4c^4(1+2h_{00})\left(\nabla-\frac{ie}{\hbar c}\mathbf{A}\right)^4+\hbar^2ec^2(1+2h_{00})B^2\\
&+2\hbar^4c^4h_{ij}\left(\nabla-\frac{ie}{\hbar c}\mathbf{A}\right)^2\left(\partial^i-\frac{ie}{\hbar c}A^i\right)\left(\partial^j-\frac{ie}{\hbar c}A^j\right)+\frac{\hbar^3ec^3}{2}\epsilon^{ijl}\left\{\left(\nabla-\frac{ie}{\hbar c}\mathbf{A}\right)^2,h_{jk}F_i{}^k\right\}\Sigma_l\\
&-\frac{\hbar^3ec^3}{2}\epsilon^{ijl}h_{jm}F_i{}^mB^k\{\Sigma_k,\Sigma_l\}-\hbar^3ec^3(1+2h_{00})\left\{\left(\nabla-\frac{ie}{\hbar c}\mathbf{A}\right)^2,B^k\right\}\Sigma_k\bigg]\\
&-\frac{\hbar^2}{8m}\partial_i(h_{00})\left(\partial^i-\frac{ie}{\hbar c}A^i\right)\gamma^0-\frac{\hbar^2}{16m}\partial^i\partial_i(h_{00})\gamma^0+\frac{i\hbar c}{4}\partial_i(h_0^i)\\
&+\frac{\hbar c}{4}\epsilon^{ijk}\partial_i(h_{0j})\Sigma_k-\frac{3i\hbar}{8}\partial_t(tr(h))+\frac{i\hbar}{4}\partial_t(h_{00})\\
&+\gamma^0\bigg[\frac{\hbar^2}{2m}\partial^i(h_{00})\nabla_i-\frac{\hbar^2}{4m}\partial^i(h_{ij})\nabla^j-\frac{\hbar^2}{2m}\partial_i\left(\frac{tr(h)}{2}-h_{00}\right)\nabla^i\\
&-\frac{i\hbar^2}{4m}\epsilon^{ijk}(\partial_i(h_{00})\nabla_j-\partial_i(h_{jl})\nabla^l)\Sigma_k-\frac{\hbar^2}{4m}\partial^i\partial_i\left(\frac{tr(h)}{2}-h_{00}\right)\bigg]\\
&+H_{dd}+O(h^2)+O\left(\frac{v^5}{c^5}\right),
\end{aligned}
\tag{C26}
$$

with

$$
\begin{aligned}
H_{dd} = &-\frac{\hbar^2 e}{16m^2c^2}\partial_i(h_{00})E^i - \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ijk}\partial_i(h_{00})E_j\Sigma_k - \frac{\hbar^2 e}{16m^2c^2}\partial^i(h_{ij})E^j + \frac{\hbar^2}{8m}\partial_i(h_{00})\left(\partial^i - \frac{ie}{\hbar c}A^i\right) \\
&+ \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ijl}\partial_i(h_{jk})E^k\Sigma_l - \frac{i\hbar^2}{8m}\epsilon^{ijk}\partial_i(h_{00})\left(\partial_j - \frac{ie}{\hbar c}A_j\right)\Sigma_k - \frac{\hbar^2 e}{8m^2c^2}\partial^i(h_{0j})F_i{}^j + \frac{i\hbar^2 e}{8m^2c^2}\epsilon^{ijl}\partial_i(h_{0k})F_j{}^k\Sigma_l \\
&- \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ijk}\partial_i\left(\frac{tr(h)}{2} - h_{00}\right)E_j\Sigma_k - \frac{i\hbar^3}{16m^2c}\epsilon^{jkl}\epsilon^{ima}\partial_m(h_{0j})\left\{\left(\partial_i - \frac{ie}{\hbar c}A_i\right), \left(\partial_k - \frac{ie}{\hbar c}A_k\right)\right\}\Sigma_a\Sigma_l \\
&- \frac{i\hbar^3}{8m^2c}\epsilon^{jkl}\epsilon_l{}^{im}\partial_m(h_{0j})\left(\partial_k - \frac{ie}{\hbar c}A_k\right)\left(\partial_k - \frac{ie}{\hbar c}A_i\right)\Sigma_a\Sigma^a + \frac{i\hbar^2 e}{8m^2c^2}\epsilon^{jkl}\partial^i(h_{0j})F_{ki}\Sigma_l \\
&- \frac{\hbar^3}{8m^2c}\epsilon^{jkl}\partial^i(h_{0j})\left(\partial_k - \frac{ie}{\hbar c}A_k\right)\left(\partial_i - \frac{ie}{\hbar c}A_i\right)\Sigma_l + \frac{\hbar^3}{16m^2c}\epsilon^{jkl}\partial^i\partial_i(h_{0j})\left(\partial_k - \frac{ie}{\hbar c}A_k\right)\Sigma_l - \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ikl}\partial_i(h_{0j})F^j{}_k\Sigma_l \\
&- \frac{i\hbar^3}{8m^2c}\partial^i\partial_i(h_{0j})\left(\partial^j - \frac{ie}{\hbar c}A^j\right) + \frac{\hbar^3}{8m^2c}\epsilon^{jki}\partial_i(h_{0j})\left(\partial_k - \frac{ie}{\hbar c}A_k\right)\left(\partial^l - \frac{ie}{\hbar c}A^l\right)\Sigma_l \\
&- \frac{\hbar^3}{16m^2c}\epsilon^{kjl}\partial_k\partial_i(h_{0j})\left(\partial^i - \frac{ie}{\hbar c}A^i\right)\Sigma_l + \frac{h^3}{16m^2c}\epsilon^{jkl}\partial^i\partial_j(h_{0k})\left(\partial_i - \frac{ie}{\hbar c}A_i\right)\Sigma_l \\
&+ \frac{i\hbar^3}{16m^2c}\epsilon^{jkl}\epsilon^{ima}\partial_m\partial_j(h_{0k})\left(\partial_i - \frac{ie}{\hbar c}A_i\right)\Sigma_l\Sigma_a - \frac{\hbar^3}{16m^2c}\epsilon^{jkl}\partial^i\partial_i\partial_j(h_{0k})\Sigma_l \\
&- \frac{\hbar^2}{8m}\partial_i(h_{00})\left(\partial^i - \frac{ie}{\hbar c}A^i\right)\gamma^0 - \frac{\hbar^2}{16m}\partial^i\partial_i(h_{00})\gamma^0 - \frac{i\hbar^3}{16m^2c^2}\partial^i\partial_t(h_{00})\left(\partial^i - \frac{ie}{\hbar c}A^i\right) \\
&- \frac{\hbar^3}{16m^2c^2}\epsilon^{ijk}\partial_j\partial_t(h_{00})\left(\partial_i - \frac{ie}{\hbar c}A_i\right)\Sigma_k - \frac{i\hbar^2 e}{8m^2c^3}\epsilon^{ijl}\partial_t(h_{jk})F_i{}^k\Sigma_l \\
&+ \frac{i\hbar^3}{16m^2c^2}\epsilon^{ijk}\partial_t\partial_j(tr(h) - h_{00})\left(\partial_i - \frac{ie}{\hbar c}A_i\right)\Sigma_k + \frac{i\hbar^3}{32m^2c^2}\partial^i\partial_i\partial_t(tr(h) - h_{00}) + \frac{i\hbar^3}{32m^2c^2}\partial^i\partial_t(h_{ij})\left(\partial^j - \frac{ie}{\hbar c}A^j\right) \\
&+ \frac{i\hbar^3}{32m^2c^2}\epsilon^{ijl}\partial_i\partial_t(h_{jk})\left(\partial^k - \frac{ie}{\hbar c}A^k\right)\Sigma_l + \frac{i\hbar^3}{32m^2c^2}\partial_t\partial^i(tr(h) - h_{00})\left(\partial_i - \frac{ie}{\hbar c}A_i\right) \\
&- \frac{\gamma^0}{8m^3c^6}\left[\frac{i\hbar^4c^4}{2}\epsilon^{ijk}\left\{\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2, \partial_i(h_{00})\left(\partial_j - \frac{ie}{\hbar c}A_j\right)\right\}\Sigma_k - \frac{i\hbar^4c^4}{2}\left\{\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2, \partial^i(h_{00})\left(\partial_i - \frac{ie}{\hbar c}A_i\right)\right\}\right. \\
&+ \frac{\hbar^4c^4}{2}\left\{\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2, \partial_i(h_{ij})\left(\partial^j - \frac{ie}{\hbar c}A^j\right)\right\} - \frac{\hbar^4c^4}{2}\epsilon^{ijl}\left\{\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2, \partial_i(h_{jk})\left(\partial^k - \frac{ie}{\hbar c}A^k\right)\right\}\Sigma_l \\
&+ \hbar^4c^4\left\{\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2, \partial_i\left(\frac{tr(h)}{2} - h_{00}\right)\partial_i\right\} \\
&+ \frac{\hbar^4c^4}{2}\left\{\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}\right)^2, \partial^i\partial_i\left(\frac{tr(h)}{2} - h_{00}\right)\right\} + \frac{\hbar^3ec^3}{2}\left\{B^k, \partial^i(h_{ij})\left(\partial^j - \frac{ie}{\hbar c}A^j\right)\right\}\Sigma_k \\
&+ i\hbar^3ec^3\epsilon^{ijl}\left\{B^k\Sigma_k, \partial_i(h_{00})\left(\partial_j - \frac{ie}{\hbar c}A_j\right)\Sigma_l\right\} - i\hbar^3ec^3\left\{B^k, \partial^i(h_{00})\left(\partial_i - \frac{ie}{\hbar c}A_i\right)\right\}\Sigma_k \\
&+ i\hbar^3ec^3\epsilon^{ijl}\left\{B^k\Sigma_k, \partial_i(h_{ja})\left(\partial^a - \frac{ie}{\hbar c}A^a\right)\Sigma_l\right\} + \hbar^3ec^3\left\{B^k, \partial_i\left(\frac{tr(h)}{2} - h_{00}\right)\partial^i\right\}\Sigma_k \\
&\left.+ \frac{\hbar^3ec^3}{2}\left\{B^k, \partial_i\partial^i\left(\frac{tr(h)}{2} - h_{00}\right)\right\}\Sigma_k\right]. \qquad \text{(C27)}
\end{aligned}
$$

By neglecting the terms containing derivatives of the gravitational field of order $\frac{v^3}{c^3}$ or higher (namely the term H_{dd}), we recover Eq. (18) of the main text.

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[22] For the sake of compactness we relegate such terms to Appendix D. Note also that in most experimental situations such contributions are negligible in the case of gravity, and they wouldn't add any further informative content to the analysis in any case.
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[26] Note however that one needs to be careful when applying the results of this section to a real experimental situations, as the term $\hat{H}_r$ might dominate over $\hat{H}_p$ (depending on the size of $\mathbf{E}$,$\mathbf{B}$, and $h_{\mu\nu}$) and should therefore be taken in consideration.
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[28] Recall that our choice for the Fourier transform is:

$$f(\mathbf{x}) = \frac{1}{(\sqrt{2\pi}\hbar)^3}\int d^3q\tilde{f}(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}$$

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