Gravitational Bremsstrahlung in the post-Minkowskian effective field theory

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We study the gravitational radiation emitted during the scattering of two spinless bodies in the post-Minkowskian effective field theory approach. We derive the conserved stress-energy tensor linearly coupled to gravity and the classical probability amplitude of graviton emission at leading and next-to-leading order in the Newton's constant G. The amplitude can be expressed in compact form as one-dimensional integrals over a Feynman parameter involving Bessel functions. We use it to recover the leading-order radiated angular momentum expression. Upon expanding it in the relative velocity between the two bodies v, we compute the total four-momentum radiated into gravitational waves at leading-order in G and up to an order v,[8] finding agreement with what was recently computed using scattering amplitude methods. Our results also allow us to investigate the zero frequency limit of the emitted energy spectrum.

I. INTRODUCTION

The understanding of the dynamics of binary systems and their gravitational wave emission has been crucial for the extraordinary discovery of LIGO/Virgo [1,2]. This field has recently received a renewed attention, particularly in the application of the so-called post-Minkowskian (PM) framework [3–12], which consists of expanding the gravitational dynamics in the Newton's constant G while keeping the velocities fully relativistic. This is complementary to the post-Newtonian approach (see [13,14] and references therein), where one expands in both velocity and G, since in a bound state these two are related by the virial theorem.

Recently, progress has been made within the PM framework, thanks to the application of several complementary approaches: in particular, the effective one-body method [10,11,15,16], the use of scattering amplitude technics, such as the double copy [17–19], generalized unitarity [20–22], and effective field theory (EFT) [23–30] (see [31–39] for the quantum field theoretic description of gravity), and worldline EFT approaches [40–44]. These developments concern the scattering of unbound states, but results can be extended to bound states by applying an analytic continuation between hyperbolic and elliptic motion [45,46]. The progress has addressed the conservative binary dynamics up to 3PM order [47–50], as well as tidal [51–57], spin [58–62], and radiation effects [63–70], and have spurred other new interesting results (see, e.g., [71–73] for an incomplete list).

The culminating product of the scattering amplitude program is the recent derivation of the 4PM two-body Hamiltonian [74]. At this order, a tail effect is present [75–77] and manifests an infrared divergence proportional

to the leading-order (G^3) energy of the radiated Bremsstrahlung, the gravitational waves emitted during the scattering of two masses approaching each other from infinity. Studies on the leading-order gravitational Bremsstrahlung include [9,78–83]. The full leading-order energy spectrum found in [74] was independently obtained in [84] using the formalism of [27], which derives classical observables from scattering amplitudes and their unitarity cuts.

In this paper, we study the gravitational Bremsstrahlung using a worldline approach inspired by nonrelativistic general relativity (NRGR) [85] (see [86–90] for reviews) and recently applied to the PM expansion [40–42,50,91]. In particular, we first define the Feynman rules that allow us to derive the leading and next-to-leading order stress-energy tensor linearly coupled to gravity. From this, we compute the classical probability amplitude of graviton emission, which is directly related to the waveform in Fourier space. The amplitude is the basic ingredient for the computation of observables such as the radiated four-momentum and angular momentum, which we discuss in various limits and compare to the literature.

II. POST-MINKOWSKIAN EFFECTIVE FIELD THEORY

We consider the scattering of two gravitationally interacting spinless bodies with mass m_1 and m_2 approaching each other from infinity. The gravitational dynamics is described by the usual Einstein-Hilbert action. Neglecting finite size effects, which would contribute at higher order in G (see, e.g., [42,51]), the bodies are treated as external sources described by point-particle actions. We use the

Polyakov-like parametrization of the action and fix the vielbein to unity. This has the advantage of simplifying the gravitational coupling to the matter sources [42,92,93]. Therefore, using the mostly minus metric signature, setting $\hbar = c = 1$ and defining the Planck mass as $m_{\rm Pl} \equiv 1/\sqrt{32\pi G}$, we have

$$S = -2m_{\rm Pl}^2 \int d^4x \sqrt{-g}R$$
$$- \sum_{a=1,2} \frac{m_a}{2} \int d\tau_a [g_{\mu\nu}(x_a)\mathcal{U}_a^\mu(\tau_a)\mathcal{U}_a^\nu(\tau_a) + 1], \quad (1)$$

where, for each body a, τ_a is its proper time and $\mathcal{U}_a^\mu \equiv dx_a^\mu/d\tau_a$ is its four-velocity.

To compute the waveform, we need the (pseudo) stress-energy tensor $T^{\mu\nu}$, defined as the linear term sourcing the gravitational field in the effective action [33,85,94], i.e.,

$$\Gamma[x_a, h_{\mu\nu}] = -\frac{1}{2m_{\rm Pl}} \int d^4x T^{\mu\nu}(x)h_{\mu\nu}(x). \quad (2)$$

In this equation, $h_{\mu\nu} \equiv m_{\rm Pl}(g_{\mu\nu} - \eta_{\mu\nu})$ denotes a radiated field propagating on shell, while $T^{\mu\nu}$ must include the contribution of both potential modes, i.e., off-shell modes responsible for the conservative forces in the two-body system, and radiation modes. (We will come back to this split below.)

From the Fourier transform of $T^{\mu\nu}$, defined by $\tilde{T}^{\mu\nu}(k) = \int d^4x T^{\mu\nu}(x)e^{ik\cdot x}$, one can compute the (classical) probability amplitude of one graviton emission with momentum $\mathbf{k}$ and helicity $\lambda = \pm 2$ [85],

$$i\mathcal{A}_\lambda(k) = -\frac{i}{2m_{\rm Pl}} \epsilon_{\mu\nu}^{*\lambda}(\mathbf{k})\tilde{T}^{\mu\nu}(k), \quad (3)$$

where $\epsilon_{\mu\nu}^\lambda(\mathbf{k})$ is the transverse-traceless helicity-2 polarization tensor, with the normalization $\epsilon_{\mu\nu}^{*\lambda}(\mathbf{k})\epsilon_{\lambda'}^{\mu\nu}(\mathbf{k}) = \delta_{\lambda'}^\lambda$ (see definition in the Appendix A). At distances r much larger than the interaction region, the waveform is given in terms of the amplitude as (see, e.g., [95])

$$h_{\mu\nu}(x) = -\frac{1}{4\pi r} \sum_{\lambda=\pm 2} \int \frac{dk^0}{2\pi} e^{-ik^0 u} \epsilon_{\mu\nu}^\lambda(\mathbf{k}) \mathcal{A}_\lambda(k)|_{k^\mu=k^0 n^\mu}, \quad (4)$$

where $u \equiv t - r$. The amplitude is evaluated on shell, i.e., $k^\mu = k^0 n^\mu$, with $n^\mu \equiv (1, \mathbf{n})$ and $\mathbf{n}$ the unitary vector pointing along the graviton trajectory.

We can obtain the stress-energy tensor defined above by matching Eq. (2) to the effective action computed order by order in G using Feynman diagrams. Let us now introduce the Feynman rules. Adding the usual de Donder gauge-fixing term to Eq. (1),

$$S_{\rm gf} = \int d^4x \left[\frac{1}{2}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} - \frac{1}{4}\partial_\rho h\partial^\rho h\right], \quad (5)$$

where $h \equiv \eta^{\mu\nu}h_{\mu\nu}$, from the quadratic part of the gravitational action, one can extract the graviton propagator,

$$\overset{\mu\nu}{\bullet}\!\!\!\underset{}{\text{\large wwww}}\!\!\!\overset{\rho\sigma}{\bullet} \; = \; \frac{i}{k^2}P_{\mu\nu;\rho\sigma}, \quad (6)$$

where $P_{\mu\nu;\rho\sigma} \equiv \frac{1}{2}(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma})$. As usual, we must specify the contour of integration in the complex k^0 plane by suitable boundary conditions. This is customarily done by splitting the gravitons into potential and radiation modes (see, e.g., [42,85]). Potential modes never hit the pole $k^2 = 0$, so the choice of boundary conditions does not affect the calculations. For radiation modes, one must impose retarded boundary conditions, i.e., $[(k^0 + i\epsilon)^2 - |\mathbf{k}|^2]^{-1}$, to account only for outgoing gravitons. Even though they are not relevant at the order in G at which we work here, in general, one must treat with care the pole of radiation modes since they play a key role for hereditary effects at higher orders [96].

Finally, from the gravitational action, one can derive the cubic interaction vertex, which is the only one relevant for this paper. In the de Donder gauge, it can be found, for instance, in [33,97].

Thanks to the Polyakov-like form, the point-particle action contains only a linear interaction vertex. However, in order to isolate the powers of G, we parametrize the worldline by expanding around straight trajectories [42,50], i.e.,

$$x_a^\mu(\tau_a) = b_a^\mu + u_a^\mu \tau_a + \delta^{(1)}x_a^\mu(\tau_a) + \dots, \quad (7)$$

$$\mathcal{U}_a^\mu(\tau_a) = u_a^\mu + \delta^{(1)}u_a^\mu(\tau_a) + \dots \quad (8)$$

Here, u_a is the (constant) asymptotic incoming velocity and b_a is the body displacement orthogonal to it, $b_a \cdot u_a = 0$, while $\delta^{(1)}x_a^\mu$ and $\delta^{(1)}u_a^\mu$ are, respectively, the deviation from the straight trajectory and constant velocity of body a at order G, induced by the gravitational interaction. Moreover, we define the impact parameter as $b^\mu \equiv b_1^\mu - b_2^\mu$ and the relative Lorentz factor as

$$\gamma \equiv u_1 \cdot u_2 = \frac{1}{\sqrt{1-v^2}}, \quad (9)$$

where v is the relativistic relative velocity between the two bodies.

The expansion of the worldline action in the second line of Eq. (1) generates two Feynman interaction rules that differ by their order in G. At zeroth order, we have [with $\int_q \equiv \int \frac{d^4q}{(2\pi)^4}$]

$$\overset{\tau_a}{\bullet}\!\!\text{e e e e e} = -\frac{im_a}{2m_{\text{Pl}}} u_a^\mu u_a^\nu \int d\tau_a \int_q e^{-iq\cdot(b_a + u_a\tau_a)} , \quad (10)$$

where a filled dot denotes the point particle evaluated using the straight worldline. At first order in G, we have

$$\overset{\tau_a}{\textcircled{1}}\!\!\text{e e e e e} = -\frac{im_a}{2m_{\text{Pl}}} \int d\tau_a \int_q e^{-iq\cdot(b_a+u_a\tau_a)}$$
$$\times \left(2\delta^{(1)} u_a^{(\mu}(\tau_a) u_a^{\nu)} - i(q\cdot\delta^{(1)}x_a(\tau_a)) u_a^\mu u_a^\nu \right) , \tag{11}$$

where the correction $\mathcal{O}(G^n)$ to the trajectory is denoted by the order n inside the circle. Following [42], the $\mathcal{O}(G)$ correction to the velocity and the trajectory can be computed by solving the geodesic equation obtained from the effective Lagrangian at order G. In the de Donder gauge, it reads, for particle 1,

$$\delta^{(1)} u_1^\mu(\tau) = \frac{m_2}{4m_{\text{Pl}}^2} \int_q \bar{\delta}(q\cdot u_2) \frac{e^{-iq\cdot b - iq\cdot u_1\tau}}{q^2} B_1^\mu, \quad (12)$$

$$\delta^{(1)} x_1^\mu(\tau) = \frac{im_2}{4m_{\text{Pl}}^2} \int_q \bar{\delta}(q\cdot u_2) \frac{e^{-iq\cdot b - iq\cdot u_1\tau}}{q^2(q\cdot u_1 + i\varepsilon)} B_1^\mu, \quad (13)$$

where we use the notation $\bar{\delta}^{(n)}(x) \equiv (2\pi)^n \delta^{(n)}(x)$, and $B_1^\mu \equiv \frac{2\gamma^2-1}{2} \frac{q^\mu}{q\cdot u_1 + i\varepsilon} - 2\gamma u_2^\mu + u_1^\mu$. (An analogous expression holds for particle 2.) The $+i\varepsilon$ in the above equations ensures to recover straight motion in the asymptotic past, i.e., $\delta^{(1)} u_1^\mu(-\infty) = 0$ and $\delta^{(1)} x_1^\mu(-\infty) = 0$. At our order in G, the deflected trajectories are completely determined by potential gravitons, but, in general, one must take into account also radiation modes with appropriate boundary conditions. Note also that at higher order it can be
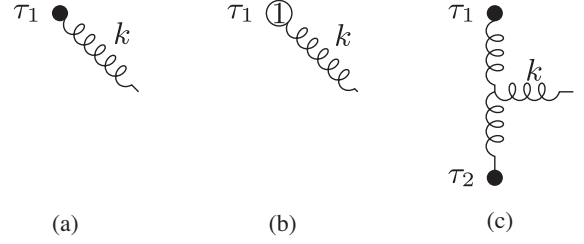

(a)　　　　　(b)　　　　　(c)

FIG. 1. The three Feynman diagrams needed for the computation of the stress-energy tensor up to NLO order in G. To compute the symmetric one, it is enough to exchange $1 \leftrightarrow 2$.

convenient to use different gauge-fixing conditions to simplify the graviton vertices [42].

III. STRESS-ENERGY TENSOR

The radiated field can be computed in powers of G in terms of the diagrams shown in Fig. 1. The leading stress-energy tensor is obtained from Fig. 1(a) and corresponds to the one of free point-particles, i.e.,

$$\tilde{T}^{\mu\nu}_{\text{Fig. 1a}}(k) = \sum_a m_a u_a^\mu u_a^\nu e^{ik\cdot b_a} \bar{\delta}(\omega_a), \quad (14)$$

where for convenience, we define

$$\omega_a \equiv k \cdot u_a, \quad a = 1, 2. \quad (15)$$

This generates a static and nonradiating contribution to the amplitude, proportional to $\bar{\delta}(\omega_a)$. While this contribution can be neglected when computing the radiated momentum, it must be crucially included for the computation of the angular momentum, as shown below.

At the next order, we find

$$\tilde{T}^{\mu\nu}_{\text{Fig. 1b}}(k) = \frac{m_1 m_2}{4m_{\text{Pl}}^2} \int_{q_1,q_2} \mu_{1,2}(k) \frac{1}{q_2^2} \left[\frac{2\gamma^2-1}{\omega_1+i\varepsilon} q_2^{(\mu} u_1^{\nu)} - 4\gamma u_2^{(\mu} u_1^{\nu)} - \left(\frac{2\gamma^2-1}{2} \frac{k\cdot q_2}{(\omega_1+i\varepsilon)^2} - \frac{2\gamma\omega_2}{\omega_1+i\varepsilon} - 1 \right) u_1^\mu u_1^\nu \right], \quad (16)$$

$$\tilde{T}^{\mu\nu}_{\text{Fig. 1c}}(k) = \frac{m_1 m_2}{4m_{\text{Pl}}^2} \int_{q_1,q_2} \mu_{1,2}(k) \frac{1}{q_1^2 q_2^2} \left[\frac{2\gamma^2-1}{2} q_2^\mu q_2^\nu + (2\omega_2^2 - q_1^2) u_1^\mu u_1^\nu + 4\gamma\omega_2 q_2^{(\mu} u_1^{\nu)} \right.$$
$$\left. - \eta^{\mu\nu} \left(\gamma\omega_1\omega_2 + \frac{2\gamma^2-1}{4} q_2^2 \right) + 2(\gamma q_1^2 - \omega_1\omega_2) u_1^{(\mu} u_2^{\nu)} \right], \quad (17)$$

where

$$\mu_{1,2}(k) \equiv e^{i(q_1\cdot b_1 + q_2\cdot b_2)} \bar{\delta}^{(4)}(k - q_1 - q_2) \bar{\delta}(q_1\cdot u_1) \bar{\delta}(q_2\cdot u_2), \quad (18)$$

and we have used momentum conservation, on-shell, and harmonic-gauge conditions to simplify the final expression. Of course, we must also include the analogous diagrams with bodies 1 and 2 exchanged. The contribution

in Fig. 1(b) comes from evaluating the worldline along deflected trajectories, while the one in Fig. 1(c) comes from the gravitational cubic interaction. We have checked that the sum of these two contributions is transverse for on shell

momenta, i.e., $k_\mu \tilde{T}^{\mu\nu} = 0$ for $k^2 = 0$, as expected for radiated gravitons. We have also verified that the finite part of the stress-energy tensor agrees with that computed in [40] once the contribution from the dilaton is removed.

IV. AMPLITUDES AND WAVEFORMS

We expand the amplitude defined in Eq. (3) in powers of G, $\mathcal{A}_\lambda = \mathcal{A}_\lambda^{(1)} + \mathcal{A}_\lambda^{(2)} + \cdots$. Given the definition (3)

$$\mathcal{A}_\lambda^{(2)}(k) = -\frac{m_1 m_2}{8m_{\mathrm{Pl}}^3} \epsilon_{\mu\nu}^{*\lambda}(\mathbf{n}) \left\{ e^{ik \cdot b_1} \left[\left(-\frac{2\gamma^2 - 1}{2} \frac{k \cdot I_{(1)}}{(\omega_1 + i\epsilon)^2} + \frac{2\gamma\omega_2}{\omega_1 + i\epsilon} I_{(0)} + 2\omega_2^2 J_{(0)} \right) u_1^\mu u_1^\nu \right. \right.$$
$$\left. \left. + \left(\frac{2\gamma^2 - 1}{\omega_1 + i\epsilon} I_{(1)}^\mu + 4\gamma\omega_2 J_{(1)}^\mu \right) u_1^\nu - 2(\gamma I_{(0)} + \omega_1\omega_2 J_{(0)}) u_1^\mu u_2^\nu + \frac{2\gamma^2 - 1}{2} J_{(2)}^{\mu\nu} \right] \right\} + (1 \leftrightarrow 2), \quad (20)$$

where we have defined the following integrals:

$$I_{(n)}^{\mu_1 \dots \mu_n} \equiv \int_q \tilde{\delta}(q \cdot u_1 - \omega_1) \tilde{\delta}(q \cdot u_2) \frac{e^{-iq \cdot b}}{q^2} q^{\mu_1} \dots q^{\mu_n}, \quad (21)$$

$$J_{(n)}^{\mu_1 \dots \mu_n} \equiv \int_q \tilde{\delta}(q \cdot u_1 - \omega_1) \tilde{\delta}(q \cdot u_2) \frac{e^{-iq \cdot b}}{q^2 (k-q)^2} q^{\mu_1} \dots q^{\mu_n}. \quad (22)$$

[The indices inside these integrals must be changed when evaluating the symmetric contribution $(1 \leftrightarrow 2)$.] As detailed in Appendix B, the first set of integrals in Eq. (21) can be solved in terms of Bessel functions. The second set of integrals in Eq. (22) comes exclusively from the gravitational cubic interaction in Fig. 1(c). Unfortunately, we were not able to come up with an explicitly solution to these integrals. However, we can express them as one-dimensional integrals over a Feynman parameter, involving Bessel functions.

To simplify the treatment, from now on, we choose a frame in which one of the two bodies, say 2, is at rest. Moreover, for convenience, we can set $b_2^\mu = 0$ and $b_1^\mu = b^\mu$ and define the unit spatial vectors in the direction of $\mathbf{v}$ and of the impact parameter $\mathbf{b}$, respectively, $\mathbf{e}_v \equiv \mathbf{v}/v$ and $\mathbf{e}_b = \mathbf{b}/|\mathbf{b}|$, with $\mathbf{e}_v \cdot \mathbf{e}_b = 0$. We also define $v^\mu \equiv (1, v\mathbf{e}_v)$ so that

$$u_2^\mu = \delta_0^\mu, \qquad u_1^\mu = \gamma v^\mu = \gamma(1, v\mathbf{e}_v). \quad (23)$$

The energies of the radiated gravitons measured by the two bodies become, respectively, $\omega_2 = k^0 \equiv \omega$ and $\omega_1 = \gamma \omega n \cdot v$. The amplitude simplifies to the following compact forms:

$$\mathcal{A}_\lambda^{(1)}(k) = -\frac{m_1}{2m_{\mathrm{Pl}}} \frac{\gamma v^2}{n \cdot v} \epsilon_{ij}^{*\lambda} \mathbf{e}_v^i \mathbf{e}_v^j \tilde{\delta}(\omega) e^{ik \cdot b}, \quad (24)$$

and the stress-energy tensor (14), the leading order reads

$$\mathcal{A}_\lambda^{(1)}(k) = -\frac{1}{2m_{\mathrm{Pl}}} \sum_a m_a \epsilon_{\mu\nu}^{*\lambda}(\mathbf{n}) u_a^\mu u_a^\nu e^{ik \cdot b_a} \tilde{\delta}(\omega_a). \quad (19)$$

The NLO can be obtained by summing Eqs. (16) and (17) and inserting the result in Eq. (3). Integrating over one of the internal momenta,

$$\mathcal{A}_\lambda^{(2)}(k) = -\frac{Gm_1 m_2}{m_{\mathrm{Pl}}\gamma v} \epsilon_{ij}^{*\lambda} \mathbf{e}_I^i \mathbf{e}_J^j A_{IJ}(k) e^{ik \cdot b}, \quad (25)$$

where the functions A_{IJ} can be obtained after solving the integrals (21) and (22). We find

$$A_{vv} = c_1 K_0(z(n \cdot v)) + ic_2[K_1(z(n \cdot v)) - i\pi\delta(z(n \cdot v))]$$
$$+ \int_0^1 dy e^{iyzv\mathbf{n} \cdot \mathbf{e}_b} [d_1(y) z K_1(zf(y)) + c_0 K_0(zf(y))], \quad (26)$$

$$A_{vb} = ic_0[K_1(z(n \cdot v)) - i\pi\delta(z(n \cdot v))]$$
$$+ i \int_0^1 dy e^{iyzv\mathbf{n} \cdot \mathbf{e}_b} d_2(y) z K_0(zf(y)), \quad (27)$$

$$A_{bb} = \int_0^1 dy e^{iyzv\mathbf{n} \cdot \mathbf{e}_b} d_0(y) z K_1(zf(y)), \quad (28)$$

where K_0 and K_1 are modified Bessel functions of the second kind, and we have introduced

$$z \equiv \frac{|\mathbf{b}|\omega}{v}, \quad (29)$$

and

$$f(y) \equiv \sqrt{(1-y)^2(n \cdot v)^2 + 2y(1-y)(n \cdot v) + y^2/\gamma^2}. \quad (30)$$

The coefficients c_0, c_1, and c_2 depend on v and on the relative angles between the graviton direction and the basis $(\mathbf{e}_v, \mathbf{e}_b)$. Moreover, d_0, d_1, and d_2 depend also on the integration parameter y. Their explicit form is given in Appendix C. In Eqs. (26) and (27), we have also included the nonradiating contribution proportional to a delta

function,[1] which may become relevant, for instance, when computing the radiated angular momentum at NLO.

For small velocities, we find agreement between our amplitude and the waveform in Fourier space of [82]. In this limit, $f(y) \to 1$, $e^{iyzv\mathbf{n}\cdot\mathbf{e}_b} \to 1$, $\gamma \to 1$, and thus,[2]

$$A_{vv} \underset{v\to 0}{\to} zK_1(z) + K_0(z), \qquad (32)$$

$$A_{vb} \underset{v\to 0}{\to} -i[K_1(z) + zK_0(z) - i\pi\delta(z)], \qquad (33)$$

$$A_{bb} \underset{v\to 0}{\to} -zK_1(z). \qquad (34)$$

We have also checked that we recover their amplitude in the forward and backward limit (i.e., $\mathbf{n}$ along the direction of $\mathbf{e}_v$), for which $\mathbf{n} \cdot \mathbf{e}_b \to 0$ and the integral in y can be solved exactly. The waveform can be computed by replacing the amplitude in Eq. (4) and integrating in k^0. We discuss this calculation in Appendix D.

V. RADIATED FOUR-MOMENTUM

In terms of the asymptotic waveform, the radiated four-momentum at infinity ($r \to \infty$) is given by [69,82][3]

$$P^\mu_{\mathrm{rad}} = \int d\Omega du r^2 n^\mu \dot{h}_{ij} \dot{h}_{ij}, \qquad (35)$$

where a dot denotes the derivative with respect to the retarded time u and $d\Omega$ is the integration surface element.

Using Eq. (4) for the waveform, this can be expressed in a manifestly Lorentz-invariant way in terms of the amplitude (3) as [40]

$$P^\mu_{\mathrm{rad}} = \sum_\lambda \int_k \bar{\delta}(k^2)\theta(k^0)k^\mu |\mathcal{A}_\lambda(k)_{\mathrm{finite}}|^2, \qquad (36)$$

where θ is the Heaviside step function, and on the right-hand side, we take only the finite part of the amplitude, excluding the terms proportional to a delta function that do not contribute to $\dot{h}_{ij}$. Thus, at leading order $|\mathcal{A}_\lambda(k)_{\mathrm{finite}}|^2 = |\mathcal{A}^{(2)}_\lambda(k)_{\mathrm{finite}}|^2 + \cdots$, and hence, the radiated four-momentum starts at order G^3.

Since the modulo squared of the amplitude is symmetric under $\mathbf{k} \to -\mathbf{k}$, the four-momentum cannot depend on the spatial direction b^μ. Moreover, the energy measured in

the frame of one body is the same as the one measured in the frame of the other one; hence, the final result must be proportional to $u^\mu_1 + u^\mu_2$. Using Eq. (25), we can write it as

$$P^\mu_{\mathrm{rad}} = \frac{G^3 m_1^2 m_2^2}{|\mathbf{b}|^3} \frac{u^\mu_1 + u^\mu_2}{\gamma + 1} \mathcal{E}(\gamma) + \mathcal{O}(G^4), \qquad (37)$$

which confirms that at this order the result has homogeneous mass dependence and is thus fixed by the probe limit [76,82,84]. The function $\mathcal{E}(\gamma)$ can be found by integrating over the phase space the modulo squared of the amplitude,

$$\mathcal{E}(\gamma) = \int d\Omega \int_0^\infty dz \frac{d\mathcal{E}}{dzd\Omega}(z, \Omega; \gamma) \qquad (38)$$

with

$$\frac{d\mathcal{E}}{dzd\Omega} \equiv \frac{2vz^2}{\pi^2\gamma^2} \sum_\lambda |\epsilon^{*\lambda}_{ij} \mathbf{e}^i_I \mathbf{e}^j_J A_{IJ}(z, \Omega)|^2. \qquad (39)$$

A more explicit but long expression of this function is reported in Appendix E; see Eq. (E1).

Due to the involved structure of the y integrals in Eq. (25), we were unable to compute $\mathcal{E}$ explicitly. Nevertheless, we can first compute the integrals in y in the $v \ll 1$ regime at any order. Then we can perform the phase-space integral expressing the angular dependence in a particular coordinate system. We have computed the energy up to order $\mathcal{O}(v^8)$, obtaining

$$\frac{\mathcal{E}}{\pi} = \frac{37}{15}v + \frac{2393}{840}v^3 + \frac{61703}{10080}v^5 + \frac{3131839}{354816}v^7 + \mathcal{O}(v^9). \qquad (40)$$

The radiated energy in center-of-mass frame, $P_{\mathrm{rad}} \cdot u_{\mathrm{CoM}}$, where

$$u^\mu_{\mathrm{CoM}} = \frac{m_1 u^\mu_1 + m_2 u^\mu_2}{\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \gamma}}, \qquad (41)$$

agrees with the 2PN results [76,82,98] while Eq. (40) matches the expansion of the fully relativistic result recently found in [84]. This is a nontrivial check of our NLO amplitude (25).

As an extra check, we can compute the leading-order energy spectrum in the soft limit, which is obtained by considering only wavelengths of the emitted gravitons much larger than the interaction region, i.e., $|\mathbf{b}|\omega/v \ll 1$. For $E_{\mathrm{rad}} \equiv P^0_{\mathrm{rad}}$, this is given by

$$\frac{dE_{\mathrm{rad}}}{d\omega}\bigg|_{\omega\to 0} = \frac{1}{2(2\pi)^3} \sum_\lambda \int d\Omega |\omega\mathcal{A}_\lambda(k)_{\omega\to 0}|^2. \quad (42)$$

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[1]To compute this contribution, we have used this integral,

$$\int_q \delta(q \cdot u_1)\delta(q \cdot u_2) \frac{e^{-iq\cdot b}q^\mu}{q^2} = \frac{b^\mu}{2\pi\gamma v|\mathbf{b}|^2}. \qquad (31)$$

[2]The signs in front of K_0 and K_1 of the last term of Eqs. (2.9b) and (2.9c) of [82] are opposite to ours because of a different Fourier transform convention.

[3]We are using a different normalization of $h_{\mu\nu}$ with respect to these references, which explains the absence of the prefactor $(32\pi G)^{-1}$.

In this limit, the amplitude at order G^2 receives contributions exclusively from the diagram in Fig. 1(b), so it is not affected by the gravitational self-interactions. From Eqs. (25)–(28), it reads

$$i\mathcal{A}_\lambda^{(2)}(k)_{\omega\to 0} = \frac{Gm_1 m_2}{m_{\text{Pl}}|\mathbf{b}|}\frac{1}{\gamma\omega n\cdot v}\epsilon_{ij}^{*\lambda}(c_2\mathbf{e}_v^i\mathbf{e}_v^j + 2c_0\mathbf{e}_v^i\mathbf{e}_b^i). \tag{43}$$

Integrating Eq. (42) over the angles by fixing some angular coordinate system and introducing the function $\mathcal{I}(v) \equiv -\frac{16}{3} + \frac{2}{v^2} + \frac{2(3v^2-1)}{v^3}\text{arctanh}(v)$ [69], we obtain

$$\left.\frac{dE_{\text{rad}}}{d\omega}\right|_{\omega\to 0} = \frac{4}{\pi}\frac{(2\gamma^2-1)^2}{\gamma^2 v^2}\frac{G^3 m_1^2 m_2^2}{|\mathbf{b}|^2}\mathcal{I}(v) + \mathcal{O}(G^4), \tag{44}$$

which agrees with [70,99]. We will come back to this result below.

VI. RADIATED ANGULAR MOMENTUM

The angular momentum lost by the system is another interesting observable as it can be related to the correction to the scattering angle due to radiation reaction [69]. In terms of the asymptotic waveform, this is given by [69,100]

$$J_{\text{rad}}^i = \epsilon^{ijk}\int d\Omega du r^2(2h_{jl}\dot{h}_{lk} - x^j\partial_k h_{lm}\dot{h}_{lm}). \tag{45}$$

As pointed out in [69], the waveform at order G is static and can be pulled out of the time integration leaving with the computation of the gravitational wave memory $\Delta h_{ij} \equiv \int_{-\infty}^{+\infty} du\dot{h}_{ij}$. This can be related to the classical amplitude by Eq. (4),

$$\Delta h_{ij} = \frac{i}{4\pi r}\sum_\lambda\int\frac{d\omega}{2\pi}\epsilon_{ij}^\lambda\bar{\delta}(\omega)\omega\mathcal{A}_\lambda(k)_{\omega\to 0}, \tag{46}$$

where from the right-hand side it is clear that only the soft limit contributes to the gravitational wave memory. Moreover, since at this order the soft limit is uniquely determined by the diagram in Fig. 1(b), the radiated angular momentum does not depend on the gravitational self-interaction, confirming [69].

To compute the radiated angular momentum, it is convenient to introduce a system of polar coordinates, where $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and an orthonormal frame tangent to the sphere, with $\mathbf{e}_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$ and $\mathbf{e}_\phi = (-\sin\phi, \cos\phi, 0)$. To express Eq. (45) in terms of the amplitudes, we can rewrite the angular dependence in the polarization tensors of the first term inside the parenthesis using $2\epsilon^{ijk}\epsilon_{jl}^\lambda\epsilon_{lk}^{*\lambda'} = -i\lambda n^i\delta^{\lambda\lambda'}$. The second term can be rewritten by noticing that $\epsilon^{ijk}x^j\partial_k = i\hat{L}^i$, where $\hat{L}^i$ is the usual orbital angular momentum operator, expressed

in terms of the angles and their derivatives (see Appendix A). Using $\epsilon_{lm}^{*\lambda'}\hat{\mathbf{L}}\epsilon_{lm}^\lambda = \lambda\cot\theta\mathbf{e}_\theta\delta^{\lambda\lambda'}$, we obtain

$$\mathbf{J}_{\text{rad}} = \sum_\lambda\int\frac{d\Omega}{(4\pi)^2}\omega\mathcal{A}_\lambda^{(2)*}(k)_{\omega\to 0}\hat{\mathbf{J}}\bar{\mathcal{A}}_\lambda^{(1)} + \mathcal{O}(G^3), \tag{47}$$

where $\hat{\mathbf{J}} \equiv \lambda(\mathbf{n} + \cot\theta\mathbf{e}_\theta) + \hat{\mathbf{L}}$, and we have introduced $\bar{\mathcal{A}}_\lambda^{(1)}$ as the leading-order amplitude striped off of the delta function, i.e., defined by

$$\mathcal{A}_\lambda^{(1)}(k) = \bar{\mathcal{A}}_\lambda^{(1)}\bar{\delta}(\omega)e^{ik\cdot b}. \tag{48}$$

One can perform the angular integral in Eq. (47) by aligning $\mathbf{e}_v$ and $\mathbf{e}_b$ along any (mutually orthogonal) directions and eventually obtains

$$\mathbf{J}_{\text{rad}} = \frac{2(2\gamma^2-1)}{\gamma v}\frac{G^2 m_1 m_2 J}{|\mathbf{b}|^2}\mathcal{I}(v)(\mathbf{e}_b\times\mathbf{e}_v), \tag{49}$$

where $J = m_1\gamma v|\mathbf{b}|$ is the angular momentum at infinity. This result agrees with [69].

As noticed in [70], from Eqs. (44) and (47), we observe an intriguing proportionality between the energy spectrum in the soft limit and the total emitted angular momentum. We leave a more thorough exploration of this result for the future.

VII. CONCLUSION

We have studied the gravitational Bremsstrahlung using a worldline approach. In particular, we have computed through the use of Feynman diagrams, expanding perturbatively in G, the leading and next-to-leading order classical probability amplitude of graviton emission and consequently, the waveform in Fourier space. The next-to-leading order amplitude receives two contributions: one from the deviation from straight orbits, which can be expressed in terms of modified Bessel functions of the second kind; another from the cubic gravitational self-interaction, which we could rewrite as one-dimensional integrals over a Feynman parameter of modified Bessel functions. When comparison was possible, we found agreement with earlier calculations of the waveforms [78,81] in different limits.

We have used the amplitude to compute the leading-order radiated angular momentum, recovering the result of [69]. Moreover, we have computed the total emitted four-momentum expanded in small velocities up to order v,[8] and we found agreement with the recent results of [74,84]. Unfortunately, we were not able to reproduce their fully relativistic result, which we leave for the future. Nevertheless, we have built the foundations for an alternative derivation of the recent results obtained with amplitude techniques.

Another interesting limit is for small gravitational wave frequencies, where the amplitude does not receive contributions from the gravitational interaction. We have

computed the soft energy spectrum recovering an intriguing relation with the emitted angular momentum [70]. Future directions include the study of spin and finite-size effects and a more thorough investigation of the relations between differential observables.

ACKNOWLEDGMENTS

Note added.—Another article [101], whose content overlaps with ours, appeared recently.

APPENDIX A: ANGULAR DEPENDENCE

We can introduce the transverse-traceless helicity-2 tensors, normalized to unity, in terms of the orthonormal frame tangent to the sphere, $\mathbf{e}_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$ and $\mathbf{e}_\phi = (-\sin\phi, \cos\phi, 0)$, used in the main text. We define

$$\epsilon_i^\pm \equiv \frac{1}{\sqrt{2}}(\pm\mathbf{e}_\theta^i + i\mathbf{e}_\phi^i), \qquad \epsilon_{ij}^{\pm 2} = \epsilon_i^\pm \epsilon_j^\pm. \quad \text{(A1)}$$

We can relate these tensors to the (real) plus and cross parametrization often used in the literature by

$$\epsilon_{ij}^{\text{plus}} = \epsilon_{ij}^+ + \epsilon_{ij}^-, \qquad \epsilon_{ij}^{\text{cross}} = -i(\epsilon_{ij}^+ - \epsilon_{ij}^-). \quad \text{(A2)}$$

For convenience, here we also explicitly report the expression of the (orbital) angular momentum operator in terms of the same polar coordinates,

$$\hat{L}^x = i(\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi), \quad \text{(A3)}$$

$$\hat{L}^y = -i(\cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi), \quad \text{(A4)}$$

$$\hat{L}^z = -i\partial_\phi. \quad \text{(A5)}$$

APPENDIX B: INTEGRALS

To compute the integrals in Eq. (21), we first need the master integral $I_{(0)}$, which can be solved by going to the frame of body 2 as in Eq. (23) and by removing the delta functions by integrating in q^0 and in the spatial momentum along **v**. This leaves us with

$$I_{(0)} = -\frac{1}{\gamma v}\int \frac{d^2\mathbf{q}_\perp}{(2\pi)^2}\frac{e^{i\mathbf{q}_\perp \cdot \mathbf{b}}}{|\mathbf{q}_\perp|^2 + \frac{\omega_1^2}{\gamma^2 v^2}} = -\frac{1}{2\pi\gamma v}K_0\left(\frac{|\mathbf{b}|\omega_1}{\gamma v}\right),$$

$$\text{(B1)}$$

where we can write $|\mathbf{b}| = \sqrt{-b^2}$ in a Lorentz-invariant fashion.

We use this result to compute the descendant integrals $I_{(n)}^{\mu_1\ldots\mu_n}$ (see analogous examples in [27]). For instance, by the presence of $\delta(q \cdot u_2)$ in the integrand, $I_{(1)}^\mu$ can only be a sum of two pieces, one proportional to b^μ and another proportional to $u_1^\mu - \gamma u_2^\mu$. The piece proportional to b^μ can be computed by taking the derivative of $I_{(0)}$ with respect to b^μ and projecting it along b^μ with proper normalization. It is easy to see that the other piece is proportional to $I_{(0)}$ upon projecting $I_{(1)}^\mu$ along u_1^μ and taking into account the first delta function.

To compute the integrals in Eq. (22), we can proceed analogously. Although we were not able to solve the master integral $J_{(0)}$ in close form, we can express it in terms of an integral over a Feynman parameter as

$$\begin{aligned}J_{(0)} &= \int_0^1 dy\, e^{-iyk\cdot b}\int_q \tilde{\delta}(q \cdot u_1 + (y-1)\omega_1)\\&\quad \times \tilde{\delta}(q \cdot u_2 + y\omega_2)e^{-iq\cdot b}/q^4\\&= \frac{|\mathbf{b}|^2}{4\pi\gamma v}\int_0^1 dy\, e^{-iyk\cdot b}\frac{K_1(zf(y))}{zf(y)},\end{aligned}\quad \text{(B2)}$$

where the integral in q has been solved similarly to $I_{(0)}$.

APPENDIX C: COEFFICIENTS

The coefficients in Eqs. (26), (27), and (28) are

$$c_0 = 1 - 2\gamma^2, \quad c_1 = -c_0 + \frac{3-2\gamma^2}{n \cdot v}, \quad c_2 = vc_0\frac{\mathbf{n} \cdot \mathbf{e}_b}{n \cdot v},$$

$$d_0(y) = f(y)c_0,$$

$$d_1(y) = v^2\frac{4\gamma^2(y-1)(n \cdot v) - c_0(y-1)^2 - 2y - 1}{f(y)} - d_0(y),$$

$$d_2(y) = -1 + (1-y)c_0(n \cdot v - 1). \quad \text{(C1)}$$

APPENDIX D: WAVEFORM IN DIRECT SPACE

In this paper, we focus on computing the emitted energy and angular momentum, obtained from the waveform in *Fourier space* or, equivalently, the amplitude of graviton emission. In this Appendix, which was added in v2 of the

paper to address one of the reviewer's comments, we show how to find the waveform in *direct space* from the expression of our amplitude.

First, we replace the NLO amplitude (25) in Eq. (4), we go in the rest frame of particle 2 and integrate over q^0, removing $\delta(q \cdot u_2)$. Then we can get rid of the other delta function by integrating over k^0, which leads to a three-dimensional integral over $\mathbf{q}$. More explicitly, at order G^2, one can find (with $h_\lambda \equiv \epsilon_\lambda^{*\mu\nu} h_{\mu\nu}$),

$$
h_{\pm 2}^{(2)} = \frac{m_1 m_2 G}{8 m_{\mathrm{Pl}} r} \int_{\mathbf{q}} e^{i\mathbf{q} \cdot \tilde{\mathbf{b}}} \left[\frac{q^i \mathcal{N}_\pm^i}{\mathbf{q}^2 (\mathbf{q} \cdot \mathbf{e}_v - i\epsilon)} \right.
$$
$$
\left. + \frac{q^i q^j \mathcal{M}_\pm^{ij}}{\mathbf{q}^2 (\mathbf{q}^2 + \mathbf{q} \cdot L \cdot \mathbf{q})} \right], \tag{D1}
$$

where [101][4]

$$
\mathcal{N}_\pm^i \equiv 4 \frac{\gamma v}{(n \cdot v)^2} (\boldsymbol{\epsilon}^\pm \cdot \mathbf{e}_v)^2 [(1 + v^2) n^i - 4 v e_v^i]
$$
$$
+ 8 \frac{\gamma(1 + v^2)}{n \cdot v} (\boldsymbol{\epsilon}^\pm \cdot \mathbf{e}_v) \epsilon_\pm^i, \tag{D2}
$$

$$
\mathcal{M}_\pm^{ij} \equiv 16 \frac{\gamma v^4}{(n \cdot v)^3} (\boldsymbol{\epsilon}^\pm \cdot \mathbf{e}_v)^2 e_v^i e_v^j + 8 \frac{\gamma(1 + v^2)}{n \cdot v} \epsilon_\pm^i \epsilon_\pm^j
$$
$$
- 32 \frac{\gamma v^2}{(n \cdot v)} (\boldsymbol{\epsilon}^\pm \cdot \mathbf{e}_v) e_v^{(i} \epsilon_\pm^{j)}, \tag{D3}
$$

and we have introduced

$$
\tilde{\mathbf{b}} \equiv \mathbf{b} + \frac{v}{n \cdot v} (u + \mathbf{b} \cdot \mathbf{n}), \qquad L^{ij} \equiv 2 \frac{v}{n \cdot v} \mathbf{e}_v^{(i} \mathbf{n}^{j)}. \tag{D4}
$$

The integrations in $\mathbf{q}$ can be performed following [101]. Eventually, one finds an expression of the waveform equivalent to that of this reference, which agrees with [82].

APPENDIX E: ENERGY AND SPECTRAL DEPENDENCE

The spectral and angular dependence of the radiated four-momentum is given by Eq. (39). Using the expressions for the functions A_{IJ} in Eqs. (26)–(28) and summing over the helicities, we find

$$
\frac{\pi^2 \gamma^2}{2 v z^2} \frac{d\mathcal{E}}{dz d\Omega} = 2 a_{vv}^2 c_1^2 K_0^2(z(n \cdot v)) + 2 a_{vv} [4 a_{bb} c_0^2 + c_2(4 a_{vb} c_0 + a_{vv} c_2)] K_1^2(z(n \cdot v))
$$
$$
+ 4 c_1 K_0(z(n \cdot v)) [(2 a_{vb}^2 - a_{vv} a_{bb}) I_0^{(c)} + a_{vv}^2 I_1^{(c)} - 2 a_{vv} a_{vb} I_2^{(s)}]
$$
$$
+ 4 K_1(z(n \cdot v)) [(2 a_{bb} a_{vb} c_0 + (2 a_{vb}^2 - a_{vv} a_{bb}) c_2) I_0^{(s)} + (2 a_{vv} a_{vb} c_0 + a_{vv}^2 c_2) I_1^{(s)} + 2 a_{vv} (2 a_{bb} c_0 + a_{vb} c_2) I_2^{(c)}]
$$
$$
+ 2 a_{bb}^2 [(I_0^{(c)})^2 + (I_0^{(s)})^2] + 2 a_{vv}^2 [(I_1^{(c)})^2 + (I_1^{(s)})^2] + 8 a_{vv} a_{bb} [(I_2^{(c)})^2 + (I_2^{(s)})^2]
$$
$$
+ 4(2 a_{vb}^2 - a_{vv} a_{bb})(I_0^{(c)} I_1^{(c)} + I_0^{(s)} I_1^{(s)}) + 8 a_{vb} I_2^{(c)}(a_{bb} I_0^{(s)} + a_{vv} I_1^{(s)}) - 8 a_{vb} I_2^{(s)}(a_{bb} I_0^{(c)} + a_{vv} I_1^{(c)}), \tag{E1}
$$

where we have defined $a_{IJ} \equiv [(\mathbf{e}_\theta \cdot \mathbf{e}_I)(\mathbf{e}_\theta \cdot \mathbf{e}_J) + (\mathbf{e}_\phi \cdot \mathbf{e}_I)(\mathbf{e}_\phi \cdot \mathbf{e}_J)]/2,$[5] and the two sets of integrals,

$$
I_i^{(s)}(z, \Omega) \equiv \int_0^1 dy \sin(y z v \mathbf{n} \cdot \mathbf{e}_b) g_i(z, \Omega; y),
$$

$$
I_i^{(c)}(z, \Omega) \equiv \int_0^1 dy \cos(y z v \mathbf{n} \cdot \mathbf{e}_b) g_i(z, \Omega; y), \tag{E2}
$$

with

$$
g_0(z, \Omega; y) \equiv d_0(y) z K_1(z f(y)),
$$
$$
g_1(z, \Omega; y) \equiv c_0 K_0(z f(y)) + d_1(y) z K_1(z f(y)),
$$
$$
g_2(z, \Omega; y) \equiv d_2(y) z K_0(z f(y)). \tag{E3}
$$

It is straightforward to integrate analytically over the polar angle, while we were not able to integrate over the azimuthal one. Because of its length, we prefer not to report the integrated expression here.

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[4]To compare these expressions with those in [101], one must replace $\mathbf{e}_v \to \hat{\mathbf{e}}_1$, $\mathbf{e}_b \to \hat{\mathbf{e}}_2$, $\mathbf{e}_\theta \to \hat{\boldsymbol{\theta}}$, $\mathbf{e}_\phi \to \hat{\boldsymbol{\phi}}$, $n^\mu \to \rho^\mu$, $v^\mu \to v_2^\mu/\gamma$ and use Eq. (A1).

[5]Choosing $\mathbf{e}_v$ along z and $\mathbf{e}_b$ along x, we have $a_{vv} = \sin^2\theta/2$, $a_{vb} = -\sin\theta\cos\theta\cos\phi/2$, $a_{bb} = [\cos^2\theta\cos^2\phi + \sin^2\phi]/2$.

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