# Ghost- and tachyon-free Weyl gauge theories: A systematic approach

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We investigate the particle content of parity-preserving Weyl gauge theories of gravity ($\mathrm{WGT}^+$) about a Minkowski background. Within a subset of the full theory, we use a systematic method previously presented by Lin *et al.* [Phys. Rev. D **99**, 064001 (2019)] to determine 862 critical cases for which the parameter values in the action lead to changes of particle contents or additional gauge invariances. We find that 168 of these cases are free of ghosts and tachyons, provided the parameters satisfy certain conditions that we also determine. We further identify 40 of these cases that are also propagating power-counting renormalizable and determine the corresponding conditions on the parameters. Of these theories, 11 have only massless tordion propagating particles, 23 have only a massive tordion propagating mode, and 6 have both. We also repeat our analysis for $\mathrm{WGT}^+$ with vanishing torsion or curvature, respectively. We compare our findings with the very few previous results in the literature.

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## I. INTRODUCTION

In recent papers [1,2], we presented a systematic method for identifying the ghost-and-tachyon-free critical cases of parity-preserving gauge theories of gravity and applied it to parity-preserving Poincaré gauge theory ($\mathrm{PGT}^+$). We found 450 critical cases (some of which possess additional gauge invariances) that are free of ghosts and tachyons. We also considered the superficial renormalizability by power counting of a subset of these unitary theories for which there are no terms in the gauge-fixed Lagrangian that mix different fields. While not stated explicitly in Ref. [2], four of the theories in that paper (cases 9, 10, 11, and 13, which have only massless modes) satisfy the original criterion used by Sezgin and van Nieuwenhuizen in Ref. [3] to be power-counting renormalizable (PCR). Moreover, we found a further 54 theories that satisfy a less restrictive criterion, which in addition permits the presence of modes that are nonpropagating at large momenta (for which the propagator decays no faster than a constant), since these should then completely decouple from the rest of the theory; this is termed "the alternative PCR criterion" in Ref. [2], but here (and henceforth), we shall instead refer to it as "propagating power-counting renormalizable" (PPCR) to avoid confusion with the well-established notion in the literature of PCR. The relationship between these two approaches is discussed at length in Ref. [2] and also briefly in Sec. IV C below. In Ref. [2], we also analyzed the simpler cases of $\mathrm{PGT}^+$ with vanishing torsion or curvature, which are not merely special cases of the full $\mathrm{PGT}^+$ Lagrangian, because additional constraints are placed not only on Lagrangian coefficients but also on the fields. Although a number of unitary critical cases were identified, no case was found that is also PPCR.

In seeking gravitational gauge theories that are renormalizable, one promising route is to demand local scale invariance *a priori*, since such theories contain no dimensionful parameters, and hence no absolute energy scale. Thus, rather than gauging the Poincaré group, one may instead gauge the Weyl group so that the action is also invariant under local dilations. The resulting Weyl gauge theories (WGTs) were first discussed in Refs. [4–6]. In this article, we apply our systematic method for identifying ghost-and-tachyon-free critical cases to parity-preserving Weyl gauge theory ($\mathrm{WGT}^+$), the ground-state particle spectrum of which has rarely been discussed in the literature before.

This paper is arranged as follows. In Sec. II, we give a brief introduction to $\mathrm{WGT}^+$, and in Sect. III, we consider the unitarity of the "root" theory, where none of the critical conditions is satisfied. In Sec. IV, we apply our systematic approach to investigating its critical cases and accommodating the associated additional source constraints as well as identifying some unitary critical cases that are also propagating power-counting renormalizable. We repeat our analysis for $\mathrm{WGT}^+$ with vanishing torsion in Sec. V and for $\mathrm{WGT}^+$ with vanishing curvature in Sec. VI. We conclude in Sec. VII.

We use the Landau-Lifshitz "mostly minus" metric signature $(+,-,-,-)$ throughout this paper.

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## II. WEYL GAUGE THEORIES

The action of an infinitesimal element of the Weyl group $W(1,3)$ on Cartesian coordinates in Minkowski spacetime has the form

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu + \omega^\mu{}_\nu x^\nu + \rho x^\mu, \qquad (1)$$

where $\epsilon^\mu$ denotes a translation, $\omega^\mu{}_\nu$ denotes a Lorentz rotation, and $\rho$ denotes a dilation. The corresponding form variation $\delta_0\varphi(x) \equiv \varphi'(x) - \varphi(x)$ of a field $\varphi$ (belonging to an irreducible representation of the Lorentz group) is $\delta_0\varphi = \delta_0^{\mathrm{P}}\varphi + w\rho\varphi$, where $\delta_0^{\mathrm{P}}$ means the variation under a Poincaré transformation and $w$ is a dimensionless constant known as the (Weyl) weight of the field.

One gauges the Weyl group $W(1,3)$ by demanding that the action be invariant with respect to (infinitesimal, passively interpreted) general coordinate transformations (GCTs) and the local action of the subgroup $H(1,3)$ (the homogeneous Weyl group), obtained by setting the translation parameters $\epsilon^\mu$ of $W(1,3)$ to zero (which leaves the origin $x^\mu = 0$ invariant) and allowing the remaining group parameters to become independent arbitrary functions of position. In this way, one is led to the introduction of the gravitational gauge fields $h_A{}^\mu$, $A^{AB}{}_\mu$, and $B_\mu$, corresponding to the translational, rotational, and dilational parts of the Weyl group, respectively, which transform under the gauged Weyl group as $\delta_0 h_A{}^\mu = \delta_0^{\mathrm{P}} h_A{}^\mu - \rho h_A{}^\mu$, $\delta_0 A^{AB}{}_\mu = \delta_0^{\mathrm{P}} A^{AB}{}_\mu$, and $\delta_0 B_\mu = -\partial_\mu\rho$.

The gauge fields are used to assemble the WGT covariant derivative [7,8]

$$\mathcal{D}^*_A\varphi = h_A{}^\mu\mathcal{D}^*_\mu\varphi = h_A{}^\mu\left(\partial_\mu + \frac{1}{2}A^{AB}{}_\mu\Sigma_{AB} + wB_\mu\right)\varphi, \qquad (2)$$

where $w$ is the weight of $\varphi$ and $\Sigma_{AB} = -\Sigma_{BA}$ are the generator matrices of the $\mathrm{SL}(2,C)$ representation to which $\varphi$ belongs. The asterisk on the derivative operators is a common notation used in WGT to distinguish these operators from their PGT counterparts (to which they reduce if $w$ or $B_\mu$ vanishes). The corresponding commutators become

$$[\mathcal{D}^*_\mu, \mathcal{D}^*_\nu]\varphi = \frac{1}{2}\mathcal{R}^{AB}{}_{\mu\nu}\Sigma_{AB}\varphi + \mathcal{H}_{\mu\nu}w\varphi, \qquad (3)$$

$$[\mathcal{D}^*_A, \mathcal{D}^*_B]\varphi = \frac{1}{2}\mathcal{R}^{CD}{}_{AB}\Sigma_{CD}\varphi - \mathcal{T}^{*C}{}_{AB}\mathcal{D}^*_C\varphi + \mathcal{H}_{AB}w\varphi, \qquad (4)$$

where the field strengths have the forms

$$\mathcal{R}^{AB}{}_{\mu\nu} = 2(\partial_{[\mu}A^{AB}{}_{\nu]} + A^A{}_{E[\mu}A^{EB}{}_{\nu]}), \qquad (5)$$

$$\mathcal{H}_{\mu\nu} = 2\partial_{[\mu}B_{\nu]}, \qquad (6)$$

$$\mathcal{T}^{*C}{}_{AB} = \mathcal{T}^C{}_{AB} + 2B_{[A}\delta^C_{B]} \qquad (7)$$

and $\mathcal{T}^C{}_{\mu\nu} = 2\mathcal{D}_{[\mu}b^C{}_{\nu]}$ is the usual expression for the translational gauge field strength in PGT. In the above expressions, latin and greek indices are related by $h_A{}^\nu$ and its inverse $b^A{}_\nu$, with the relation

$$g_{\mu\nu}h_A{}^\mu h_B{}^\mu = \eta_{AB}, \qquad \eta_{AB}b^A{}_\mu b^B{}_\mu = g_{\mu\nu}. \qquad (8)$$

One may show that the weights of the translational and rotational gauge fields are $w(h_A{}^\mu) = -1$ and $w(A^{AB}{}_\mu) = 0$ so that $w(b^A{}_\mu) = 1$ and the weight of its determinant is $w(b) = 4$, but the dilatational gauge field $B_\mu$ itself transforms inhomogeneously under dilations, as expected. The weights of the corresponding field strengths are $w(\mathcal{R}^{CD}{}_{AB}) = w(\mathcal{H}_{AB}) = -2$ and $w(\mathcal{T}^{*C}{}_{AB}) = -1$.

In the action $S = \int b\mathcal{L}\mathrm{d}^4x$, the Lagrangian $\mathcal{L}$ is the sum of terms corresponding to the free gravitational fields and terms containing the matter fields, and has the general form[1]

$$\mathcal{L} = \mathcal{L}_{\mathrm{G}}(\mathcal{R}^{CD}{}_{AB}, \mathcal{T}^{*C}{}_{AB}, \mathcal{H}_{AB}) + \mathcal{L}_{\mathrm{M}}(\varphi, \mathcal{D}^*_A\varphi). \qquad (9)$$

For $S$ to be scale invariant (i.e., of weight 0), the weights of both $\mathcal{L}_{\mathrm{G}}$ and $\mathcal{L}_{\mathrm{M}}$ must be $-4$. Restricting our attention to terms in $\mathcal{L}_{\mathrm{G}}$ that are at most quadratic in the field strengths, these may thus be quadratic in $\mathcal{R}^{CD}{}_{AB}$ and $\mathcal{H}_{AB}$, or consist of the product of the two, but may not include terms linear in $\mathcal{R}^{CD}{}_{AB}$ or quadratic in $\mathcal{T}^{*C}{}_{AB}$.

One can, however, include further terms in the Lagrangian by introducing an additional massless scalar field (or fields) $\phi$ with Weyl weight $w(\phi) = -1$, often termed the compensator(s) [7], which is usually nonminimally (conformally) coupled to the field strength tensors of the gravitational gauge fields. For example, terms proportional to $\phi^2\mathcal{R}$ or $\phi^2\mathcal{L}_{\mathcal{T}^{*2}}$, where $\mathcal{L}_{\mathcal{T}^{*2}}$ consists of terms quadratic in $\mathcal{T}^{*C}{}_{AB}$, have weight $w = -4$ and so may be added to the total Lagrangian [9–12]. One should also include a free kinetic term $(\mathcal{D}^*\phi)^2$ for the scalar field and may also add a self-interaction term $\phi^4$, but we shall not consider the latter here. Thus, also requiring parity invariance, the Lagrangian for free WGT$^+$ has the form

[1]Note that in Refs. [1,2], the definition of $\mathcal{L}$ sometimes included $b$.

$$\mathcal{L}_{\rm G} = -\lambda\phi^2\mathcal{R} + \frac{1}{6}(2r_1+r_2)\mathcal{R}^{ABCD}\mathcal{R}_{ABCD} + \frac{2}{3}(r_1-r_2)\mathcal{R}^{ABCD}\mathcal{R}_{ACBD} + \frac{1}{6}(2r_1+r_2-6r_3)\mathcal{R}^{ABCD}\mathcal{R}_{CDAB}$$
$$+ (r_4+r_5)\mathcal{R}^{AB}\mathcal{R}_{AB} + (r_4-r_5)\mathcal{R}^{AB}\mathcal{R}_{BA} - c_1\mathcal{R}^{AB}\mathcal{H}_{AB} + \xi\mathcal{H}^{AB}\mathcal{H}_{AB} + \frac{1}{2}\nu\mathcal{D}^*_A\phi\mathcal{D}^{*A}\phi$$
$$+ \frac{1}{12}(4t_1+t_2+3\lambda)\phi^2\mathcal{T}^{*ABC}\mathcal{T}^*_{ABC} - \frac{1}{6}(2t_1-t_2+3\lambda)\phi^2\mathcal{T}^{*ABC}\mathcal{T}^*_{BCA} - \frac{1}{3}(t_1-2t_3+3\lambda)\phi^2\mathcal{T}^*{}_B{}^{AB}\mathcal{T}^*{}_{CA}{}^C, \quad (10)$$

where $\mathcal{R}^A{}_B = \mathcal{R}^{AC}{}_{BC}$, $\mathcal{R} = \mathcal{R}^A{}_A$, and $\mathcal{D}^*_A\phi = \partial_A\phi - B_A\phi$. The parameters in the Lagrangian are dimensionless and set in combinations that enable a straightforward comparison with our previous studies of PGT$^+$ [1,2]. Note that the Gauss-Bonnet identity has been used to remove the term proportional to $\mathcal{R}^2$.

Provided $\phi(x)$ does not vanish anywhere, one can use local scale invariance to set the field to a constant value $\phi_0$, which is known as the Einstein gauge and is usually interpreted as breaking the scale symmetry. This interpretation is questioned in Ref. [8]; however, since it is shown that if one rewrites the Lagrangian in terms of a set of scale-invariant variables [6], then the resulting equations of motion are the same as those of Einstein gauge, yet this approach involves no breaking of the scale symmetry. In any case, we will adopt the Einstein gauge $\phi = \phi_0$ here, the most significant effect of which is that the term $\frac{1}{2}\nu\mathcal{D}^*_A\phi\mathcal{D}^{*A}\phi$ in the Lagrangian becomes $\frac{1}{2}\nu\phi_0^2B_AB^A$. We then absorb the $\phi_0^2$ factor into the now dimensionful parameters $\lambda$, $\nu$, $t_1$, $t_2$, and $t_3$, without loss of generality. Note that a $\phi^4$ potential term for the compensator scalar field was not included in the Lagrangian, since it becomes a constant in the Einstein gauge, acting like an effective cosmological constant, which would be inconsistent with considering a Minkowski background.

WGT is most naturally interpreted as a field theory in Minkowski spacetime [8,13,14], in the same way as the gauge field theories describing the other fundamental interactions. It is more common, however, to reinterpret it geometrically in terms of a Weyl-Cartan spacetime ($W_4$), which generalises the Riemann-Cartan spacetime ($U_4$) underlying the geometric interpretation of PGT by incorporating local scale invariance [7].

Weyl-Cartan spacetime is a manifold with linear connection ($\Gamma$) and metric ($g_{\mu\nu}$), which satisfy

$$\mathcal{D}^*_\rho(\Gamma)g_{\mu\nu} = 0, \quad (11)$$

where the covariant derivative of a field $\varphi$ with weight $w$ is defined by

$$\mathcal{D}^*_\mu(\Gamma)\varphi \equiv (\mathcal{D}_\mu(\Gamma) + wB_\mu)\varphi, \quad (12)$$

in which $\mathcal{D}_\mu(\Gamma) = \partial_\mu + \Gamma^\sigma{}_{\rho\mu}\mathsf{X}^\rho{}_\sigma$ is the $U_4$ covariant derivative and $\mathsf{X}^\rho{}_\sigma$ are the GL$(4,R)$ generator matrices appropriate to the GCT tensor character of the field to which the operator is applied. The semimetricity condition (11) replaces the metricity condition in $U_4$. Since $w(g_{\mu\nu}) = 2$, the semimetricity condition can also be written as $\mathcal{D}_\rho(\Gamma)g_{\mu\nu} = -2B_\rho g_{\mu\nu}$, from which one finds that the infinitesimal change of length of a parallel transported vector is proportional to the length itself, $\mathcal{D}_\rho(\Gamma)V^2 = -2B_\rho V^2$. One may solve for the connection $\Gamma$, which is given by

$$\Gamma^\mu{}_{\nu\rho} = \left\{ {\mu \atop \nu\rho} \right\} + \delta^\mu_\nu B_\rho + \delta^\mu_\rho B_\nu - g_{\nu\rho}B^\mu + K^\mu{}_{\nu\rho}, \quad (13)$$

where $\left\{ {\mu \atop \nu\rho} \right\}$ is the ordinary Christoffel symbol and $K^\mu{}_{\nu\rho}$ is the contorsion tensor (discussed further below).

A local Lorentz frame at each point on the manifold describes the tangent space and is determined by the tetrad basis $h_A{}^\mu$ with its inverse $b^A{}_\mu$; these quantities may be used to convert between coordinate and local Lorentz indices. The Minkowski metric $\eta_{AB}$ is invariant under Weyl transformation, so $w(\eta_{AB}) = 0$ and $w(h_A{}^\mu) = -1$. The local frame has a connection $A^{AB}{}_\mu$, and the covariant derivative $\mathcal{D}^*_A(A)$ has properties similar to (12), where

$$\mathcal{D}^*_\rho(A)\eta_{AB} = 0, \quad (14)$$

$$\mathcal{D}^*_\rho(A)\varphi \equiv (\mathcal{D}_\rho(A) + wB_\rho)\varphi, \quad (15)$$

and $\mathcal{D}_\rho(A)$ is the covariant derivative in $U_4$. One may also define the "total covariant derivative" $\mathcal{D}^*_\rho(\Gamma + A)$ to act on quantities with both coordinate and local Lorentz indices

$$\mathcal{D}^*_\rho(\Gamma + A)\varphi = (\mathcal{D}_\rho(\Gamma) + \mathcal{D}_\rho(A) - \partial_\rho - wB_\rho)\varphi. \quad (16)$$

Since the total covariant derivative $\mathcal{D}^*_\rho(\Gamma + A)V^A$ of the local Lorentz components of a vector is a coordinate tensor in Weyl-Cartan spacetime, the relation $\mathcal{D}^*_\rho(\Gamma + A)V^A = b^A{}_\mu\mathcal{D}^*_\rho(\Gamma + A)V^\mu$ should hold, from which one obtains the so-called tetrad postulate

$$D^*_\mu(\Gamma + A)b^A{}_\nu \equiv \partial^*_\mu b^A{}_\nu + A^A{}_{B\mu}b^B{}_\nu - \Gamma^\sigma{}_{\nu\mu}b^A{}_\sigma = 0, \quad (17)$$

where $\partial^*_\mu \equiv \partial_\mu + wB_\mu$. One can therefore express the affine connection in the quantities corresponding to gauge fields as

$$\Gamma^{\lambda}{}_{\nu\mu} = h_A{}^{\lambda}(\partial^*_\mu b^A{}_\nu + A^A{}_{B\mu} b^B{}_\nu) \tag{18}$$

and hence show that the translational gauge field strength is equivalent to (minus) the geometric torsion tensor

$$\mathcal{T}^{*\rho}{}_{\mu\nu} = \Gamma^{\rho}{}_{\nu\mu} - \Gamma^{\rho}{}_{\mu\nu}, \tag{19}$$

in terms of which the contorsion is given by

$$K_{\mu\lambda\nu} = -\frac{1}{2}(\mathcal{T}_{\mu\lambda\nu} - \mathcal{T}_{\nu\mu\lambda} + \mathcal{T}_{\lambda\nu\mu}). \tag{20}$$

From (18), (19), and (20), one also obtains

$$A_{AB\mu} = \Delta^*_{AB\mu} + K_{AB\mu}, \tag{21}$$

where we define the quantities

$$\Delta^*_{AB\mu} \equiv \Delta_{AB\mu}|_{\partial\to\partial^*} = \Delta_{AB\mu} - B_A b_{B\mu} + B_B b_{A\mu}, \tag{22}$$

$$\Delta_{AB\mu} \equiv \frac{1}{2}(c_{ABC} - c_{CAB} + c_{BCA}) b^C{}_\mu, \tag{23}$$

$$c^A{}_{\mu\nu} \equiv \partial_\mu b^A{}_\nu - \partial_\nu b^A{}_\mu. \tag{24}$$

One then finds that, in contrast to the torsion, the geometric (Riemann) curvature tensor differs from the rotational gauge field strength $\mathcal{R}^{\rho}{}_{\sigma\mu\nu}$, so we denote the former by

$$\begin{aligned}\tilde{\mathcal{R}}^{\rho}{}_{\sigma\mu\nu} &= \mathcal{R}^{\rho}{}_{\sigma\mu\nu} + H_{\mu\nu}\delta^{\rho}_{\sigma},\\ &= \partial_\mu \Gamma^{\rho}{}_{\sigma\nu} - \partial_\nu \Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\rho}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\sigma\nu} - \Gamma^{\rho}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\sigma\mu}.\end{aligned} \tag{25}$$

Unlike $\mathcal{R}_{\rho\sigma\mu\nu}$, the curvature tensor $\tilde{\mathcal{R}}_{\rho\sigma\mu\nu}$ is not antisymmetric in $(\rho,\sigma)$, while both are antisymmetric in $(\mu,\nu)$ [7,8]. Indeed, one may take advantage of these symmetry properties by using $\mathcal{R}_{\rho\sigma\mu\nu}$ to perform calculations instead of $\tilde{\mathcal{R}}_{\rho\sigma\mu\nu}$. One should note, however, that, unlike the curvature tensor in Riemann spacetime $V_4$ familiar from general relativity, neither $\mathcal{R}_{\rho\sigma\mu\nu}$ nor $\tilde{\mathcal{R}}_{\rho\sigma\mu\nu}$ is symmetric in $(\rho\sigma,\mu\nu)$.

## III. ROOT THEORY

We now apply the method described in Ref. [1] to the "root" theory (10), where none of the critical conditions is satisfied. We first linearize the Lagrangian around the Minkowski background using $A_{ABC} \sim O(t)$, $B_A \sim O(t)$, $h_A{}^\mu = \delta_A{}^\mu + f_A{}^\mu$, and $f_{AB} = \mathfrak{s}_{AB} - \mathfrak{a}_{AB} \sim O(t)$, where $\mathfrak{s}$ and $\mathfrak{a}$ denote the symmetric and antisymmetric parts of $f$, respectively. Note that we cannot perturb $\phi$ as $\phi_0 + \epsilon$, for some excitation $\epsilon$, because we have already fixed the gauge on $\phi$. The Lagrangian then becomes

$$b\mathcal{L}_{\mathrm{G}} = 2\lambda\partial_A A^{BA}{}_B + \mathcal{O}(t^2), \tag{26}$$

where the linear term is just a total derivative. We then decompose the quadratic part into

$$b\mathcal{L}_{\mathrm{G}} = \sum_{J,P,i,j} a(J^P)_{ij}\hat{\zeta}^\dagger \cdot \hat{P}(J^P)_{ij} \cdot \hat{\zeta}, \tag{27}$$

using the spin projection operators (SPOs) $\hat{P}(J^P)_{ij}$ [15–17]. Section II of Ref. [1] contains a description of our notation (note that Eq. (52) in Ref. [1] contains a typographical error and should read $f_{AB} = \mathfrak{s}_{AB} - \mathfrak{a}_{AB}$, as here, but this does not affect the remaining contents in Refs. [1,2]). The SPOs for WGT$^+$ are given in the Appendix A. One then obtains the $a$ matrices:

$$a(0^-) = \begin{array}{c|c} & A \\ \hline A & \left(2(k^2 r_2 + t_2)\right) \end{array}, \tag{28}$$

$$a(0^+) = \begin{array}{c|cccc} & A & \mathfrak{s} & \mathfrak{s} & B \\ \hline A & 2(2k^2(r_1 - r_3 + 2r_4) + t_3) & 2i\sqrt{2}kt_3 & 0 & -2\sqrt{6}(t_3-\lambda) \\ \mathfrak{s} & -2i\sqrt{2}kt_3 & 4k^2(t_3-\lambda) & 0 & 4i\sqrt{3}k(t_3-\lambda) \\ \mathfrak{s} & 0 & 0 & 0 & 0 \\ B & -2\sqrt{6}(t_3-\lambda) & -4i\sqrt{3}k(t_3-\lambda) & 0 & 4(3t_3 - 3\lambda + \frac{\nu}{4}) \end{array}, \tag{29}$$

$$a(1^-) = \begin{array}{c|ccccc} & A & A & \mathfrak{s} & \mathfrak{a} & B \\ \hline A & 2[k^2(r_1+r_4+r_5)+\frac{1}{6}(t_1+4t_3)] & -\frac{\sqrt{2}}{3}(t_1-2t_3) & -\frac{\sqrt{2}}{3}ik(t_1-2t_3) & \frac{\sqrt{2}}{3}ik(t_1-2t_3) & -c_1k^2+4t_3-4\lambda \\ A & -\frac{\sqrt{2}}{3}(t_1-2t_3) & \frac{2}{3}(t_1+t_3) & \frac{2}{3}ik(t_1+t_3) & -\frac{2}{3}ik(t_1+t_3) & 2\sqrt{2}(t_3-\lambda) \\ \mathfrak{s} & \frac{\sqrt{2}}{3}ik(t_1-2t_3) & -\frac{2}{3}ik(t_1+t_3) & \frac{2}{3}k^2(t_1+t_3) & -\frac{2}{3}k^2(t_1+t_3) & -2i\sqrt{2}k(t_3-\lambda) \\ \mathfrak{a} & -\frac{\sqrt{2}}{3}ik(t_1-2t_3) & \frac{2}{3}ik(t_1+t_3) & -\frac{2}{3}k^2(t_1+t_3) & \frac{2}{3}k^2(t_1+t_3) & 2i\sqrt{2}k(t_3-\lambda) \\ B & -c_1k^2+4t_3-4\lambda & 2\sqrt{2}(t_3-\lambda) & 2i\sqrt{2}k(t_3-\lambda) & -2i\sqrt{2}k(t_3-\lambda) & 4(3t_3-3\lambda+\frac{\nu}{4}+k^2\xi) \end{array}, \tag{30}$$

$$a(1^+) = \begin{array}{c} \\ A \\ A \\ \mathfrak{a} \end{array} \begin{array}{c} \begin{array}{ccc} A & A & \mathfrak{a} \end{array} \\ \begin{pmatrix} \frac{1}{3}(6k^2(2r_3+r_5)+t_1+4t_2) & \frac{1}{3}\sqrt{2}(t_1-2t_2) & -\frac{1}{3}i\sqrt{2}k(t_1-2t_2) \\ \frac{1}{3}\sqrt{2}(t_1-2t_2) & \frac{2}{3}(t_1+t_2) & -\frac{2}{3}ik(t_1+t_2) \\ \frac{1}{3}i\sqrt{2}k(t_1-2t_2) & \frac{2}{3}ik(t_1+t_2) & \frac{2}{3}k^2(t_1+t_2) \end{pmatrix} \end{array}, \tag{31}$$

$$a(2^-) = \begin{array}{c} \\ A \end{array} \begin{array}{c} A \\ (2(k^2 r_1 + \frac{t_1}{2})) \end{array}, \tag{32}$$

$$a(2^+) = \begin{array}{c} \\ A \\ \mathfrak{s} \end{array} \begin{array}{c} \begin{array}{cc} A & \mathfrak{s} \end{array} \\ \begin{pmatrix} 2(k^2(2r_1-2r_3+r_4)+\frac{t_1}{2}) & i\sqrt{2}kt_1 \\ -i\sqrt{2}kt_1 & 2k^2(t_1+\lambda) \end{pmatrix} \end{array}. \tag{33}$$

In general, if any of the matrices $a(J^P)$ in the decomposition (27) are singular, then the theory possesses gauge invariances. One may fix these gauges by deleting rows and columns of the $a$ matrices such that they become nonsingular. The elements of the resulting matrices are usually denoted by $b_{ij}(J^P)$. For WGT$^+$, some of the $a$ matrices given above are indeed singular. In particular, one may delete the third row/column of $a(0^+)$, the third and fourth row/column of $a(1^-)$, and the third row/column of $a(1^+)$ to obtain the corresponding nonsingular $b$ matrices. The singular nature of these three $a$ matrices results in them having both null right and left eigenvectors, which give us gauge invariance and source constraints, respectively. For each spin-parity sector, the null left eigenvectors are given by

$$0^+\!: \ (0,0,1,0) \tag{34}$$

$$1^-\!: \ (0,-ik,0,1,0), (0,ik,1,0,0) \tag{35}$$

$$1^+\!: \ (0,-ik,1), \tag{36}$$

where one should note that the $B$ field is not involved, since the corresponding vector component is always zero, and the remaining components are the same as those found for PGT$^+$. This is no surprise, since the dilation gauge invariance has been fixed by adopting the Einstein gauge, and the remaining symmetry should indeed be local Poincaré invariance.

The null eigenvectors may be used to derive the form of the associated gauge invariances and the corresponding source constraints for WGT$^+$, which are found to be the same as those in PGT$^+$, as expected. The gauge invariances are given by

$$\delta h_{AB} = u_{[AB]} + ik_B v_A \tag{37}$$

$$\delta A_{ABC} = ik_C u_{[AB]}, \tag{38}$$

where $u_{[AB]}$ and $v_A$ are some arbitrary fields, and the source constraints have the form

$$k^A \sigma_{AB} = 0 \tag{39}$$

$$ik^A \tau_{ABC} - \sigma_{[AB]} = 0, \tag{40}$$

where $\sigma_{AB}$ is the source current of $f_{AB}$ and $\tau_{ABC}$ is the source current of $A_{ABC}$.

The requirement that a theory is free from ghosts and tachyons places conditions on the $b$ matrices, and one must consider the massless and massive particle sectors separately. For the massless modes, one requires only that there be no ghosts. As discussed in Ref. [1], this is determined by considering the coefficient matrices $\mathbf{Q}_{2n}$ in a Laurent series expansion of the saturated propagator about the origin in momentum space. For WGT$^+$, one finds that all of the entries $\mathbf{Q}_{2n}$ vanish identically for $n > 1$, and so the saturated propagator does not have a higher pole at $k^2 = 0$. The nonzero eigenvalues of $\mathbf{Q}_2$ are found to be

$$\frac{1+6|\vec{k}|^2}{\lambda}, \qquad \frac{1+8|\vec{k}|^2}{2\lambda}, \tag{41}$$

and so there are 2 degrees of freedom in the propagating massless particle sector.[2] The massless no-ghost condition is that all eigenvalues of $\mathbf{Q}_{2n}$ are non-negative, and so one requires simply that

$$\lambda > 0. \tag{42}$$

Turning to the massive particle sector, one must first determine the particle masses by calculating the determinants of the $b$ matrices,

[2]Note that the expression for the eigenvalues is not unique but depends on the form chosen for the source constraints. To be precise, one can obtain another set of the null vectors $\mathbf{n}_i$ in Eq. (30) of Ref. [1] by linear combination.

$$\det[b(0^-)] = 2k^2 r_2 + 2t_2, \tag{43}$$

$$\det[b(0^+)] = 16(r_1 - r_3 + 2r_4)(t_3 - \lambda)\nu k^4 - 8\lambda[12(t_3 - \lambda)\lambda + t_3\nu]k^2, \tag{44}$$

$$\det[b(1^-)] = -\frac{2}{3}(t_1 + t_3)[c_1^2 - 8(r_1 + r_4 + r_5)\xi]k^4 + \frac{4}{3}\{6c_1 t_1(t_3 - \lambda) + (r_1 + r_4 + r_5) \times [12(t_3 - \lambda)(t_1 + \lambda) + (t_1 + t_3)\nu] + 6t_1 t_3\xi\}k^2 + 2t_1[12\lambda(t_3 - \lambda) + t_3\nu], \tag{45}$$

$$\det[b(1^+)] = \frac{4}{3}(2r_3 + r_5)(t_1 + t_2)k^2 + 2t_1 t_2, \tag{46}$$

$$\det[b(2^-)] = 2r_1 k^2 + t_1, \tag{47}$$

$$\det[b(2^+)] = 4(2r_1 - 2r_3 + r_4)(t_1 + \lambda)k^4 + 2t_1\lambda k^2, \tag{48}$$

from which one finds that there is no massive mode in the $0^+$ sector, and the particle masses in the other sectors are given by

$$m^2(0^-) = -\frac{t_2}{r_2}, \tag{49}$$

$$m^2(0^+) = \frac{12\lambda^2(t_3 - \lambda) + t_3\lambda}{2(r_1 - r_3 + 2r_4)(t_3 - \lambda)\nu}, \tag{50}$$

$$m^2(1^-) = (\text{the two roots of } \det[b(1^-)]), \tag{51}$$

$$m^2(1^+) = -\frac{3t_1 t_2}{2(2r_3 + r_5)(t_1 + t_2)}, \tag{52}$$

$$m^2(2^-) = -\frac{t_1}{2r_1}, \tag{53}$$

$$m^2(2^+) = -\frac{t_1\lambda}{2(2r_1 - 2r_3 + r_4)(t_1 + \lambda)}. \tag{54}$$

The no-tachyon conditions are then simply $m^2(J^P) > 0$. We give the conditions for the $1^-$ sector in Appendix B because of the length of the expressions involved. Note also for the $1^-$ sector that one requires the two roots of (45) to be distinct in order to avoid a dipole ghost. Hence, in each sector, the masses are distinct, and so one can apply Eq. (45) in Ref. [1] directly to obtain the massive no-ghost conditions,

$$0^-:\ r_2 < 0, \tag{55}$$

$$0^+:\ (r_1 - r_3 + 2r_4)(t_3 - \lambda)\lambda\nu^2\{24(t_3 - \lambda)\lambda^3 + 12(r_1 - r_3 + 2r_4)(t_3 - \lambda)\lambda\nu + [(r_1 - r_3 + 2r_4)t_3 + t_3\lambda - \lambda^2]\nu^2\} > 0, \tag{56}$$

$$1^+:\ (2r_3 + r_5) > 0, \tag{57}$$

$$2^-:\ r_1 < 0, \tag{58}$$

$$2^+:\ \lambda(2r_1 - 2r_3 + r_4)(\lambda + t_1) \times [(2r_1 - 2r_3 + r_4)t_1 - \lambda^2 - \lambda t_1] < 0, \tag{59}$$

where again we do not write out the condition for $1^-$ because of its length but instead give the relevant expression in Appendix B.

The combined no-ghost-and-tachyon conditions for each sector other than $1^-$ are then

$$0^-:\ t_2 > 0, \qquad r_2 < 0 \tag{60}$$

$$0^+:\ r_1 + 2r_4 > r_3, (t_3 - \lambda)\lambda\nu[12\lambda(t_3 - \lambda) + t_3\nu] > 0 \tag{61}$$

$$1^+:\ 2r_3 + r_5 > 0, \qquad t_1 t_2(t_1 + t_2) < 0 \tag{62}$$

$$2^-:\ t_1 > 0, \qquad r_1 < 0 \tag{63}$$

$$2^+:\ 2r_1 + r_4 > 2r_3, \qquad \lambda t_1(\lambda + t_1) < 0. \tag{64}$$

For the $1^-$ sector, we give the combined condition in Appendix B and show that it does allow some ranges of the parameters, but we are unable to obtain a simplified expression for it. Note that, except for the $0^+$ and $1^-$ sectors, the combined condition in each of the other spin-parity sectors is exactly the same as originally found in Ref. [3] for PGT$^+$.

Finally, if we consider all the no-tachyon and no-ghost conditions from all the massive sectors, we find that they cannot be satisfied simultaneously. Thus, the root theory must contain a massive ghost or tachyon.

## IV. CRITICAL CASES

If the parameters in the action satisfy certain "critical conditions," the particle masses (49)–(54) can become zero or infinite, and the resulting critical cases may possess additional gauge invariances, so one may have to re-evaluate the no-tachyon and no-ghost conditions for both the massless and massive sectors.

### A. Unitarity

In attempting to apply the method in Ref. [1] to obtain all the critical cases of the root WGT$^+$ theory, one finds that some of the coefficients in Eqs. (44) and (45) cannot be factorized into linear combinations of the parameters. Consequently, the method in Ref. [1] cannot be applied straightforwardly to obtain all the critical cases, and one must check carefully where it is applicable. For example, one of the factors in the coefficient of the $k^2$ term in (44) is

$$12(t_3-\lambda)\lambda + t_3\nu, \quad (65)$$

which cannot be written as the product of factors that are linear in the Lagrangian parameters. Indeed, for (65) to equal zero, one has the two solutions:

$$\nu = -\frac{12(t_3-\lambda)\lambda}{t_3} \quad \text{with} \quad t_3 \neq 0, \quad (66)$$

$$t_3 = \lambda = 0. \quad (67)$$

It is therefore not as straightforward to apply the condition $12(t_3-\lambda)\lambda + t_3\nu = 0$ by substitution. Moreover, the second solution (67) requires one to eliminate 2 degrees of freedom in the parameters simultaneously and thus breaks the hierarchy of the "tree" of critical cases discussed in Ref. [1].

In general, one finds that allowing any of the Lagrangian parameters $\nu$, $\xi$, or $c_1$ in (10) to be nonzero introduces similar problems. It requires further improvement of our systematic method to accommodate such cases, and so here we set $\nu = \xi = c_1 = 0$ to avoid these difficulties. Thus, for the remainder of this section, the "root theory" refers to (10) with $\nu = \xi = c_1 = 0$. As we will show below, however, one may nevertheless construct a theory with $\nu \neq 0$ and/or $\xi \neq 0$ from a theory with $\nu = \xi = 0$, provided its $a$ matrices are "nonmixing."

Starting from the root theory, we systematically identify 862 critical cases (excluding the "vanishing" Lagrangian, for which all parameters are zero). Of these critical cases, we find 168 are free of ghosts and tachyons, provided the parameters in each case satisfy some additional conditions that preclude them from generating another critical case; this general issue is discussed in detail in Appendix C.

### B. Comparison with previous results

We now compare our results with the only other example of a unitary WGT$^+$ theory of which we are aware in the literature [18]. This has the Lagrangian

$$\mathcal{L} = -\lambda\phi^2\mathcal{R} + a\mathcal{R}^2 - \frac{1}{4}H^{\mu\nu}H_{\mu\nu} + \frac{1}{2}\mathcal{D}^*_\mu\phi\mathcal{D}^{*\mu}\phi, \quad (68)$$

which on adopting the Einstein gauge becomes

$$\mathcal{L} = -\lambda\phi_0^2\mathcal{R} + a\mathcal{R}^2 - \frac{1}{4}H^{\mu\nu}H_{\mu\nu} + \frac{1}{2}\phi_0^2 B_\mu B^\mu. \quad (69)$$

Thus, the $B$ field is decoupled from the other gauge fields, and so the theory can be viewed as the combination of PGT$^+$ with $\mathcal{L} = -\lambda\phi_0^2\mathcal{R} + a\mathcal{R}^2$ and Proca theory $\mathcal{L}_{\rm Pr} = -\frac{1}{4}H^{\mu\nu}H_{\mu\nu} + \frac{1}{2}\phi_0^2 B_\mu B^\mu$ for a massive vector field. The Proca part is well known to be unitary. Using the Gauss-Bonnet identity, the PGT$^+$ part may be shown to correspond to the critical case $r_1 = r_2 = 2r_3 - r_4 = 2r_3 + r_5 = t_1 + t_2 = t_1 + t_3 = t_1 + \lambda = 0, r_3 \neq 0, \lambda \neq 0$. This a type C critical case of the root PGT$^+$ theory with no massive mode and massless modes with 2 degrees of freedom; the no-ghost-and-tachyon condition is simply $\lambda > 0$. Therefore, provided this condition is satisfied, the theory (68) is indeed unitary.

One should note that the presence of the kinetic terms for the $B$ and $\phi$ fields means that (68) is not a critical case of our redefined WGT$^+$ with $\nu = \xi = c_1 = 0$ in (10) but is a critical case of the "full" WGT$^+$ root theory without this constraint on the Lagrangian parameters. In particular, Eq. (68) belongs to an extended set of theories with $\nu \neq 0$ and $\xi \neq 0$ that can be separated into a PGT$^+$ part and a dilaton part, which we discuss below in the context of propagating power-counting renormalizability. We note, however, that the PGT$^+$ part of (68) is not listed in Ref. [2] because one cannot obtain nonmixing $b$ matrices by deleting rows and columns from its $a$ matrices.

### C. Propagating power-counting renormalizability

In addition to possessing no ghosts or tachyons, a healthy physical theory should also be renormalizable. The first step in assessing whether this is possible is to determine whether the theory is power-counting renormalizable.

As discussed in Refs. [1,2], the key quantity for determining whether a theory is PCR is the propagator

$$\hat{D} = \sum_{J,P,i,j} b^{-1}_{ij}\hat{P}(J^P)_{ij}. \quad (70)$$

In particular, if the $b$ matrices are block diagonal, with each block containing only one of the fields $A$, $\mathfrak{s}$, $\mathfrak{a}$, and $B$, then there are no mixing terms in the (gauge-fixed) Lagrangian, and it is straightforward to obtain the propagators for these fields separately from $\hat{D}$. Extending the original PCR criterion used by Sezgin and van Nieuwenhuizen in Ref. [3] would require the propagator of the $A$ and $B$ fields to decay at least as quickly as $k^{-2}$ at high energy, and those of the $\mathfrak{s}$ and $\mathfrak{a}$ fields to fall off at least as $k^{-4}$ (see Appendix D). By contrast, we proposed an alternative criterion in Refs. [1,2], which we now term propagating power-counting renormalizability, that in addition allows the presence of nonpropagating fields at high momenta (for which the propagator decays no faster than a constant). Since the physical basis of power-counting renormalizability relates to the divergence at large momenta of integrals describing the propagation of particles around closed loops in Feynman diagrams, it seems physically reasonable to allow for the presence of modes that do not propagate at large momenta, since these should be integrated out and not contribute to the loop integrals. PPCR is less restrictive than PCR, and it may therefore retain some theories that are eliminated by PCR erroneously. The ultimate consistency of these two approaches in identifying

TABLE I. Parameter conditions for the PPCR critical cases that are ghost- and tachyon-free and cannot be constructed directly from PGT. The parameters listed in "Additional conditions" must be nonzero to prevent the theory becoming a different critical case.

| No. | Critical condition | Additional conditions | No-ghost-and-tachyon condition |
|---|---|---|---|
| 1 | $r_1, \frac{r_3}{2} - r_4, t_1, \lambda = 0$ | $r_2, r_3, 2r_3 + r_5, r_3 + 2r_5, t_2, t_3$ | $t_2 > 0, r_2 < 0, r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$ |
| 2 | $r_2, r_1 - r_3, r_4, t_1, t_2, \lambda = 0$ | $r_1, r_1 + r_5, 2r_1 + r_5, t_3$ | $r_1(r_1 + r_5)(2r_1 + r_5) < 0$ |
| 3 | $r_1, r_2, \frac{r_3}{2} - r_4, t_1, t_2, \lambda = 0$ | $r_3, 2r_3 + r_5, r_3 + 2r_5, t_3$ | $r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$ |
| 4 | $r_1, \frac{r_3}{2} - r_4, t_1, t_2, \lambda = 0$ | $r_2, r_3, 2r_3 + r_5, r_3 + 2r_5, t_3$ | $r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$ |
| 5 | $r_1, r_2, \frac{r_3}{2} - r_4, t_1, \lambda = 0$ | $r_3, 2r_3 + r_5, r_3 + 2r_5, t_2, t_3$ | $r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$ |
| 6 | $r_1, r_3, r_4, r_5, \lambda = 0$ | $r_2, t_1, t_2, t_1 + t_2, t_3$ | $t_2 > 0, r_2 < 0$ |
| 7 | $r_1, r_3, r_4, r_5, t_1 + t_2, \lambda = 0$ | $r_2, t_1, t_3$ | $r_2 < 0, t_1 < 0$ |
| 8 | $r_2, r_1 - r_3, r_4, r_1 + r_5, t_1 + t_2, \lambda = 0$ | $r_1, t_1, t_3$ | $t_1 > 0, r_1 < 0$ |
| 9 | $r_1, r_3, r_4, r_5, t_1, \lambda = 0$ | $r_2, t_2, t_3$ | $t_2 > 0, r_2 < 0$ |
| 10 | $r_1, r_3, r_4, t_1, \lambda = 0$ | $r_2, r_5, t_2, t_3$ | $t_2 > 0, r_2 < 0$ |
| 11 | $r_1 - r_3, r_4, 2r_1 + r_5, t_1, \lambda = 0$ | $r_1, r_2, t_2, t_3$ | $t_2 > 0, r_2 < 0$ |
| 12 | $r_1, \frac{r_3}{2} - r_4, 2r_3 + r_5, t_1, \lambda = 0$ | $r_2, r_3, t_2, t_3$ | $t_2 > 0, r_2 < 0$ |
| 13 | $r_1, \frac{r_3}{2} - r_4, \frac{r_3}{2} + r_5, t_1, \lambda = 0$ | $r_2, r_3, t_2, t_3$ | $t_2 > 0, r_2 < 0$ |

particular theories as PCR and PPCR is discussed at length in Ref. [2], although the second approach is preferred since it identifies further critical cases that reduce to those identified by Sezgin and van Nieuwenhuizen's criterion at linear level after integrating out any nonpropagating modes. We therefore again adopt the latter method here, which is consistent with our previous work.

On performing this analysis, one finds that most of the critical cases identified as PPCR are identical to those listed in Table I, III, or V in Ref. [2] or are a $\mathrm{PGT}^+$ without any propagating mode (which were not listed in Ref. [2]) but with an additional propagating dilaton. One may understand the reason for this by first expanding the $\mathcal{T}^{*2}$ terms in (10) to obtain

$$\mathcal{T}^*_{ABC}\mathcal{T}^{*ABC} = \mathcal{T}_{ABC}\mathcal{T}^{ABC} + 4B_A\mathcal{T}^{CA}{}_C + 6B^AB_A, \quad (71)$$

$$\mathcal{T}^*_{ABC}\mathcal{T}^{*BCA} = \mathcal{T}_{ABC}\mathcal{T}^{BCA} - 2B_A\mathcal{T}^{CA}{}_C - 3B^AB_A, \quad (72)$$

$$\mathcal{T}^{*B}{}_{BA}\mathcal{T}^{*C}{}_C{}^A = \mathcal{T}^B{}_{BA}\mathcal{T}^C{}_C{}^A + 6B_A\mathcal{T}^{CA}{}_C + 9B^AB_A. \quad (73)$$

The $B\mathcal{T}$ terms are the only possible origin for mixing terms containing the $B$ field after linearization, and so there will be no mixing terms in the $a$ matrices if these terms vanish, for which the condition on the Lagrangian parameters is

$$t_3 = \lambda. \quad (74)$$

Moreover, the same condition ensures that the $B^2$ terms from $\mathcal{T}^{*2}$ also vanish. Hence, if $t_3 = \lambda$, the $\mathcal{R} + \mathcal{R}^2 + \mathcal{T}^{*2}$ part of the $\mathrm{WGT}^+$ Lagrangian is identical to its $\mathrm{PGT}^+$ counterpart with the replacement $\mathcal{T}^* \to \mathcal{T}$.

The $\mathrm{PGT}^+$ critical cases identified as PPCR in Ref. [2] and having $t_3 = \lambda$ are:

(1) $\mathrm{PGT}^+$ with 2 massless degrees of freedom and a massive mode: cases 1, 3, 4, 6, and 7 in Table I of Ref. [2];
(2) $\mathrm{PGT}^+$ with only 2 massless degrees of freedom: cases 9–13, 17, and 19 in Table III of Ref. [2][3];
(3) $\mathrm{PGT}^+$ with only massive mode(s): cases 26–28, 30–36, 38–40, 55, and 58 in Table V of Ref. [2] (these cases all have 1 massive mode, either $0^-$ or $2^-$).

If the $\mathrm{PGT}^+$ part of a $\mathrm{WGT}^+$ satisfying $t_3 = \lambda$ has no propagating mode, then the corresponding $\mathrm{WGT}^+$ can at most have a propagating $B$ field. There are 37 critical cases of $\mathrm{PGT}^+$ satisfying $t_3 = \lambda$ and containing no propagating mode (these are not listed in Refs. [1,2]). Requiring $\xi \neq 0$ in the corresponding $\mathrm{WGT}^+$ Lagrangian (10) ensures that they contain a propagating dilaton. The dilaton part of $\mathrm{WGT}^+$ Lagrangians satisfying $t_3 = \lambda$ is simply

$$\mathcal{L}_B = \xi\mathcal{H}^{AB}\mathcal{H}_{AB}, \quad (75)$$

which is that of a massless $1^-$ vector.

For all cases for which the $a$ matrices are nonmixing, there are no cross-terms of $B$ and the other fields, and so adding a mass term for $B$ in the Lagrangian does not affect the other fields. Hence, if one adds the term $\frac{1}{2}\nu\mathcal{D}^*_A\phi\mathcal{D}^{*A}\phi$ to such a case, the only effect is either to make an already propagating $B$ field massive or to add a nonpropagating $B$ field. In the former (and more interesting) case, the corresponding dilaton Lagrangian is a Proca theory in the Einstein gauge ($\phi_0 = 1$)

$$\mathcal{L}_B = \xi\mathcal{H}^{AB}\mathcal{H}_{AB} + \frac{1}{2}\nu B_\mu B^\mu, \quad (76)$$

and the corresponding no-ghost-and-tachyon condition is $\xi < 0$ and $\nu > 0$. With these extensions, one can thus

[3]We note that cases 9, 10, 11, and 13 in Ref. [2] satisfy the original criterion used by Sezgin and van Nieuwenhuizen in Ref. [3] to be PCR and are discussed further in Appendix E

TABLE II. Particle content of the PPCR critical cases that are ghost- and tachyon-free and cannot be constructed directly from PGT. The column "$b$ sectors" describes the diagonal elements in the $b^{-1}$ matrix of each spin-parity sector in the sequence $\{0^-, 0^+, 1^-, 1^+, 2^-, 2^+\}$. Here, it is notated as $\varphi_v^n$ or $\varphi_l^n$, where $\varphi$ is the field, $-n$ is the power of $k$ in the element in the $b^{-1}$ matrix when $k$ goes to infinity, $v$ means a massive pole, and $l$ means a massless pole. If $n = \infty$, it represents that the diagonal element is zero. If $n \leq 0$, the field is not propagating. The "|" notation denotes the different form of the elements of the $b^{-1}$ matrices in different choices of gauge fixing, and the "&" connects the diagonal elements in the same $b^{-1}$ matrix. The superscript "N" represents that there is nonzero off-diagonal term in the $b^{-1}$ matrix.

| No. | Massless mode degrees of freedom | Massive mode | $b$ sectors |
|---|---|---|---|
| 1 | 2 | $0^-$ | $\{A_{\mathrm{v}}^2, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\vert B^0, (A_{\mathrm{l}}^2\ \&\ A_{\mathrm{l}}^0)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{s}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ B_{\mathrm{l}}^0)^{\mathrm{N}}, (A_{\mathrm{l}}^2\ \&\ A_{\mathrm{l}}^0)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}, \times, A_{\mathrm{l}}^2\}$ |
| 2 | 2 | $\times$ | $\{\times, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\vert B^0, (A_{\mathrm{l}}^2\ \&\ A_{\mathrm{l}}^0)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{s}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ B_{\mathrm{l}}^0)^{\mathrm{N}}, A_{\mathrm{l}}^2, A_{\mathrm{l}}^2, \times\}$ |
| 3 | 2 | $\times$ | $\{\times, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\vert B^0, (A_{\mathrm{l}}^2\ \&\ A_{\mathrm{l}}^0)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{s}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ B_{\mathrm{l}}^0)^{\mathrm{N}}, A_{\mathrm{l}}^2, \times, A_{\mathrm{l}}^2\}$ |
| 4 | 2 | $\times$ | $\{A_{\mathrm{l}}^2, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\vert B^0, (A_{\mathrm{l}}^2\ \&\ A_{\mathrm{l}}^0)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{s}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ B_{\mathrm{l}}^0)^{\mathrm{N}}, A_{\mathrm{l}}^2, \times, A_{\mathrm{l}}^2\}$ |
| 5 | 2 | $\times$ | $\{A^0, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\vert B^0, (A_{\mathrm{l}}^2\ \&\ A_{\mathrm{l}}^0)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{s}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ B_{\mathrm{l}}^0)^{\mathrm{N}}, (A_{\mathrm{l}}^2\ \&\ A_{\mathrm{l}}^0)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}, \times, A_{\mathrm{l}}^2\}$ |
| 6 | 0 | $0^-$ | $\{A_{\mathrm{v}}^2, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\vert B^0, (A^0\ \&\ A^0)^{\mathrm{N}}\vert(A^0\ \&\ \mathfrak{s}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A^0\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A^0\ \&\ B^0)^{\mathrm{N}}\vert(\mathfrak{s}_{\mathrm{l}}^2\ \&\ B^0)^{\mathrm{N}}\vert(\mathfrak{a}_{\mathrm{l}}^2\ \&\ B^0)^{\mathrm{N}}, (A^0\ \&\ A^0)^{\mathrm{N}}\vert(A^0\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\}$ |
| 7 | 0 | $0^-$ | $\{A_{\mathrm{v}}^2, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\vert B^0, (A^0\ \&\ A^0)^{\mathrm{N}}\vert(A^0\ \&\ \mathfrak{s}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A^0\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A^0\ \&\ B^0)^{\mathrm{N}}\vert(\mathfrak{s}_{\mathrm{l}}^2\ \&\ B^0)^{\mathrm{N}}\vert(\mathfrak{a}_{\mathrm{l}}^2\ \&\ B^0)^{\mathrm{N}}, (A^\infty\ \&\ A^0)^{\mathrm{N}}\vert(A^\infty\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}, A^0, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\}$ |
| 8 | 0 | $2^-$ | $\{A^0, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\vert B^0, (A^0\ \&\ A^0)^{\mathrm{N}}\vert(A^0\ \&\ \mathfrak{s}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A^0\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A^0\ \&\ B^0)^{\mathrm{N}}\vert(\mathfrak{s}_{\mathrm{l}}^2\ \&\ B^0)^{\mathrm{N}}\vert(\mathfrak{a}_{\mathrm{l}}^2\ \&\ B^0)^{\mathrm{N}}, (A^\infty\ \&\ A^{-2})^{\mathrm{N}}\vert(A^\infty\ \&\ \mathfrak{a}_{\mathrm{l}}^0)^{\mathrm{N}}, A_{\mathrm{v}}^2, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\}$ |
| 9 | 0 | $0^-$ | $\{A_{\mathrm{v}}^2, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\vert B^0, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\vert\mathfrak{a}_{\mathrm{l}}^2\vert B^0, A^0\vert\mathfrak{a}_{\mathrm{l}}^2, \times, \times\}$ |
| 10 | 0 | $0^-$ | $\{A_{\mathrm{v}}^2, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\vert B^0, (A_{\mathrm{l}}^2\ \&\ A_{\mathrm{l}}^0)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{s}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ B_{\mathrm{l}}^0)^{\mathrm{N}}, (A_{\mathrm{l}}^2\ \&\ A_{\mathrm{l}}^0)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}, \times, \times\}$ |
| 11 | 0 | $0^-$ | $\{A_{\mathrm{v}}^2, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\vert B^0, (A_{\mathrm{l}}^2\ \&\ A_{\mathrm{l}}^0)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{s}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ B_{\mathrm{l}}^0)^{\mathrm{N}}, A^0\vert\mathfrak{a}_{\mathrm{l}}^2, A_{\mathrm{l}}^2, \times\}$ |
| 12 | 0 | $0^-$ | $\{A_{\mathrm{v}}^2, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\vert B^0, (A_{\mathrm{l}}^2\ \&\ A_{\mathrm{l}}^0)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{s}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ B_{\mathrm{l}}^0)^{\mathrm{N}}, A^0\vert\mathfrak{a}_{\mathrm{l}}^2, \times, A_{\mathrm{l}}^2\}$ |
| 13 | 0 | $0^-$ | $\{A_{\mathrm{v}}^2, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\vert B^0, A^0\vert\mathfrak{s}_{\mathrm{l}}^2\vert\mathfrak{a}_{\mathrm{l}}^2\vert B^0, (A_{\mathrm{l}}^2\ \&\ A_{\mathrm{l}}^0)^{\mathrm{N}}\vert(A_{\mathrm{l}}^2\ \&\ \mathfrak{a}_{\mathrm{l}}^2)^{\mathrm{N}}, \times, A_{\mathrm{l}}^2\}$ |

construct more tachyon-and-ghost-free and PPCR cases for $\mathrm{WGT}^+$ from the $\mathrm{PGT}^+$ cases with $t_3 = \lambda$.

There are, however, some PPCR critical cases of $\mathrm{WGT}^+$ that cannot be constructed directly from $\mathrm{PGT}^+$ in the manner described above. These cases have nonmixing $b$ matrices, but their $a$ matrices contain mixing terms. In particular, this occurs when there are $BA$ mixing terms in the linearized Lagrangian. Since the $B$ field can be fixed using the additional gauge invariance of the critical case, there are no $BA$ terms in the $b$ matrices. We list these further PPCR critical cases in Tables I and II. Note that none of these cases is PCR.

## V. TORSION-FREE WGT⁺

In addition to the general case of $\mathrm{WGT}^+$, one may also consider the simpler cases with vanishing torsion or curvature, which are *not* merely special cases of the general $\mathrm{WGT}^+$ action, because additional constraints are placed not only the coefficients but also on the fields. In this section, we consider the case of vanishing torsion.

If one sets the torsion $\mathcal{T}^{*\rho}{}_{\mu\nu}$ to zero, then one sees from (21) that the gauge fields $A^{AB}{}_\mu$, $h_a{}^\mu$, and $B_\mu$ are no longer independent. Indeed, Eq. (21) gives an explicit expression for the $A$ field in terms of the $B$ and $b$ fields. On making this substitution in the Lagrangian, one may then apply the same method as in the previous section to investigate torsion-free $\mathrm{WGT}^+$ and its critical cases. In this simpler theory, one need not set $\nu = \xi = c_1 = 0$, since one does not encounter critical conditions that are nonlinear in the Lagrangian parameters. Hence, we do not adopt this restriction in this section.

### A. Root theory

In this case, the $a$ matrices of the root theory (10) are

$$a(0^+) = \begin{array}{c} \\ \mathfrak{s} \\ \mathfrak{s} \\ B \end{array} \begin{array}{c} \begin{array}{ccc} \quad\quad\quad\mathfrak{s} & \quad\quad\quad\quad\mathfrak{s} & \quad\quad\quad\quad B \end{array} \\ \begin{pmatrix} 8(r_1 - r_3 + 2r_4)k^4 - 4\lambda k^2 & 0 & 8i\sqrt{3}(r_1 - r_3 + 2r_4)k^3 \\ 0 & 0 & 0 \\ -8i\sqrt{3}(r_1 - r_3 + 2r_4)k^3 & 0 & 24k^2(r_1 - r_3 + 2r_4) + 12\lambda + \nu \end{pmatrix} \end{array}, \tag{77}$$

$$a(1^-) = \begin{array}{c} \\ \mathfrak{s} \\ \mathfrak{a} \\ B \end{array} \begin{array}{c} \begin{array}{ccc} \mathfrak{s} & \mathfrak{a} & B \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4k^2(c_1+2r_1+2r_4+2r_5+\xi)+12\lambda+\nu \end{pmatrix} \end{array}, \tag{78}$$

$$a(1^+) = \mathfrak{a} \; \overset{\mathfrak{a}}{(0)}, \tag{79}$$

$$a(2^+) = \mathfrak{s} \; \overset{\mathfrak{s}}{\left(4(2r_1-2r_3+r_4)k^4+2\lambda k^2\right)}, \tag{80}$$

where the SPOs are obtained from those listed in Appendix A by simply deleting the rows and columns corresponding to the $A$ field. The $a$ matrices for $0^-$ and $2^-$ sectors have no element, so we do not list them. One can fix the gauge simply by removing the rows and columns whose elements are all zeros from the $a$ matrices, to obtain the corresponding $b$ matrices. These may then be inverted to obtain the saturated propagator.

Considering first the massless sector, the nonzero eigenvalues of the Laurent series coefficient matrix $\mathbf{Q}_2$ are

$$\frac{1}{\lambda}, \quad \frac{1}{2\lambda}. \tag{81}$$

Thus, the theory has two massless degrees of freedom, and the no-ghost condition for the massless sector is simply

$$\lambda > 0. \tag{82}$$

Turning to the massive sector, the determinants of the $b$ matrices are

$$\det[b(0^+)] = 8(r_1-r_3+2r_4)\nu k^4 - 4\lambda(12\lambda+\nu)k^2, \tag{83}$$

$$\det[b(1^-)] = 4(c_1+2r_1+2r_4+2r_5+\xi)k^2+12\lambda+\nu, \tag{84}$$

$$\det[b(2^+)] = 4(2r_1-2r_3+r_4)k^4+2\lambda k^2, \tag{85}$$

from which one obtains the masses

$$m^2(0^+) = \frac{\lambda(12\lambda+\nu)}{2(r_1-r_3+2r_4)\nu}, \tag{86}$$

$$m^2(1^-) = \frac{-12\lambda-\nu}{4(c_1+2r_1+2r_4+2r_5+\xi)}, \tag{87}$$

$$m^2(2^+) = -\frac{\lambda}{2(2r_1-2r_3+r_4)}. \tag{88}$$

The no-tachyon conditions $m^2(J^P) > 0$ may then be read off from the above expressions. In each sector, the masses are distinct, and so one can again apply Eq. (45) in Ref. [1] directly to obtain the massive no-ghost conditions

$$0^+: \; \frac{1}{4\lambda} + \frac{6\lambda^2}{(r_1-r_3+2r_4)\nu^2} + \frac{3}{\nu} > 0, \tag{89}$$

$$1^-: \; c_1 + 2(r_1+r_4+r_5) + \xi < 0, \tag{90}$$

$$2^+: \; \lambda < 0. \tag{91}$$

One thus finds that the combined no-ghost-and-tachyon conditions for the massive sector are

$$0^+: r_1 + 2r_4 > r_3, \lambda\nu(12\lambda+\nu) > 0, \tag{92}$$

$$1^-: 12\lambda+\nu > 0, c_1 + 2(r_1+r_4+r_5)+\xi < 0, \tag{93}$$

$$2^+: 2r_1 + r_4 > 2r_3, \qquad \lambda < 0. \tag{94}$$

Since the conditions in the massive $2^+$ sector contradict the condition (82) in the massless sector, the theory must have a ghost or tachyon.

### B. Critical cases

We now consider the critical cases of torsion-free WGT$^+$. As discussed in detail in Ref. [1], one finds all conditions that cause a theory to be a critical case. While some conditions may cause criticality in more than one way, one can still divide all the critical conditions into three categories, which we called type A, B, and C conditions.

Considering first the root theory, it becomes critical and thereby loses 1 degree of freedom in the Lagrangian parameter space if any of the following expressions vanishes:

$$\text{type B}: \; \lambda, 12\lambda+\nu, \tag{95}$$

$$\text{type C}: \; 2r_1-2r_3+r_4, \; r_1-r_3+2r_4, \nu, \; c_1+\xi+2r_1+2r_4+2r_5. \tag{96}$$

The two critical cases resulting from the type B conditions (95) of the root theory contain ghosts or tachyons, but some

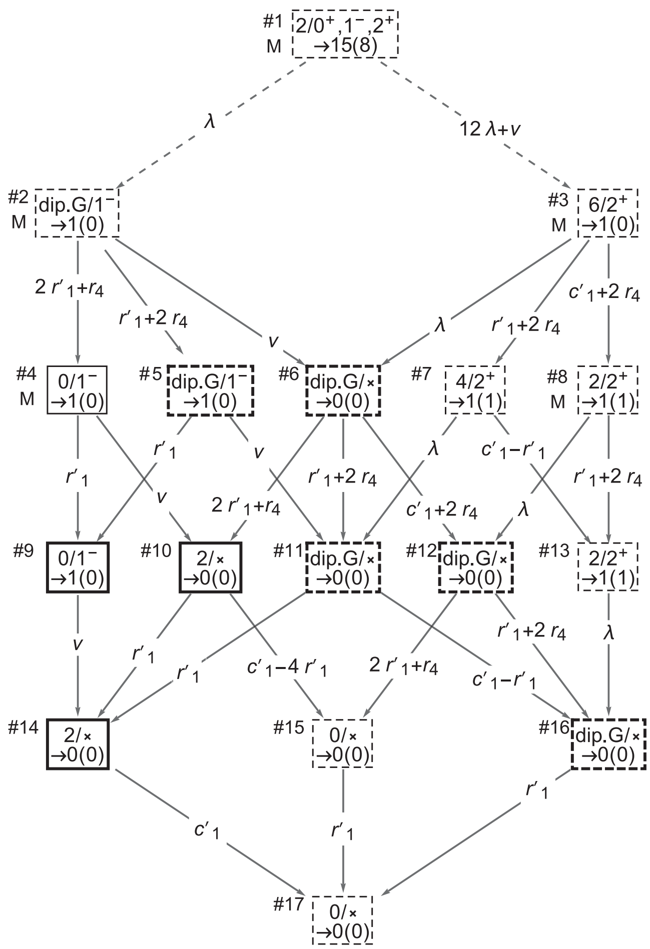


FIG. 1. Critical cases of torsionless WGT$^+$ resulting from type A or type B conditions. Each node represents a critical case, except the top and bottom nodes, which represent the root theory and the zero Lagrangian, respectively. Each arrow points from a node to one of its critical cases. A solid arrow represents type A critical condition, and a dashed arrow represents type B. The labels on the arrows are the critical parameters; for brevity, the variables $r_1' = r_1 - r_3$ and $c_1' = c_1 + 2r_1 + 2r_5 + \xi$ have been defined. The critical condition of a node can be obtained by setting all the critical parameters to zero in the path from the root theory to that node, and the conditions are path independent. In each node, the first line is in the format “degree of freedom of massless mode or ‘dip.G’ if there are massless dipole ghosts/ massive mode,” and the second line is “number of child critical cases resulting form type C conditions (number of no-ghost-and-tachyon cases among them),” which are not shown but are listed in Table III. The dashed/solid frames indicate those cases that contain any/no ghost or tachyon. The thick frames indicate PPCR cases, and the thin frames indicate those that are non-PPCR or have mixing $b$ matrices. The M symbols under the numbers at the left of the nodes indicate that those nodes must have mixing $b$ matrices.

of their descendant critical cases, all of which result from type A or C conditions, are free from ghosts and tachyons. The critical cases resulting from type A and type B conditions of torsion-free WGT$^+$ are shown in Fig. 1, whereas those

TABLE III. Critical cases of torsion-free WGT$^+$ resulting from type C conditions. The first numbers in the column “No.” correspond to the numbers in Fig. 1, and the corresponding nodes are the parent critical cases of the rows. The “Critical condition” column indicates the critical condition with respect to the parent case. For example, “1-3” is the third critical case resulting from type C conditions of case 1. The symbols $\circ/\times$ indicate whether it is possible for the theory to be free of ghosts and tachyons. The – symbols denote that there is no propagating mode, and the M symbols indicate the cases with mixing $b$ matrices.

| No. | Critical condition | Massive mode | No ghost nor tachyon | PPCR |
|---|---|---|---|---|
| 1–1 | $\nu$ | $1^-, 2^+$ | $\times$ | M |
| 1–2 | $r_1' + 2r_4$ | $1^-, 2^+$ | $\times$ | $\times$ |
| 1–3 | $r_1' + 2r_4, \nu$ | $1^-, 2^+$ | $\times$ | $\times$ |
| 1–4 | $c_1' + 2r_4$ | $0^+, 2^+$ | $\times$ | M |
| 1–5 | $\nu, c_1' + 2r_4$ | $2^+$ | $\times$ | M |
| 1–6 | $r_1' + 2r_4, c_1' + 2r_4$ | $2^+$ | $\times$ | $\times$ |
| 1–7 | $r_1' + 2r_4, \nu, c_1' + 2r_4$ | $2^+$ | $\times$ | $\times$ |
| 1–8 | $2r_1' + r_4$ | $0^+, 1^-$ | $\circ$ | M |
| 1–9 | $2r_1' + r_4, \nu$ | $1^-$ | $\circ$ | M |
| 1–10 | $2r_1' + r_4, r_1' + 2r_4$ | $1^-$ | $\circ$ | $\times$ |
| 1–11 | $2r_1' + r_4, r_1' + 2r_4, \nu$ | $1^-$ | $\circ$ | $\times$ |
| 1–12 | $2r_1' + r_4, c_1' + 2r_4$ | $0^+$ | $\circ$ | M |
| 1–13 | $2r_1' + r_4, \nu, c_1' + 2r_4$ | $\times$ | $\circ$ | M |
| 1–14 | $2r_1' + r_4, r_1' + 2r_4, c_1' + 2r_4$ | $\times$ | $\circ$ | $\times$ |
| 1–15 | $2r_1' + r_4, r_1' + 2r_4, \nu, c_1' + 2r_4$ | $\times$ | $\circ$ | $\times$ |
| 2–1 | $c_1' + 2r_4$ | $\times$ | $\times$ | M |
| 3–1 | $2r_1' + r_4$ | $\times$ | $\times$ | M |
| 4–1 | $c_1' - 4r_1'$ | $\times$ | – | – |
| 5–1 | $c_1' - r_1'$ | $\times$ | $\times$ | $\circ$ |
| 7–1 | $r_1'$ | $\times$ | $\circ$ | $\times$ |
| 8–1 | $2r_1' + r_4$ | $\times$ | $\circ$ | M |
| 9–1 | $c_1'$ | $\times$ | – | – |
| 13–1 | $r_1'$ | $\times$ | $\circ$ | $\times$ |

arising from type C critical conditions are listed in Table III; those cases that are ghost- and tachyon-free are indicated, as described in the captions. One sees that four cases in Fig. 1 are free from ghosts and tachyons, and nine critical cases in Table III share this property. We also note that there are 15 critical cases of the root theory in total that result from type C conditions, which correspond to self-consistent combinations of those in (96). As is clear from (88), those critical cases resulting from type C conditions and for which $2r_1 - 2r_3 + r_4 = 0$ are free from ghosts and tachyons because the $2^+$ massive mode is not propagating.

### C. Comparison with previous results

The particle spectrum of a subset of torsion-free Weyl-invariant higher-curvature gravity theories has been studied previously by Ref. [19], both in (anti-)de Sitter and Minkowski backgrounds (to our knowledge, this is the only other investigation of a torsionless WGT ground state in the literature). For $n=4$ spacetime dimensions, the coefficients $(\alpha,\beta,\gamma,\epsilon,\sigma)$ in their Lagrangian (see Eqs. (1), (7), and (14) in Ref. [19]) are related to those in our notation used in (10) by

$$\begin{aligned}&\alpha=-\frac{1}{2}r_1+r_3=\frac{1}{4}(r_4-r_5),\\&\beta=r_4+r_5=-\frac{1}{2}c_1,\\&\gamma=\frac{1}{2}r_1,\\&\epsilon=\xi-(r_4+r_5+2r_1),\\&\sigma=\lambda,\end{aligned}\tag{97}$$

together with the conditions

$$r_1=r_2,\quad \nu=-1.\tag{98}$$

In particular, one should note that the Lagrangian in Ref. [19] is written in terms of the curvature tensor $\tilde{\mathcal{R}}_{\mu\nu\rho\sigma}$. As discussed in Sec. II, this has even fewer symmetry properties than the rotational gauge field strength tensor $\mathcal{R}_{\mu\nu\rho\sigma}$ used in (10). Consequently, there are further quadratic combinations of $\tilde{\mathcal{R}}_{\mu\nu\rho\sigma}$ that could appear in the Lagrangian in Ref. [19], but only three such terms are included. Consequently, there are fewer degrees of freedom in the parameters of their Lagrangian, as compared with our Lagrangian in (10), as is evident from the above parameter identifications. Moreover, since $\tilde{\mathcal{R}}_{\mu\nu\rho\sigma}$ has many fewer symmetries than the standard curvature tensor in Riemannian spacetime $V_4$, the appropriate form of the Gauss-Bonnet identity differs from the usual formula that is assumed in Eq. (34) of Ref. [19] (see, for example, Refs. [8,20]); fortunately, most of the conclusions presented in Ref. [19] do not depend on this expression.

The constraints on our parameters in (97)–(98) do not coincide with any of the critical conditions in any critical case, so the structure of our “criticality tree” of torsion-free WGT is not affected. In Ref. [19], it is found that about a four-dimensional Minkowski background the WGTs considered are unitary, provided (in terms of our parameters)

$$2(r_1-r_3)+r_4=0,\tag{99}$$

$$r_1-r_3+2r_4=0,\tag{100}$$

$$\lambda>0.\tag{101}$$

Both equalities coincide with our type C critical conditions, and they eliminate $2^+$ and $0^+$ massive modes, leaving a $1^-$ massive mode. The condition on $\lambda$ also matches ours, so their result is consistent with our critical case 1–10 of the root theory, listed in Table III.

It is concluded in Ref. [19], however, that the theory has a massless spin-2 field and a massless spin-0 field, and so the massless sector has 3 degrees of freedom, whereas we find just 2. This difference may result from the fact that they employ a gauge fixing condition $\mathcal{D}^*_\mu B^\mu=0$ on the $B^\mu$ field (their $A^\mu$ field), described in their Eq. (30), but then treat this field as if it is unconstrained when reading off the particle content from their Eq. (59). This situation is analogous to that in Stueckelberg theory, as discussed in Appendix B in Ref. [2]. If one fixes the gauge by setting $\partial\cdot B=0$, then the Lagrangian appears to describe a massive vector $B$ and a massless scalar $\phi$ without interaction. Conversely, if one instead sets $\phi=0$, the Lagrangian contains only a massive vector without constraint. Thus, one should interpret the theory as containing either a massive vector or a massive vector with a Stueckelberg ghost and a Faddeev-Popov ghost.

Also, it is claimed in Ref. [19] that unitarity requires both (99) and (100) to hold, whereas we require only the former condition, if no type A or B critical condition is satisfied. The condition (100) is necessary in Ref. [19] because the authors do not adopt the Einstein gauge and so require the higher-derivative Pais-Uhlenbeck term $(\bar{\Box}\Phi_L)^2$ to vanish, where $\Phi_L$ is the linearized $\phi$. By contrast, all the higher-order poles in our saturated propagator vanish due to the source constraints, and so the condition (100) is not necessary in our case. This difference may be worthy of further investigation.

### D. Propagating power-counting renormalizability

We determine whether each critical case is PPCR using the same method as discussed in Sec. IV C. The results are presented in Fig. 1 and Table III. In particular, we find three critical cases in Fig. 1 that are both PPCR and contain no ghost nor tachyon; these are indicated by nodes with thick, solid frames. We note that each of these theories can be gauge fixed to contain only the $B$ gauge field. It is also worth highlighting that, perhaps as a consequence of this, there is no simultaneously unitary and PPCR case in torsion-free PGT$^+$ [1], and so these three theories may be worthy of further investigation. No critical case in Table III is both PPCR and unitary.

## VI. CURVATURE-FREE WGT$^+$

In this section, we consider WGT$^+$ with vanishing curvature. This is a more subtle condition than the equivalent case in PGT$^+$, which was discussed in Ref. [1].[4] As mentioned in Sec. II, the geometric (Riemann) curvature

[4]There is a typographical error in Fig. 2 in Ref. [1]. The node $t_1+t_2=0$ has 3 massless degrees of freedom rather than 2. The correction does not affect the remaining contents in Ref. [1].

tensor $\tilde{\mathcal{R}}^{\rho}{}_{\sigma\mu\nu}$ in Weyl-Cartan spacetime differs from the rotational gauge field strength $\mathcal{R}^{\rho}{}_{\sigma\mu\nu}$, so it is unclear which should be set to zero. Here, we consider only the case in which the latter vanishes, since this may imposed in the same way as in PGT by simply setting $A_{AB\mu} = 0$, since the expression for the rotational gauge field strength in terms of the rotational gauge field are identical in PGT and WGT. In this simpler theory, one sees from (10) that one requires only the Lagrangian parameters $\xi$, $\nu$, $t_1$, $t_2$, and $t_3$, since one can set $\lambda = 0$ without loss of generality.

### A. Root theory

In this case, the $a$ matrices of the root theory are

$$a(0^+) = \begin{array}{c} \\ \mathfrak{s} \\ \mathfrak{s} \\ B \end{array} \begin{array}{c} \begin{array}{ccc} \mathfrak{s} & \mathfrak{s} & B \end{array} \\ \begin{pmatrix} 4k^2 t_3 & 0 & 4i\sqrt{3}kt_3 \\ 0 & 0 & 0 \\ -4i\sqrt{3}kt_3 & 0 & 12t_3 + \nu \end{pmatrix} \end{array}, \tag{102}$$

$$a(1^-) = \begin{array}{c} \\ \mathfrak{s} \\ \mathfrak{a} \\ B \end{array} \begin{array}{c} \begin{array}{ccc} \mathfrak{s} & \mathfrak{a} & B \end{array} \\ \begin{pmatrix} \frac{2}{3}k^2(t_1 + t_3) & -\frac{2}{3}k^2(t_1 + t_3) & -2i\sqrt{2}kt_3 \\ -\frac{2}{3}k^2(t_1 + t_3) & \frac{2}{3}k^2(t_1 + t_3) & 2i\sqrt{2}kt_3 \\ 2i\sqrt{2}kt_3 & -2i\sqrt{2}kt_3 & 12t_3 + \nu + 4k^2\xi \end{pmatrix} \end{array}, \tag{103}$$

$$a(1^+) = \begin{array}{c} \\ \mathfrak{a} \end{array} \begin{array}{c} \mathfrak{a} \\ \left(\frac{2}{3}k^2(t_1 + t_2)\right) \end{array}, \tag{104}$$

$$a(2^+) = \begin{array}{c} \\ \mathfrak{s} \end{array} \begin{array}{c} \mathfrak{s} \\ (2k^2 t_1) \end{array}. \tag{105}$$

As in the torsion-free theory, the SPOs are obtained from those listed in Appendix A by deleting the rows and columns corresponding to the $A$ field, and the $a$ matrices for the $0^-$ and $2^-$ sectors contain no elements. After fixing the gauge by deleting rows and columns, one obtains the nonsingular $b$ matrices, which may be inverted to obtain saturated propagator.

Considering first the massless sector, one finds that the Laurent series coefficient matrix $\mathbf{Q}_4$ is nonzero in this case, and the condition for it to vanish is

$$\nu = -\frac{12t_1(t_1 - 2t_2)t_3}{t_1^2 - 2t_1t_2 + 4t_1t_3 + t_2t_3}. \tag{106}$$

One further finds that the Laurent coefficient matrix $\mathbf{Q}_2$ cannot be positive definite and contains eight nonzero eigenvalues, which are too complicated to give here. Consequently, the root theory must contain ghosts in the massless sector.

One can, however, continue to analyze the massive sector. The determinants of the $b$ matrices are

$$\det[b(0^+)] = 4t_3\nu k^2, \tag{107}$$

$$\det[b(1^-)] = \frac{2}{3}[t_3\nu + t_1(12t_3 + \nu)]k^2 \tag{108}$$

$$+\frac{8}{3}(t_1 + t_3)\xi k^4, \tag{109}$$

$$\det[b(1^+)] = \frac{2}{3}(t_1 + t_2)k^2, \tag{110}$$

$$\det[b(2^+)] = 2t_1k^2. \tag{111}$$

Only the $1^-$ sector contains a massive mode, with mass

$$m^2(1^-) = \frac{-12t_1t_3 - (t_1 + t_3)\nu}{4(t_1 + t_3)\xi}, \tag{112}$$

and the no-tachyon condition is $m^2(1^-) > 0$. Applying Eq. (45) in Ref. [1] directly, in this case, the no-ghost condition is

$$1^-\colon\ (t_1 + t_3)[12t_1t_3 + (t_1 + t_3)\nu]\xi\{(t_1 + t_3) \times [12t_1t_3 + (t_1 + t_3)\nu] - 72t_3^2\xi\} < 0. \tag{113}$$

The combined no-ghost-nor-tachyon conditions for the massive sector are thus

$$\xi < 0, \qquad \nu > -\frac{12t_1t_3}{t_1 + t_3}, \tag{114}$$

but one should recall that the massless sector always contains a ghost.

### B. Critical cases

The critical cases of the root theory occur when any of the following expressions vanish:

$$\text{type A}\colon\ t_1, t_1 + t_2, t_3, \nu, \tag{115}$$

$$\text{type B}\colon\ 12t_1t_3 + t_1\nu + t_3\nu, \tag{116}$$

$$\text{type C}\colon\ t_1 + t_3, \xi. \tag{117}$$

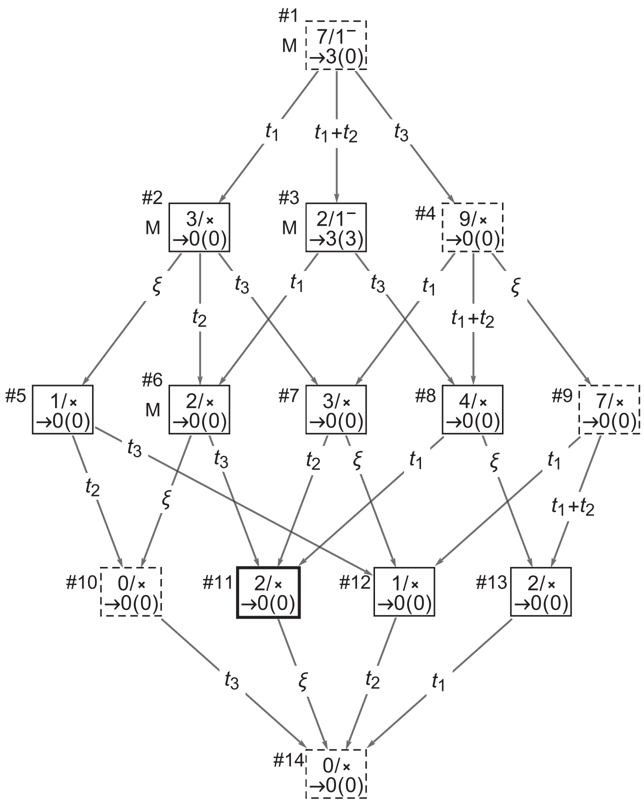


FIG. 2. Critical cases resulting from type A critical conditions of curvature-free WGT$^+$. The notation follows that of Fig. 2.

However, since $12t_1t_3 + t_1\nu + t_3\nu$ cannot be factorized into a linear combination of the parameters, one cannot apply our algorithm to find all the critical cases directly. Below, we therefore consider the critical case $\nu = 0$, which removes the kinetic term of the scalar field $\phi$, as the simplified root theory and instead find its critical cases. Before turning to these, we note that the massless sector of this simplified root theory requires $t_1 - 2t_2 = 0$ to make its Laurent series coefficient matrix $\mathbf{Q}_4$ vanish, and thus prevent the presence of dipole ghosts, but in any case, the matrix $\mathbf{Q}_2$ has seven nonzero eigenvalues and cannot be made be positive definite. Therefore, the massless sector must contain a ghost. The conditions for the massive sector of the simplified root theory to be ghost- and tachyon-free may be obtained from (112)–(114) by setting $\nu = 0$.

Turning now to the critical cases of the simplified root theory, the critical conditions are given by (115)–(117) with $\nu = 0$. One should note that this results in the simplified root theory containing no type B critical condition, since the resulting condition that $t_1t_3$ should vanish is trivially factorized and the separate requirements that $t_1$ or $t_3$ should vanish are already included in the type A critical conditions, and it turns out that there is no type B critical condition in the descendants. The critical cases resulting from type A and type C conditions are summarized in Fig. 2 and Table IV, respectively. Cases that are ghost- and tachyon-free are indicated, as described in the captions.

TABLE IV. Critical cases resulting from type C critical conditions of curvature-free WGT$^+$. The notation follows that of Table III.

| No. | Critical condition | Massive mode | No ghost nor tachyon | PPCR |
|---|---|---|---|---|
| 1–1 | $\xi$ | × | × | M |
| 1–2 | $t_1 + t_3$ | × | × | M |
| 1–3 | $t_1 + t_3, \xi$ | × | × | M |
| 3–1 | $\xi$ | × | ◦ | M |
| 3–2 | $t_1 + t_3$ | × | ◦ | M |
| 3–3 | $t_1 + t_3, \xi$ | × | ◦ | M |

In particular, we note that there are nine critical cases in Fig. 2 that are free from ghosts and tachyons and three such critical cases in Table IV.

### C. Propagating power-counting renormalizability

We determine whether each critical case is PPCR using the same method as discussed in Sec. IV C. The results are presented in Fig. 2 and Table IV. In particular, we find that there is just a single critical case in Fig. 2, which is just the pure dilaton Lagrangian $\mathcal{L} \sim \mathcal{H}^2$, that is both PPCR and unitary; this is indicated by the node with a thick, solid frame. No such critical case is found in Table IV.

## VII. CONCLUSIONS

We have used the systematic method in Ref. [1] to determine the no-ghost-nor-tachyon conditions for the most general WGT$^+$ (the root theory) and found it must contain a ghost or tachyon. For a subset of the theory, with the restriction $\nu = \xi = c_1 = 0$ on the parameters in the Lagrangian (10), which removes the kinetic terms for the scalar field $\phi$ and dilational gauge field $B$, and the only "cross-term" $\mathcal{R}^{AB}\mathcal{H}_{AB}$ between gauge field strengths, we found and categorized all 862 critical cases and identified 168 that are free from ghosts and tachyons. We compared our findings with the only other example of a unitary WGT$^+$ of which we are aware in the literature [18] and found the results to be consistent. We further identified those critical cases of WGT$^+$ that are also PPCR and introduce a method to construct more PPCR cases outside the 862 critical cases. Most of these are identical to or can be constructed from those in PGT$^+$ listed in Ref. [2] or a PGT$^+$ without any propagating mode (which were not listed in Ref. [2]). Nonetheless, we also identified a further 13 PPCR and ghost-and-tachyon-free critical cases of WGT$^+$ that cannot be constructed directly from PGT$^+$.

We repeated our analysis for the simpler cases of torsion-free and curvature-free WGT$^+$, which are not merely special cases of the general WGT$^+$ action, because additional constraints are placed not only on the coefficients but also on the fields. For the torsion-free case, we found that the root theory (without any further conditions on the Lagrangian parameters) must contain a ghost or tachyon.

Nonetheless, we identify 13 critical cases that are free from ghosts and tachyons. We also compare our results with the only other investigation of the ground state of a torsionless WGT$^+$ of which we are aware in the literature. We find our results to be consistent, apart from a minor issue related to the number of propagating degrees of freedom in the massless sector, most probably resulting from the different approaches to gauge fixing used in the two analyses. Of our 13 ghost-and-tachyon-free critical cases, we further identified three that are also PPCR, each of which can be gauge fixed to contain only the $B$ gauge field. This may explain the sharp contrast with torsion-free PGT$^+$, for which there is no unitary and PPCR critical case, and suggests that these three theories may be worthy of further investigation.

For curvature-free WGT$^+$, we find that the massless sector of the root theory (again with no further conditions on the Lagrangian parameters) must contain a ghost. For the simplified root theory with $\nu = 0$, which has no kinetic term for the scalar field $\phi$ in the Lagrangian and is itself found to have a ghost in the massless sector, we find 13 critical cases that are free from ghosts and tachyons, of which just a single case is found also to be PPCR, which corresponds to the pure dilaton Lagrangian $\mathcal{L} \sim \mathcal{H}^2$.

All the restrictions on Lagrangian parameters mentioned above are necessary to avoid critical conditions that cannot be written as the product of real linear terms, which is required by the systematic method in Ref. [1]. We plan to improve our approach to accommodate such cases in future work and also apply the method to more general gauge theories, such as metric affine gravities, whose unitarity was recently investigated by Ref. [21] using SPOs.

Finally, we point out that gauge theories of gravity can yield interesting phenomenology. In particular, in a cosmological context, recent investigations of some of the PGT$^+$ cases that were identified in Refs. [1,2] as being unitary and PPCR have been carried out in Ref. [22] and are found to have rich background solutions that support the concordance $\Lambda$CDM background cosmology up to an optional, effective dark radiation, which shows considerable promise in alleviating the Hubble tension. These theories have been shown to map to a noncanonical biscalar-tensor theory in the Jordan frame, which provides a unified framework for future investigation by the broader community, and for many parameter choices, the noncanonical term reduces to a Cuscuton field [23]. Moreover, one of the cases yields two dark energy solutions: accelerated expansion from a negative bare cosmological constant whose magnitude is screened and emergent dark energy to replace vanishing bare cosmological constant in $\Lambda$CDM. Further investigation of the unitary and PPCR cases of PGT$^+$ and WGT$^+$ is ongoing.

The full set of results, displayed in an interactive form, can be found at http://www.mrao.cam.ac.uk/projects/gtg/wgt/.

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## APPENDIX A: SPIN PROJECTION OPERATORS FOR WGT$^+$

The block matrices $\mathsf{P}(J^P)$ containing the spin projection operators for WGT$^+$ used in this paper are as follows:

$$\mathsf{P}(0^-) = \begin{array}{cc} & A_{ABC} \\ A^*_{IJK} & \left(\tfrac{2}{3}\Theta_{IC}\Theta_{JA}\Theta_{KB} + \tfrac{1}{3}\Theta_{IA}\Theta_{JB}\Theta_{KC}\right) \end{array}, \tag{A1}$$

$$\mathsf{P}(0^+) = \begin{array}{c} \\ A^*_{IJK} \\ s^*_{IJ} \\ s^*_{IJ} \\ B^*_K \end{array} \begin{array}{c} \begin{array}{cccc} A_{ABC} & s_{AB} & s_{AB} & B_C \end{array} \\ \begin{pmatrix} \frac{2}{3}\Theta_{CB}\Theta_{KJ}\Omega_{IA} & \frac{\sqrt{2}}{3}\tilde{k}_J\Theta_{AB}\Theta_{KI} & \sqrt{\frac{2}{3}}\tilde{k}_J\Theta_{KI}\Omega_{BA} & -\sqrt{\frac{2}{3}}\Theta_{KJ}\Omega_{IC} \\ \frac{\sqrt{2}}{3}\tilde{k}_B\Theta_{CA}\Theta_{IJ} & \frac{1}{3}\Theta_{AB}\Theta_{IJ} & \frac{1}{\sqrt{3}}\Theta_{IJ}\Omega_{AB} & \frac{1}{\sqrt{3}}\tilde{k}_C\Theta_{IJ} \\ \sqrt{\frac{2}{3}}\tilde{k}_B\Theta_{CA}\Omega_{JI} & \frac{1}{\sqrt{3}}\Theta_{AB}\Omega_{IJ} & \Omega_{AB}\Omega_{IJ} & \tilde{k}_C\Omega_{IJ} \\ -\sqrt{\frac{2}{3}}\Theta_{CB}\Omega_{AK} & \frac{1}{\sqrt{3}}\tilde{k}_K\Theta_{AB} & \tilde{k}_K\Omega_{AB} & \Omega_{KC} \end{pmatrix} \end{array}, \tag{A2}$$

$$\mathsf{P}(1^-) = \begin{array}{c} \\ A^*_{IJK} \\ A^*_{IJK} \\ s^*_{IJ} \\ a^*_{IJ} \\ B^*_K \end{array} \begin{array}{c} \begin{array}{ccccc} A_{ABC} & A_{ABC} & s_{AB} & a_{AB} & B_C \end{array} \\ \begin{pmatrix} \Theta_{CB}\Theta_{IA}\Theta_{KJ} & \sqrt{2}\Theta_{IA}\Theta_{KJ}\Theta_{CB} & \sqrt{2}\tilde{k}_B\Theta_{IA}\Theta_{KJ} & \sqrt{2}\tilde{k}_B\Theta_{IA}\Theta_{KJ} & \Theta_{IC}\Theta_{KJ} \\ \sqrt{2}\Theta_{AI}\Theta_{CB}\Omega_{KJ} & 2\Theta_{IA}\Omega_{CB}\Omega_{KJ} & 2\tilde{k}_J\Theta_{IA}\Omega_{KB} & 2\tilde{k}_J\Theta_{IA}\Omega_{KB} & \sqrt{2}\Theta_{IC}\Omega_{KJ} \\ \sqrt{2}\tilde{k}_J\Theta_{AI}\Theta_{CB} & 2\tilde{k}_B\Theta_{AI}\Omega_{CJ} & 2\Theta_{IA}\Omega_{JB} & 2\Theta_{IA}\Omega_{JB} & \sqrt{2}\tilde{k}_J\Theta_{IC} \\ \sqrt{2}\tilde{k}_J\Theta_{AI}\Theta_{CB} & 2\tilde{k}_B\Theta_{IA}\Omega_{CJ} & 2\Theta_{IA}\Omega_{JB} & 2\Theta_{IA}\Omega_{JB} & \sqrt{2}\tilde{k}_J\Theta_{IC} \\ \Theta_{AK}\Theta_{CB} & \sqrt{2}\Theta_{AK}\Omega_{CB} & \sqrt{2}\tilde{k}_B\Theta_{AK} & \sqrt{2}\tilde{k}_B\Theta_{AK} & \Theta_{KC} \end{pmatrix} \end{array}, \tag{A3}$$

$$\mathsf{P}(1^+) = \begin{array}{c} A^*_{IJK} \\ A^*_{IJK} \\ a^*_{IJ} \end{array} \overset{\begin{array}{ccc} A_{ABC} & A_{ABC} & a_{AB} \end{array}}{\begin{pmatrix} \Theta_{IC}\Theta_{KB}\Omega_{JA} + \Theta_{IA}\Theta_{KC}\Omega_{JB} & -\sqrt{2}\Theta_{JA}\Theta_{KB}\Omega_{IC} & \sqrt{2}\tilde{k}_J\Theta_{IA}\Theta_{KB} \\ -\sqrt{2}\Theta_{BI}\Theta_{CJ}\Omega_{AK} & \Theta_{IA}\Theta_{JB}\Omega_{KC} & \tilde{k}_K\Theta_{IA}\Theta_{JB} \\ \sqrt{2}\tilde{k}_B\Theta_{AI}\Theta_{CJ} & \tilde{k}_C\Theta_{AI}\Theta_{BJ} & \Theta_{AI}\Theta_{BJ} \end{pmatrix}}, \tag{A4}$$

$$\mathsf{P}(2^-) = A^*_{IJK} \overset{A_{ABC}}{\left(\tfrac{2}{3}\Theta_{IC}\Theta_{JB}\Theta_{KA} + \tfrac{2}{3}\Theta_{IA}\Theta_{JB}\Theta_{KC} - \Theta_{CB}\Theta_{IA}\Theta_{KJ}\right)}, \tag{A5}$$

$$\mathsf{P}(2^+) = \begin{array}{c} A^*_{IJK} \\ s^*_{IJ} \end{array} \overset{\begin{array}{cc} A_{ABC} & s_{AB} \end{array}}{\begin{pmatrix} -\tfrac{2}{3}\Theta_{CB}\Theta_{KJ}\Omega_{IA} + \Theta_{IC}\Theta_{KA}\Omega_{JB} + \Theta_{IA}\Theta_{KC}\Omega_{JB} & \sqrt{2}\tilde{k}_J(\Theta_{IA}\Theta_{KB} - \tfrac{1}{3}\Theta_{AB}\Theta_{KI}) \\ \sqrt{2}\tilde{k}_B(\Theta_{CJ}\Theta_{IA} - \tfrac{1}{3}\Theta_{CA}\Theta_{IJ}) & -\tfrac{1}{3}\Theta_{AB}\Theta_{IJ} + \Theta_{IA}\Theta_{JB} \end{pmatrix}}. \tag{A6}$$

These SPOs differ from those used in Ref. [1] for PGT$^+$ by having one additional row/column in both the $0^+$ and $1^-$ sectors, which are related to the extra vector gauge field $B_A$ present in WGT$^+$. For more details about SPOs in general, please refer to Ref. [1].

## APPENDIX B: NO-TACHYON AND NO-GHOST CONDITIONS FOR THE $1^-$ SECTOR

First, to avoid tachyons and a dipole ghost, one requires the roots of (45) to be real and distinct, such that

$$\begin{aligned} &\{6c_1t_1(t_3-\lambda) + (r_1+r_4+r_5)[12(t_3-\lambda)(t_1+\lambda) + (t_1+t_3)\nu] + 6t_1t_3\xi\}^2 \\ &\quad + 3t_1(t_1+t_3)[12(t_3-\lambda)\lambda + t_3\nu][c_1^2 - 8(r_1+r_4+r_5)\xi] > 0. \end{aligned} \tag{B1}$$

The no-tachyons conditions that both of the roots are positive then read

$$(t_1+t_3)[c_1^2 - 8(r_1+r_4+r_5)\xi]\{6c_1t_1(t_3-\lambda) + (r_1+r_4+r_5)[12(t_3-\lambda)(t_1+\lambda) + (t_1+t_3)\nu] + 6t_1t_3\xi\} > 0, \tag{B2}$$

$$t_1(t_1+t_3)(12(t_3-\lambda)\lambda + t_3\nu)(c_1^2 - 8(r_1+r_4+r_5)\xi) < 0. \tag{B3}$$

The no-ghost condition is

$$\begin{aligned} &[c_1^2 - 8(r_1+r_4+r_5)\xi][3c_1(t_1-2t_3)(t_3-\lambda) - r_5(t_1^2 + 2t_1t_3 + 19t_3^2 - 36t_3\lambda + 18\lambda^2) \\ &\quad - r_1(t_1^2 + 2t_1t_3 + 19t_3^2 - 36t_3\lambda + 18\lambda^2) - r_4(t_1^2 + 2t_1t_3 + 19t_3^2 - 36t_3\lambda + 18\lambda^2) - 3(t_1^2 + 2t_3^2)\xi] < 0, \end{aligned} \tag{B4}$$

$$\begin{aligned} &(t_1+t_3)[c_1^2 - 8(r_1+r_4+r_5)\xi]\{9(t_1+t_3)\{2t_1(7t_3^2 - 12t_3\lambda + 6\lambda^2) + t_1^2(14t_3 - 12\lambda + \nu) + 2t_3[12(t_3-\lambda)\lambda + t_3\nu]\}^2 \\ &\quad \times [c_1^2 - 8(r_1+r_4+r_5)\xi] - 48t_1[12(t_3-\lambda)\lambda + t_3\nu][r_5t_1^2 - 3c_1t_1t_3 + 2r_5t_1t_3 + 6c_1t_3^2 + 19r_5t_3^2 \\ &\quad + 3c_1t_1\lambda - 6c_1t_3\lambda - 36r_5t_3\lambda + 18r_5\lambda^2 + r_1(t_1^2 + 2t_1t_3 + 19t_3^2 - 36t_3\lambda + 18\lambda^2) + r_4(t_1^2 + 2t_1t_3 + 19t_3^2 \\ &\quad - 36t_3\lambda + 18\lambda^2) + 3t_1^2\xi + 6t_3^2\xi]^2 + 16\{2t_1(7t_3^2 - 12t_3\lambda + 6\lambda^2) + t_1^2(14t_3 - 12\lambda + \nu) + 2t_3[12(t_3-\lambda)\lambda + t_3\nu]\} \\ &\quad \times \{9c_1t_1(-t_3+\lambda) + \frac{3}{2}(r_1+r_4+r_5)[-12(t_3-\lambda)(t_1+\lambda) - (t_1+t_3)\nu] - 9t_1t_3\xi\}[-r_5t_1^2 + 3c_1t_1t_3 - 2r_5t_1t_3 \\ &\quad - 6c_1t_3^2 - 19r_5t_3^2 - 3c_1t_1\lambda + 6c_1t_3\lambda + 36r_5t_3\lambda - 18r_5\lambda^2 - r_1(t_1^2 + 2t_1t_3 + 19t_3^2 - 36t_3\lambda + 18\lambda^2) - r_4(t_1^2 + 2t_1t_3 \\ &\quad + 19t_3^2 - 36t_3\lambda + 18\lambda^2) - 3(t_1^2 + 2t_3^2)\xi]\} < 0. \end{aligned} \tag{B5}$$

Combining the requirements for no tachyons and no ghosts, there exists at least one parameter set satisfying all five conditions above, for example,

$$c_1 = -9, \quad r_1 = -1, \quad r_4 = 0, \quad r_5 = 0, \quad t_1 = \frac{1}{2}, \quad t_3 = -1, \quad \lambda = -4, \quad \nu = -142, \quad \xi = -18, \tag{B6}$$

where the other parameters may take arbitrary values, provided they do not make the theory a critical case.

## APPENDIX C: COMPLETENESS OF THE CRITICAL CASES

An "additional condition" is defined as the condition(s) to prevent a theory from being critical. In our previous paper [1], the additional condition was the requirement that the "sibling critical conditions" should not be satisfied, and we will call this the "sibling additional condition." For example, consider a theory that has the critical conditions that the (linear) parameter combinations $X$, $Y$, and $Z$ should vanish; we will call $X$, $Y$, and $Z$ the "critical parameters" of the theory. In the case, the sibling critical parameters for the critical case $X = 0$ are $Y$ and $Z$. To prevent a theory from being critical, one can require the critical parameters not equal to zero. We will call this kind of condition a "child additional condition." In PGT, as discussed in Ref. [1], the sibling additional condition is identical to the child additional condition, except for the root case. This occurs because we add only one linear condition at a time for cases resulting from type A or B critical conditions, but we attempt to use all possible combinations of conditions simultaneously for type C critical parameters (which we term "combining" the conditions). We then recursively find the child critical cases of cases resulting from type A and B critical conditions (the "uncombined" cases) but stop doing that for those from type C critical conditions (the combined cases). If type C critical conditions are treated in the same way as type A and type B, then the statement is not valid for PGT.

There are two situations in which the statement is invalid. The first is the occurence of "hidden" critical parameters. Consider a theory with only a $1 \times 1$ $b$ matrix $(XY + Zk^2)$. The theory has type B critical parameters, $X$ and $Y$, and a type C one, $Z$. For the critical case $X = 0$, the $b$ matrix becomes $(Zk^2)$, so there is only one critical parameter $Z$. To prevent the theory being critical (child additional condition), one requires $Z \neq 0$. However, its sibling critical parameters are $Y$ and $Z$, which are different. The critical parameter $Y$ is hidden in this case. If there are hidden parameters and one is requiring only child additional conditions, then a point in the parameter space may belong to more than one critical case. For example, the critical case $X = 0$, $Z \neq 0$ and the case $Y = 0$, $Z \neq 0$ have the overlap $X = Y = 0, Z \neq 0$, and they actually have the same $b$ matrix $(Zk^2)$ and represent the same theory. If we use the sibling additional condition instead, the two cases become $X = 0, Y \neq 0, Z \neq 0$ and $Y = 0, X \neq 0, Z \neq 0$, and there is no overlap. Hidden parameters do not occur in PGT or any of the critical cases discussed in this paper, if we combine all the type C critical cases as in Ref. [1]. While the overlapping and redundancy do no real harm to the correctness of our results, it may be worth modifying our algorithm to accommodate the situation for simplicity.

The second reason is the occurrence of "emergent" critical parameters. Some critical parameters appear after a $b$ matrix becomes singular and a new $b$ matrix forms, which may happen in critical cases resulting from a type A critical parameter (it is worth noting that critical parameters of the root theory are always emergent because it has no parent or sibling critical cases). In PGT$^+$ and torsion-free or simplified curvature-free WGT$^+$, either the new $b$ matrix is $0 \times 0$ or its critical parameters are already included in the sibling critical parameters, and so there is no emergent critical parameter. However, in simplified full WGT$^+$, this is not the case. For example, the $b(0^+)$ matrix of the simplified root WGT$^+$ is

$$\begin{pmatrix} 2[2k^2(r_1 - r_3 + 2r_4) + t_3] & 2i\sqrt{2}kt_3 & -2\sqrt{6}(t_3 - \lambda) \\ -2i\sqrt{2}kt_3 & 4k^2(t_3 - \lambda) & 4i\sqrt{3}k(t_3 - \lambda) \\ -2\sqrt{6}(t_3 - \lambda) & -4i\sqrt{3}k(t_3 - \lambda) & 12(t_3 - \lambda) \end{pmatrix}, \tag{C1}$$

which has $\det[b(0^+)] = -96(t_3 - \lambda)\lambda^2k^2$. Its critical case $\lambda = 0$ has

$$\begin{pmatrix} 2[2k^2(r_1 - r_3 + 2r_4) + t_3] & 2i\sqrt{2}kt_3 \\ -2i\sqrt{2}kt_3 & 4k^2t_3 \end{pmatrix} \tag{C2}$$

with $\det[b(0^+)] = 16(r_1 - r_3 + 2r_4)t_3k^4$. The critical parameter $(r_1 - r_3 + 2r_4)$ is neither a critical parameter of the root theory nor among the sibling critical parameters of case $\lambda = 0$. However, the emergent parameters will not affect our algorithm if we apply the child additional condition, which already includes the emergent parameters.

In conclusion, as long as there is no hidden critical parameter in critical cases resulting from type A and B critical parameters, and the cases resulting from type C critical parameters are combined, then we can apply the child additional conditions for the uncombined cases and the sibling additional conditions for the combined cases as the "(extended) additional condition" (this is also equivalent to combining the sibling and child additional conditions as the additional condition for all cases). This is what the term "additional condition" actually means in this paper. Our algorithm then holds, and each parameter set corresponds to one critical case. We have also checked that all the critical cases in Ref. [1] and this paper cover the entire parameter space and the critical cases have no overlap.

## APPENDIX D: POWER-COUNTING RENORMALIZABILITY

Since the PCR criterion for $\mathrm{PGT}^+$ is merely stated by Sezgin and van Nieuwenhuizen [3], rather than derived, and we also wish to extend the criterion to $\mathrm{WGT}^+$, we give a brief outline derivation here. Before doing so, however, we note that power-counting is not the ultimate criterion for renormalizability. Some PCR theories may be nonrenormalizable because of some deeper problems such as anomalies, and non-PCR theories may turn out to be renormalizable (for example, see Ref. [24]).

We consider a quantum field theory in $d$-dimensional spacetime with some fields labeled by $i$ and assume for each field the propagator $\to k^{-l_i}$ as $k \to \infty$. We also define the canonical dimension [25] of the field $\varphi_i$ as $[\varphi_i] \equiv (d - l_i)/2$, which only sometimes coincides with the mass dimension of the field in natural units. The latter can be inferred from the fact that each term in the Lagrangian density has mass dimension $d$. One may always ensure that the two dimensions coincide by making a field redefinition in which the original field is multiplied by a constant. If the interactions are labeled by $a$, with coupling constants $\lambda_a$, then the general criterion for a theory to be PCR is that there is no coupling constant with negative canonical dimension [25], so that $[\lambda_a] \geq 0\ \forall\ a$.

For $\mathrm{WGT}^+$, in terms of the linearized fields introduced in Sec. III, the most general Lagrangian in the Einstein gauge with $\phi_0$ absorbed into the coefficients is given schematically by

$$
\begin{aligned}
b\mathcal{L}_G &\sim b(\lambda\mathcal{R} + r\mathcal{R}^2 + t\mathcal{T}^{*2} + \xi\mathcal{H}^2 + c_1\mathcal{R}\mathcal{H} + \nu B^2)\\
&\sim (1 + f + f^2 + \cdots)\{\lambda(1+f)^2(\partial A + A^2) + r(1+f)^4(\partial A + A^2)^2\\
&\quad + t(1+f)^2[\partial(f + f^2 + \cdots) + (1 + f + f^2 + \cdots)(A + B)]^2\\
&\quad + \xi(1+f)^4(\partial B)^2 + c_1(1 + f + f^2 + \cdots)(\partial A + A^2)\partial B + \nu(1+f)^2B^2\},
\end{aligned}
\tag{D1}
$$

where we do not show the detailed structures of the indices and coefficients. The mass dimensions of the parameters and fields are $[\lambda]_\mathrm{M} = 2$, $[r]_\mathrm{M} = 0$, $[t]_\mathrm{M} = 2$, $[\xi]_\mathrm{M} = 0$, $[c_1]_\mathrm{M} = 0$, $[A]_\mathrm{M} = 1$, $[f]_\mathrm{M} = 0$, and $[B]_\mathrm{M} = 1$. Assuming the propagators of $h$, $A$, and $B$ behave as $k^{-l_h}$, $k^{-l_A}$, and $k^{-l_B}$, respectively, we need to redefine the fields as $\tilde{h} = M_h^{2-l_h/2}h$, $\tilde{A} = M_A^{1-l_A/2}A$, and $\tilde{B} = M_B^{2-l_B/2}B$. Therefore, we require $l_h \geq 4$, $l_A \geq 2$, and $l_B \geq 2$ for the theory to be PCR.[5] The original PCR criterion in Ref. [3] for $\mathrm{PGT}^+$ is obtained immediately by setting $B = 0$.

## APPENDIX E: PCR CRITICAL CASES

There exists a "folk theorem" dating back to the 1970s, a version of which is presented in the introduction of Sezgin and van Nieuwenhuizen's paper [3], that suggests that any gravity theory that is unitary cannot also be PCR. The argument is not based on any rigorous no-go theorem but instead on the following simple observation: as shown in Appendix D, for a $\mathrm{PGT}^+$ to be PCR, the propagator of the $A$ field must decay at least as quickly as $k^{-2}$ at high energy, and those of the $\mathfrak{s}$ and $\mathfrak{a}$ fields must fall off at least as $k^{-4}$, but the resulting total propagator, in general, contains terms of opposite sign when expressed in partial fractions, and so the theory is not unitary. This viewpoint has never subsequently been seriously challenged, and so our claim to have found counterexamples is in conflict with the accepted wisdom. We therefore take the opportunity here to elucidate the four unitary critical cases that also satisfy the original criterion used by Sezgin and van Nieuwenhuizen in Ref. [3] to be PCR. These cases coincide with the $\mathrm{PGT}^+$ cases 9, 10, 11, and 13, first identified in Ref. [1] and listed in Table III of Ref. [2]. In particular, we explain how these theories, each of which contains only 2 massless degrees of freedom, evade the argument in Ref. [3].

The key relevant property of these theories, at least in the linearized approximation considered here, is that they contain no "graviton" (degree of freedom associated with the $\mathfrak{s}$ and $\mathfrak{a}$ fields), but only "tordions" (degrees of freedom associated with the $A$ field), as originally discussed in Ref. [1] (and no dilaton degree of freedom associated with the $B$ field, since we are considering only $\mathrm{PGT}^+$ here). In other words, for these four theories, the $a$ matrices (28)–(33) contain nonzero entries only in the rows/columns corresponding to the $A$ field. As a result, the propagator in each case needs only to decay at least as quickly as $k^{-2}$ at high energy, and so the partial fractions argument outlined above does not necessarily apply.

One may verify directly by explicit calculation of their propagators that this indeed occurs for cases 9, 10, 11, and 13. We consider each case in turn, where the $a$ matrices for each case may be found by substituting its critical condition into (28)–(33):

[5] If $r = 0$, then the interaction terms with the highest degree of $A$ are $A^2$ with coefficients of dimension 2. Hence, in this case, we may have a looser condition $l_A \geq 0$. However, there is no dynamical term for $A$ if $r = 0$, so we consider $A$ not propagating.

(1) For case 9, the critical condition is $r_2 = r_1 - r_3 = r_4 = t_1 = t_2 = t_3 = \lambda = 0$, the resulting propagator of the $A$ field is

$$\hat{D}_A = \frac{1}{2(r_1 + r_5)k^2}\hat{P}_{11}(1^-) + \frac{1}{2(2r_1 + r_5)k^2}\hat{P}_{11}(1^+) + \frac{1}{2r_1 k^2}\hat{P}_{11}(2^-), \tag{E1}$$

and the condition for no ghost nor tachyon is $r_1(r_1 + r_5)(2r_1 + r_5) < 0$.

(2) For case 10, the critical condition is $r_2 = r_1 = r_3/2 - r_4 = t_1 = t_2 = t_3 = \lambda = 0$, the propagator is

$$\hat{D}_A = \frac{1}{(r_3 + 2r_5)k^2}\hat{P}_{11}(1^-) + \frac{1}{2(2r_3 + r_5)k^2}\hat{P}_{11}(1^+) - \frac{1}{3r_3 k^2}\hat{P}_{11}(2^+), \tag{E2}$$

and the condition for no ghost nor tachyon is $r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$.

(3) For case 11, the critical condition is $r_1 = r_3/2 - r_4 = t_1 = t_2 = t_3 = \lambda = 0$, the propagator is

$$\hat{D}_A = \frac{1}{2r_2 k^2}\hat{P}_{11}(0^-) + \frac{1}{(r_3 + 2r_5)k^2}\hat{P}_{11}(1^-) + \frac{1}{2(2r_3 + r_5)k^2}\hat{P}_{11}(1^+) - \frac{1}{3r_3 k^2}\hat{P}_{11}(2^+), \tag{E3}$$

and the condition for no ghost nor tachyon is $r_3(2r_3 + r_5)(r_3 + 2r_5) < 0$.

(4) For case 13, the critical condition is $r_2 = 2r_1 - 2r_3 + r_4 = t_1 = t_2 = t_3 = \lambda = 0$, the propagator is

$$\hat{D}_A = \frac{1}{-12(r_1 - r_3)k^2}\hat{P}_{11}(0^+) + \frac{1}{2(-r_1 + 2r_3 + r_5)k^2}\hat{P}_{11}(1^-) + \frac{1}{2(2r_3 + r_5)k^2}\hat{P}_{11}(1^+) + \frac{1}{2r_1 k^2}\hat{P}_{11}(2^-), \tag{E4}$$

and the condition for no ghost nor tachyon is $r_1(r_1 - 2r_3 - r_5)(2r_3 + r_5) > 0$.

Since $\Theta_{AB} = \eta_{AB} - \frac{k_A k_B}{k^2}$ and $\Omega_{AB} = \frac{k_A k_B}{k^2}$, all the SPOs behave as constants at high $k^2$. Therefore, in each case, the propagator of the $A$ field goes as $k^{-2}$ at high energy, and so the theory is PCR. We also note that, for each case, the additional conditions that prevent the theory from becoming a different critical case are that none of the denominators of the coefficients of the SPOs may vanish.

The absence of a graviton does not, however, preclude the possibility that the 2 tordion massless degrees of freedom are in the spin-$2^+$ sector, and indeed this may occur for cases 10 and 11, although not for cases 9 and 13, as discussed in Ref. [1]; this is also apparent from the above propagator for each theory. Thus, in cases 10 and 11, aspects of the gravitational interaction may still be mediated by a massless spin-$2^+$ particle, despite it corresponding to degrees of freedom of the $A$ field rather than of the $\mathfrak{h}$ and $\mathfrak{a}$ fields. As mentioned in Ref. [1], it is worth pointing out again here that the actions of cases 10 and 11 both reduce in the absence of torsion to that of conformal gravity, which is well known to be PCR but not unitary; it is claimed that one can nonetheless construct a unitary quantum theory of conformal gravity by redefining its Fock space [26], although this suggestion is controversial [27].


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