Nonsingular extension of the Kerr-NUT–(anti–)de Sitter spacetimes

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Due to the conical singularity along the symmetry axis Taub-NUT spacetimes suffer from a long and problematic history of physical interpretation. In 1969 Misner proposed a nonsingular interpretation taking advantage of the spacetime's topology and its underlying group-theoretic structure. We extend and refine his method to include a broader family of solutions and completely solve the outstanding issue of a nonsingular extension of the Kerr-NUT–(anti–)de Sitter solutions to Einstein's equations. Our approach relies on an observation that in 2 dimensional algebra of Killing vector fields there exist two distinguished vector fields that may be used to define $U(1)$-principal bundle structure over the nonsingular spaces of non-null orbits. For all admissible parameters we derive appropriate Killing vector fields and discuss limits to spacetimes with less parameters. The global structure of spacetime, together with nonsingular conformal geometry of the infinities is presented and (possibly also projectively nonsingular) Killing horizons is presented.

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I. INTRODUCTION

In 1963 Ezra Ted Newman and his two students Louis A. Tamburino, and Theodore W. J. Unti introduced a deformation of the Schwarzschild spacetime that made it twisting [1]. In the cosmological context, an equivalent solution to Einstein's equation had been found earlier, in 1951, by Abraham Haskel Taub [2]. Due to the twisting Taub-NUT solution was an early candidate for the relativistic description of a rotating black hole. Even though this proposal did not turn out to be true the NUT paper had an enormous impact on the theory of exact solutions to the Einstein equations. In particular it led to the correct description in the form of Kerr solution [3].

The NUT-like modification may be generalized to the Kerr solution to the vacuum Einstein equations by adding a parameter l called the NUT parameter. The resulting Kerr-NUT spacetime is still Ricci flat, but its topology is considerably different than that of the Kerr spacetime. Due to the Misner's method of compactifying the symmetry group [4], the global structure of those spacetimes is obtained as $\mathbb{R} \times S^3$ and it contains closed timelike curves. On the other hand, for sufficiently large value of l, the Kerr singularity is smoothed out, although the spacetime still contains horizons. Due to the unquestionably growing relevance of the cosmological constant in physics, it is natural to generalize the family of the Kerr-NUT solutions by adding a constant Λ. That has been done a long time ago. These solutions to the vacuum Einstein equations with a cosmological constant are referred to as the Kerr-NUT-(anti) de Sitter spacetimes and set a 4-dimensional family and since then has been of an interest [5–9].

In the original case of the Taub-NUT spacetime the recipe for Misner's gluing consisted of connecting two patches Taub-NUT spacetime into a nonsingular one. This had the consequence of compactifying the orbits of ∂_t Killing vector to circles. However, in the case of Kerr-NUT–(anti–)de Sitter, a generalized Misner's gluing does not work properly—there still persists a conical singularity irremovable at least from one of the axis of the rotational symmetry. We completely solve that problem in the current paper, thus our approach generalizes the Misner's compactification both in used methods and the class of applicable spacetimes.

We recognize a geometric mechanism of the problem—it is hidden in the spaces of non-null orbits of Killing vectors of Kerr-NUT–(anti–)de Sitter spacetimes. Only some distinguished Killing vectors define a nonsingular geometry and we find all of them. Next, one of those fields is used to perform a nonsingular generalization of Misner's gluing. We study the global properties of the resulting spacetime from the past conformal infinity $\mathcal{I}^-$ to the future one $\mathcal{I}^+$, as well as the contained Killing horizons.

This paper is the third in the series concerning the nonsingular interpretation of Kerr-NUT-de Sitter spacetimes. However, it is a completely self contained continuation. In the previous papers we focused on the geometry of the Killing horizons contained in that family of spacetimes. We introduced a notion of a projectively nonsingular horizon, i.e., the horizon is said to be projectively nonsingular if its space of null generators in nonsingular, and

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derived a 3-dimensional subfamily of the Kerr-NUT-de Sitter spacetimes, each of which contains a projectively nonsingular horizon. For those Kerr-NUT-de Sitter spacetimes (of a well tuned value of the cosmological constant) we were able to introduce a generalized Misner's construction in a nonsingular manner. This is a special case of the generalization derived in the current paper that is valid for all the values of Λ independently of the remaining three parameters, regardless of the projective properties of the horizons. If a Kerr-NUT-de Sitter spacetime happens to contain a projectively nonsingular horizon, then the current construction of nonsingular spacetimes may be reduced to the one presented in [10]. Thus the previous results fit neatly into the new ones.

II. KERR-NUT–(ANTI–)DE SITTER SPACETIMES AND OUR APPROACH

A. Kerr-NUT–(anti–)de Sitter spacetimes and their problems

The Kerr-NUT–(anti–)de Sitter metric in the simplified form first derived by Griffith and Podolsky [7] can be expressed in the Boyer–Lindquist-like coordinates as

$$ds^2 = -\frac{\mathcal{Q}}{\Sigma}(dt - Ad\phi)^2 + \frac{\Sigma}{\mathcal{Q}}dr^2 + \frac{\Sigma}{P}d\theta^2 + \frac{P}{\Sigma}\sin^2\theta(adt - \rho d\phi)^2, \tag{1}$$

where

$$\begin{aligned} \Sigma &= r^2 + (l + a\cos\theta)^2, \\ A &= a\sin^2\theta + 4l\sin^2\frac{1}{2}\theta, \\ \rho &= r^2 + (l+a)^2 = \Sigma + aA, \\ \mathcal{Q} &= (a^2 - l^2) - 2mr + r^2 \\ &\quad - \Lambda\left((a^2 - l^2)l^2 + \left(\frac{1}{3}a^2 + 2l^2\right)r^2 + \frac{1}{3}r^4\right), \\ P &= 1 + \frac{4}{3}\Lambda al\cos\theta + \frac{\Lambda}{3}a^2\cos^2\theta. \end{aligned} \tag{2}$$

Above, l and a denote the NUT and the Kerr parameters, respectively, Λ is a cosmological constant of any value and m stands for the mass parameter (when $\Lambda \neq 0$, m is proportional to the conserved quantity corresponding to time translation symmetry [11], while for $\Lambda = 0$ it is exactly the mass.

Throughout this paper we use a generalization of the (ingoing) Eddington-Finkelstein coordinates adopted to the rotating spacetime

$$dv := dt + \frac{\rho}{\mathcal{Q}}dr, \qquad d\tilde{\phi} := d\phi + \frac{a}{\mathcal{Q}}dr. \tag{3}$$

This provides an extension of the metric (1) that covers the roots of the function $\mathcal{Q}$. Then the metric tensor takes the following form

$$ds^2 = -\frac{\mathcal{Q}}{\Sigma}(dv - Ad\tilde{\phi})^2 + 2dr(dv - Ad\tilde{\phi}) + \frac{\Sigma}{P}d\theta^2 + \frac{P}{\Sigma}\sin^2\theta(adv - \rho d\tilde{\phi})^2. \tag{4}$$

The above metric shows singularities (apparent or true) familiar from the analysis of the standard Kerr and Kerr-(anti–)de Sitter solutions, which are special cases of the considered metrics.

We emphasize now the consequences of the presence of the NUT parameter l. A helpful consequence of

$$|l| > |a|$$

is that the function Σ never vanishes. Otherwise, if

$$|l| \leq |a|$$

the function Σ takes the value zero at $r = 0$ and $\theta = \theta_c$ such that

$$\cos\theta_c = -\frac{l}{a}.$$

That is a source of a nonremovable curvature singularity [8]. The singularity has a similar structure to that of Kerr, in particular, there are continues curves that pass from the $r > 0$ region to the region of $r < 0$ such that Σ is finite along them, hence they avoid the singular regions. Therefore, this singularity does not split spacetime into two disconnected components corresponding to $r > 0$, and $r < 0$, by the analogy to the Kerr singularity. In the current paper we admit all the values of a and l, hence the vanishing Σ singularity either appears or not.

If the NUT parameter l is large enough while a and Λ are kept constant the function $P(\theta)$ changes the sign for some $\theta \in [0, \pi]$. That is accompanied by a change of the signature of the metric tensor. To avoid this pathology we allow in the current paper only those values of l, a, and Λ that ensure

$$P(\theta) > 0, \quad \text{for every } \theta \in [0, \pi]. \tag{5}$$

This amounts to

$$P > 0 \Longleftrightarrow \left(\left(2\left|\frac{l}{a}\right| \le 1 \wedge 0 < \Lambda < \frac{3}{4l^2} \right) \vee \left(2\frac{l}{a} > 1 \wedge 0 < \Lambda < \frac{-3}{a^2 - 4al} \right) \vee \left(-2\frac{l}{a} > 1 \wedge 0 < \Lambda < \frac{-3}{a^2 + 4al} \right) \vee \left(\frac{-3}{a^2 - 4al} < \Lambda < 0 \wedge al < 0 \right) \vee \left(\frac{-3}{a^2 + 4al} < \Lambda < 0 \wedge al > 0 \right) \right). \tag{6}$$

It is conceivable, that this assumption could be carefully relaxed by making the inequality nonsharp, but this lies beyond the scope of this paper.

The nonvanishing NUT parameter $l \neq 0$, introduces a notable difficulty that eventually has topological consequences. What is peculiar about this case is the singularity of the differential 1-form $Ad\tilde{\phi}$ that is caused by the term

$$4l \sin^2 \frac{1}{2}\theta d\tilde{\phi}.$$

Indeed, when considered on a sphere parametrized by $(\theta, \tilde{\phi}) \in [0, \pi] \times [0, 2\pi)$, that term is discontinuous at the pole $\theta = \pi$. That singularity can be cured by introducing another chart that covers the pole $\theta = \pi$. It was defined first by Misner in the Taub-NUT case, i.e.,

$$a = 0 = \Lambda$$

and can be easily generalized to arbitrary values of the parameters a and Λ. The Misner charts give rise to the topology $\mathbb{R} \times S^3$ of spacetime, and an action of the U(1) group generated by the Killing vector ∂_t of the metric tensor (1) that induces a principal fiber bundle structure

$$\mathbb{R} \times S^3 \to \mathbb{R} \times S^2, \tag{7}$$

that for each sphere S^3, defined by a fixed value of the variable r, reduces to

$$S^3 \to S^2. \tag{8}$$

In the case of $\Lambda = 0$ the gluing solves the problem at $\theta = \pi$. However, at a generic case of

$$al\Lambda \neq 0,$$

one more obstacle appears. As long as the fibers of the bundle (8) contained in the spacetime $\mathbb{R} \times S^3$ are not null and the spacetime is twice differentiable, the geometry induced on S^2 should be continues and differentiable. The latter one is defined by the angular part of the spacetime metric tensor (4) (or equivalently of the metric tensor (1) except for the horizons)

$$\frac{\Sigma}{P} d\theta^2 + \frac{P}{\Sigma} \sin^2 \theta \rho^2 d\tilde{\phi}^2 \tag{9}$$

while the remaining parts are differentiable on S^2 on their own and the term $(4l \sin^2 \frac{1}{2}\theta d\tilde{\phi})^2$ is cured by the Misner gluing. It is easy to see [10], that the tensor (9) gives rise to a well-defined and differentiable metric tensor on entire S^2 including the poles $\theta = 0$ and $\theta = \pi$ by a suitable rescaling of the variable $\tilde{\phi}$, if and only if

$$P(0) = P(\pi), \tag{10}$$

that is the case (see definitions (1) and (2), if and only if

$$al\Lambda = 0.$$

Otherwise, the angular part of the metric tensor (9) has an irremovable conical singularity at least at one of the poles. In the spacetime $\mathbb{R} \times S^3$, the corresponding singularity takes the form of a 2-dimensional surface on which the variable r takes all the values in $\mathbb{R}$. Hence, in the case $al\Lambda \neq 0$, the metric tensor (1) can not be extended to an analytic metric tensor defined on $\mathbb{R} \times S^3$ such that the Killing vector ∂_t generates the fibers of the projection (7). The above naive approach to the solution of the problem is further justified in the following chapters using the broader geometrical picture of the spaces with NUT parameters.

In the current paper we solve the problem of a nonsingular generalization of Misner's gluing to a general case of the Kerr-NUT–(anti–)de Sitter spacetime. The resulting spacetime still has the U(2)-bundle structure (7) and the only possible singularities corresponding to zeros of the function Σ if $|l| < |a|$. Otherwise the spacetime is completely singularity free. Thus adding large enough NUT parameter may be seen as a complete and smooth (as will be demonstrated later) regularization of Schwarzschild and Kerr solutions, which also happens to satisfy Einstein Equations.

The metric tensor is well defined and analytic in the following range of the variables

$$-\infty < v, \quad r < \infty, \quad 0 \le \theta < \pi, \qquad 0 \le \tilde{\phi} < 2\pi P(0),$$

while we still have to take care of the half-axis $\theta = \pi$.

B. Our approach to the problem

Throughout this paper we use the convention that objects with bar (e.g., $\bar{g}$, $\bar{q}$, $\bar{\xi}$, $\bar{\omega}$) are globally defined on the whole $\mathbb{R}\times S^3$. We wish to construct a spacetime metric $\bar{g}$, such that $\bar{g}$:

(i) is locally isometric to (4) with $l\neq 0$,

(ii) admits an isometric action of the group U(1) that induces the principal fiber structure

$$\Pi\colon\ \mathbb{R}\times S^3\to\mathbb{R}\times S^2. \tag{11}$$

We start with addressing necessary conditions for a choice of a Killing vector ξ of (4) whose orbits will be compactified. If the desired metric $\bar{g}$ exists then, the extension $\bar{\xi}$ of the vector field ξ is the generator of the U(1) action, and as such has to be a nowhere vanishing Killing vector field. The projection (11) induces a metric tensor $\bar{q}$ on an open subset

$$U\subset\mathbb{R}\times S^2,$$

of the non-null (with respect to $\bar{g}$) orbits of the action of U(1) in $\mathbb{R}\times S^3$. In every point of U the metric tensor $\bar{q}$ should be well defined and (at least) differentiable. Therefore, in Sec. III we study the spaces of orbits of Killing vectors of the form

$$\partial_v+b\partial_{\tilde{\phi}}$$

in the spacetime (4). We determine those Killing vectors that define a nonsingular metric tensor q on U. It turns out, that in the presence of $l\neq 0$ there are allowed exactly two values of the parameter f for each triple of values of a, Λ and l. As far as we know this consequence of the presence of the NUT parameter $l\neq 0$ has not been described in the literature before. Indeed, in the $l=0$ case for every value of the parameter b, the corresponding Killing vector field defines a nonsingular geometry q on the space of non-null orbits and the problem becomes trivial.

The element of the desired metric tensor $\bar{g}$ on $\mathbb{R}\times S^3$ encoding the nontrivial structure of the bundle (11) is the 1-form of the rotation-connection of the Killing vector $\bar{\xi}$, namely

$$\bar{\omega}\coloneqq\frac{\bar{\xi}_\mu dx^\mu}{\bar{\xi}_\alpha\bar{\xi}^\alpha}, \tag{12}$$

valid wherever

$$\bar{\xi}_\alpha\bar{\xi}^\alpha\neq 0.$$

If the bundle extension (11) of the spacetime (4) exists, then the part of the spacetime $\mathbb{R}\times S^3$ described by (4) is a trivialization of (11) that covers the pole $\theta=0$ of S^2. Therefore, in Sec. IV, for each of the Killing vectors ξ derived in Sec. III we derive the rotation-connection 1-form ω. The analysis of the discontinuity of ω as $\theta\to\pi$ leads to a complementary trivialization of (11) that covers the pole $\theta=\pi$. Remarkably, the key limit properties of ω at $\theta=\pi$ are independent of r. Hence, the second trivialization covers also the null orbits. The transformation law between the trivializations becomes a recipe for bundle reconstruction implemented in Sec. V. The trivializations come with metric tensors g and g', respectively. The former one is the original metric tensor (4), and the latter is a new, transformed metric. On the overlap of the trivializations the metric tensors g and g' are consistent with each other according to the trivialization transformations. In that way they consistently make up a uniquely defined metric tensor $\bar{g}$ on the entire manifold $\mathbb{R}\times S^3$ that satisfies all of the desired properties.

III. NONSINGULAR SPACE OF KILLING ORBITS AND NONSINGULAR KN(A)DS SPACETIME

In this section we consider the geometries of the spaces of orbits of the Killing vector fields [12] in the spacetime (4). In a case of generic parameters, the most general form of nowhere vanishing Killing vector field is

$$\xi=\partial_v+b\partial_{\tilde{\phi}},\qquad b=\text{const}. \tag{13}$$

For $a=0$, i.r. in the Taub-NUT–(anti–)de Sitter case the metric has richer symmetry group generated by $\mathfrak{su}(2)\oplus\mathfrak{u}(1)$, where the summands correspond right action by a 3D rotations and right action by a time translation, respectively.[1] Consequently instead of the combination (13) we could equivalently consider its rotation.

In an adapted coordinates system, that is

$$(x^\mu)=(\tau,x^i)=(v,r,\theta,\hat{\phi}\coloneqq -bv+\tilde{\phi}) \tag{14}$$

the Killing vector field ξ takes a simple form

$$\xi=\partial_\tau.$$

The three coordinates

$$(x^i)=(r,\theta,\hat{\phi})$$

are adopted in the sense that they satisfy

$$\xi(x^i)=0,$$

hence they set a coordinate system on the space of the orbits. To find the metric tensor induced thereon, we use the rotation-connection 1-form

[1] In fact every Ricci flat spacetime admitting $SU(2)\times U(1)$ isometry group corresponds to a generalized Taub-NUT spacetime with the topology of $\mathbb{R}\times L(n,1)$, where $L(n,1)$ is a lens space [13].

$$\omega := \frac{\xi_\mu dx^\mu}{\xi_\alpha \xi^\alpha}, \tag{15}$$

and decompose the spacetime metric (4) in the following manner

$$g = \xi^\alpha \xi_\alpha \omega^2 + q. \tag{16}$$

Then, the part q of the spacetime metric satisfies

$$\xi^\mu q_{\mu\nu} = 0 = \mathcal{L}_\xi q_{\mu\nu},$$

hence, it is expressed purely in terms of the three coordinates (x^i),

$$q = q_{ij}(r, \theta, \hat{\phi}) dx^i dx^j. \tag{17}$$

As a matter of fact, q is the pullback to the spacetime of the metric tensor induced on the space of the non-null orbits of ξ. When the variables r, θ, and $\hat{\phi}$ denote both the spacetime and the space of the orbits coordinates, the pullback of the metric on the space of the orbits is given exactly by (17). We calculated q for the metric tensor (4) transformed to the adapted coordinate system (14), and the result reads as

$$\begin{aligned} q = {} & \frac{(1-Ab)^2\Sigma}{\mathcal{Q}(1-Ab)^2 - P\sin^2\theta(a-b\rho)^2} dr^2 \\ & + \frac{P\sin^2\theta\Sigma(b\rho - a)}{\mathcal{Q}(1-Ab)^2 - P\sin^2\theta(a-b\rho)^2} 2drd\hat{\phi} \\ & + \frac{\Sigma}{P} d\theta^2 + \frac{P\mathcal{Q}\sin^2\theta\Sigma}{\mathcal{Q}(1-Ab)^2 - P\sin^2\theta(a-b\rho)^2} d\hat{\phi}^2. \end{aligned} \tag{18}$$

The above metric is well defined for $(r, \theta, \hat{\phi}) \in \mathbb{R} \times]0, \pi[\times [0, 2\pi c[$, as long as the denominators do not vanish. The parameter c represents the rescaling freedom that will be used to fix the metric at the poles. The function P is positive everywhere by the assumption, and the other denominators are proportional to $\xi^\mu \xi_\mu$. More precisely

$$g(\xi, \xi) = g_{\tau\tau} = \Sigma^{-1}(P\sin^2\theta(a-b\rho)^2 - \mathcal{Q}(1-Ab)^2). \tag{19}$$

For general values of the parameters m, a, Λ, l, b it does vanish for some values of (r, θ), where we do not expect the metric q to be well defined. Hence we consider the metric q only where

$$P\sin^2\theta(a-b\rho)^2 - \mathcal{Q}(1-Ab)^2 \neq 0. \tag{20}$$

The degeneracies of q that we do worry about are the half-axis p_0 and p_π corresponding to $\theta = 0$ and $\theta = \pi$, respectively. The term proportional to dr^2 is manifestly regular, so is the term proportional to $drd\hat{\phi}$ because it can be viewed as a regular 1-form $\sin^2\theta d\hat{\phi}$ times an analytic function times dr. Now we turn to the purely angular part and consider the pullbacks of q to the surfaces of $r = \text{const}$, that is

$${}^{(2)}q = \frac{\Sigma}{P} d\theta^2 + \frac{P\mathcal{Q}\sin^2\theta\Sigma}{\mathcal{Q}(1-Ab)^2 - P\sin^2\theta(a-b\rho)^2} d\hat{\phi}^2. \tag{21}$$

One of the tools we use for the analysis are closed curves of $\theta = \theta_0$, which can be view as circles around either pole (notice, that the $\hat{\phi} = \text{const}$ curves are geodesic with respect to ${}^{(2)}q$). The radii as seen from either pole (R_0 or R_π, respectively) and circumference (L_0) are defined as

$$\begin{aligned} & R_0(\theta_0) = \int_0^{\theta_0} \sqrt{{}^{(2)}q_{\theta\theta}} d\theta, \qquad R_\pi(\theta_0) = \int_{\theta_0}^{\pi} \sqrt{{}^{(2)}q_{\theta\theta}} d\theta, \\ & L(\theta_0) = \int_0^{2\pi c} \sqrt{{}^{(2)}q_{\hat{\phi}\hat{\phi}}} d\hat{\phi}. \end{aligned} \tag{22}$$

Then the condition for removing the conical singularity is recovering the expected limit of 2π of ratio of the circumference to radius of the aforementioned cures as we tend to the poles

$$\lim_{\theta_0 \to 0} \frac{L(\theta_0)}{R_0(\theta_0)} = 2\pi = \lim_{\theta_0 \to \pi} \frac{L(\theta_0)}{R_\pi(\theta_0)}. \tag{23}$$

The above amounts to[2]

$$P(0) = \frac{P(\pi)}{|1 - 4lb|}, \qquad c = 1/P(0). \tag{24}$$

We would be in trouble, if this condition involved the coordinate r, however, this is not the case because the function P depends only on θ.

Due of the absolute value in the denominator above, for $l \neq 0$, there are 2 possible branches of solutions, each depending on the parameters of the spacetime. For the further convenience let us denote

$$\sigma := \text{sgn}(1 - 4lb).$$

Either we have $\sigma = 1$ and then we find the solution $b = b_+$, such that

$$0 < 1 - 4lb_+ = \frac{P(\pi)}{P(0)}, \qquad b_+ = \frac{2a\Lambda}{3 + a^2\Lambda + 4al\Lambda} \tag{25}$$

or $\sigma = -1$ in which case the solution $f = f_-$, satisfies

[2]One may also consider an extension of the Kerr-NUT–(anti–) de Sitter spacetimes to the case with the acceleration parameter. Then the condition (23) is formally the same, although with more complicated function P. See [14] for a discussion of the nonsingularity if ξ develops the horizon.

$$0 < 4lb_- - 1 = \frac{P(\pi)}{P(0)}, \qquad b_- = \frac{3 + a^2\Lambda}{2l(3 + a^2\Lambda + 4al\Lambda)}. \tag{26}$$

We note, that the assumed inequalities rewritten in (25) and (26) are consistent with the overall assumption that the function P does not vanish for $\theta \in [0, \pi]$.

In either case, the rescaled angle variable ranging from 0 to 2π is

$$\varphi = P(0)\hat{\phi}. \tag{27}$$

It is instructing to test our results on the special cases that are encountered in the literature. A very special case when the Killing vector field ξ develops a horizon was studied extensively in [10,14]. Then the coefficient b is related to the value r_0 taken by the coordinate r at the horizon, namely

$$b = \frac{a}{r_0^2 + (a + l)^2}. \tag{28}$$

Then it follows that

$$1 - 4lb = 1 - \frac{4la}{r_0^2 + (a + l)^2} = \frac{r_0^2 + (a - l)^2}{r_0^2 + (a + l)^2} > 0, \tag{29}$$

hence this choice falls in the very case (25). The conditions (28) and (25) determine the value of Λ, namely

$$\Lambda = \frac{3}{a^2 + 2l^2 + 2r_0^2}. \tag{30}$$

That is exactly the value found in [14] when the horizon can be made projectively nonsingular, i.e., its space of the null generators is nonsingular. This can be done using the same rescaled coordinate as for the surrounding spacetime. The horizon is then necessarily cosmological (more precisely: the outermost, possibly with a negative mass parameter) and nonextremal.

Another compelling choice of the Killing vector is simply

$$\xi = \partial_v, \qquad \text{meaning} \qquad b = 0,$$

resembling the original Misner's choice is his nonsingular interpretation of the Taub-NUT metric tensor. Upon this choice, the condition (24) amounts to the constraint

$$P(0) = P(\pi),$$

which is met iff $\Lambda al = 0$. We have discussed that case in Sec. II A. However now, in view of our general result derived in this section, the value $b = 0$ falls into the b_+ case (25), while there is yet another solution, the one of the b_- type (26), namely

$$b = \frac{1}{2l}.$$

The corresponding the Killing vector field is

$$\xi = \partial_v + \frac{1}{2l}\partial_{\hat{\phi}}. \tag{31}$$

Finally, when the NUT parameter is switched off, that is when $l = 0$, then every Killing vector field

$$\xi = \partial_v + b\partial_{\hat{\phi}}, \qquad b = \text{const}$$

defines a nonsingular geometry on the orbit space wherever $\xi^\mu \xi_\mu \neq 0$. That is why we never encounter that issue while considering spacetimes without the NUT parameter.

Remark. *An intriguing and useful observation are the following general identities:*

$$b_+ + b_- = \frac{1}{2l},$$

and

$$b_- - b_+ = \frac{P(\pi)}{2lP(0)}. \tag{32}$$

Although from the conceptual point of view satisfying the constraint (23) guarantees only the continuity of the metric, we also recover that the metric is smooth. The suspicious parts of the decomposition (16) are connection 1-form ω and the orbit metric ${}^{(2)}q$ induced on sphere $r = \text{const}$. The explicit formulas for the before-mentioned tensors are given by (21) after the substitution (27) and (35). Using the coordinates corresponding to orthogonal projections of a hemisphere covering one of the poles to the plane, we check by inspection that the components of those tensors are smooth. The details of this procedure are analogous to those described in the Appendix of [14].

IV. GENERALIZATION OF MISNER'S GLUING

The starting point for this section is one of the Killing vector fields ξ (13) found in the previous section, that is such that the constant b satisfies one of the conditions: either (25) or (26). In terms of the adapted coordinates (14) with the rescaled angle variable (27) the metric tensor (4) takes the following form

$$\begin{aligned} ds^2 = &-\frac{Q}{\Sigma}\left((1 - bA)d\tau - \frac{A}{P(0)}d\varphi\right)^2 \\ &+ 2dr\left((1 - bA)d\tau - \frac{A}{P(0)}d\varphi\right) + \frac{\Sigma}{P}d\theta^2 \\ &+ \frac{P}{\Sigma}\sin^2\theta\left((a - \rho b)d\tau - \frac{\rho}{P(0)}d\varphi\right)^2, \end{aligned} \tag{33}$$

and the Killing vector field is simply

$$\xi = \partial_\tau. \tag{34}$$

If this spacetime is a trivialization of the principal fibre bundle (11) and ξ is a generator of the structure group action, as we want it to be, then the orbits of ξ are closed curves, and the parameter τ takes values in a finite interval

$$\tau \in [0, \tau_0).$$

The relation of τ_0 and the NUT parameter will follow as a consistency condition for a transformation between the given one and a new, complementary trivialization that will cover the half-axis $\theta = \pi$. It is the rotation-connection 1-form (12) that will tell us, how to construct this complementary trivialization. The explicit formula for ω reads

$$\omega = d\tau - \frac{(1-Ab)\Sigma dr + (A\mathcal{Q}(1-Ab) - P\sin^2\theta\rho(a-b\rho))d\varphi/P(0)}{\mathcal{Q}(1-Ab)^2 - P\sin^2\theta(a-b\rho)^2}. \tag{35}$$

It is well defined at $\theta = 0$, however it fails to be so at $\theta = \pi$,

$$\omega_\varphi(r, \theta = 0) = 0, \qquad \omega_\varphi(r, \theta = \pi) = -\frac{4l}{1-4lb}\frac{1}{P(0)} = \sigma\frac{-4l}{P(\pi)}. \tag{36}$$

The obstruction is nonvanishing of the component ω_φ at the second half-axis.

Along with ω the metric tensor is not well defined at $\theta = \pi$, what can be seen from the formula (16). From the limit of ω_φ at $\theta = \pi$, we deduce a coordinate transformation that cures ω at that half-axis at the cost of $\theta = 0$, namely

$$\tau = \tau' + \sigma\frac{4l}{P(\pi)}\varphi', \qquad r = r', \qquad \theta = \theta', \qquad \varphi = \varphi' \qquad \text{for,} \quad \theta, \theta' \neq 0, \pi. \tag{37}$$

The condition for the constant τ_0 is hidden behind the transformation of τ. If φ and $\varphi + 2\pi$ correspond to a same point of spacetime for every value of τ, r, θ and ϕ, and the same is true for φ' and $\varphi' + 2\pi$, also τ and τ' must parametrize circles with the period

$$\tau_0 = 2\pi\frac{4l}{1-4lb}\frac{1}{P(0)} = 2\pi\sigma\frac{4l}{P(\pi)}, \tag{38}$$

alternatively the period may be a $\frac{1}{n}$ fraction of the above. Hence, the coordinates $(\tau, r, \theta, \varphi)$ parametrize $S_1 \times \mathbb{R} \times (S^2 \backslash \{p_\pi\})$, and the coordinates $(\tau', r', \theta', \varphi')$ parametrize $S_1 \times \mathbb{R} \times (S^2 \backslash \{p_0\})$, where p_0 and p_π are the poles of S^2 corresponding to $\theta = 0$ and $\theta = \pi$, respectively. The transformation (37) defines gluing of the patches and the vector fields ∂_τ and $\partial_{\tau'}$ give rise to a uniquely defined vector field

$$\partial_\tau = \bar{\xi} = \partial_{\tau'}.$$

The manifold defined by the two charts is diffeomorphic to $\mathbb{R} \times S^3$ and the flow of $\bar{\xi}$ makes it the bundle (11). The transformation (37) maps the 1-form ω into ω', which is extendable by the continuity to $\theta' = \pi$. It is analytic 1-form in the subset of the second chart corresponding to $\xi'^\mu \xi'_\nu \neq 0$. Finally, the 2-metric tensor q is invariant with respect to the transformation (37).

Applying the transformation (37) to the metric tensor (33) we obtain a metric, which is well defined on the chart containing the pole $\theta = \pi$

$$\begin{aligned} ds'^2 = &-\frac{\mathcal{Q}}{\Sigma}\left((1-bA)d\tau' - \frac{A'}{P(\pi)}\sigma d\varphi'\right)^2 \\ &+ 2dr'\left((1-bA)d\tau' - \frac{A'}{P(\pi)}d\varphi'\right) + \frac{\Sigma}{P}d\theta'^2 \\ &+ \frac{P}{\Sigma}\sin^2\theta'\left((a-\rho b)d\tau' - \frac{\rho'}{P(\pi)}\sigma d\varphi'\right)^2, \end{aligned} \tag{39}$$

where $A'(\theta') \coloneqq a\sin^2\theta' - 4l\cos^2\frac{1}{2}\theta'$, $\rho'(r') \coloneqq r'^2 + (a-l)^2$. Note that $A'd\varphi'$ is dual to $Ad\varphi$ is the sense that $A'd\varphi'$ vanishes at $\theta = \pi$ and is singular at $\theta = 0$. Another way of discovering these functions would be, instead of starting with metric (1) well defined at p_0, to start with a metric well defined at p_π. This can be achieved by a transformation $t' = t + 4l\phi$ and replacing A and ρ with A' and ρ' in (1) and (2).

Finally, we can turn to the nonsingularity of the resulting metric tensor $\bar{g}$. This issue amounts to showing the nonsingularity of the metric tensors (33) and (39) in their charts. By construction, each of the metric tensors is automatically nonsingular as long as

$$\bar{\xi}^\mu \bar{\xi}_\mu \neq 0, \tag{40}$$

owing to the decomposition

$$g = \xi^\alpha \xi_\alpha \omega^2 + q, \qquad g' = \xi'^\alpha \xi'_\alpha \omega'^2 + q \tag{41}$$

and the nonsingularity of $\xi^\alpha \xi_\alpha$, $\xi'^\alpha \xi'_\alpha$, ω, ω' and q in the corresponding charts. Notice, that the missing prime at the second q is intentional—indeed, at that stage of the construction we use the single 3-metric tensor q, the same for each chart.

As was argued in the previous section the above components are smooth and thus metric g is smooth whenever the decomposition (41) is valid. To also cover the surfaces of $\bar{\xi}^\mu \bar{\xi}_\mu = 0$ we repeat the procedure used for the analysis of the smoothness of the decomposition. Given the metric tensors in the form (33) and (39) we relax the assumption (40) and decompose the formulas into another set of nonsingular elements. First of all, except for the half-axes $\theta = 0$ and $\theta' = \pi$, all of the coefficients are nonsingular. To analyse the metrics at the poles we decompose them into the following way. First consider the purely angular parts

$$\frac{\Sigma}{P} d\theta^2 + \frac{P}{\Sigma}\sin^2\theta \left(\frac{\rho}{P(0)}\right)^2 d\varphi^2 = \frac{\Sigma}{P}\left(d\theta^2 + \frac{P^2}{\Sigma^2}\sin^2\theta\left(\frac{\rho}{P(0)}\right)^2 d\varphi^2\right), \quad \text{at } \theta = 0, \tag{42}$$

$$\frac{\Sigma}{P} d\theta'^2 + \frac{P}{\Sigma}\sin^2\theta' \left(\frac{\rho'}{P(\pi)}\right)^2 d\varphi\prime 2 = \frac{\Sigma}{P}\left(d\theta'^2 + \frac{P^2}{\Sigma^2}\sin^2\theta'\left(\frac{\rho'}{P(\pi)}\right)^2 d\varphi'^2\right) \quad \text{at } \theta' = \pi. \tag{43}$$

Employing again the orthogonal projection (as described in Appendix A of [14] one can see that in the parentheses, the coefficients at $\sin^2\theta d\varphi^2$ and $\sin^2\theta' d\varphi'^2$ are smooth (analytic) and tend to 1 at $\theta = 0$ and, $\theta' = \pi$, respectively.

Next, consider the differential 1-forms appearing in (33) and (39)

$$\alpha_1 = A d\varphi, \qquad \alpha_2 = \sin^2\theta d\varphi,$$
$$\alpha'_1 = A' d\varphi', \qquad \alpha'_2 = \sin^2\theta' d\varphi'. \tag{44}$$

Clearly, they are smooth (analytic) in their domains including the poles $\theta = 0$, and $\theta' = \pi$ respectively. The remaining elements used in the definitions of the metric tensors g and g' are smooth (analytic) functions in their domains, also at the respective half-axes.

In conclusion, the metric tensors g and g' give rise to a metric tensor $\bar{g}$ uniquely defined on the manifold constructed above, diffeomorphic to $\mathbb{R} \times S^3$, and admitting the U(1) bundle structure induced by the flow of the Killing vector field $\bar{\xi}$.

For the construction above we have used one of the two possible choices (25) or (26) of the parameter b. Does the other vector have any special meaning in the resulting spacetime? Does the outcome of this section depend on that choice? To answer those questions suppose that

$$b = b_+,$$

with the corresponding vector field ξ renamed as ξ_+ for consistency. Next, consider the other vector field ξ_- corresponding to b_-. Now we transform it to the coordinates adapted to ξ_+

$$\xi_- = \partial_\tau + (b_- - b_+)\partial_{\hat{\phi}} = \partial_\tau + \frac{P(\pi)}{2l}\partial_\varphi.$$

It is convenient to consider to consider a rescaled version of ξ_-

$$\hat{\xi}_- \coloneqq \partial_\varphi + \frac{2l}{P(\pi)}\partial_\tau.$$

Upon the triviality transformation (37) it is expressed as

$$\hat{\xi}_- = \partial_{\varphi'} - \frac{2l}{P(\pi)}\partial_{\tau'}.$$

That symmetry indicates a special character of this vector field. Indeed, we can introduce on every surface $r = \text{const}$ an auxiliary structure of the group SU(2), such that the vector field ξ_+ generates a left invariant vector field, the vector field ξ_- a right invariant vector field, and the vector fields coincide at the circles corresponding to $\theta = 0$, while they equal minus each other along the circles $\theta = \pi$. For this reason the two points of intersection of orbits of ξ_+ and ξ_- in Figs. 1 and 2 are in fact two circles. An important consequence of that symmetry between ξ_+ and

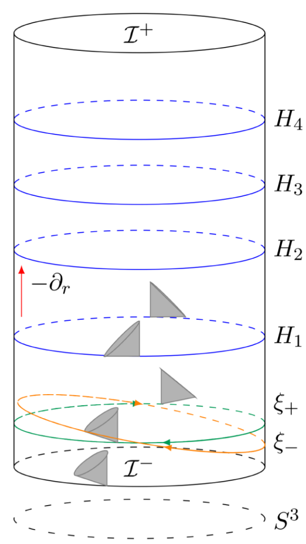

FIG. 1. Global structure of the spacetime for $\Lambda > 0$. Each point on the cylinder corresponds to a S^2. Each circle corresponds to a S^3 although via different sections. Four uppermost, blue circles H_i correspond to surfaces of constant $r = r_i$ for $i = 1, 2, 3, 4$, i.e., Killing horizons. Green, lowermost circle is S^3 projected along the orbit of ξ_+. The orbit is closed and spacelike near both of the $\mathcal{I}$. Similarly the orbit of ξ_- is shown orange. Points of intersection of the orbits are circles. Future oriented (in the direction of ∂_{-r}) parts of lightcones are shown.

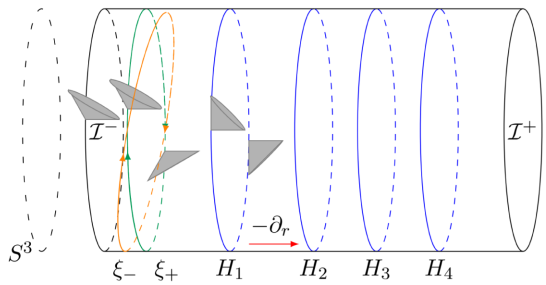

FIG. 2. Global structure of the spacetime for $\Lambda < 0$. Each point on the cylinder corresponds to a S^2. Each circle corresponds to a S^3 although via different sections. Four rightmost, blue circles H_i correspond to surfaces of constant $r = r_i$ for $i = 1, 2, 3, 4$, i.e., Killing horizons. Green, leftmost circle is S^3 projected along the orbit of ξ. The orbit is manifestly closed and timelike near both of the $\mathcal{I}$. Similarly the orbit of ξ_- is shown orange. Points of intersection of the orbits are circles. Future oriented (in the direction of ∂_{-r}) parts of lightcones are shown.

ξ_- is that the glued spacetime is independent of whether we chose ξ_+ or ξ_- in order to define the generalized Misner gluing. Alternatively it is straightforward to explicitly check that after performing the gluing with ξ_+ it is possible to find four coordinate systems such that two of them are compatible [in the sense of (14)] with ξ_+, one covering $\theta = 0$ pole and the other $\theta = \pi$ pole, and the other two coordinate systems are an analogue for ξ_-. Then the transformation between the coordinates compatible with ξ_- satisfy precisely (37) with $\sigma = -1$, thus showing that spacetimes constructed with either ξ_+ or ξ_- are equivalent.

V. THE GLOBAL STRUCTURE

The spacetime manifold is the entire $\mathbb{R} \times S^3$ provided $|l| > |a|$. Otherwise, if

$$|l| \leq |a|, \tag{45}$$

the vanishing of Σ produces an nonremovable singularity at

$$(r, \theta) = (0, \theta_c) = (r', \theta'),$$

where the critical value θ_c is defined by

$$\cos\theta_c = -\frac{l}{a}.$$

The type of the singularity can be characterized as a "ring" one, except for the case

$$l = \pm a,$$

where the ring is shrunk to a point. Notice however that, as in the Kerr spacetime, a curve going between the $r > 0$ to $r < 0$ regions has to cross the surface

$$r = 0, \qquad \theta, \theta' \neq \theta_c,$$

which has a nonvanishing 3-dimensional volume, hence it connects the two spacetimes regions making a connected spacetime.

The vector field corresponding to ∂_r in unprimed chart, and to $\partial_{r'}$ in the primed chart is globally defined and everywhere null

$$\bar{g}(\partial_r, \partial_r) = 0.$$

A time orientation of spacetime can be defined by either declaring

(i) ∂_r and $\partial_{r'}$ to be future directed, or
(ii) $-\partial_r$ and $-\partial_{r'}$ to be future directed.

The coordinate transformation

$$(\tau'', r'') \coloneqq (-\tau, -r)$$

maps the first case into the second (and vice versa), hence, without lack of generality we can assume that $-\partial_r$ is a future pointing vector field. It should be emphasized that in doing so we allow for an arbitrary sign of the parameters (a, l, m, Λ).

The Killing orbits are (generically) two dimensional surfaces endowed with the induced geometry

$$ds^2 = -\frac{\mathcal{Q}}{\Sigma}\left((1 - bA)d\tau - \frac{A}{P(0)}d\varphi\right)^2 + \frac{P}{\Sigma}\sin^2\theta(\left(a - b\rho)d\tau - \frac{\rho}{P(0)}d\varphi\right)^2, \tag{46}$$

The signature of the above is

(i) $(-, +)$, if $\mathcal{Q} > 0$,
(ii) $(+, +)$, if $\mathcal{Q} < 0$,
(iii) $(0, +)$, i.e., null, if $\mathcal{Q} = 0$.

If an orbit is timelike (or null) at a given point, than a vector field

$$\eta = \partial_\tau + \frac{P(0)}{\rho(r_c)}(a - b\rho(r_c))\partial_\varphi, \tag{47}$$

where $r = r_c$ is constant, is timelike (or null). Its time orientation is encoded by the scalar product

$$g(\eta_c, -\partial_r) = -\frac{\Sigma}{\rho} < 0$$

hence it is always future pointing.

Let r_i, $i = 1, 2, 3, 4$ be the roots of the polynomial $\mathcal{Q}$. Then the surface of $r = r_i$ determines a Killing horizon (see [10]) developed by the Killing vector η_i with r_c replaced with corresponding r_i. Similarly to the Kerr-(anti–)de Sitter space time, all of the roots cannot have the same sign. This follows from Viete's formulae: because there is no r^3 term in $\mathcal{Q}$ the roots must sum to 0. Another constraint is that $r_i = 0$ corresponds to singularity and not a Killing horizon [10]. It also follows, that this Killing vector is always future pointing.

In the very special case when

$$\Lambda = \frac{3}{a^2 + 2l^2 + 2r_i^2}, \tag{48}$$

the coefficient at ∂_φ of (47) vanishes. Then the horizon is developed by our Killing vector $\bar{\xi}$ itself. Hence, the null generators are closed and the space of the null generators is diffeomorphic to S^2. The geometry induced thereon is the limit of the metric tensor ${}^{(2)}q$ (21), it is nonsingular and smooth. That case was discovered and described in detail in [10] along with their relation to solutions of Type D equation on Hopf bundle and isolated horizons [15,16]

Another nongeneric case is when the ratio of the coefficient at ∂_φ to $4l/P(\pi)$ is rational. Then the null generators will be finite and each such a case requires individual characterization. For a nonrational value of the ratio, all the null generators are infinite and each of them is dense in a 2-manifold contained in S^3. The quotient space of those null generators is non-Hausdorff and lacks a differential structure.

We can introduce a coordinate

$$\Omega \coloneqq \frac{1}{r}$$

valid for either $r < 0$ or $r > 0$. The metric tensor g can be written as

$$\begin{aligned} g = \frac{1}{\Omega^2}\Bigg(&\frac{\Lambda}{3}\left((1 - bA)d\tau - \frac{A}{P(0)}d\varphi\right)^2 + \frac{d\theta^2}{P} \\ &+ P\sin^2\theta\left(bd\tau + \frac{d\varphi}{P(0)}\right)^2 \\ &- 2d\Omega\left((1 - bA)d\tau - \frac{A}{P(0)}d\varphi\right) + O(\Omega)\Bigg). \end{aligned}$$

The surfaces of $\Omega = 0$, corresponding to $r = \pm\infty$, define the future / past infinity equipped with an induced geometry

$$\begin{aligned} &\frac{\Lambda}{3}\left((1 - bA)d\tau - \frac{A}{P(0)}d\varphi\right)^2 + \frac{d\theta^2}{P} \\ &\quad + P\sin^2\theta\left(bd\tau + \frac{d\varphi}{P(0)}\right)^2 \end{aligned}$$

of the signature depending on Λ in the known way. Applying a similar procedure to the metric tensor (39) one arrives at

$$\begin{aligned} g' = \frac{1}{\Omega^2}\Bigg(&\frac{\Lambda}{3}\left((1 - bA)d\tau' - \frac{A'}{P(\pi)}d\varphi'\right)^2 + \frac{d\theta'^2}{P} \\ &+ P\sin^2\theta'\left(bd\tau' + \frac{d\varphi'}{P(0)}\right)^2 \\ &- 2d\Omega\left((1 - bA)d\tau' - \frac{A'}{P(\pi)}d\varphi'\right) + O(\Omega)\Bigg), \end{aligned}$$

which on the surfaces $\Omega = 0$ gives the following geometry

$$\begin{aligned} &\frac{\Lambda}{3}\left((1 - bA)d\tau' - \frac{A'}{P(\pi)}d\varphi'\right)^2 + \frac{d\theta'^2}{P} \\ &\quad + P\sin^2\theta'\left(bd\tau' + \frac{d\varphi'}{P(\pi)}\right)^2. \end{aligned}$$

There are still two discrete degrees of freedom we have not discussed yet. The first one is the causal orientation. The second would be using the outgoing Eddington-Finkelstein coordinates

$$dv \coloneqq dt - \frac{\rho}{\mathcal{Q}}dr, \qquad d\tilde{\phi} \coloneqq d\phi + \frac{a}{\mathcal{Q}}dr, \tag{49}$$

rather than incoming ones (3). However, the latter is equivalent to $(v, a, l) \to (-v, -a, -l)$. Of course, there is still the symmetry of reversing signs of both: the r coordinate and mass $(r, m) \to (-r - m)$.

The schematics of global structure summarizing the above constructions and conventions is shown in Figs. 1 and 2 for positive and negative values of cosmological constant, respectively.

VI. SUMMARY

The result of this paper is a 4 dimensional family [parametrized by the quadruple (m, a, l, Λ)] of globally defined spacetimes that are locally isometric to the Kerr-NUT-(anti) de Sitter metric tensors (1), however, they do not suffer the singularities along the axis $\theta = 0$ and $\theta = \pi$. The spacetime manifold is obtained by gluing the manifolds

$$S_1 \times \mathbb{R} \times (S^2 \backslash \{p_\pi\}) \tag{50}$$

parametrized by $(\tau, r, \theta, \varphi)$, and

$$S_1 \times \mathbb{R} \times (S^2 \backslash \{p_0\}) \tag{51}$$

parametrized by $(\tau', r', \theta', \varphi')$, together with the transformation (37). The coordinates (θ, φ) and (θ', φ') are the standard spherical coordinates on S^2, while the variables τ and τ' parametrize circles. The points p_0 and p_π are the poles of S^2 corresponding $\theta = 0$ and $\theta' = \pi$.

For every choice of the parameters (m, a, l, Λ), the spacetime metric tensor is defined by (33) and (39), with the functions $\mathcal{Q}, \Sigma, \rho, P > 0$ and A defined by (1). The only possible singularity of our spacetime metric tensor may be caused (depending on the ratio l/a, see (45)) by vanishing of the function Σ. The singularity has a similar character to that of Kerr spacetime—in particular it does not restrict the domain of the r coordinate $-\infty < r < \infty$. What is new about our result, is the simultaneous presence of the Kerr parameter $a \neq 0$, the NUT parameter $l \neq 0$ and the cosmological constant $\Lambda \neq 0$.

An underlying structure for the construction was the assumed isometric action of the group U(1) that makes the spacetime a principal fiber bundle

$$\mathbb{R} \times S^3 \to \mathbb{R} \times S^2$$

(modulo the possible singularities discussed above). The key element of our method was determining a suitable candidate Killing vector field in the Kerr-NUT–(anti–)de Sitter metric tensor (1) that could become a generator of that nonsingular action of U(1) on $\mathbb{R} \times S^3$. We have achieved that by studying the geometry of the spaces of non-null orbits of each Killing vector field of the Kerr-NUT–(anti–)de Sitter spacetime, and selecting those that induce nonsingular 3-geometry.

We studied the global structure of the constructed spacetimes. Depending on the value of the cosmological constant Λ, our spacetime is asymptotically de Sitter or anti–de Sitter. We derive the conformal geometry of the conformal infinity and find it is nonsingular as well (topologically, the conformal infinities are two copies of 3-sphere). In particular, in the case $\Lambda > 0$ the signature of the infinity is $(+++)$ and a spacetime neighborhood seems to be hyperbolic. The spacetime contains up to four Killing horizons corresponding to the roots of the function $\mathcal{Q}$. Generically, the null generators of the horizons are infinite curves and each of them densely covers a 2-surface. Hence the space of the null generators is not a differentiable 2-dimensional manifold. For special values of (m, a, l, Λ) [see (30)] the null generators of one of the horizons coincide with the fibers of the bundle (51). Then, the horizon is projectively nonsingular, that is the space of the null generators has a nonsingular geometry diffeomorphic to S^2 [10].

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