Nonlocality and gravitoelectromagnetic duality

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The weak-field Schwarzschild and NUT solutions of general relativity are gravitoelectromagnetically dual to each other, except on the positive z-axis. The presence of nonlocality weakens this duality and violates it within a smeared region around the positive z-axis, whose typical transverse size is given by the scale of nonlocality. We restore an exact nonlocal gravitoelectromagnetic duality everywhere via a manifestly dual modification of the linearized nonlocal field equations. In the limit of vanishing nonlocality we recover the well-known results from weak-field general relativity. Nonlocality, as a possible ultraviolet completion of gravity, does not pose a fundamental impediment to duality structures at the linear level.

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I. INTRODUCTION

The formal similarities between general relativity and electrodynamics were observed long ago by Einstein himself and others: After the early work on the structure of four-dimensional curvature by Bach [1], Rainich [2], Einstein [3], Lanczos [4] (see also Ref. [5]) as well as Ruse [6] in the 1920s-1940s, it was Matte [7] and Bel [8,9] who sought to express the description of gravitational waves in a language similar to vacuum electrodynamics. This framework has since provided useful mathematical and conceptual foundations for the study of gravitational waves, and this field of research has been dubbed gravitoelectromagnetism ("GEM" in what follows) with many fruitful applications in the context of classical general relativity [10–13].

In a second step, GEM *dualities* allow the mapping of gravitoelectric phenomena into gravitomagnetic ones, and vice versa. They have been studied in a wide range of publications; see Refs. [14–19] and references therein. These dualities proved to be a useful tool that can simplify calculations considerably, and it is hence no surprise that they play an important role not only in electrodynamics, but also black hole physics [20] as well as quantum field theory, for example in double copy theory [21–23]. Since it has recently been shown that scattering methods can be employed to derive aspects of the full, nonlinear gravitational theory [24,25]—in the spirit of Sakharov [26]—the role of GEM dualities is under active investigation in that context as well [27].

With GEM dualities playing a central role not only in classical gravity but also in quantum field theoretical considerations, it is natural to ask how GEM generalizes to modified gravity theories. After all, GEM dualities prove useful for the study of gravitational waves, which may very well emanate from astrophysical regions with strong gravitational fields, where such modifications of gravitational theory are thought to become relevant. Moreover, from the perspective of quantum field theory, general relativity is not ultraviolet (UV) complete, so it is desirable to consider duality structures of a modified gravitational theory with improved UV behavior.

As is well known, higher-curvature gravity has UV-improved behavior but is typically accompanied by ghosts [28]. A certain class of nonlocal theories [29–34] sidesteps this problem by introducing infinite-derivative kinetic terms that do not introduce new poles in the propagator and are hence devoid of ghost states. In the present paper we would like to focus our attention on this class of *infinite-derivative* nonlocal gravity.

Nonlocal theories of this type have received considerable attention in the recent literature as well [35,36]. While a few exact classical solutions have been found in the context of gravitational waves [37,38] and cosmology [39,40], the complexity of the nonlocal gravitational field equations has so far prohibited a deeper study of the nonlinear regime; a notable exception is the recent work on almost universal spacetimes [41]. At the weak-field level, however, a plethora of solutions has been constructed in the past years [42–53]. The common feature of these linearized solutions lies in two main aspects:

(i) At the location of sharp δ-shaped sources, such as point particles, strings, or branes, the gravitational field is smoothed out and manifestly regular.

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(ii) At distances much larger than the scale of nonlocality ℓ, the solutions typically approach the solution encountered in local theory.

For this reason we expect that GEM duality should be asymptotically recovered in nonlocal theories. However, at small distance scales this may not be the case, and this paper is devoted to a test of this hypothesis.

We focus our attention on the well-known GEM duality between the Schwarzschild solution [54,55]—which can be regarded as the gravitational field of a point particle—and the somewhat more enigmatic Taub–NUT solution [56,57]—which may be interpreted as a semi-infinite rotating string [58], but see also the related discussion in Refs. [20,23,59–68]. Here, the Schwarzschild solution serves as the archetypical gravitoelectric monopole, whereas the Taub–NUT solution plays the role of the gravitomagnetic monopole. Using their weak-field approximations in nonlocal infinite-derivative gravity we ask: are these geometries still dual to one another?

This paper is organized as follows: In Sec. II we briefly introduce the framework for weak-field infinite-derivative gravity and discuss the role of the Weyl tensor and Ricci tensor in such theories. In Sec. III we introduce GEM quantities for stationary spacetimes, define our notion of GEM duality, and present the weak-field Schwarzschild and NUT solutions in infinite-derivative gravity. Section IV is devoted to a study of the putative GEM duality between the two solutions. Evaluating the electric and magnetic parts of the Weyl curvature we show that an exact GEM duality is spoiled in the presence of nonlocality and becomes exact everywhere except on the positive z-axis when nonlocality vanishes. The duality can be made exact in the local theory, and in the final part of Sec. IV we prove that this remains true in the nonlocal case, and propose a manifestly self-dual nonlocal model. Therein, any two solutions that are dual in the local model are mapped into dual nonlocal solutions, and this duality is applicable to a wide range of stationary nonlocal infinite-derivative gravity theories. In Sec. V we summarize our findings and address potential future work.

II. WEAK-FIELD NONLOCAL INFINITE-DERIVATIVE GRAVITY

Let us work in Cartesian coordinates $x^\mu = (t, x^i)$ and $x^i = (x, y, z)$ such that the Minkowski metric takes the form

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{1}$$

Moreover, let us parametrize a perturbation $h_{\mu\nu}$ such that the full metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{2}$$

For later convenience we also define the trace of the metric perturbation,

$$h = \eta^{\mu\nu} h_{\mu\nu}. \tag{3}$$

Last, let us define the totally antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$ as the volume element on Minkowski spacetime. To linear order, the spacetime curvature is

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \partial_\nu \partial_{[\rho} h_{\sigma]\mu} - \partial_\mu \partial_{[\rho} h_{\sigma]\nu}, \\ R_{\mu\nu} &= \eta^{\rho\sigma} R_{\mu\rho\nu\sigma} \\ &= \partial_\rho \partial_{(\mu} h_{\nu)}{}^\rho - \frac{1}{2}(\partial_\mu \partial_\nu h + \Box h_{\mu\nu}), \\ R &= \eta^{\mu\nu} R_{\mu\nu} = \partial_\rho \partial_\sigma h^{\rho\sigma} - \Box h, \end{aligned} \tag{4}$$

where we denoted the d'Alembert and Laplace operators

$$\Box = \eta^{\mu\nu}\partial_\mu \partial_\nu = -\partial_t^2 + \triangle, \qquad \triangle = \partial_x^2 + \partial_y^2 + \partial_z^2. \tag{5}$$

With the geometric setup in place, let us now study the weak-field model of nonlocal infinite-derivative gravity.

A. Field equations

Let us now briefly comment on the model of infinite-derivative gravity [35,36]. The starting point to derive the linearized nonlocal field equations is an action of the form [36]

$$S = \frac{1}{2\kappa}\int \sqrt{-g}\mathrm{d}^D x \left[R + \frac{1}{2} R_{\mu\nu\rho\sigma} \mathcal{O}^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}\right], \tag{6}$$

where the operator $\mathcal{O}^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta}$ contains arbitrary (infinite) orders of covariant derivatives ∇_μ as well as covariant d'Alembert operators $\Box$; if it is a purely local object without derivatives one recovers the higher-curvature action considered by Stelle [28]. Substituting the small perturbation $h_{\mu\nu}$ in the above action, and only keeping quadratic terms as justified by the weak-field approximation, one arrives at the following form (for more details see, e.g., Appendix A in Ref. [44]):

$$\begin{aligned} S = \frac{1}{2\kappa}\int \mathrm{d}^D x \bigg(&\frac{1}{2} h^{\mu\nu} a(\Box)\Box h_{\mu\nu} - h^{\mu\nu} a(\Box)\partial_\mu \partial_\alpha h^\alpha{}_\nu \\ &+ h^{\mu\nu} c(\Box)\partial_\mu \partial_\nu h - \frac{1}{2} h c(\Box)\Box h \\ &+ \frac{1}{2} h^{\mu\nu} \frac{a(\Box) - c(\Box)}{\Box} \partial_\mu \partial_\nu \partial_\alpha \partial_\beta h^{\alpha\beta} \bigg). \end{aligned} \tag{7}$$

The resulting field equations take the form

$$\begin{aligned} &a(\Box)[\Box h_{\mu\nu} - 2\partial_\rho \partial_{(\mu} h_{\nu)}{}^\rho] \\ &\quad + c(\Box)[\eta_{\mu\nu}(\partial_\rho \partial_\sigma h^{\rho\sigma} - \Box h) + \partial_\mu \partial_\nu h] \\ &\quad + \frac{a(\Box) - c(\Box)}{\Box} \partial_\mu \partial_\nu \partial_\rho \partial_\sigma h^{\rho\sigma} = -2\kappa T_{\mu\nu}, \end{aligned} \tag{8}$$

where $\kappa = 8\pi G$ stands for Einstein's gravitational constant, and parentheses denote symmetrization,

$$\partial_{(\mu}h_{\nu)\alpha} = \frac{1}{2}(\partial_\mu h_{\nu\alpha} + \partial_\nu h_{\mu\alpha}). \tag{9}$$

One may verify that the field equations are consistent with $\partial^\mu T_{\mu\nu} = 0$. The functions $a(\Box)$ and $c(\Box)$ are called *form factors* and parametrize the nonlocality of the field equations. They are subject to the constraint

$$a(0) = c(0) = 1, \tag{10}$$

which guarantees a proper Newtonian limit.

B. Ricci curvature

Using Eq. (4) the field equations can be recast in terms of the Ricci curvature tensor as follows:

$$a(\Box)R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}c(\Box)R - \frac{a(\Box)-c(\Box)}{2\Box}\partial_\mu\partial_\nu R = \kappa T_{\mu\nu}. \tag{11}$$

Note that $\partial^\mu T_{\mu\nu} = 0$ implies that

$$a(\Box)\partial^\mu\left(R_{\mu\nu} - \frac{1}{2}R\eta_{\mu\nu}\right) = 0. \tag{12}$$

This corresponds to the usual contracted Bianchi identity for the Einstein tensor (in the weak-field limit), since in infinite-derivative nonlocal field theories discussed here we assume that the form factors are strictly nonvanishing such that they can be inverted.

For general $a(\Box)$ and $c(\Box)$ the field equation (11) is not algebraic in the Ricci tensor, unlike in general relativity. In momentum space, however, it is possible to express the Ricci tensor via the energy-momentum tensor directly,

$$R_{\mu\nu} = \left[\frac{1}{a_k}\delta^\alpha_\mu\delta^\beta_\nu + \frac{c_k}{a_k^2 - 3a_k c_k}\eta_{\mu\nu}\eta^{\alpha\beta} + \frac{a_k - c_k}{a_k^2 - 3a_k c_k}\frac{k_\mu k_\nu}{k^2}\eta^{\alpha\beta}\right]\kappa T_{\alpha\beta}, \tag{13}$$

where we defined $a_k = a(-k^2)$ and $c_k = c(-k^2)$ for convenience, and k^2 denotes the square of the 4-momentum.

This implies that even at the linearized level, the interpretation of the Ricci curvature as the "matter curvature" is no longer valid in nonlocal theories of the above type. In particular, the above considerations also show that Ricci flat spacetimes, $R_{\mu\nu} = 0$, are always vacuum spacetimes, $T_{\mu\nu} = 0$, but the converse is no longer true: it appears possible to construct vacuum spacetimes that have nonvanishing Ricci curvature.

From now on we shall focus on a special class of nonlocal theories where

$$a(\Box) = c(\Box). \tag{14}$$

Then the field equations (11) simplify to

$$a(\Box)\left(R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R\right) = \kappa T_{\mu\nu}, \tag{15}$$

such that the Ricci curvature can be expressed as

$$R_{\mu\nu} = a^{-1}(\Box)\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right). \tag{16}$$

Recall that in the local theory one has $a = 1$ and hence the Ricci tensor and the energy-momentum tensor are linked *algebraically*. In nonlocal theories, even at the linear level, this is no longer the case. The inverse operator $a^{-1}(\Box)$ always exists in nonlocal theories of this class since $a(\Box)$ has no zeroes. In the literature it has been shown that this inverse operator can act as a smearing operator on sharply localized objects, mostly in the static case but also in the time-dependent case [45,53].

This allows for the tentative interpretation of the Ricci curvature as the "smeared out matter curvature" in this class of nonlocal theories. Moreover, this emphasizes the special role of the Weyl curvature in this class of theories as the only part of curvature that is not directly specified by the field equations. In this linear setting, the Weyl tensor can be written as

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \eta_{\mu[\rho}R_{\sigma]\nu} + \eta_{\nu[\rho}R_{\sigma]\mu} + \frac{1}{3}R\eta_{\mu[\rho}\eta_{\sigma]\nu}, \tag{17}$$

where square brackets denote antisymmetrization,

$$\eta_{\mu[\rho}\eta_{\sigma]\nu} = \frac{1}{2}(\eta_{\mu\rho}\eta_{\sigma\nu} - \eta_{\mu\sigma}\eta_{\rho\nu}). \tag{18}$$

The Weyl tensor can hence be interpreted as the difference between the full Riemann curvature and the smeared out matter curvature, and for that reason we shall refer to the Weyl tensor as the "vacuum curvature."

C. $\mathrm{GF_N}$ model for nonlocal theories

In what follows we will focus our considerations on so-called $\mathrm{GF_N}$ theory wherein

$$a(\Box) = c(\Box) = \exp[(-\ell^2\Box)^N], \qquad N \in \mathbb{N}, \tag{19}$$

and $\ell > 0$ denotes the *scale of nonlocality*. Clearly this form factor satisfies $a(0) = 1$, which also guarantees that one recovers the local theory in the limit $\ell \to 0$. In the time-independent case, which we study in this paper, this simplifies further to

$$a(\triangle) = \exp[(-\ell^2\triangle)^N], \qquad N \in \mathbb{N}, \tag{20}$$

with the final form of the field equations

$$\exp[(-\ell^2\triangle)^N]\left(R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R\right) = \kappa T_{\mu\nu}. \tag{21}$$

It is well known that in this case the field equations can be interpreted as the local Einstein equations with a smeared matter source,

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R &= \kappa T^{\mathrm{eff}}_{\mu\nu}, \\ T^{\mathrm{eff}}_{\mu\nu} &\equiv \exp[-(-\ell^2\triangle)^N]T_{\mu\nu}. \end{aligned} \tag{22}$$

As has been shown in the literature, if $T_{\mu\nu}$ describes a sharply concentrated matter distribution, then the effective energy-momentum tensor is smeared out [69]. In order to formalize this notion somewhat, as well as for later convenience, let us introduce the concept of a smeared δ function and a smeared Heaviside function as follows:

$$\begin{aligned} \delta^{(d)}_\ell(\boldsymbol{x}) &\equiv \exp[-(-\ell^2\triangle)^N]\delta^{(d)}(\boldsymbol{x}), \\ \theta_\ell(x) &\equiv \exp[-(-\ell^2\triangle)^N]\theta(x). \end{aligned} \tag{23}$$

Note that these functions are related via

$$\partial_x\theta_\ell(x) = \delta^{(1)}_\ell(x), \tag{24}$$

which follows from the formal identity $\partial_x\theta(x) = \delta^{(1)}(x)$ which can be verified in the distributional sense within an integral. In the limiting case of $\ell \to 0$ one recovers

$$\lim_{\ell\to 0}\delta^{(d)}_\ell(\boldsymbol{x}) = \delta^{(d)}(\boldsymbol{x}), \qquad \lim_{\ell\to 0}\theta_\ell(x) = \theta(x) \tag{25}$$

In the simplest case of $N = 1$ one finds the expressions

$$\begin{aligned} \delta^{(d)}_\ell(\boldsymbol{x}) &= \frac{1}{(4\pi\ell^2)^{d/2}}\,e^{-\boldsymbol{x}^2/(4\ell^2)}, \\ \theta_\ell(x) &= \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x}{2\ell}\right)\right], \end{aligned} \tag{26}$$

and one may verify that they satisfy Eqs. (24) and (25). Let us mention that this smeared δ-function appears in the definition of static nonlocal Green functions,

$$\begin{aligned} a(\triangle)\triangle\mathcal{G}_d(\boldsymbol{x}) &= -\delta^{(d)}(\boldsymbol{x}), \\ \Leftrightarrow \triangle\mathcal{G}_d(\boldsymbol{x}) &= -\delta^{(d)}_\ell(\boldsymbol{x}). \end{aligned} \tag{27}$$

Due to spherical symmetry $\mathcal{G}_d(\boldsymbol{x}-\boldsymbol{y})$ is a function of $r = |\boldsymbol{x}-\boldsymbol{y}|$ and hence in what follows we may abbreviate $\mathcal{G}_d(\boldsymbol{x}-\boldsymbol{y}) = \mathcal{G}_d(r)$. Last, let us note that the static Green functions are related via [44]

$$\mathcal{G}_{d+2}(r) = -\frac{1}{2\pi r}\frac{\partial\mathcal{G}_d(r)}{\partial r}. \tag{28}$$

This allows a successive construction of nonlocal static Green functions from just two "seed functions," and for a more in-depth reference on nonlocal spatial Green functions we refer to Ref. [47]. In the simplest case of $N = 1$ a sufficient set of seed functions is

$$\mathcal{G}_3(r) = \frac{1}{4\pi r}\mathrm{erf}\left(\frac{r}{2\ell}\right), \tag{29}$$

$$\mathcal{G}_4(r) = \frac{1}{4\pi^2 r^2}[1 - e^{-r^2/(4\ell^2)}]. \tag{30}$$

III. STATIONARY WEAK-FIELD METRICS

In the present paper we are interested in gravitoelectromagnetic properties of stationary geometries, which are defined by the presence of a timelike Killing vector $\boldsymbol{\xi} = \partial_t$ such that

$$\mathcal{L}_{\boldsymbol{\xi}}h_{\mu\nu} = 0, \tag{31}$$

where $\mathcal{L}_{\boldsymbol{\xi}}$ denotes the Lie derivative along $\boldsymbol{\xi}$.

A. Gravitoelectromagnetic duality

Using this Killing vector we may define the *electric* and *magnetic* part of the Weyl tensor as follows [8,9,19,70]:

$$\begin{aligned} E_{ij} &= C_{\mu i\nu j}\xi^\mu\xi^\nu = C_{titj}, \\ B_{ij} &= \frac{1}{2}\epsilon_{\mu i\rho\sigma}C^{\rho\sigma}{}_{\nu j}\xi^\mu\xi^\nu = \frac{1}{2}\epsilon_{ti\rho\sigma}C^{\rho\sigma}{}_{tj}. \end{aligned} \tag{32}$$

It follows from the antisymmetry in the pairs of indices of the Weyl tensor and the ϵ-symbol that these tensors have no timelike components. Moreover, by the fundamental symmetry properties of the Weyl tensor these tensors are symmetric and tracefree,

$$E_{[ij]} = B_{[ij]} = 0, \qquad \eta^{ij}E_{ij} = \eta^{ij}B_{ij} = 0. \tag{33}$$

Therefore, they each encompass five independent components which encode the ten independent tensorial components of the four-dimensional Weyl tensor.

Suppose now that one calculates E_{ij} and B_{ij} for a Weyl tensor $C_{\mu\nu\rho\sigma}$. We define a duality transformation

$$\tilde{C}_{\mu\nu\rho\sigma} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}C_{\alpha\beta\rho\sigma}, \tag{34}$$

which maps the Weyl tensor into its left dual. Calculating the electric and magnetic pieces for this left dual of the Weyl tensor one finds

$$\tilde{E}_{ij} \equiv \tilde{C}_{titj} = B_{ij}, \tag{35}$$

$$\tilde{B}_{ij} \equiv \frac{1}{2}\epsilon_{ti\rho\sigma}\tilde{C}^{\rho\sigma}{}_{tj} = -E_{ij}, \tag{36}$$

which follows from the four-dimensional relation[1]

$$\epsilon_{\mu\nu\alpha\beta}\epsilon^{\alpha\beta\rho\sigma} = -2\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]}. \tag{37}$$

This implies that, up to a sign, a duality transformation (34) maps gravitoelectric and gravitomagnetic quantities into each other.[2]

We can use this observation to define duality between two distinct geometries $h_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$ in a strict sense. Calculate E_{ij} and B_{ij} for a metric $h_{\mu\nu}$, and $\tilde{E}_{ij}$ and $\tilde{B}_{ij}$ for a metric $\tilde{h}_{\mu\nu}$. Then, the metrics $h_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$ are dual to each other if

$$E_{ij} = \tilde{B}_{ij}, \qquad B_{ij} = -\tilde{E}_{ij}. \tag{38}$$

Since we are interested in weak-field stationary metrics, let us make the ansatz

$$h_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = \phi\mathrm{d}t^2 + 2A_i\mathrm{d}x^i\mathrm{d}t + h_{ij}\mathrm{d}x^i\mathrm{d}x^j, \tag{39}$$

where $\partial_t\phi = \partial_t A_i = \partial_t h_{ij} = 0$ due to Eq. (31). It is also useful to define the quantities

$$\epsilon_{ijk} = \epsilon_{tijk}, \qquad F_{ij} = \partial_i A_j - \partial_j A_i, \qquad \bar{R} = \delta^{ij}R_{ij}. \tag{40}$$

Then, the electric and magnetic parts of the Weyl tensor can be written as

$$E_{ij} = -\frac{1}{2}\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\triangle\right)\phi + \frac{1}{2}\left(R_{ij} - \frac{1}{3}\bar{R}\delta_{ij}\right), \tag{41}$$

$$B_{ij} = \frac{1}{4}(\partial_j\epsilon_i{}^{kl}F_{kl} - \epsilon_{ij}{}^{l}\partial^m F_{ml}) = \frac{1}{4}\partial_{(i}\epsilon_{j)}{}^{kl}F_{kl}. \tag{42}$$

Clearly, E_{ij} is symmetric and tracefree. B_{ij} is also tracefree since $\partial_{[i}F_{jk]} = 0$ by construction, and it is symmetric because its antisymmetric part vanishes:

$$\begin{aligned} 8B_{[ij]} &= (\partial_j\epsilon_i{}^{kl} - \partial_i\epsilon_j{}^{kl} - 2\epsilon_{ij}{}^{l}\partial^k)F_{kl} \\ &= (\partial_j\epsilon_i{}^{kl} - \partial_i\epsilon_j{}^{kl} - 2\epsilon_{ij}{}^{a}\partial^b\delta^l_{[a}\delta^k_{b]})F_{kl} \\ &= (\partial_j\epsilon_i{}^{kl} - \partial_i\epsilon_j{}^{kl} - 2\epsilon_{ij}{}^{a}\partial^b\epsilon_{cab}\epsilon^{clk})F_{kl} = 0, \end{aligned} \tag{43}$$

where we have employed the three-dimensional identity $\epsilon_{cab}\epsilon^{cij} = +\delta^i_{[a}\delta^j_{b]}$. Note that in case of spherical symmetry one has $h_{ij} = \psi\delta_{ij}$ and hence one can further simplify the structure of E_{ij}. One finds

$$E_{ij} = -\frac{1}{4}\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\triangle\right)(\phi + \psi). \tag{44}$$

B. Schwarzschild and NUT solutions

Let us now apply this formalism to study the GEM duality properties of weak-field solutions in nonlocal infinite-derivative gravity. In what follows we will consider the gravitational fields of a point particle ("Schwarzschild solution") and that of a spinning semi-infinite string ("NUT solution"). In the derivation we assume the Lorenz gauge $\partial_\mu h^\mu_\nu = \frac{1}{2}\partial_\nu h$.

1. Schwarzschild solution

The weak-field Schwarzschild geometry is sourced by the distributional energy-momentum tensor

$$T^{\mathrm{SCHW}}_{\mu\nu} = m\delta^t_\mu\delta^t_\nu\delta^{(3)}(\boldsymbol{x}), \tag{45}$$

which describes a static particle of mass $m > 0$ at rest in the coordinate origin. Since the energy-momentum tensor is proportional to a three-dimensional δ-function, the solution of Eq. (21) is proportional to the nonlocal three-dimensional static Green function. For $N = 1$ it takes the form [35,36,46,47]

$$\begin{aligned} h^{\mathrm{SCHW}}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu &= \phi\mathrm{d}t^2 + \psi(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \\ \phi = \psi &= \kappa m\mathcal{G}_3(r) = \frac{2Gm}{r}\mathrm{erf}\left(\frac{r}{2\ell}\right). \end{aligned} \tag{46}$$

As has been discussed elsewhere in great detail, this solution is manifestly regular at $r = 0$ and one asymptotically recovers the weak-field Schwarzschild solution of linearized general relativity as $r/\ell \to \infty$. Since the solution is given by the nonlocal Green function directly, it can readily be generalized to different $\mathrm{GF_N}$ theories.

2. NUT solution

The weak-field NUT solution, in its massless limit, is sourced by the following energy-momentum tensor:

$$T^{\mathrm{NUT}}_{\mu\nu} = -\delta^t_{(\mu}\delta^i_{\nu)}n^j_i\partial_j\delta^{(1)}(x)\delta^{(1)}(y)\theta(z), \tag{47}$$

where $n_{ij} = -n_{ji}$ is an antisymmetric tensor with

$$n \equiv n_{xy} = -n_{yx}. \tag{48}$$

The solution of Eq. (21) can be found analytically in the case of $N = 1$ [52]. Here we rewrite it in terms of the smeared δ-function and Heaviside function as follows:

[1]This is the tensorial equivalent of the relation $\star\star = -1$ one encounters for the Hodge dual acting on differential forms on Lorentzian manifolds. It gives rise to an *almost complex structure* and allows the notion of duality.

[2]In general relativity, this transformation maps the mass to the NUT parameter, and the angular momentum to the rotational parameter of the NUT solution [19].

$$h^{\rm NUT}_{\mu\nu}{\rm d}x^\mu{\rm d}x^\nu = 2A_x{\rm d}x{\rm d}t + 2A_y{\rm d}y{\rm d}t,$$
$$A_i = \kappa n\frac{\epsilon_{ijk}x^jL^k}{\rho^2}V, \qquad L^\mu = \delta^\mu_z,$$
$$V = \frac{1}{4\pi} + z\mathcal{G}_3(r) - 2\ell^2\theta_\ell(z)\delta^{(2)}_\ell(\rho), \tag{49}$$

The third term in V is interesting since it corresponds to a smeared positive z-axis; it vanishes identically in the local limit due to the ℓ^2-prefactor. The compact and universal form of this solution suggests that it may be possible to construct this metric for other $\mathrm{GF_N}$ theories as well.[3] The metric reduces to the general relativistic expression as $\rho/\ell \to \infty$, and one recovers the previously found metric of a slowly spinning string as $z \to +\infty$ [50].

IV. GRAVITOELECTROMAGNETIC SCHWARZSCHILD–NUT DUALITIES

As we have shown above, in nonlocal $\mathrm{GF_N}$ theories the Ricci tensor can be interpreted as a smeared matter curvature. Its tensorial structure is hence dictated by those of the energy-momentum tensor. In analogy to the local case we hence study the duality properties of the Weyl tensor alone. This step sets $\mathrm{GF_N}$ theories apart from (i) higher-derivative theories (even at the linear level) as well as (ii) nonlocal models at the nonlinear level, where such an interpretation of the Ricci curvature is in general not possible.

A. Broken duality

With the weak-field solutions at our disposal, we can now determine if they are dual to each other in the sense of Eq. (38). The electric and magnetic parts of their respective Weyl tensors via Eq. (44) and (42) are

$$E^{\rm SCHW}_{ij} = -\frac{\kappa m}{2}\left[\partial_i\partial_j\mathcal{G}_3(r) + \frac{1}{3}\delta_{ij}\delta^{(3)}_\ell(r)\right], \tag{50}$$

$$B^{\rm NUT}_{ij} = +\frac{\kappa n}{2}[\delta_{ij}L^k\partial_k - (3 + x^k\partial_k)L_{(i}\partial_{j)} + x_{(i}\partial_{j)}L^k\partial_k]\frac{V}{\rho^2}, \tag{51}$$

$$B^{\rm SCHW}_{ij} = E^{\rm NUT}_{ij} = 0. \tag{52}$$

One may verify that these tensors are indeed tracefree. The difference of the magnetic Schwarzschild part and the electric NUT part is zero (because both expressions vanish identically), but the difference between the electric Schwarzschild part and the magnetic NUT part for $n \to m$ takes the form

$$\Xi_{ij} \equiv E^{\rm SCHW}_{ij}(m) - B^{\rm NUT}_{ij}(n\to m) = \frac{\kappa m}{2}\left[L_{(i}\partial_{j)}\theta_\ell(z) - \frac{1}{3}\delta_{ij}\delta^{(1)}_\ell(z)\right]\delta^{(2)}_\ell(\rho) \neq 0. \tag{53}$$

According to our definition of GEM duality (38), the above relation implies that these solutions are *not* dual to one another. If the duality was exact, then one would have $\Xi_{ij} = 0$. Because it does not vanish, the GEM duality between the Schwarzschild and massless NUT solution is *broken* at the linear level in the nonlocal theory.

In the local theory, however, the situation is different. Utilizing the relations (25) in the limiting case of $\ell \to 0$ one finds instead

$$\Xi^{\ell\to 0}_{ij} = \frac{\kappa m}{2}\left[L_{(i}\partial_{j)}\theta(z) - \frac{1}{3}\delta_{ij}\delta(z)\right]\delta^{(2)}(\rho), \tag{54}$$

which is a distributional quantity that is nonvanishing on the positive z-axis. This corresponds to the sometimes overlooked fact that in weak-field general relativity the Schwarzschild and NUT solution are only dual to each other away from the positive z-axis, as was pointed out some time ago by Argurio and Dehouck [17].

This calculation justifies the interpretation of the scale of nonlocality as a regulator, since the distributional quantities only appear in the local limit $\ell \to 0$. Hence, even if physics turns out to be ultimately local, "nonlocal regularization" may simply serve as a tool.

B. Exact duality

As just seen, in weak-field general relativity the GEM quantities exhibit distributional character on the positive z-axis. The study is hence mathematically more involved since, in principle, one would be required to employ distributional calculus to make sense of derivatives of distributions as encountered in Eq. (55). However, in the nonlocal theory this is not the case, and all functions encountered are smooth and differentiable for finite $\ell > 0$. At any rate, in both setups there *is no exact duality.*

In the local weak-field theory, Bunster *et al.* [15] propose a modified set of gravitational equations that is manifestly invariant under duality transformations similar to (34), albeit applied to the full Riemann tensor,

$$\tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta\rho\sigma}. \tag{55}$$

Let us call this model the "BCHP model" after its inventors. In spirit, this is similar to the inclusion of magnetic monopoles into the Maxwell equations; see Edelen [71] and references therein. Within this BCHP model, as Argurio and Dehouck demonstrate [17], the weak-field Schwarzschild–NUT duality becomes exact everywhere, including the positive z-axis. Here we would like to extend this conclusion to our nonlocal $\mathrm{GF_1}$ model. This step is

[3]Formally it is possible to derive the NUT solution for any $\mathrm{GF_N}$ theory, see Appendix.

nontrivial since the GEM duality is manifestly violated by nonlocality.

In the BCHP model, just as in general relativity, the fundamental variable is the metric tensor. There are, however, *two* sources of gravity. The energy-momentum tensor $T_{\mu\nu}$ as well as an additional symmetric tensor $\Theta_{\mu\nu}$ which may be viewed as a gravitomagnetic monopole source. The gravitational equations take the form

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{56}$$

$$3R_{\mu[\nu\alpha\beta]} = -\kappa\epsilon_{\nu\alpha\beta\gamma}\Theta^{\gamma}{}_{\mu}, \tag{57}$$

$$R_{\mu\nu[\alpha\beta,\gamma]} = 0. \tag{58}$$

These equations are manifestly invariant under the transformation $(R, \tilde{R}, T, \Theta) \to (\tilde{R}, -R, \Theta, -T)$. This is a duality at the level of the field equations, not to be confused with the duality between solutions discussed earlier. Because the equations remain form-invariant under this transformation we call these equations "self-dual," and we may also write the equivalent expression

$$\tilde{G}_{\mu\nu} = \kappa\Theta_{\mu\nu}, \tag{59}$$

$$3\tilde{R}_{\mu[\nu\alpha\beta]} = +\kappa\epsilon_{\nu\alpha\beta\gamma}T^{\gamma}{}_{\mu}, \tag{60}$$

$$\tilde{R}_{\mu\nu[\alpha\beta,\gamma]} = 0. \tag{61}$$

Here, $\tilde{G}_{\mu\nu}$ denotes the Einstein tensor derived from the dual tensor $\tilde{R}_{\mu\nu\rho\sigma}$. However, note that in the above $R_{\mu\nu\rho\sigma}$ does not admit the interpretation as a Riemannian curvature tensor because it does not satisfy the algebraic Bianchi identity as per Eq. (57).

In order to interpret $\Theta_{\mu\nu}$ as a proper source term, it should be conserved. This can be achieved by expressing it as a divergence of an auxiliary object $\Phi^{\mu\nu}{}_{\rho}$ such that

$$\Theta^{\mu}{}_{\nu} = -\frac{1}{2\kappa}\partial_{\alpha}\Phi^{\alpha\mu}{}_{\nu}, \qquad \Phi^{\mu\nu}{}_{\rho} = -\Phi^{\nu\mu}{}_{\rho}. \tag{62}$$

The antisymmetry of $\Phi^{\mu\nu}{}_{\rho}$ implies the conservation law $\partial_{\mu}\Theta^{\mu}{}_{\nu} = 0$. Then, the object $R_{\mu\nu\rho\sigma}$ is related to the curvature tensor (called $r_{\mu\nu\rho\sigma}$ in this section) via

$$\begin{aligned} R_{\mu\nu\rho\sigma} &\equiv r_{\mu\nu\rho\sigma} + \delta R_{\mu\nu\rho\sigma}, \\ \delta R_{\mu\nu\rho\sigma} &\equiv \frac{1}{4}\epsilon_{\mu\nu\alpha\beta}(\partial_{\rho}\bar{\Phi}^{\alpha\beta}{}_{\sigma} - \partial_{\sigma}\bar{\Phi}^{\alpha\beta}{}_{\rho}), \\ \bar{\Phi}^{\mu\nu}{}_{\rho} &\equiv \Phi^{\mu\nu}{}_{\rho} + \frac{1}{2}(\delta^{\mu}_{\rho}\Phi^{\nu} - \delta^{\nu}_{\rho}\Phi^{\mu}), \qquad \Phi^{\nu} = \Phi^{\nu\alpha}{}_{\alpha}. \end{aligned} \tag{63}$$

Recall that $G_{\mu\nu}$ in Eq. (56) is the Einstein tensor calculated from $R_{\mu\nu\rho\sigma}$. For our present discussion we simply note that the curvature tensor is modified by the presence of a putative conserved $\Theta_{\mu\nu}$ monopole source. Just as the Schwarzschild solution is sourced by the energy-momentum tensor

$$T_{\mu\nu} = m\delta^{t}_{\mu}\delta^{t}_{\nu}\delta^{(3)}(\boldsymbol{x}), \tag{64}$$

in the BCHP model the NUT solution is sourced by

$$\Theta_{\mu\nu} = n\delta^{t}_{\mu}\delta^{t}_{\nu}\delta^{(3)}(\boldsymbol{x}). \tag{65}$$

In order to check whether this mathematical setup solves the duality problem, we may simply calculate the contribution of the additional curvature term $\delta R_{\mu\nu\rho\sigma}$ to the electromagnetic pieces of the Weyl tensor. To that end, the monopole source (65) corresponds to

$$\Phi^{zt}{}_{t} = -\Phi^{tz}{}_{t} = 2\kappa n\delta^{(1)}(x)\delta^{(1)}(y)\theta(z). \tag{66}$$

Since the nonlocal GF_N theory, at the linear level, is equivalent to the local theory with smeared out sources, in what follows we consider the influence of the source

$$\Theta^{\mathrm{eff}}_{\mu\nu} = n\delta^{t}_{\mu}\delta^{t}_{\nu}\delta^{(3)}_{\ell}(\boldsymbol{x}), \tag{67}$$

mediated via

$$\Phi^{\mathrm{eff}\,zt}{}_{t} = -\Phi^{\mathrm{eff}\ tz}{}_{t} = 2\kappa n\delta^{(1)}_{\ell}(x)\delta^{(1)}_{\ell}(y)\theta_{\ell}(z). \tag{68}$$

The resulting contributions to the electric and magnetic parts of the Weyl tensor can be readily computed:

$$\delta E_{ij} \equiv \Pi^{kl}_{ij}\delta R_{tktl} = 0, \tag{69}$$

$$\begin{aligned} \delta B_{ij} &\equiv \frac{1}{2}\Pi^{kl}_{ij}\epsilon_{tk\rho\sigma}\delta R^{\rho\sigma}{}_{tl} \\ &= \frac{\kappa n}{2}\left[L_{(i}\partial_{j)}\theta_{\ell}(z) - \frac{1}{3}\delta_{ij}\delta^{(1)}_{\ell}(z)\right]\delta^{(2)}_{\ell}(\rho), \end{aligned} \tag{70}$$

where we defined the projection operator

$$\Pi^{kl}_{ij} = \delta^{k}_{(i}\delta^{l}_{j)} - \frac{1}{3}\eta_{ij}\eta^{kl}, \tag{71}$$

which extracts the symmetric and traceless part of a rank-2 tensor.[4] This result for δB_{ij} precisely coincides with the discrepancy Ξ_{ij} found in Eq. (53) and thereby *manifestly restores* the exact GEM duality.

The same is true for the local case, as already worked out by Argurio and Dehouck [17]. We can recover their solution via the limiting procedure

[4]We did not calculate the full Weyl tensor for the modified Riemann tensor $R_{\mu\nu\rho\sigma}$ since it violates the equality $R_{\mu[\nu\rho\sigma]} = 0$ and has hence more irreducible pieces.

$$\delta B_{ij}^{\ell\to 0} = \frac{\kappa n}{2}\left[L_{(i}\partial_{j)}\theta(z) - \frac{1}{3}\delta_{ij}\delta^{(1)}(z)\right]\delta^{(2)}(\rho), \tag{72}$$

which is a distributional quantity nonvanishing only on the positive z-axis. Let us emphasize that in our nonlocal GF_1 model no such distributional quantities appear.

This construction shows that nonlocality, at the linear level, exacerbates the violation of GEM duality into regions away from the positive z-axis. However, as we just demonstrated, it can be restored *precisely* by the same procedure that is required in the local case.

C. A nonlocal BCHP model

Based on the successful application of the *local* BCHP model to the weak-field sector with smeared sources, we would like to propose the following nonlocal generalization of the BCHP model:

$$f(\triangle)G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{73}$$

$$3f(\triangle)R_{\mu[\nu\alpha\beta]} = -\kappa\epsilon_{\nu\alpha\beta\gamma}\Theta^{\gamma}_{\mu}, \tag{74}$$

$$R_{\mu\nu[\alpha\beta,\gamma]} = 0. \tag{75}$$

Here, $f(\triangle)$ is a nonlocal operator that satisfies $f(0) = 1$ and is formally given as a power series of the Laplace operator. Equivalently, due to their manifest GEM duality, we may write the field equations as

$$f(\triangle)\tilde{G}_{\mu\nu} = \kappa\Theta_{\mu\nu}, \tag{76}$$

$$3f(\triangle)\tilde{R}_{\mu[\nu\alpha\beta]} = +\kappa\epsilon_{\nu\alpha\beta\gamma}T^{\gamma}{}_{\mu}, \tag{77}$$

$$\tilde{R}_{\mu\nu[\alpha\beta,\gamma]} = 0. \tag{78}$$

Based on our previous considerations, the following metric is a manifestly self-dual solution in this framework:

$$\begin{aligned} h_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu &= \phi(\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \\ &\quad + 2A_x \mathrm{d}x\mathrm{d}t + 2A_y \mathrm{d}y\mathrm{d}t, \\ \phi &= \frac{2Gm}{r}\mathrm{erf}\left(\frac{r}{2\ell}\right), \qquad A_i = \kappa n\frac{\epsilon_{ijk}x^j L^k}{\rho^2}V, \\ V &= \frac{1}{4\pi} + z\mathcal{G}_3(r) - 2\ell^2\theta_\ell(z)\delta^{(2)}_\ell(\rho), \qquad L^\mu = \delta^\mu_z. \end{aligned} \tag{79}$$

It is sourced by the expressions

$$T_{\mu\nu} = m\delta^t_\mu\delta^t_\nu\delta^{(3)}(\boldsymbol{x}), \qquad \Theta_{\mu\nu} = n\delta^t_\mu\delta^t_\nu\delta^{(3)}(\boldsymbol{x}). \tag{80}$$

Interestingly, the restoration of GEM duality did not require any change in the structure of the metric functions or the source terms, and has solely been accomplished by a modification of the field equations. The price to pay was the interpretation of $R_{\mu\nu\rho\sigma}$ as a curvature tensor: since it no longer satisfies the algebraic Bianchi identity, it may perhaps be regarded as a torsionful curvature tensor [72]; see also Ref. [73].

Even though the explicit considerations of this paper are devoted to an understanding of the linearized Schwarzschild and NUT solutions, it is clear from the manifestly self-dual form of the nonlocal BCHP equations that similar relations hold for many other nonlocal solutions. In fact, two static solutions that are dual in the local BCHP model remain dual in the nonlocal extension.

D. Harnessing duality structures

In this last section we would like to briefly mention possible applications where the duality structures can be harnessed. To that end, recall that solutions with a given $\Theta_{\mu\nu}$-source can always be mapped into solutions of the regular Einstein equations with a $T_{\mu\nu}$-source. In other words, the modification term $\delta R_{\mu\nu\rho\sigma}$, as per Eq. (56), can be moved to the right-hand side and viewed as a contribution to the energy-momentum tensor,

$$\begin{aligned} \delta T_{\mu\nu} = -\frac{1}{4\kappa}(&\partial_\nu\epsilon_{\mu\alpha\beta\gamma}\Phi^{\alpha\beta\gamma} - \epsilon_{\mu\alpha\beta\gamma}\partial^\alpha\Phi^{\beta\gamma}_\nu \\ &+ \epsilon_{\mu\nu\alpha\beta}\partial^\alpha\Phi^\beta - \eta_{\mu\nu}\epsilon_{\alpha\beta\gamma\delta}\partial^\alpha\Phi^{\beta\gamma\delta}). \end{aligned} \tag{81}$$

Is this contribution always symmetric? The answer is yes, if and only if $\Theta_{\mu\nu}$ is symmetric, which we assume throughout in accordance with Ref. [15]. The easiest way to prove this is from considering the cyclic Bianchi identity (57), from which one may derive an antisymmetric part of the Riemann tensor

$$\begin{aligned} R_{\mu\nu\alpha\beta} - R_{\alpha\beta\mu\nu} &= \delta R_{\mu\nu\alpha\beta} - \delta R_{\alpha\beta\mu\nu} \\ &= -\frac{\kappa}{2}(\epsilon_{\mu\nu\alpha\lambda}\Theta^\lambda{}_\beta - \epsilon_{\mu\nu\beta\lambda}\Theta^\lambda{}_\alpha \\ &\quad - \epsilon_{\alpha\beta\mu\lambda}\Theta^\lambda{}_\nu + \epsilon_{\alpha\beta\nu\lambda}\Theta^\lambda{}_\mu). \end{aligned} \tag{82}$$

Note that the above expression vanishes for a pure Riemann tensor $r_{\mu\nu\alpha\beta}$, which is why this contribution is proportional to the gravitomagnetic source term $\Theta_{\mu\nu}$. It induces a potentially antisymmetric part to the Ricci tensor according to

$$\delta R_{[\mu\alpha]} = \eta^{\nu\beta}(\delta R_{\mu\nu\alpha\beta} - \delta R_{\alpha\beta\mu\nu}) = -\kappa\epsilon_{\mu\alpha\gamma\delta}\Theta^{\gamma\delta}. \tag{83}$$

However, since $\Theta_{\mu\nu} = \Theta_{\nu\mu}$, this antisymmetric part of the Ricci tensor modification vanishes. This constitutes an important consistency check of the resulting effective Einstein equations.

We can use this duality structure as follows. Start with the energy-momentum tensor $T_{\mu\nu}$ of a seed metric, of which the solution to the nonlocal Einstein equations is known. Then, by means of the duality, set $\Theta_{\mu\nu} = T_{\mu\nu}$ and use the relations above to determine the resulting energy-momentum tensor from that choice. The solution of the resulting nonlocal Einstein equation will yield the dual solution for the original seed metric.

In the context of our previous example, we began with a point particle solution where $T_{\mu\nu} \sim \delta^t_\mu \delta^t_\nu \delta^{(3)}(\boldsymbol{x})$. The weak-field solution is the nonlocal Schwarzschild metric. Then, one may stipulate instead a monopole source of the same form, $\Theta_{\mu\nu} \sim \delta^t_\mu \delta^t_\nu \delta^{(3)}(\boldsymbol{x})$, which gives rise to nonvanishing components δT_{ti} with $i = x, y$. Then, the resulting Einstein equations are solved by the massless NUT solution. Hence the interesting features of the BCHP model and its nonlocal extension therefore lie in the clever distribution of matter sources in the field equations, whereas the differential properties of the field equations remain essentially unchanged.

While a systematic survey of self-dual nonlocal solutions is beyond the scope of this paper we believe that the tools presented here serve as an ideal starting point for such inquiries.

V. CONCLUSIONS

In this paper we have studied the fate of GEM duality for weak-field nonlocal gravity. As a testing ground, we considered the gravitational field of a point particle (Schwarzschild solution, "gravitoelectric monopole") and a semi-infinite spinning string (massless NUT solution, "gravitomagnetic monopole"). In the case of linearized general relativity, these solutions are dual to each other everywhere except on the positive z-axis, where the duality is violated explicitly by distributional expressions. Since the realm of violation coincides with the location of matter sources, it may still be regarded as exact.

In this paper we showed that nonlocality smears this violation of exact GEM duality to finite transverse distances away from the z-axis, the characteristic scale being the scale of nonlocality ℓ. In other words: nonlocality *spoils* any exact GEM duality.

Viewed from a different perspective, the existence of δ-sources in general relativity has long been an active field of investigation; see the seminal work by Geroch and Traschen [74], or the more recent discussion by Pantoja and Rago [75]. Here we demonstrated that nonlocality can serve as a regulator that turns distributional expressions (δ-functions and derivatives thereof) into smooth functions. We emphasized this feature by introducing a notion of emergent δ-functions and Heaviside functions. In the limiting case of $\ell \to 0$, we recover the results of linearized general relativity.

However, since the GEM duality is not exact even in linearized general relativity due to distributional quantities on the positive z-axis, Bunster *et al.* [15] developed a manifestly dual set of gravitational field equations that involves an additional gravitational source term. Applying their model to the nonlocal setup with smeared matter sources, we demonstrated that this procedure indeed *solves the duality problem* in the class of nonlocal theories under consideration in this paper. In our calculations we relied heavily on the notion of effective δ-functions, which in the mathematical literature are sometimes referred to as *nascent δ-functions*: these functions depend on the scale of nonlocality $\ell > 0$, and reduce to their usual behavior in the limiting case of $\ell \to 0$.

Last, guided by the successful adoption of the local gravitational model by Bunster *et al.* to the nonlocal case, we extended their field equations to a nonlocal model by including infinite-derivative nonlocal form factors. We demonstrated that this nonlocal model maps dual solutions of the local theory into dual solutions of the nonlocal theory, which significantly extends the conclusions from the simple nonlocal Schwarzschild–NUT duality to far more general scenarios. Finally, we commented on how this self-duality structure of our nonlocal model can be employed to construct dual solutions to well-known nonlocal geometries.

Even though the considerations presented in this paper are only applicable to the weak-field regime, they present an important consistency check of nonlocal infinite-derivative gravity. Nonlocality, as a possible ultraviolet completion of gravity, does not pose a fundamental impediment to duality structures at the linear level. In close proximity to matter sources one may expect that the full, nonlinear nonlocal theory will lead to further modifications of GEM dualities, but we shall leave that discussion open for the future.

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APPENDIX: DERIVATION OF THE NUT SOLUTION

The NUT solution for weak-field nonlocal gravity has been constructed via Laplace transform methods in Ref. [52]. Here we would like to briefly delineate a possibly simpler derivation of the NUT solution that simultaneously extends to more general nonlocal theories of the $\mathrm{GF_N}$ type. Let us recall the energy-momentum tensor of the NUT source,

$$T^{\mathrm{NUT}}_{\mu\nu} = -\delta^t_{(\mu}\delta^i_{\nu)} n^j_i \partial_j \delta^{(1)}(x)\delta^{(1)}(y)\theta(z),$$

with $n = n_{xy} = -n_{yx}$. Inserting the following ansatz into the stationary field equations (21),

$$h_{\mu\nu}^{\rm NUT} dx^\mu dx^\nu = 2A_x dx dt + 2A_y dy dt, \tag{A1}$$

and differentiating with respect to z one finds

$$e^{(-\ell^2\triangle)^N} \triangle A_i' = n_i^j \partial_j \delta^{(3)}(\boldsymbol{x}), \qquad A_i' \equiv \partial_z A_i. \tag{A2}$$

This is solved by a rotating solution, recently discussed in Ref. [51], taking the form

$$A_i' = -\kappa n_i^j \partial_j \mathcal{G}_3(r), \qquad r^2 = x^2 + y^2 + z^2, \tag{A3}$$

$$e^{(-\ell^2\triangle)^N} \triangle \mathcal{G}_3(\boldsymbol{x}) = -\delta^{(3)}(\boldsymbol{x}). \tag{A4}$$

The form of $\mathcal{G}_3(r)$ is known for various N and has been given in the literature, see e.g., Ref. [47]. The final solution is hence obtained via integration over z,

$$A_i = -\kappa n_i{}^j \partial_j \int_{-\infty}^{z} d\tilde{z}\, \mathcal{G}_3\left(\sqrt{x^2 + y^2 + \tilde{z}^2}\right). \tag{A5}$$

In the simplest case for $N = 1$ this integral can be performed analytically and one precisely recovers Eq. (49). Employing the recursion relation (28) for nonlocal static Green functions one may write the equivalent

$$A_i = 2\pi\kappa\, n_{ij} x^j \int_{-\infty}^{z} d\tilde{z} \mathcal{G}_5\left(\sqrt{x^2 + y^2 + \tilde{z}^2}\right). \tag{A6}$$

[1] R. Bach, Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des Krümmungsbegriffs, Math. Z. **9**, 110 (1921); English Translation: 'On Weyl's theory of relativity and Weyl's extension of the notion of curvature'.

[2] G. Y. Rainich, Electricity in curved space-time, Nature (London) **115**, 498 (1925).

[3] A. Einstein, Über die formale Beziehung des Riemannschen Krümmungstensors zu den Feldgleichungen der Gravitation, Math. Ann. **97**, 99 (1927); English Translation: 'On the formal relation between the Riemannian curvature tensor and the field equations of gravity'.

[4] C. Lanczos, A remarkable property of the Riemann–Christoffel tensor in four dimensions, Ann. Math. **39**, 842 (1938).

[5] C. Lanczos, The splitting of the Riemann tensor, Rev. Mod. Phys. **34**, 379 (1962).

[6] H. S. Ruse, The self-polar Riemann complex for a V4, Proc. London Math. Soc. **50**, 75 (1944).

[7] A. Matte, Sur de nouvelles solutions oscillatoires des équations de la gravitation, Can. J. Math. **5**, 1 (1953); English Translation: 'On the new oscillatory solutions of the equations of gravitation'.

[8] L. Bel, Les états de radiation et le problme de l'énergie en relativité générale, Cahiers de Physique **16**, 59 (1962); English Translation: 'Radiation states and the problem of energy in general relativity'.

[9] L. Bel, Radiation states and the problem of energy in general relativity, Gen. Relativ. Gravit. **32**, 2047 (2000); English Translation of the French original of 1962.

[10] R. Maartens and B. A. Bassett, Gravitoelectromagnetism, Classical Quantum Gravity **15**, 705 (1998).

[11] B. Mashhoon, J. C. McClune, and H. Quevedo, The gravitoelectromagnetic stress energy tensor, Classical Quantum Gravity **16**, 1137 (1999).

[12] J. M. M. Senovilla, General electric magnetic decomposition of fields, positivity and Rainich-like conditions, in *Spanish Relativity Meeting on Reference Frames and Gravitomagnetism (EREs2000)* (2000) [arXiv:gr-qc/0010095].

[13] L. F. O. Costa and J. Natario, Gravito-electromagnetic analogies, Gen. Relativ. Gravit. **46**, 1792 (2014).

[14] M. Henneaux and C. Teitelboim, Duality in linearized gravity, Phys. Rev. D **71**, 024018 (2005).

[15] C. W. Bunster, S. Cnockaert, M. Henneaux, and R. Portugues, Monopoles for gravitation and for higher spin fields, Phys. Rev. D **73**, 105014 (2006).

[16] G. Barnich and C. Troessaert, Manifest spin 2 duality with electric and magnetic sources, J. High Energy Phys. 01 (2009) 030.

[17] R. Argurio and F. Dehouck, Gravitational duality and rotating solutions, Phys. Rev. D **81**, 064010 (2010).

[18] G. Castillo and J. Sanchez, Gravitomagnetic mass in the linearized einstein theory, Rev. Mex. Fis. **42**, 498 (1996).

[19] J. Boos, Plebański–Demiański solution of general relativity and its expressions quadratic and cubic in curvature: Analogies to electromagnetism, Int. J. Mod. Phys. D **24**, 1550079 (2015).

[20] R. A. Hennigar, D. Kubizňák, and R. B. Mann, Thermodynamics of Lorentzian Taub-NUT spacetimes, Phys. Rev. D **100**, 064055 (2019).

[21] Z. Bern, J. J. M. Carrasco, and H. Johansson, Perturbative Quantum Gravity as a Double Copy of Gauge Theory, Phys. Rev. Lett. **105**, 061602 (2010).

[22] R. Monteiro, D. O'Connell, and C. D. White, Black holes and the double copy, J. High Energy Phys. 12 (2014) 056.

[23] A. Luna, R. Monteiro, D. O'Connell, and C. D. White, The classical double copy for Taub–NUT spacetime, Phys. Lett. B **750**, 272 (2015).

[24] N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Planté, and P. Vanhove, General Relativity from Scattering Amplitudes, Phys. Rev. Lett. **121**, 171601 (2018).

[25] N. Arkani-Hamed, Y.-t. Huang, and D. O'Connell, Kerr black holes as elementary particles, J. High Energy Phys. 01 (2020) 046.

[26] A. D. Sakharov, Vacuum quantum fluctuations in curved space and the theory of gravitation, Dokl. Akad. Nauk Ser. Fiz. **177**, 70 (1967).

[27] Y.-T. Huang, U. Kol, and D. O'Connell, Double copy of electric-magnetic duality, Phys. Rev. D **102**, 046005 (2020).

[28] K. S. Stelle, Renormalization of higher-derivative quantum gravity, Phys. Rev. D **16**, 953 (1977).

[29] A. Pais and G. E. Uhlenbeck, On field theories with non-localized action, Phys. Rev. **79**, 145 (1950).

[30] G. V. Efimov, Non-local quantum theory of the scalar field, Commun. Math. Phys. **5**, 42 (1967).

[31] G. V. Efimov, On a class of relativistic invariant distributions, Commun. Math. Phys. **7**, 138 (1968).

[32] G. V. Efimov, On the construction of nonlocal quantum electrodynamics, Ann. Phys. (N.Y.) **71**, 466 (1972).

[33] N. V. Krasnikov, Non-local gauge theories, Theor. Math. Phys. **73**, 1184 (1987).

[34] Y. V. Kuzmin, The convergent non-local gravitation (in Russian), Sov. J. Nucl. Phys. **50**, 1011 (1989).

[35] L. Modesto, Super-renormalizable quantum gravity, Phys. Rev. D **86**, 044005 (2012).

[36] T. Biswas, E. Gerwick, T. Koivisto, and A. Mazumdar, Towards Singularity and Ghost-Free Theories of Gravity, Phys. Rev. Lett. **108**, 031101 (2012).

[37] E. Kilicarslan, pp-waves as exact solutions to ghost-free infinite-derivative gravity, Phys. Rev. D **99**, 124048 (2019).

[38] S. Dengiz, E. Kilicarslan, I. Kolář, and A. Mazumdar, Impulsive waves in ghost-free infinite-derivative gravity in anti-de Sitter spacetime, Phys. Rev. D **102**, 044016 (2020).

[39] T. Biswas, A. Mazumdar, and W. Siegel, Bouncing universes in string-inspired gravity, J. Cosmol. Astropart. Phys. 03 (2006) 009.

[40] T. Biswas, T. Koivisto, and A. Mazumdar, Towards a resolution of the cosmological singularity in non-local higher-derivative theories of gravity, J. Cosmol. Astropart. Phys. 11 (2010) 008.

[41] I. Kolář, T. Málek, and A. Mazumdar, Exact solutions of non-local gravity in class of almost universal spacetimes, arXiv:2103.08555 [Phys. Rev. D (to be published)].

[42] V. P. Frolov, A. Zelnikov, and T. de Paula Netto, Spherical collapse of small masses in the ghost-free gravity, J. High Energy Phys. 06 (2015) 107.

[43] V. P. Frolov, Mass-Gap for Black Hole Formation in Higher Derivative and Ghost-Free Gravity, Phys. Rev. Lett. **115**, 051102 (2015).

[44] V. P. Frolov and A. Zelnikov, Head-on collision of ultra-relativistic particles in ghost-free theories of gravity, Phys. Rev. D **93**, 064048 (2016).

[45] V. P. Frolov and A. Zelnikov, Radiation from an emitter in the ghost-free scalar theory, Phys. Rev. D **93**, 105048 (2016).

[46] J. Edholm, A. S. Koshelev, and A. Mazumdar, Behavior of the Newtonian potential for ghost-free gravity and singularity-free gravity, Phys. Rev. D **94**, 104033 (2016).

[47] J. Boos, V. P. Frolov, and A. Zelnikov, Gravitational field of static p-branes in linearized ghost-free gravity, Phys. Rev. D **97**, 084021 (2018).

[48] L. Buoninfante, A. S. Koshelev, G. Lambiase, J. Marto, and A. Mazumdar, Conformally-flat, non-singular static metric in infinite-derivative gravity, J. Cosmol. Astropart. Phys. 06 (2018) 014.

[49] L. Buoninfante, G. Harmsen, S. Maheshwari, and A. Mazumdar, Nonsingular metric for an electrically charged point-source in ghost-free infinite-derivative gravity, Phys. Rev. D **98**, 084009 (2018).

[50] J. Boos, Angle deficit and non-local gravitoelectromagnetism around a slowly spinning cosmic string, Int. J. Mod. Phys. D **29**, 2043027 (2020).

[51] J. Boos, J. Pinedo Soto, and V. P. Frolov, Ultrarelativistic spinning objects in non-local ghost-free gravity, Phys. Rev. D **101**, 124065 (2020).

[52] I. Kolář and A. Mazumdar, NUT charge in linearized infinite-derivative gravity, Phys. Rev. D **101**, 124005 (2020).

[53] I. Kolář and J. Boos, Retarded field of a uniformly accelerated source in non-local scalar field theory, Phys. Rev. D **103**, 105004 (2021).

[54] J. Droste, The field of a single centre in Einstein's theory of gravitation, and the motion of a particle in that field, Gen. Relativ. Gravit. **34**, 1545 (2002), English Translation of the Dutch original of 1916.

[55] K. Schwarzschild, Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl. **189**, 189 (1916); (English Translation: 'On the gravitational field of a mass point according to Einstein's theory').

[56] A. H. Taub, Empty space-times admitting a three parameter group of motions, Ann. Math. **53**, 472 (1951).

[57] E. Newman, L. Tamburino, and T. Unti, Empty space generalization of the Schwarzschild metric, J. Math. Phys. (N.Y.) **4**, 915 (1963).

[58] C. W. Misner, The flatter regions of Newman, Unti and Tamburino's generalized Schwarzschild space, J. Math. Phys. (N.Y.) **4**, 924 (1963).

[59] J. S. Dowker and J. A. Roche, The gravitational analogues of magnetic monopoles, Proc. Phys. Soc. **92**, 1 (1967).

[60] W. B. Bonnor, A new interpretation of the nut metric in general relativity, Math. Proc. Cambridge Philos. Soc. **66**, 145 (1969).

[61] A. Sackfield, Physical interpretation of N. U. T. metric, Math. Proc. Cambridge Philos. Soc. **70**, 89 (1971).

[62] W. B. Bonnor, Physical interpretation of vacuum solutions of Einstein's equations. Part i. Time-independent solutions, Gen. Relativ. Gravit. **24**, 551 (1992).

[63] V. S. Manko and E. Ruiz, Physical interpretation of the NUT family of solutions, Classical Quantum Gravity **22**, 3555 (2005).

[64] J. B. Griffiths and J. Podolský, *Exact Space-Times in Einstein's General Relativity*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, Cambridge, England, 2009).

[65] G. Clément, D. Gal'tsov, and M. Guenouche, Rehabilitating space-times with NUTs, Phys. Lett. B **750**, 591 (2015).

[66] S. Gera and S. Sengupta, Taming Dirac strings and timelike loops in vacuum gravity, Phys. Rev. D **99**, 124038 (2019).

[67] A. B. Bordo, F. Gray, R. A. Hennigar, and D. Kubizňák, Misner gravitational charges and variable string strengths, Classical Quantum Gravity **36**, 194001 (2019).

[68] I. Kolář and P. Krtouš, Symmetry axes of Kerr-NUT-(A)dS spacetimes, Phys. Rev. D **100**, 064014 (2019).
[69] B. L. Giacchini and T. de Paula Netto, Effective delta sources and regularity in higher-derivative and ghost-free gravity, J. Cosmol. Astropart. Phys. 07 (2019) 013.
[70] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations*, 2nd ed. (Cambridge University Press, Cambridge, England, 2003).
[71] D. G. B. Edelen, A metric-free electrodynamics with electric and magnetic charges, Ann. Phys. (N.Y.) **112**, 366 (1978).
[72] F. W. Hehl, P. Von Der Heyde, G. D. Kerlick, and J. M. Nester, General relativity with spin and torsion: Foundations and prospects, Rev. Mod. Phys. **48**, 393 (1976).
[73] U. Kol, Dual Komar mass, torsion and Riemann–Cartan manifolds, arXiv:2010.07887.
[74] R. P. Geroch and J. H. Traschen, Strings and other distributional sources in general relativity, Phys. Rev. D **36**, 1017 (1987).
[75] N. R. Pantoja and H. Rago, Distributional sources in general relativity: Two point-like examples revisited, Int. J. Mod. Phys. D **11**, 1479 (2002).