Generalized split monopole magnetospheres: The effects of current sheets

Huiquan Li[*] and Jiancheng Wang

Yunnan Observatories, Chinese Academy of Sciences, 650216 Kunming, China, Key Laboratory for the Structure and Evolution of Celestial Objects, Chinese Academy of Sciences, 650216 Kunming, China and Center for Astronomical Mega-Science, Chinese Academy of Sciences, 100012 Beijing, China

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We show that energy should be dissipated or extracted in the current sheet (CS) of a split magnetosphere deviating from the Michel split monopole, with the CS heating up or cooling down; but the electromagnetic energy remains unchanged everywhere. Based on the decentered monopole solution generated by symmetry in flat spacetime, we construct two generalized split monopole configurations, in which the field lines intersect with the CS at arbitrary angles. One configuration resembles the outer geometry of the so-called "new pulsar magnetosphere model", for which up to 47% of the spin down energy is transferred to the Joule heating process in the CS. In the other configuration, we observe that negative energy is dissipated in the CS, which is usually observed in magnetospheres on rotating black holes. This means that energy is extracted simultaneously from the central star and the CS to power the output Poynting flux at infinity. We interpret the extraction of energy from the CS as that thermal energy of charged particles in the CS is transferred to the ordered kinetic energy of these particles drifting in the force-free (FF) electromagnetic fields. Hence, the CS is like an "air conditioner" in the sky, which can heat up or cool down, depending on the configurations .

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I. INTRODUCTION

To avoid the appearance of a magnetic monopole, the splitting technique by introducing an equatorial current sheet (CS) is usually adopted in constructing a monopole magnetosphere on a compact object. In the popular pulsar magnetosphere model at present, the near-star region is of a dipole structure [1,2]. The magnetic field lines around the poles can extend beyond the light cylinder (LC) to infinity. They asymptotically approach a split monopole in the outer region, whose analytical solution is the one found by Michel [3]. This split monopole results from a splitting and then gluing of two centered monopoles with opposite magnetic charge. The discontinuity of the fields across the equator gives rise to a singular CS. In the Michel split monopole, the surface of the CS is parallel to the magnetic field lines neighbor to the equator. So the Lorentz force vanishes and no dissipation occurs in the CS. This structure is part of the standard pulsar magnetosphere model first realized numerically by [4].

But this splitting method is not unique to obtain a split monopole. It is possible that the magnetic field lines are not necessarily parallel to the infinitely thin CS [5–7]. This modification may cause nontrivial effects. As shown in the numerical solutions [5,6], the nonparallel splitting leads to a CS where the spin down energy is dissipated. It is unclear wether this dissipation process consumes the electromagnetic energy in the CS. If it does (like the magnetic reconnection case), the magnetosphere should evolve in time.

In some other numerical simulations on rotating black holes, an alternative role of the CS is explored. In [8–11], it is found that the energy dissipated in the CS developed within the ergosphere is gained by the force-free (FF) fields to power the jet formation. The origin of the negative dissipation energy remains vague. It looks like that this phenomenon is specific to a gravitational system.

In this work, we construct two analytical split monopole models that the magnetic field lines intersect with the infinitely thin CS at arbitrary angles. Using these exact configurations, we can clarify the current and energy flows in the systems in detail, and specify the precise effects of the CS. The paper is organized as follows. In terms of the translational freedom mentioned in Sec. II, we give the general monopole solution whose center can be shifted along the spin axis in Sec. III. In Sec. IV we present the generalized split monopole configurations based on the decentered solution and discuss the effects of the CS by calculating the exact amount of flows in them. Finally, we summarize and discuss in the last section.

[*]lhq@ynao.ac.cn

II. TRANSLATION OF THE MAGNETOSPHERE

We consider the force-free magnetosphere on an axisymmetric rotator. The fields satisfy the following FF condition

$$\rho\vec{E} + \vec{j} \times \vec{B} = 0. \tag{1}$$

This condition implies $\vec{j} \cdot \vec{E} = 0$ and $\vec{E} \cdot \vec{B} = 0$.

Under this condition, the Maxwell's equations can be reduced to a simple system described by three correlated functions: the flux ψ, the angular velocity $\Omega(\psi)$ of field lines, and the poloidal electric current $I(\psi)$. In terms of them, the electromagnetic fields in the unit basis of spherical coordinates can be expressed as

$$\vec{E} = -\vec{V}_\phi \times \vec{B} = -\frac{\Omega(\psi)}{r}(r\partial_r\psi, \partial_\theta\psi, 0), \tag{2}$$

$$\vec{B} = \frac{1}{r^2 \sin\theta}(\partial_\theta\psi, -r\partial_r\psi, rI(\psi)), \tag{3}$$

where $V_\phi = r\sin\theta\Omega$. The charge and current densities are respectively

$$\rho = -\frac{1}{4\pi}\vec{\nabla} \cdot (\Omega\vec{\nabla}\psi), \tag{4}$$

$$\vec{j} = \rho r \sin\theta\Omega\vec{e}_\phi + \frac{1}{4\pi}I'\vec{B}. \tag{5}$$

With the above relations and equations, we arrive at the so-called force-free pulsar magnetosphere equation. By redefining $z = r\cos\theta$ and $x = r\sin\theta$, we express the equation in cylindrical coordinates as

$$(1 - \Omega^2x^2)(\partial_x^2\psi + \partial_z^2\psi) - \frac{1}{x}(1 + \Omega^2x^2)\partial_x\psi + (\partial_z\psi)^2] = -I(\psi)I'(\psi), \tag{6}$$

where the primes stand for the derivative with respect to ψ.

It is easy to see that the differential equation is invariant under the shift along the symmetry axis,

$$z \to z' = z - \epsilon. \tag{7}$$

So any solution ψ shifted along the rotation axis is still a solution to the pulsar equation (6). Under this translation, the functional relations $\Omega(\psi)$, $I(\psi)$, and the global features are kept the same. In what follows, we consider the translated version of Michel's monopole solution.

III. DECENTERED MONOPOLE SOLUTION

The exact monopole solution found by Michel [3] is quite simple, only relying on the angle θ in spherical coordinates,

$$\psi(\theta) = -q\cos\theta = -q\frac{z}{\sqrt{x^2 + z^2}}. \tag{8}$$

For an arbitrary $\Omega(\psi)$, the electric current is

$$I(\psi) = \frac{1}{q}\Omega(\psi)(\psi^2 - q^2), \tag{9}$$

where q is the charge of the monopole. The solution gives rise to a magnetosphere with magnetic domination and null current.

The above solution (8) under the translation (7) becomes

$$\psi(r,\theta) = -q\frac{z - \epsilon}{\sqrt{x^2 + (z - \epsilon)^2}} = -q\frac{r\cos\theta - \epsilon}{\sqrt{r^2 - 2\epsilon r\cos\theta + \epsilon^2}}. \tag{10}$$

The solution is now dependent on both poloidal coordinates in the spherical coordinates. The functional relations between $I(\psi)$, $\Omega(\psi)$, and ψ remain invariant, the same as shown in Eq. (9). The solution describes a monopole magnetosphere whose center is shifted away from the origin (the center of the star) along the rotation axis by a distance ϵ, which generalizes the coincident Michel's solution.

For the solution, the electromagnetic fields are

$$\vec{E} = \frac{qr\sin\theta\Omega}{D^3}(\epsilon\sin\theta, -(r - \epsilon\cos\theta), 0), \tag{11}$$

$$\vec{B} = \frac{q}{D^3}(r - \epsilon\cos\theta, \epsilon\sin\theta, -r\sin\theta\Omega D), \tag{12}$$

where $D = \sqrt{r^2 - 2\epsilon r\cos\theta + \epsilon^2}$. Thus, the invariant is

$$\vec{B}^2 - \vec{E}^2 = \frac{q^2}{D^4} > 0, \tag{13}$$

so it is also magnetically dominated.

The relations $B_\phi = E_\theta = V_\phi B_r$ for the original Michel's solution are replaced by

$$B_\phi = -\sqrt{E_r^2 + E_\theta^2} = V_\phi\sqrt{B_r^2 + B_\theta^2}. \tag{14}$$

But, at large distances $r \gg |\epsilon|$, the former is still a good approximation.

The Poynting flux is

$$\vec{S} = \frac{1}{4\pi}\vec{E} \times \vec{B} = \frac{(qr\sin\theta\Omega)^2}{4\pi D^5}\left(r - \epsilon\cos\theta, \epsilon\sin\theta, \frac{D}{r\sin\theta\Omega}\right). \tag{15}$$

From this, we can find that the drift velocity $\vec{v}_D = 4\pi\vec{S}/B^2$ gets a nonvanishing component at the θ direction. The four-current is

$$J^\mu \equiv (\rho, \vec{j}) = -\frac{q\Omega(r\cos\theta - \epsilon)}{2\pi D^4}(D, r - \epsilon\cos\theta, \epsilon\sin\theta, 0). \tag{16}$$

So it is null with $J^2 = 0$, meaning the particle travels at speed of light. This is the same as Michel's centered solution. But the null surface where the charge density and current vanish is not located on the equator any more, but on the plane shifted by a distance ϵ; $z = r\cos\theta = \epsilon$. It is noticed that the poloidal components of the magnetic field, Poynting flux, the drift velocity and the current are all parallel to each other.

IV. GENERALIZED SPLIT MONOPOLES

Since the magnetic monopole has not yet been confirmed, the splitting technique is usually adopted in constructing a monopole magnetosphere. For the Michel solution, the split monopole configuration on the two half-planes is expressed as

$$\psi(\theta) = \begin{cases} q(1 - \cos\theta), & \theta \in [0, \pi/2) \\ q(1 + \cos\theta). & \theta \in (\pi/2, \pi]. \end{cases} \tag{17}$$

The splitting results in a discontinuity on the equatorial plane, giving rise to an infinitely thin CS there. In this case, the magnetic field lines are parallel to the surface of the CS.

With the decentered solution (10), we make a similar splitting. We denote the solution on the upper half-plane by $\psi_\vee$ and the one on the lower hemisphere by $\psi_\vee$. On the upper hemisphere $\theta \in [0, \pi/2)$, we choose $\epsilon = \pm d$ ($d > 0$) and express the solution as

$$\psi_\vee^{(\pm)} = q\left(1 - \frac{r\cos\theta \pm d}{\sqrt{r^2 \pm 2dr\cos\theta + d^2}}\right). \tag{18}$$

On the lower one $\theta \in (\pi/2, \pi]$, the solution is

$$\psi_\wedge^{(\pm)} = q\left(1 + \frac{r\cos\theta \pm d}{\sqrt{r^2 \pm 2dr\cos\theta + d^2}}\right). \tag{19}$$

There are two trivial cases: $\psi = (\psi_\vee^{(+)}, \psi_\wedge^{(+)})$ and $\psi = (\psi_\vee^{(-)}, \psi_\wedge^{(-)})$, which just describe decentered versions of the Michel split monopole magnetosphere, shifted as a whole respectively downward and upward by a distance d. We are more interested in the two nontrivial configurations that will be discussed as follows.

A. $\psi = (\psi_\vee^{(-)}, \psi_\wedge^{(+)})$

This configuration is shown in the left panel of Fig. 1. The profile of the configuration resembles the outer geometry in the "new pulsar magnetosphere model" obtained numerically in [6]. The valid region of the configuration is restricted to be $r > d$, where the integral of the magnetic field over any closed surface leads to zero magnetic charge.

The expanded form of the split solution can be obtained in terms of the general expanded form in the Appendix. It is clearly seen that the full expanded solution of this spit monopole in the outer range $d/r < 1$ is a summation of a

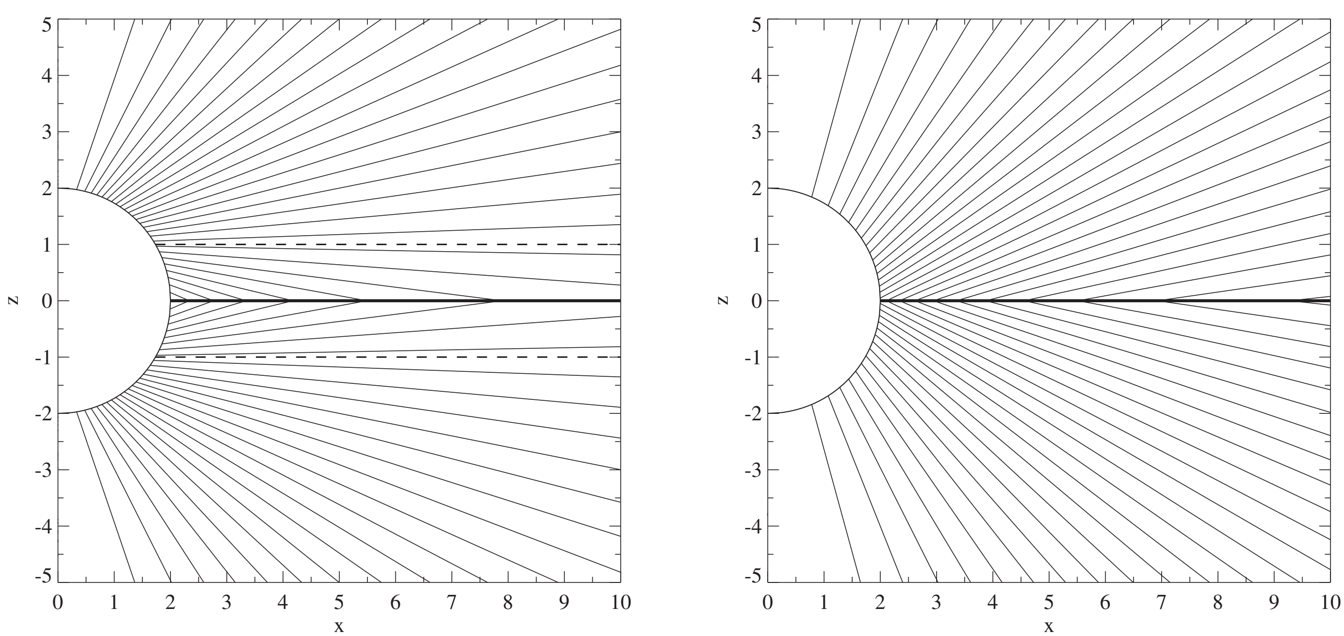

FIG. 1. Magnetic field lines for the configurations defined in Sec. IV A (left) and IV B (right) with $q = 1$, $r_0 = 2$ and $d = 1$. The bold lines on the equators represent the infinitely thin CS. The dashed lines on the left panel represent the null surfaces (with $\rho = 0$), across which the charge density ρ changes sign. In a realistic pulsar magnetosphere model, these split monopole configurations are relevant only to the geometry outside the LC.

closed and an open field line component; the odd order terms of r are continuous across the equatorial plane, contributing to a closed field line component, while the even order terms are discontinuous, contributing an open field line component. The discontinuity in the latter gives rise to a current sheet on the equatorial plane.

For $d/r < 1$, the solution expanded to the order $\mathcal{O}(d^2/r^2)$ is

$$\psi \simeq \begin{cases} q[1-\cos\theta + d\sin^2\theta r^{-1} + (3d^2/2)\cos\theta\sin^2\theta r^{-2}], & \theta\in[0,\pi/2) \\ q[1+\cos\theta + d\sin^2\theta r^{-1} - (3d^2/2)\cos\theta\sin^2\theta r^{-2}] & \theta\in(\pi/2,\pi]. \end{cases} \tag{20}$$

At infinity $r\to\infty$, the solution is asymptotically the Michel split monopole solution. In the near region with small d/r (but not too small), the dipole part becomes more important. Besides, there are two null surfaces in this split monopole solution. This is coincident to the case of the corotating dipole inside the LC in the standard pulsar magnetosphere model, where two straight null surfaces extend from the star surface to the LC. Hence, the new split solution here can better describe a smooth transition from a dipole to a monopole. By adjusting the parameter d, the two null lines in the corotating dipole and in the new split monopole can be matched. In particular, the expanded solution (20) is exactly the solution outside the light torus of the exact dipole magnetosphere [12].

Let us now examine the dynamical consequence in this configuration. From the solution, the force-free fields approaching the equator $\theta\to\pi/2$ from either side are given by

$$\vec{E}_\vee \to \frac{qr\Omega}{(r^2+d^2)^{\frac{3}{2}}}(d,-r,0), \tag{21}$$

$$\vec{E}_\wedge \to \frac{qr\Omega}{(r^2+d^2)^{\frac{3}{2}}}(d,r,0), \tag{22}$$

$$\vec{B}_\vee \to \frac{q}{(r^2+d^2)^{\frac{3}{2}}}\left(r,d,-r\Omega\sqrt{r^2+d^2}\right), \tag{23}$$

$$\vec{B}_\wedge \to \frac{q}{(r^2+d^2)^{\frac{3}{2}}}\left(-r,d,r\Omega\sqrt{r^2+d^2}\right). \tag{24}$$

We denote the continuous fields at the equator as

$$E_c^r \equiv E_\vee^r\left(\theta\to\frac{\pi}{2}\right) = E_\wedge^r\left(\theta\to\frac{\pi}{2}\right),$$
$$B_c^\theta \equiv B_\vee^\theta\left(\theta\to\frac{\pi}{2}\right) = B_\wedge^\theta\left(\theta\to\frac{\pi}{2}\right). \tag{25}$$

They are the field components that are nonvanishing within the CS. The other field components are discontinuous.

The discontinuity of the perpendicular electric field E^θ leads to the surface charge density in the CS,

$$\sigma_c = \frac{qr^2\Omega}{2\pi(r^2+d^2)^{\frac{3}{2}}}. \tag{26}$$

The discontinuities of the parallel magnetic fields B^r and B^ϕ give rise to the surface current densities flowing in the CS respectively along the r and ϕ directions:

$$i_c^\phi = \frac{qr}{2\pi(r^2+d^2)^{\frac{3}{2}}}, \qquad i_c^r = \frac{qr\Omega}{2\pi(r^2+d^2)}. \tag{27}$$

In the FF regions, the total change rate of the charges through the sphere (excluding the equator) at radius r is

$$\dot{Q}^{FF}(r) = \int_\vee j_\vee^r ds^r + \int_\wedge j_\wedge^r ds^r$$
$$= [I_\vee(\psi)]_{\theta=0}^{\theta=\pi/2} + [I_\wedge(\psi)]_{\theta=\pi/2}^{\theta=\pi}, \tag{28}$$

where $ds^r = 2\pi r^2\sin\theta d\theta$ and the dot denotes the derivative with respect to time. The change rate through the section of the CS at r is

$$\dot{Q}^{CS}(r) = 2\pi r i_c^r. \tag{29}$$

Thus, it is justified that the total electric current flowing through a sphere at any radius r is zero;

$$\dot{Q}^{FF}(r) + \dot{Q}^{CS}(r) = 0. \tag{30}$$

This implies that the central star always remains neutral.

We take the value $\dot{Q}^{CS}(r=r_0)$ at some initial radius r_0 ($> d$) as the current directly from the central star and the one $\dot{Q}^{CS}(r\to\infty)$ at infinity as the output current. From the second equation of Eq. (27), the latter is given by

$$\dot{Q}^{CS}(r\to\infty) = q\Omega. \tag{31}$$

It is the same as the Michel split monopole.

Towards the equator, the perpendicular electric currents along the FF magnetic fields are

$$j_\vee^\theta\left(\theta\to\frac{\pi}{2}\right) = -j_\wedge^\theta\left(\theta\to\frac{\pi}{2}\right) = \frac{q\Omega d^2}{2\pi(r^2+d^2)^2}. \tag{32}$$

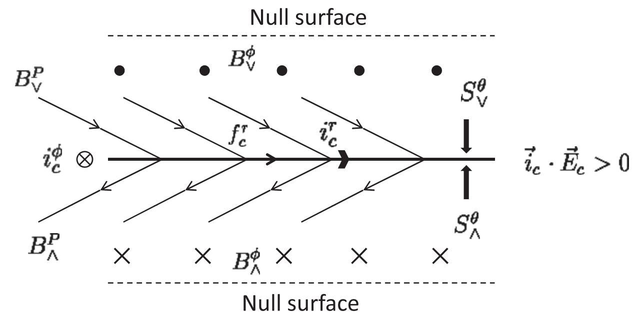

FIG. 2. The field structure and the energy flows near the CS outside the LC of the configuration in Sec. IV A. The electric field lines (not displayed) are perpendicular to the poloidal magnetic field lines B^P. A net Poynting flux flows into the CS from the upper and lower sides to provide the energy dissipated in the CS.

This means that there is a net electric current flowing into the CS from both the upper and the lower sides. Including the injected current at r_0 and the output current at $r \to \infty$, we find

$$\dot{Q}^{CS}(r = r_0) + \int_{r_0}^{\infty} 2\pi r \left[j^\theta_\vee \left(\theta \to \frac{\pi}{2} \right) - j^\theta_\wedge \left(\theta \to \frac{\pi}{2} \right) \right] dr = \dot{Q}^{CS}(r \to \infty). \tag{33}$$

This equation says that the output CS current comes from directly the central star and the FF fields.

With the continuous fields, we obtain the nonvanishing Lorentz force densities in the CS

$$f^r_c = \sigma_c E^r_c - i^\phi_c B^\theta_c = \frac{q^2 r d (r^2 \Omega^2 - 1)}{2\pi (r^2 + d^2)^3}, \tag{34}$$

$$f^\phi_c = i^r_c B^\theta_c = \frac{q^2 r d \Omega}{2\pi (r^2 + d^2)^{\frac{5}{2}}}. \tag{35}$$

Both tend to zero in the Michel split monopole solution with $d = 0$. The directions of the radial component are opposite on the two sides of the LC located at $r = r_{LC} = 1/\Omega$: the magnetic force dominates inside the LC, while the electric force dominates outside the LC. The force-free fields become electrically dominated outside LC either. It is usually assumed that the monopole solution only exists outside the LC.

The Poynting fluxes for the FF fields in the limit $\theta \to \pi/2$ are

$$\vec{S}_\vee \to \frac{(q r \Omega)^2}{4\pi (r^2 + d^2)^{\frac{5}{2}}} \left(r, d, \frac{1}{r\Omega} \sqrt{r^2 + d^2} \right), \tag{36}$$

$$\vec{S}_\wedge \to \frac{(q r \Omega)^2}{4\pi (r^2 + d^2)^{\frac{5}{2}}} \left(r, -d, \frac{1}{r\Omega} \sqrt{r^2 + d^2} \right). \tag{37}$$

The discontinuous perpendicular component indicates that there is a net Poynting flux flowing into the CS from both sides of the FF regions. Thus, the CS gains energy.

The FF regions should be dissipation free and the electromagnetic energy should always be conserved. This can be expressed as

$$\dot{\mathcal{E}}^{FF}_\vee = \dot{\mathcal{E}}^{FF}_\wedge = 0, \tag{38}$$

where the change rates of the electromagnetic energy are given by

$$\dot{\mathcal{E}}^{FF}_\vee = \oint_\vee \vec{S}_\vee \cdot d\vec{s}, \qquad \dot{\mathcal{E}}^{FF}_\wedge = \oint_\wedge \vec{S}_\wedge \cdot d\vec{s}. \tag{39}$$

The integrals go through all the boundaries of the FF regions.

We first consider the upper hemisphere. The change rate of the FF electromagnetic energy due to the Poynting influx crossing the hemisphere at the initial radius r_0 is

$$\dot{\mathcal{E}}^{FF}_\vee (r = r_0) = 2\pi r_0^2 \int_0^{\frac{\pi}{2}} \sin\theta S^r_\vee d\theta = \frac{q^2 \Omega^2}{6} \left[2 + \frac{d(3r_0^2 + 2d^2)}{(r_0^2 + d^2)^{\frac{3}{2}}} \right]. \tag{40}$$

This influx can be viewed as the one directly extracted from the central star (via the inner magnetosphere). The change rate measured at $r \to \infty$:

$$\dot{\mathcal{E}}^{FF}_\vee (r \to \infty) = -\frac{q^2 \Omega^2}{3}. \tag{41}$$

The negative sign means that the energy flows out the FF region to infinity. This result is also the same as the Michel solution. On the boundary along the equator, the energy also flows out the FF region

$$\dot{\mathcal{E}}^{FF}_\vee \left(\theta \to \frac{\pi}{2} \right) = -\int_{r_0}^{\infty} 2\pi r S^\theta_\vee \left(\theta \to \frac{\pi}{2} \right) dr = -\frac{d q^2 \Omega^2 (3r_0^2 + 2d^2)}{6 (r_0^2 + d^2)^{\frac{3}{2}}}. \tag{42}$$

The calculations on the lower hemisphere lead to identical results; $\dot{\mathcal{E}}^{FF}_\wedge = \dot{\mathcal{E}}^{FF}_\vee$ for each of the components. We denote the summation; $\dot{\mathcal{E}}^{FF} = \dot{\mathcal{E}}^{FF}_\vee + \dot{\mathcal{E}}^{FF}_\wedge = 2\dot{\mathcal{E}}^{FF}_\vee$. Then we have

$$\dot{\mathcal{E}}^{FF}(r = r_0) + \dot{\mathcal{E}}^{FF}\left(\theta \to \frac{\pi}{2} \right) + \dot{\mathcal{E}}^{FF}(r \to \infty) = 0. \tag{43}$$

Thus, the conservation law (38) is verified. This indicates that the energy extracted from the star sources the Poynting fluxes flowing into the CS and to infinity. For the Michel split monopole, the second term vanishes and so the Poynting flux is constant through any sphere.

We now turn to the energy conservation law in the CS. As given in Eq. (25), the fields that exist inside the CS are the continuous fields; E_c^r and B_c^θ. They give a toroidal Poynting flux inside the CS, which is conserved itself. The electric current flowing in and out of the CS is also conserved in terms of Eq. (33). So the only change of the energy in the CS comes from the Poynting influx $-\dot{\mathcal{E}}^{FF}(\theta \to \pi/2)$ and the dissipated energy. The latter arises from the Joule heating process due to the nonvanishing $\vec{i}_c \cdot \vec{E}_c$. It leads to an increase of the CS energy at a total rate

$$\dot{\mathcal{E}}^{CS} = \int_{r_0}^{\infty} 2\pi r i_c^r E_c^r dr. \tag{44}$$

It is clear that this dissipated energy is completely compensated by the Poynting influx from both sides of the FF fields: $i_c^r E_c^r = 2S_\vee^\theta(\theta \to \pi/2)$ or

$$-\dot{\mathcal{E}}^{FF}\left(\theta \to \frac{\pi}{2}\right) = \dot{\mathcal{E}}^{CS}. \tag{45}$$

Hence, the energy is conserved and there is also no electromagnetic energy lost in the CS.

The Eq. (45) should be a consequence of the following process; as the charged particles flow into the CS along the magnetic field lines, the perpendicular component of the drift velocity will be eventually damped to zero. The kinetic energy is transferred to the thermal internal energy in the CS (as shown in Fig. 2.).

Compared with the Michel split monopole, the spin down power is enhanced in this configuration. The ratio of dissipated energy to the total extracted energy is

$$\frac{\dot{\mathcal{E}}^{CS}}{\dot{\mathcal{E}}^{FF}(r = r_0)} = \left[1 + \frac{2(r_0^2 + d^2)^{\frac{3}{2}}}{d(3r_0^2 + 2d^2)}\right]^{-1}. \tag{46}$$

Since $r_0 > d$, the maximum energy that can be dissipated in the CS is 47% of the total spin down energy, which is close to the numerical result of [6]. For larger $r_0/d > 1$, a smaller portion of energy is dissipated.

B. $\psi = (\psi_\vee^{(+)}, \psi_\wedge^{(-)})$

The magnetic field distribution of this configuration is shown in the right panel of Fig. 1. It looks similar to the split magnetosphere in the presence of a thin accretion disk that contains magnetic fields itself (e.g., [8–10,13,14]). But here the CS is not an accretion disk since no gravity is involved.

The quantities for this configuration are given by the previous case just with the replacement $d \to -d$. In the CS, only the continuous fields E_c^r and B_c^θ exist. The discontinuous fields lead to surface charge density σ_c and current densities i_c^ϕ, and i_c^r, which are the same as the previous case. The currents close with the same forms as given in Eqs. (30) and (33). But the Lorentz force take the opposite directions,

$$f_c^r = -\frac{q^2 rd(r^2\Omega^2 - 1)}{2\pi(r^2 + d^2)^3}, \qquad f_c^\phi = -\frac{q^2 rd\Omega}{2\pi(r^2 + d^2)^{\frac{5}{2}}}. \tag{47}$$

The Poynting fluxes perpendicular to the CS are

$$S_\vee^\theta\left(\theta \to \frac{\pi}{2}\right) = -S_\wedge^\theta\left(\theta \to \frac{\pi}{2}\right) \to -\frac{d(qr\Omega)^2}{4\pi(r^2 + d^2)^{\frac{5}{2}}}. \tag{48}$$

This indicates that net Poynting fluxes flow off the CS into the FF magnetosphere on both sides. So the FF magnetosphere gains energy from the CS. Integrating the Poynting flux along the equator, we can find that the energy gained by the FF fields is exactly that lost in the CS,

$$\dot{\mathcal{E}}^{FF}\left(\theta \to \frac{\pi}{2}\right) = -\dot{\mathcal{E}}^{CS} = \frac{dq^2\Omega^2(3r_0^2 + 2d^2)}{3(r_0^2 + d^2)^{\frac{3}{2}}}. \tag{49}$$

So the energy is conserved in the CS and the electromagnetic energy density remains unchanged.

Similarly, we can show that the electromagnetic energy is conserved in the FF regions. With the above Eq. (49), the conservation law can be expressed as

$$\dot{\mathcal{E}}^{FF}(r = r_0) - \dot{\mathcal{E}}^{CS} = \dot{\mathcal{E}}^{FF}(r \to \infty), \tag{50}$$

where $\dot{\mathcal{E}}^{FF}(r \to \infty)$ is the same as the previous case (also equal to the one in the Michel split monopole). This equation means that the output energy flux at infinity is simultaneously extracted from the central star and the CS. For a given output power, the spin down energy extracted from the star can only be 11.6% of that by the Michel split monopole since $r_0/d > 1$.

Notice that here $\dot{\mathcal{E}}^{CS}$ is negative since $\vec{i}_c \cdot \vec{E}_c = i_c^r E_c^r < 0$. This mysterious negative energy has been encountered in the numerical simulations on rotating black holes [8,10,11]. It may be due to the observational effect in the gravitational system. But here no gravity is involved in our system, which may bring us new understanding on it. We think that

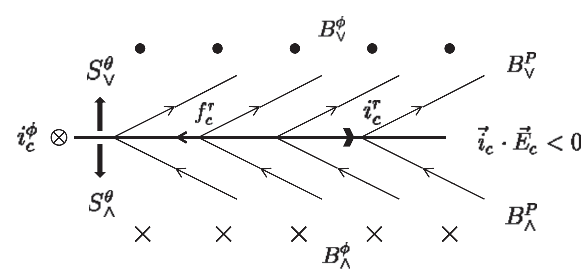

FIG. 3. The field structure and the energy flows near the CS outside the LC of the configuration in Sec. IV B. A net Poynting flux flows off the CS to the FF regions on both sides as the CS loses energy.

the negative energy here arises from the observational effect in the corotation frame, which is an acceleration frame and is similar to a gravitational system in terms of the Einstein equivalent principle between gravity and acceleration.

Following the above analysis, we can interpret this negative energy process as an inverse process of the one discussed in the previous configuration (see Fig. 3.): the charges flow away from the CS to the FF regions along the magnetic field lines on both sides to form the electric currents that constitute the split monopole configuration. Then the internal energy of thermal motion of the particles in the CS is transferred to the ordered drift motion when the particles enter into the FF regions. So the CS should cool down with the negative energy dissipated to provide the extracted energy.

V. CONCLUSIONS AND DISCUSSIONS

The Michel split monopole model is not unique and the deviation from it leads to nontrivial consequences. By varying the centered model in different ways, we illustrate how the CS plays different roles.

Based on the decentered monopole solution generated by the translational symmetry in the axisymmetric case, we construct two generalized split monopole configurations. One configuration resembles the outer geometry of a new pulsar magnetosphere model, while the other may be useful in describing the physical process in a split magnetosphere with an accretion disk. These generalized configurations can also be constructed in the oblique rotation case, since the translational symmetry still exists in the magnetosphere on an oblique rotator [15].

It is shown that the CS is a site where energy is dissipated or extracted. This will increase or decrease the spin down energy extracted from the central star, for a given output of the Poynting flux. We interpret this process as a result that the internal energy of thermal motion and the kinetic energy of drift motion are transferred into each other. The electromagnetic energy is always not lost everywhere, i.e., in the CS and the FF regions. When the Poynting flux flows in, the CS is heated up and possibly leads to synchrotron and inverse Compton radiations, which are observable [16]. On the contrary, energy is extracted as the CS cools down. So the CS can also cause temperature discontinuities in the systems. The effects of the thermal nonequilibrium on the magnetohydrodynamics need further investigations.

Our results will also apply to any variation of the split monopole in the standard pulsar magnetosphere model. In a realistic situation, the split monopole should not be exactly like the Michel model. The CS may have finite size, different geometries or even be dynamical with wavy structures. So all these variations will cause extra energy dissipation or extraction in the CS in terms of our results above.

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APPENDIX: EXPANSIONS OF THE DECENTERED MONOPOLE SOLUTION

In this Appendix, we present the expanded forms of the decentered monopole solution

$$\psi(r,\theta) = -q\frac{r\cos\theta - \epsilon}{\sqrt{r^2 - 2\epsilon r\cos\theta + \epsilon^2}} = -q\frac{\vec{\epsilon}}{|\vec{\epsilon}|}\cdot\frac{\vec{r}-\vec{\epsilon}}{|\vec{r}-\vec{\epsilon}|}, \tag{A1}$$

where $\vec{\epsilon}$ is a constant vector on the axis. This can be done by using the generating function for the Legendre polynomials

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n. \tag{A2}$$

Let us first consider the expansions in the region $r > |\epsilon|$. Using the following identity

$$\frac{1}{n+1}\sin\theta P^1_{n+1}(x) = xP_{n+1}(x) - P_n(x), \tag{A3}$$

we obtain the expansion form

$$\psi(r,\theta) = \psi_0 + \sum_{n=1}^{\infty}\psi_{-n}(\theta)r^{-n}, \tag{A4}$$

where

$$\psi_0 = -q\cos\theta, \qquad \psi_{-n} = -\frac{1}{n}q\epsilon^n\sin\theta P^1_n(\cos\theta). \tag{A5}$$

Using the identity

$$\frac{1}{n+1}\sin\theta P^1_n(x) = P_{n+1}(x) - xP_n(x), \tag{A6}$$

we have for $r < |\epsilon|$

$$\psi(r,\theta) = q + \sum_{n=2}^{\infty}\psi_n(\theta)r^n, \tag{A7}$$

where

$$\psi_n = \frac{1}{n}q\epsilon^{-n}\sin\theta P^1_{n-1}(\cos\theta). \tag{A8}$$

This branch of expansions is irrelevant in the discussions here.

With the expanded forms, it is easy to find that the generalized solution (A1) can be obtained from the pulsar equation by adopting the expansion method in [17]. In doing so, it is interesting to notice a cubic order identity for the Legendre polynomials that is not yet found elsewhere,

$$\sin\theta\cos\theta\partial_\theta\Gamma_k + (1-k\sin^2\theta)\Gamma_k = \sum_{i=0}^{k}\sum_{j=0}^{i}\Gamma_{k-i}\Gamma_{i-j}\Gamma_j, \tag{A9}$$

where

$$\Gamma_i = \epsilon^i[P_{i-1}(\cos\theta) - \cos\theta P_i(\cos\theta)] \qquad (i \geq 0), \tag{A10}$$

with the definition $P_l = 0$ for negative l.

[1] P. Goldreich and W. H. Julian, Pulsar electrodynamics, Astrophys. J. **157**, 869 (1969).
[2] F. C. Michel, Rotating magnetosphere: A simple relativistic model, Astrophys. J. **180**, 207 (1973).
[3] F. C. Michel, Rotating magnetospheres: An exact 3-D solution, Astrophys. J. Lett. **180**, L133 (1973).
[4] I. Contopoulos, D. Kazanas, and C. Fendt, The axisymmetric pulsar magnetosphere, Astrophys. J. **511**, 351 (1999).
[5] A. Gruzinov, Dissipative pulsar magnetospheres, J. Cosmol. Astropart. Phys. 11 (2008) 002.
[6] I. Contopoulos, C. Kalapotharakos, and D. Kazanas, A new standard pulsar magnetosphere, Astrophys. J. **781**, 46 (2014).
[7] S. E. Gralla and T. Jacobson, Spacetime approach to force-free magnetospheres, Mon. Not. R. Astron. Soc. **445**, 2500 (2014).
[8] S. S. Komissarov, Electrodynamics of black hole magnetospheres, Mon. Not. R. Astron. Soc. **350**, 427 (2004).
[9] D. A. Uzdensky, Force-free magnetosphere of an accretion disk-black hole system. II. Kerr geometry, Astrophys. J. **620**, 889 (2005).
[10] F. L. Carrasco and O. A. Reula, Novel scheme for simulating the force-free equations: Boundary conditions and the evolution of solutions towards stationarity, Phys. Rev. D **96**, 063006 (2017).
[11] W. E. East and H. Yang, Magnetosphere of a spinning black hole and the role of the current sheet, Phys. Rev. D **98**, 023008 (2018).
[12] S. A. Petrova, A novel look at the pulsar force-free magnetosphere, Astrophys. Space Sci. **363**, 51 (2018).
[13] R. Buckley, Some exact axisymmetric force-free magnetospheres. I, Mon. Not. R. Astron. Soc. **196**, 1021 (1981).
[14] L. Chen and B. Zhang, Analytical solution of magnetically dominated astrophysical jets/winds, Astrophys. J. **906**, 105 (2021).
[15] T. Uchida, The force-free magnetosphere around an oblique rotator, Mon. Not. R. Astron. Soc. **297**, 315 (1998).
[16] J. Pétri, Theory of pulsar magnetosphere and wind, J. Plasma Phys. **82**, 635820502 (2016).
[17] H. Li and J. Wang, Expanded solutions of force-free electrodynamics on general Kerr black holes, Phys. Rev. D **96**, 023014 (2017).