Derivation of the nonlocal collision term in the relativistic Boltzmann equation for massive spin-1/2 particles from quantum field theory

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We derive the Boltzmann equation and the collision kernel for massive spin-1/2 particles, using the Wigner-function formalism and employing an expansion in powers of $\hbar$. The phase space is enlarged to include a variable related to the spin degrees of freedom. This allows us to reduce the transport equations of the independent components of the Wigner function to one scalar equation. To next-to-leading order in $\hbar$, we find that the collision kernel contains both local and nonlocal terms. We show that off-shell contributions cancel in the Boltzmann equation. Our framework can be used to study spin-polarization phenomena induced by vorticity as recently observed in heavy-ion collisions and in condensed-matter systems.

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I. INTRODUCTION

Polarization phenomena in nuclear collisions have recently been the focus of intense research. The large orbital angular momentum in noncentral heavy-ion collisions can (at least partially) be transferred as vorticity to the hot and dense matter created in the collision zone. This, in turn, may align the spins of particles along the direction of the global orbital angular momentum, leading to a nonzero spin polarization [1–4]. Such a mechanism is rather similar to the time-honored Barnett effect [5]. In a nonrelativistic system, the alignment of spins by rotation implies the alignment of magnetic moments, and thus polarization is equivalent to magnetization. In a relativistic system, however, both particles and antiparticles are present, and while the spins of particles and antiparticles align in the same direction through rotation, the magnetic moments align in the opposite direction, which reduces the magnetization. Thus, a system with equal numbers of particles and antiparticles is polarized, but not magnetized. This effect is a prime example for the interplay between a macroscopic quantity, the rotation, and a microscopic quantity, which is inherently of quantum nature: the spin of the particles.

The STAR Collaboration found that Lambda baryons emitted in noncentral heavy-ion collisions are indeed emitted with a finite global polarization (i.e., a polarization along the direction of the global angular momentum), providing evidence of spin polarization generated by vorticity [6]. The global polarization predicted by models based on the assumption of local thermodynamic equilibrium of spin degrees of freedom turn out to be in good agreement with the experimental findings [4,7–13]. More recently, the STAR Collaboration measured the projection of the Lambda polarization along the beam direction, the so-called longitudinal polarization, as a function of the azimuthal angle of the particles [14]. Unfortunately, the same theoretical models which were able to describe the global-polarization data predict an opposite sign with respect to the experimental observations, often dubbed the "polarization sign problem" [15,16]. A number of attempts [17–24] have been made to explain the polarization sign problem, but as of yet, no definite conclusion has been reached.

The polarization sign problem suggests that spin degrees of freedom have nontrivial dynamics, which is not captured by the theoretical models used to accurately describe the global-polarization data. One possibility is that nonequilibrium effects of spin degrees of freedom have to be included in the kinetic and hydrodynamic description of the hot and dense matter. A theory of relativistic spin

hydrodynamics, first introduced in Refs. [25,26] and followed by Refs. [27–32], has also been recently derived from various approaches: kinetic theory [33–39], effective action [40–42], entropy-current analysis [43,44], and holographic duality [45,46].

From a microscopic point of view, the dynamics underlying the conversion between orbital and spin angular momentum (and vice versa) can be understood in terms of particle collisions. Such dynamics was studied in the nonrelativistic case in a seminal paper by Hess and Waldmann, where a kinetic theory for a dilute gas of particles with spin was formulated [47]. One of the main conclusions of this work is that, in order to describe polarization phenomena through rotation (e.g., the Barnett effect), one needs nonlocal particle collisions. The authors were not able to provide a first-principle derivation of the nonlocal collision kernel, and they phenomenologically added terms in their kinetic theory to describe the orbital-to-spin angular momentum conversion. A nonrelativistic Boltzmann equation with a nonlocal collision term was then discussed by Hess in Ref. [48]. Detailed derivations of nonlocal collision terms for a nonrelativistic system of spinless particles can be found, e.g., in Refs. [49–51]. In the relativistic case, a microscopic mechanism based on nonlocal scatterings between wave packets was proposed in Ref. [19] to explain the generation of the spin-vorticity coupling in heavy-ion collisions. However, to the best of our knowledge, a systematic derivation of a nonlocal collision kernel in the relativistic Boltzmann equation for particles with spin based on quantum field theory has only been performed very recently in our previous work [34] [for related efforts, see also Refs. [52,53]].

In Ref. [34], we presented a Boltzmann equation using the Wigner-function formalism, which includes the nonlocality of the scattering process between particles and established its connection with spin hydrodynamics. In this paper, we now give the details of the derivation. The Wigner-function formalism provides a first-principle formulation of kinetic theory and also turns out to be a very powerful tool for the description of anomalous transport in the quark-gluon plasma created in heavy-ion collisions (see, e.g., Refs. [54–61]). Our derivation is based on a semiclassical expansion of the Wigner function, i.e., an expansion in the Planck constant $\hbar$, where spin effects are considered to be of at least first order in $\hbar$. As it will become clear in the following, an expansion in the Planck constant is also effectively an expansion in gradients. Thus, vorticity, which is a quantity of first order in gradients, is of the same order as spin polarization. The latter can therefore be generated from the former through nonlocal scattering processes.

The structure of the paper is the following. In Sec. II, we derive the quantum transport equations from the Wigner-function formalism. In order to have a more compact transport equation for the components of the Wigner function, in Sec. III. we enlarge the phase space by introducing a variable related to spin. In Secs. IV and V, we explicitly derive the local and nonlocal parts of the collision term, respectively. Such calculations are based on the method discussed in Ref. [62]. Finally, conclusions are given in Sec. VII. We use the notation and conventions $a\cdot b = a^\mu b_\mu$, $a_{[\mu}b_{\nu]} \equiv a_\mu b_\nu - a_\nu b_\mu$, $g_{\mu\nu} = \mathrm{diag}(+,-,-,-)$, $\epsilon^{0123} = -\epsilon_{0123} = 1$, and repeated indices are summed over.

II. QUANTUM TRANSPORT EQUATIONS

We start from the Wigner function for spin-1/2 particles defined as [62–64]

$$W_{\alpha\beta}(x,p) = \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y}\langle :\bar{\psi}_\beta(x_1)\psi_\alpha(x_2):\rangle, \quad (1)$$

with $x_{1,2} = x \pm y/2$ and $\psi(x), \bar{\psi}(x)$ being Dirac spinor fields. Here, $\langle :: \rangle$ denotes the normal-ordered ensemble average. In our previous work [65] (see also related work in Refs. [66–72]), we derived general solutions of the equations of motion for the Wigner function in the free-streaming limit. Here, we extend this idea by including collisions and thus account for the effect of interactions. The Lagrangian for Dirac fields used in this paper is of the form

$$\mathcal{L}_D = \bar{\psi}\left(\frac{i\hbar}{2}\gamma\cdot\overleftrightarrow{\partial} - m\right)\psi + \mathcal{L}_I, \quad (2)$$

with $\overleftrightarrow{\partial} \equiv \vec{\partial} - \overleftarrow{\partial}$ and $\mathcal{L}_I$ being a general interaction Lagrangian. We remark that, if $\mathcal{L}_I$ contains gauge-field interactions, Eq. (1) has to be modified to include a gauge link in order to ensure gauge invariance; see, e.g., Ref. [64]. We obtain the following equation of motion,

$$(i\hbar\gamma\cdot\partial - m)\psi(x) = \hbar\rho(x), \quad (3)$$

where $\rho = -(1/\hbar)\partial\mathcal{L}_I/\partial\bar{\psi}$. From Eq. (3), one derives the transport equation for the Wigner function [62],

$$\left[\gamma\cdot\left(p + i\frac{\hbar}{2}\partial\right) - m\right]W_{\alpha\beta} = \hbar\mathcal{C}_{\alpha\beta}, \quad (4)$$

where

$$\mathcal{C}_{\alpha\beta} \equiv \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y}\langle :\bar{\psi}_\beta(x_1)\rho_\alpha(x_2):\rangle. \quad (5)$$

By acting $\gamma\cdot(p + i\frac{\hbar}{2}\partial) + m$ onto Eq. (4) and taking the real part, we obtain a modified on-shell condition for the Wigner function

$$\left(p^2 - m^2 - \frac{\hbar^2}{4}\partial^2\right) W_{\alpha\beta}(x,p) = \hbar \delta M_{\alpha\beta}(x,p), \quad (6)$$

with

$$\delta M_{\alpha\beta} \equiv \frac{1}{2}\int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y} \times \langle : [\bar{\rho}(x_1)(i\hbar\gamma\cdot\overleftarrow{\partial} + m)]_\beta \psi_\alpha(x_2) + \bar{\psi}_\beta(x_1)[(-i\hbar\gamma\cdot\partial + m)\rho(x_2)]_\alpha : \rangle. \quad (7)$$

On the other hand, from the imaginary part, we find a Boltzmann-like equation for the Wigner function,

$$p\cdot\partial W_{\alpha\beta}(x,p) = C_{\alpha\beta}(x,p), \quad (8)$$

with

$$C_{\alpha\beta} = \frac{i}{2}\int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y} \times \langle : [\bar{\rho}(x_1)(-i\hbar\gamma\cdot\overleftarrow{\partial} + m)]_\beta \psi_\alpha(x_2) - \bar{\psi}_\beta(x_1)[(i\hbar\gamma\cdot\partial + m)\rho(x_2)]_\alpha : \rangle. \quad (9)$$

We will restrict the following considerations to the positive-energy part of the Wigner function. The extension to negative energies is straightforward. Thus, in what follows, all mass-shell delta functions are implicitly accompanied by a $\theta(p_0)$, which we do not explicitly denote for the sake of simplicity.

In order to reveal the dependence on the Wigner function on the right-hand side of Eq. (9), it is convenient to calculate the ensemble average by performing the trace over the noninteracting initial n-particle states defined as [62]

$$|p_1,\ldots,p_n;r_1,\ldots,r_n\rangle_{\text{in}} \equiv a^\dagger_{\text{in},r_1}(p_1)\cdots a^\dagger_{\text{in},r_n}(p_n)|0\rangle, \quad (10)$$

where p_i and r_i, $i = 1,\ldots,n$, denote the particle momentum and spin projection, respectively, and $a^\dagger_{\text{in},r_i}(p_i)$ is the creation operator for that particle. Since we are interested in a kinetic description, we neglect initial correlations. This corresponds to the molecular-chaos assumption. Furthermore, we restrict ourselves to two-particle states; i.e., we only consider binary collisions. Hence, Eq. (9) can be written in the form [62] (see Appendix A for details)

$$C_{\alpha\beta} = \frac{1}{2(4\pi\hbar m^2)^2}\sum_{r_1,r_2,s_1,s_2}\int d^4x_1 d^4x_2 d^4p_1 d^4p_2 d^4q_1 d^4q_2 \times {}_{\text{in}}\left\langle p_1 - \frac{q_1}{2}, p_2 - \frac{q_2}{2}; r_1, r_2 \middle| \Phi_{\alpha\beta}(p) \middle| p_1 + \frac{q_1}{2}, p_2 + \frac{q_2}{2}; s_1, s_2 \right\rangle_{\text{in}} \times \prod_{j=1}^{2} \exp\left(\frac{i}{\hbar}q_j\cdot x_j\right) \bar{u}_{s_j}\left(p_j + \frac{q_j}{2}\right) W_{\text{in}}(x + x_j, p_j) u_{r_j}\left(p_j - \frac{q_j}{2}\right), \quad (11)$$

where the operator Φ is given by

$$\Phi_{\alpha\beta}(p) \equiv \frac{i}{2}\int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y} : \left\{ \left[P_\mu, \bar{\rho}\left(\frac{y}{2}\right)\gamma^\mu\right]_\beta \psi_\alpha\left(-\frac{y}{2}\right) + m\bar{\rho}_\beta\left(\frac{y}{2}\right)\psi_\alpha\left(-\frac{y}{2}\right) - \bar{\psi}_\beta\left(\frac{y}{2}\right)\left[\gamma^\mu\rho\left(-\frac{y}{2}\right), P_\mu\right]_\alpha - m\bar{\psi}_\beta\left(\frac{y}{2}\right)\rho_\alpha\left(-\frac{y}{2}\right)\right\} :, \quad (12)$$

where P^μ is the total 4-momentum operator. We also introduced the variable q_j, which is the conjugate to x_j in the Wigner transformation, but not related to the particle momenta p_i. We notice that the Boltzmann-like equation (8) with the collision kernel (11) is not a closed equation for the interacting Wigner function W, as $C_{\alpha\beta}$ is a functional of the initial Wigner function W_{in}. However, for a dilute system, we further approximate

$$W = W_{\text{in}} + \ldots, \quad (13)$$

where the ellipsis corresponds to corrections of higher order in density, which we neglect [62]. We will invert this relation and replace W_{in} in the collision term by W. Furthermore, we see that the collision term in Eq. (11) takes into account the nonlocality of the collision process, as the Wigner functions depend on $x + x_j$. If the Wigner function varies slowly in space and time on the microscopic scale corresponding to the interaction range, we can Taylor expand $W(x + x_j, p_j)$ around x and keep only terms up to first order in gradients (equivalent to first order in $\hbar$) [62], i.e.,

$$W(x + x_j, p_j) = W(x, p_j) + x_j\cdot\partial W(x, p_j). \quad (14)$$

Substituting Eqs. (13) and (14) into Eq. (11), it follows that

$$C_{\alpha\beta} = \frac{(2\pi\hbar)^6}{(2m)^4} \sum_{r_1,r_2,s_1,s_2} \int d^4p_1 d^4p_2 d^4q_1 d^4q_2 \left\langle p_1 - \frac{q_1}{2}, p_2 - \frac{q_2}{2}; r_1, r_2 \middle| \Phi_{\alpha\beta}(p) \middle| p_1 + \frac{q_1}{2}, p_2 + \frac{q_2}{2}; s_1, s_2 \right\rangle_{\text{in}}$$
$$\times \prod_{j=1}^{2} \bar{u}_{s_j}\left(p_j + \frac{q_j}{2}\right)\{W(x,p_j)\delta^{(4)}(q_j) - i\hbar[\partial^\mu_{q_j}\delta^{(4)}(q_j)]\partial_\mu W(x,p_j)\}u_{r_j}\left(p_j - \frac{q_j}{2}\right), \tag{15}$$

where we performed the integration over d^4x_1 and d^4x_2. Equation (15) is the collision kernel for the Boltzmann equation (8), which we will use as a starting point for the explicit computation of collision effects.

Following Refs. [34,65,68,69], we employ an expansion in powers of $\hbar$ for the Wigner function; i.e., we search for solutions of the form

$$W = W^{(0)} + \hbar W^{(1)} + \hbar^2 W^{(2)} + \mathcal{O}(\hbar^3). \tag{16}$$

We notice that, since gradients are always accompanied by factors of $\hbar$, such an expansion is also a gradient expansion. We also stress that the gradient expansion of the nonlocal term has to be considered as an $\hbar$ expansion, as Eq. (15) shows.

Furthermore, in our treatment, we will consider an expansion around equilibrium,

$$W = W_{\text{eq}} + \delta W, \tag{17}$$

where W_{eq} is the equilibrium Wigner function and δW the deviation from equilibrium. In our scheme, we always consider δW to be at least of first order in an expansion in gradients. As a consequence, if we take into account only the lowest-order gradient correction in the nonlocal collision term, we can neglect contributions from δW in the second term in the second line of Eq. (15), as they would be of higher order in gradients.

It is now convenient to decompose the Wigner function in terms of a basis of the generators of the Clifford algebra

$$W = \frac{1}{4}\left(\mathcal{F} + i\gamma^5\mathcal{P} + \gamma\cdot\mathcal{V} + \gamma^5\gamma\cdot\mathcal{A} + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\right), \tag{18}$$

where $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, and substitute it into Eq. (4) to obtain the equations of motion for the coefficient functions. The real part of Eq. (4) yields

$$p\cdot\mathcal{V} - m\mathcal{F} = \hbar D_{\mathcal{F}}, \tag{19a}$$

$$\frac{\hbar}{2}\partial\cdot\mathcal{A} + m\mathcal{P} = -\hbar D_{\mathcal{P}}, \tag{19b}$$

$$p^\mu\mathcal{F} - \frac{\hbar}{2}\partial_\nu\mathcal{S}^{\nu\mu} - m\mathcal{V}^\mu = \hbar D^\mu_{\mathcal{V}}, \tag{19c}$$

$$-\frac{\hbar}{2}\partial^\mu\mathcal{P} + \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}p_\nu\mathcal{S}_{\alpha\beta} + m\mathcal{A}^\mu = -\hbar D^\mu_{\mathcal{A}}, \tag{19d}$$

$$\frac{\hbar}{2}\partial^{[\mu}\mathcal{V}^{\nu]} - \epsilon^{\mu\nu\alpha\beta}p_\alpha\mathcal{A}_\beta - m\mathcal{S}^{\mu\nu} = \hbar D^{\mu\nu}_{\mathcal{S}}, \tag{19e}$$

while from the imaginary part we obtain

$$\hbar\partial\cdot\mathcal{V} = 2\hbar C_{\mathcal{F}}, \tag{20a}$$

$$p\cdot\mathcal{A} = \hbar C_{\mathcal{P}}, \tag{20b}$$

$$\frac{\hbar}{2}\partial^\mu\mathcal{F} + p_\nu\mathcal{S}^{\nu\mu} = \hbar C^\mu_{\mathcal{V}}, \tag{20c}$$

$$p^\mu\mathcal{P} + \frac{\hbar}{4}\epsilon^{\mu\nu\alpha\beta}\partial_\nu\mathcal{S}_{\alpha\beta} = -\hbar C^\mu_{\mathcal{A}}, \tag{20d}$$

$$p^{[\mu}\mathcal{V}^{\nu]} + \frac{\hbar}{2}\epsilon^{\mu\nu\alpha\beta}\partial_\alpha\mathcal{A}_\beta = -\hbar C^{\mu\nu}_{\mathcal{S}}. \tag{20e}$$

Here, we defined $D_i = \text{ReTr}(\tilde{\Gamma}_i\mathcal{C})$, $C_i = \text{ImTr}(\tilde{\Gamma}_i\mathcal{C})$, $i = \mathcal{F},\mathcal{P},\mathcal{V},\mathcal{A},\mathcal{S}$, $\tilde{\Gamma}_{\mathcal{F}} = 1$, $\tilde{\Gamma}_{\mathcal{P}} = -i\gamma_5$, $\tilde{\Gamma}_{\mathcal{V}} = \gamma^\mu$, $\tilde{\Gamma}_{\mathcal{A}} = \gamma^\mu\gamma^5$, $\tilde{\Gamma}_{\mathcal{S}} = \sigma^{\mu\nu}$. Note that each coefficient function will obey a modified on-shell condition and a Boltzmann-like equation, as can be readily seen from Eqs. (6) and (8).

We now assume that effects related to spin, and consequently to the polarization, are at least of first order in $\hbar$. This excludes the case of a large initial polarization of the system; i.e., we focus on situations where a nonzero polarization arises only through scatterings in the presence of a nonvanishing medium vorticity. Therefore, since $\mathcal{A}^\mu$ is related to the polarization vector [65], its zeroth-order contribution is assumed vanish, $\mathcal{A}^{(0)\mu} = 0$, and consequently, from Eq. (19e), $\mathcal{S}^{(0)\mu\nu} = 0$. Equation (19b) then implies that $\mathcal{P}^{(0)} = 0$. Thus, at zeroth order, all pseudoscalar quantities vanish, and, as a consequence, also the collision terms which carry pseudoscalar quantum numbers must vanish at zeroth order, $D^{(0)}_{\mathcal{P}} = C^{(0)}_{\mathcal{P}} = 0$. Using Eqs. (19b) and (20b), this, in turn, implies that

$$\mathcal{P} = \mathcal{O}(\hbar^2), \qquad p\cdot\mathcal{A} = \mathcal{O}(\hbar^2). \tag{21}$$

For the vector part, at zeroth order, the only vector at our disposal is p^μ, i.e.,

$$D^\mu_{\mathcal{V}} = p^\mu\delta V + \mathcal{O}(\hbar), \tag{22}$$

with a scalar function δV. Thus, from Eq. (19c), we obtain

$$\mathcal{V}^\mu = \frac{1}{m} p^\mu \bar{\mathcal{F}} + \mathcal{O}(\hbar^2), \tag{23}$$

where we defined $\bar{\mathcal{F}} \equiv \mathcal{F} - \hbar\delta V$. We can extend this definition to any order in $\hbar$ by setting

$$\bar{\mathcal{F}} \equiv \frac{m}{p^2}\mathrm{Tr}(p\cdot\gamma W). \tag{24}$$

We note that the assumption of polarization entering at first order in $\hbar$ implies that also the axial-vector $D_{\mathcal{A}}^{(0)\mu}$ and the antisymmetric tensors $D_{\mathcal{S}}^{(0)\mu\nu}$ and $C_{\mathcal{S}}^{(0)\mu\nu}$ must vanish.

We can now write down the modified on-shell conditions for $\bar{\mathcal{F}}$ and $\mathcal{A}^\mu$. From Eqs. (19a), (19c), and (20e), the modified on-shell condition for the vector component reads

$$(p^2 - m^2)\mathcal{V}^\mu = \hbar p^\mu D_{\mathcal{F}} + \hbar m D_{\mathcal{V}}^\mu + \mathcal{O}(\hbar^2), \tag{25}$$

which, from Eq. (23), implies

$$\begin{aligned}(p^2 - m^2)\bar{\mathcal{F}} &= \hbar\delta M_F + \mathcal{O}(\hbar^2)\\ &= \hbar m\left(D_{\mathcal{F}} + \frac{m}{p^2}p\cdot D_{\mathcal{V}}\right) + \mathcal{O}(\hbar^2),\end{aligned} \tag{26}$$

where δM_F can be expressed at any order in $\hbar$ by δM, defined in Eq. (7), via the relation

$$\delta M_F = \frac{m}{p^2}\mathrm{Tr}(p\cdot\gamma\delta M). \tag{27}$$

Furthermore, from Eqs. (19d) and (19e), we obtain

$$(p^2 - m^2)\mathcal{A}^\mu = \hbar\delta M_A^\mu + \mathcal{O}(\hbar^2) = \mathcal{O}(\hbar^2), \tag{28}$$

with

$$\delta M_A^\mu = \mathrm{Tr}(\gamma^\mu\gamma^5\delta M), \tag{29}$$

which, as a quantity with axial-vector quantum numbers, is itself of order $\mathcal{O}(\hbar)$. This shows that, under the assumptions adopted, the axial-vector component, unlike $\bar{\mathcal{F}}$, remains on the mass shell at first order in $\hbar$ even in the presence of interactions.

The Boltzmann equations are derived from Eqs. (20a) and (20e), using Eqs. (21) and (23), and read up to corrections of order $\mathcal{O}(\hbar^2)$,

$$p\cdot\partial\bar{\mathcal{F}} = mC_F, \tag{30a}$$

$$p\cdot\partial\mathcal{A}^\mu = mC_A^\mu, \tag{30b}$$

with $C_F = 2C_{\mathcal{F}}$ and $C_A^\mu \equiv -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\nu C_{\mathcal{S}\alpha\beta}$. From Eqs. (8), (24), and (30), one finds

$$C_F = \frac{1}{p^2}\mathrm{Tr}(p\cdot\gamma C), \tag{31a}$$

$$C_A^\mu = \frac{1}{m}\mathrm{Tr}(\gamma^\mu\gamma^5 C), \tag{31b}$$

which establishes the connection to C given in Eq. (15). Equations (31) will be used to determine the right-hand sides of Eqs. (30), which, together with Eq. (21), form a closed system of equations for $\bar{\mathcal{F}}$ and $\mathcal{A}^\mu$, as will be explicitly shown in the following.

III. SPIN IN PHASE SPACE

We now introduce spin as an additional variable in phase space [30,33,34,73–75]. The advantage of this concept is that it immediately connects the first-principle quantum description to a "classical" description of spin, which can be used, e.g., for hydrodynamics [34]. Furthermore, as we will see later, it combines the full dynamics of the Boltzmann-like equations (30) into one scalar equation and provides a natural interpretation for the conservation laws and the collisional invariants [34].

It is convenient to define the single-particle distribution function in the phase space extended by the additional spin variable $\mathfrak{s}$ as

$$\mathfrak{f}(x,p,\mathfrak{s}) \equiv \frac{1}{2}[\bar{\mathcal{F}}(x,p) - \mathfrak{s}\cdot\mathcal{A}(x,p)]. \tag{32}$$

This definition holds at any order in $\hbar$. We then introduce the covariant integration measure

$$\int dS(p) \equiv \sqrt{\frac{p^2}{3\pi^2}}\int d^4\mathfrak{s}\,\delta(\mathfrak{s}\cdot\mathfrak{s} + 3)\delta(p\cdot\mathfrak{s}), \tag{33}$$

which has the properties

$$\int dS(p) = 2, \tag{34a}$$

$$\int dS(p)\mathfrak{s}^\mu = 0, \tag{34b}$$

$$\int dS(p)\mathfrak{s}^\mu\mathfrak{s}^\nu = -2\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right). \tag{34c}$$

Consequently,

$$\bar{\mathcal{F}}(x,p) = \int dS(p)\mathfrak{f}(x,p,\mathfrak{s}), \tag{35a}$$

$$\mathcal{A}^\mu(x,p) = \int dS(p)\mathfrak{s}^\mu\mathfrak{f}(x,p,\mathfrak{s}). \tag{35b}$$

Higher moments of $\mathfrak{f}$ with respect to the variable $\mathfrak{s}$ can be also related to $\mathcal{F}$ and $\mathcal{A}^\mu$ and do not yield any further

information. From Eqs. (19c), (19e), and (23), we obtain relations for $\mathcal{V}^\mu$ and $\mathcal{S}^{\mu\nu}$, which are valid up to corrections of order $\mathcal{O}(\hbar^2)$,

$$\mathcal{V}^\mu(x,p) = \int dS(p)\left(\frac{1}{m}p^\mu + \frac{\hbar}{2m}\partial_\nu \Sigma^{\mu\nu}_{\mathfrak{s}}\right)\mathfrak{f}(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2),$$
$$\mathcal{S}^{\mu\nu}(x,p) = \int dS(p)\left(\Sigma^{\mu\nu}_{\mathfrak{s}} + \frac{\hbar}{2m^2}\partial^{[\mu}p^{\nu]}\right)\mathfrak{f}(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2), \tag{36}$$

where we defined

$$\Sigma^{\mu\nu}_{\mathfrak{s}} \equiv -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\alpha \mathfrak{s}_\beta. \tag{37}$$

Similarly as for the Wigner function, we can write a modified on-shell condition and the Boltzmann equation for the scalar distribution $\mathfrak{f}$. Using Eqs. (26) and (28), the on-shell condition is given by

$$(p^2 - m^2)\mathfrak{f}(x,p,\mathfrak{s}) = \hbar\mathfrak{M}(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2), \tag{38}$$

with

$$\mathfrak{M}(x,p,\mathfrak{s}) = \frac{1}{2}[\delta M_F(x,p) - \mathfrak{s}\cdot\delta M_A(x,p)]. \tag{39}$$

In order to find a solution for Eq. (38), we employ the quasiparticle approximation; i.e., we assume that the distribution $\mathfrak{f}$ is of the form

$$\mathfrak{f}(x,p,\mathfrak{s}) = m\delta(p^2 - M^2)f(x,p,\mathfrak{s}), \tag{40}$$

where $f(x,p,\mathfrak{s})$ is a function without singularity at $p^2 = M^2 \equiv m^2 + \hbar\delta m^2$, with $\delta m^2(x,p,\mathfrak{s})$ being a correction to the mass-shell condition for free particles arising from interactions. After Taylor expanding the delta function to first order in $\hbar$ and assuming that $f(x,p,\mathfrak{s})$ has no singularity at $p^2 = m^2$, i.e., $(p^2 - m^2)\delta(p^2 - m^2)f(x,p,\mathfrak{s}) = 0$, we can relate δm^2 with $\mathfrak{M}$,

$$\hbar\mathfrak{M}(x,p,\mathfrak{s}) = \hbar\delta m^2(x,p,\mathfrak{s})\delta(p^2 - m^2)mf(x,p,\mathfrak{s}) + \mathcal{O}(\hbar^2), \tag{41}$$

where we used $(p^2 - m^2)\delta'(p^2 - m^2) = -\delta(p^2 - m^2)$. As a consequence of our assumption that spin degrees of freedom enter at first order, the $\mathfrak{s}$ dependence of $\hbar\delta m^2$ appears at least at $\mathcal{O}(\hbar^2)$.

The Boltzmann equation for $\mathfrak{f}$ is derived from Eqs. (30) and (32) and reads

$$p\cdot\partial\mathfrak{f}(x,p,\mathfrak{s}) = m\mathfrak{C}, \tag{42}$$

where we introduced the collision kernel

$$\mathfrak{C} \equiv \frac{1}{2}(C_F - \mathfrak{s}\cdot C_A). \tag{43}$$

As will be shown in the following, up to first order in $\hbar$, the collision term has the following structure:

$$\mathfrak{C} = \mathfrak{C}^{(0)}_l + \hbar\{\mathfrak{C}^{(1)}_l + \mathfrak{C}^{(1)}_{nl}\} \equiv \mathfrak{C}_l + \hbar\mathfrak{C}^{(1)}_{nl}. \tag{44}$$

Here, local and nonlocal contributions are denoted by subscripts l and nl, respectively. As already mentioned, the zeroth-order contribution is purely local [76], while the first-order contribution has both local and nonlocal parts. In the next sections, we will calculate the local and nonlocal collision terms explicitly.

IV. LOCAL COLLISIONS

In order to explicitly calculate the collision term, we follow Ref. [62]. We first focus on the local part, i.e., the term proportional to $\delta^{(4)}(q_i)$ in the second line of Eq. (15). The matrix element of Φ appearing in this equation, with Φ given by Eq. (12), is calculated in Appendix B. The local contribution is thus obtained from Eq. (B10) with $q_i = 0$ [62],

$$(2\pi\hbar)^6{}_{\text{in}}\langle p_1,p_2;r_1,r_2|\Phi|p_1,p_2;s_1,s_2\rangle_{\text{in}} = \sum_{rs} u_r(p)\bar{u}_s(p)w^{rs}_{r_1r_2s_1s_2}(p_1,p_2,p), \tag{45}$$

with

$$w^{rs}_{r_1r_2s_1s_2}(p_1,p_2,p) = 2\delta(p^2-m^2)\Big\{\sum_{r'}\int dP' \times\delta(p+p'-p_1-p_2)\langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle \times\langle p_1,p_2;r_1,r_2|t^\dagger|p,p';s,r'\rangle + [i\pi\hbar p^0\delta^{(3)}(\mathbf{p}-\mathbf{p}_1)(\langle p,p_2;r,r_2|t|p,p_2;s_1,s_2\rangle\delta_{r_1s} - \langle p,p_2;r_1,r_2|t^\dagger|p,p_2;s,s_2\rangle\delta_{rs_1}) + (1\leftrightarrow 2)]\Big\}, \tag{46}$$

where the symbol $(1\leftrightarrow 2)$ denotes the exchange of the indices 1 and 2, $dP \equiv d^4p\delta(p^2 - m^2)$, and

$$\langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle \equiv -\sqrt{\frac{(2\pi\hbar)^7}{2}}\bar{u}_r(p)_{\text{out}}\langle p';r'|:\rho(0):|p_1,p_2;s_1,s_2\rangle_{\text{in}} \tag{47}$$

is the conventional scattering amplitude due to the interaction ρ, which can be computed using standard techniques from quantum field theory [62,77]. We are now ready to calculate the local part of Eq. (44). To this end, we insert Eq. (45) into Eq. (15); then Eq. (15) into Eqs. (31); and, finally, we plug Eqs. (31) into Eq. (43). In this way, the local part of the collision kernel is given by

$$\mathfrak{C}_l = \frac{1}{8m^4}\sum_{r_1,r_2,s_1,s_2}\int d^4p_1 d^4p_2 \sum_{r',s'} h_{s'r'}(p,\mathfrak{s}) \times w^{r's'}_{r_1r_2s_1s_2}(p_1,p_2,p)\prod_{j=1}^{2}\bar{u}_{s_j}(p_j)W(x,p_j)u_{r_j}(p_j), \tag{48}$$

where we have used

$$p^\mu \delta_{sr} \equiv \frac{1}{2}\bar{u}_s(p)\gamma^\mu u_r(p), \tag{49}$$

$$n^\mu_{sr}(p) \equiv \frac{1}{2m}\bar{u}_s(p)\gamma^5\gamma^\mu u_r(p) \tag{50}$$

and defined

$$h_{sr}(p,\mathfrak{s}) \equiv \delta_{sr} + \mathfrak{s}\cdot n_{sr}(p). \tag{51}$$

The factor $\delta(p^2-m^2)$ in Eq. (46) shows that the local term is always on shell. This comes from the difference $G(p)-G^\star(p) = 2\pi i\hbar^2\delta(p^2-m^2)$, with

$$G(p) = -\frac{\hbar^2}{p^2-m^2+i\epsilon p^0}, \tag{52}$$

which appears in the first line of Eq. (B11) when we set $q_i = 0$. We now use the Clifford decomposition (18) to write Eq. (48) as

$$\mathfrak{C}_l = \frac{1}{32m^2}\sum_{r_1,r_2,s_1,s_2}\int d^4p_1 d^4p_2 \sum_{r',s'} h_{s'r'}(p,\mathfrak{s}) \times w^{r's'}_{r_1r_2s_1s_2}(p_1,p_2,p)\prod_{j=1}^{2}\Big[\mathcal{F}(x,p_j)\delta_{s_jr_j} + \frac{1}{m}p\cdot\mathcal{V}(x,p_j)\delta_{s_jr_j} + n_{s_jr_j}(p_j)\cdot\mathcal{A}(x,p_j) + \frac{1}{2}\Sigma^{\mu_j\nu_j}_{s_jr_j}(p_j)\mathcal{S}_{\mu_j\nu_j}(x,p_j)\Big], \tag{53}$$

where we defined

$$\Sigma^{\mu\nu}_{rs}(p) \equiv \frac{1}{2m}\bar{u}_r(p)\sigma^{\mu\nu}u_s(p) = \frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\alpha n_{rs\beta}(p). \tag{54}$$

Using Eqs. (35b), (36), and (40), the relations $p_\mu\Sigma^{\mu\nu}_{\mathfrak{s}} = p_\mu\Sigma^{\mu\nu}_{sr} = 0$, and

$$\Sigma^{\mu\nu}_{sr}\Sigma_{\mathfrak{s}\mu\nu} = 2\frac{p^2}{m^2}\mathfrak{s}\cdot n_{sr}, \tag{55}$$

we can rewrite Eq. (53) in the form

$$\mathfrak{C}_l = \frac{1}{8}\sum_{r_1,r_2,s_1,s_2}\int d\Gamma_1 d\Gamma_2 \sum_{r',s'} h_{s'r'}(p,\mathfrak{s}) \times w^{r's'}_{r_1r_2s_1s_2}(p_1,p_2,p)\prod_{j=1}^{2}h_{s_jr_j}(p_j,\mathfrak{s}_j)f(x,p_j,\mathfrak{s}_j), \tag{56}$$

where

$$\int d\Gamma \equiv \int d^4p\,\delta(p^2-m^2)\int dS(p). \tag{57}$$

Plugging Eq. (46) into (56), the collision term reduces to

$$\mathfrak{C}_l = \delta(p^2-m^2)\mathfrak{C}_{\text{on-shell},l}[f], \tag{58}$$

where

$$\begin{aligned}\mathfrak{C}_{\text{on-shell},l}[f] = \frac{1}{4}\sum_{r_1,r_2,s_1,s_2}\sum_{r,r',s}\int d\Gamma_1 d\Gamma_2 dP' h_{sr}(p,\mathfrak{s})\,\delta^{(4)}(p+p'-p_1-p_2)\langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle \\ \times\langle p,p_2;r_1,r_2|t^\dagger|p,p';s,r'\rangle\prod_{j=1}^{2}h_{s_jr_j}(p_j,\mathfrak{s}_j)f(x,p_j,\mathfrak{s}_j) \\ + i\frac{\pi\hbar}{4}\sum_{r_2,s_1,s_2}\sum_{r,s}\int d\Gamma_2 dS_1(p)h_{s_2r_2}(p_2,\mathfrak{s}_2)f(x,p,\mathfrak{s}_1) \\ \times f(x,p_2,\mathfrak{s}_2)[h_{sr}(p,\mathfrak{s})h_{s_1s}(p,\mathfrak{s}_1)\langle p,p_2;r,r_2|t|p,p_2;s_1,s_2\rangle \\ -h_{s_1s}(p,\mathfrak{s})h_{sr}(p,\mathfrak{s}_1)\langle p,p_2;r,r_2|t^\dagger|p,p_2;s_1,s_2\rangle]\end{aligned} \tag{59}$$

is the local collision term on the mass shell. Using the identity

$$\sum_{s'} n^{\mu}_{rs'}(p) n^{\nu}_{s's}(p) = -\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{m^2}\right)\delta_{rs} + \frac{i}{m}\epsilon^{\mu\nu\alpha\beta} p_\alpha n_{rs\beta}(p), \tag{60}$$

we can simplify

$$\sum_s [h_{sr}(p,\mathfrak{s}) h_{s_1 s}(p,\mathfrak{s}_1) - h_{s_1 s}(p,\mathfrak{s}) h_{sr}(p,\mathfrak{s}_1)] = \mathfrak{s}_\mu \mathfrak{s}_{1\nu} \sum_s [n^\mu_{sr}(p) n^\nu_{s_1 s}(p) - n^\mu_{s_1 s}(p) n^\nu_{sr}(p)] = -i\frac{2}{m}\mathfrak{s}_\mu \mathfrak{s}_{1\nu}\epsilon^{\mu\nu\alpha\beta} p_\alpha n_{s_1 r\beta}(p) \tag{61}$$

to obtain

$$\begin{aligned}
\mathfrak{C}_{\text{on-shell},l}[f] = {} & \frac{1}{4}\sum_{r_1,r_2,s_1,s_2}\sum_{r,r',s}\int d\Gamma_1 d\Gamma_2 dP' h_{sr}(p,\mathfrak{s})\,\delta^{(4)}(p+p'-p_1-p_2)\langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle \\
& \times \langle p,p_2;r_1,r_2|t^\dagger|p,p';s,r'\rangle \prod_{j=1}^{2} h_{s_j r_j}(p_j,\mathfrak{s}_j) f(x,p_j,\mathfrak{s}_j) \\
& + i\frac{\pi\hbar}{8}\sum_{r_2,s_1,s_2}\sum_{r,s}\int d\Gamma_2 dS_1(p) h_{s_2 r_2}(p_2,\mathfrak{s}_2) f(x,p,\mathfrak{s}_1) f(x,p_2,\mathfrak{s}_2)[h_{sr}(p,\mathfrak{s}) h_{s_1 s}(p,\mathfrak{s}_1) + h_{s_1 s}(p,\mathfrak{s}) h_{sr}(p,\mathfrak{s}_1)] \\
& \times \langle p,p_2;r,r_2|t-t^\dagger|p,p_2;s_1,s_2\rangle + \frac{\pi\hbar}{4m}\sum_{r_2,s_1,s_2}\sum_{r}\int d\Gamma_2 dS_1(p) h_{s_2 r_2}(p_2,\mathfrak{s}_2) f(x,p,\mathfrak{s}_1) \\
& \times f(x,p_2,\mathfrak{s}_2)\mathfrak{s}_\mu \mathfrak{s}_{1\nu}\epsilon^{\mu\nu\alpha\beta} p_\alpha n_{s_1 r\beta}(p)\,\langle p,p_2;r,r_2|t+t^\dagger|p,p_2;s_1,s_2\rangle.
\end{aligned} \tag{62}$$

The expression above can be further simplified by noting that the term involving the amplitude with the operator $t - t^\dagger$ is related to the first term through the optical theorem [62]

$$i\pi\hbar\langle p,p_1;r,r_1|t-t^\dagger|p,p_1;s,s_1\rangle = -\sum_{r',r'_1}\int dP' dP'_1 \langle p,p_1;r,r_1|t|p',p'_1;r',r'_1\rangle\,\langle p',p'_1;r',r'_1|t^\dagger|p,p_1;s,s_1\rangle. \tag{63}$$

Hence, the collision term is cast into the form

$$\begin{aligned}
\mathfrak{C}_{\text{on-shell},l}[f] = {} & \frac{1}{4}\sum_{r_1,r_2,s_1,s_2}\sum_{r,r',s}\int d\Gamma_1 d\Gamma_2 dP' h_{sr}(p,\mathfrak{s})\,\delta^{(4)}(p+p'-p_1-p_2)\langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle \\
& \times \langle p,p_2;r_1,r_2|t^\dagger|p,p';s,r'\rangle \prod_{j=1}^{2} h_{s_j r_j}(p_j,\mathfrak{s}_j) f(x,p_j,\mathfrak{s}_j) - \frac{1}{8}\sum_{r_2,s_1,s_2}\sum_{r,s,r',r'_1}\int d\Gamma_2 dP' dP'_1 dS_1(p) h_{s_2 r_2}(p_2,\mathfrak{s}_2) \\
& \times f(x,p,\mathfrak{s}_1) f(x,p_2,\mathfrak{s}_2)[h_{sr}(p,\mathfrak{s}) h_{s_1 s}(p,\mathfrak{s}_1) + h_{s_1 s}(p,\mathfrak{s}) h_{sr}(p,\mathfrak{s}_1)]\langle p,p_2;r,r_2|t|p',p'_1;r',r'_1\rangle \\
& \times \langle p',p'_1;r',r'_1|t^\dagger|p,p_2;s_1,s_2\rangle + \frac{\pi\hbar}{4m}\sum_{r_2,s_1,s_2}\sum_{r}\int d\Gamma_2 dS_1(p) h_{s_2 r_2}(p_2,\mathfrak{s}_2) f(x,p,\mathfrak{s}_1) \\
& \times f(x,p_2,\mathfrak{s}_2)\mathfrak{s}_\mu \mathfrak{s}_{1\nu}\epsilon^{\mu\nu\alpha\beta} p_\alpha n_{s_1 r\beta}(p)\,\langle p,p_2;r,r_2|t+t^\dagger|p,p_2;s_1,s_2\rangle.
\end{aligned} \tag{64}$$

In order to write the collision term in a compact form, we insert factors of one for the phase-space spin variable in the form $1 = (1/2)\int dS(p)$ and obtain

$$\mathfrak{C}_{\text{on-shell},l}[f] \equiv \mathfrak{C}_{\text{p+s}}[f] + \mathfrak{C}_{\text{s}}[f], \tag{65}$$

with

$$\mathfrak{C}_{\text{p+s}}[f] \equiv \int d\Gamma_1 d\Gamma_2 d\Gamma' dS'_1(p)\mathcal{W}[f(x,p_1,\mathfrak{s}_1) f(x,p_2,\mathfrak{s}_2) - f(x,p,\mathfrak{s}'_1) f(x,p',\mathfrak{s}')], \tag{66a}$$

$$\mathfrak{C}_{\text{s}}[f] \equiv \int d\Gamma_2 dS_1(p)\mathfrak{W} f(x,p,\mathfrak{s}_1) f(x,p_2,\mathfrak{s}_2), \tag{66b}$$

where

$$\mathcal{W} \equiv \frac{1}{32}\sum_{s,r,s_1'}[h_{ss_1'}(p,\mathfrak{s}_1')h_{s_1'r}(p,\mathfrak{s}) + h_{ss_1'}(p,\mathfrak{s})h_{s_1'r}(p,\mathfrak{s}_1')] \times \sum_{s',r',s_1,s_2,r_1,r_2} h_{s'r'}(p',\mathfrak{s}')h_{s_1r_1}(p_1,\mathfrak{s}_1)h_{s_2r_2}(p_2,\mathfrak{s}_2) \times \langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle\langle p_1,p_2;r_1,r_2|t^\dagger|p,p';s,s'\rangle \times \delta^{(4)}(p+p'-p_1-p_2) \tag{67}$$

and

$$\mathfrak{W} \equiv \frac{\pi\hbar}{4m}\sum_{s_1,s_2,r,r_2}\epsilon_{\mu\nu\alpha\beta}\mathfrak{s}^\mu\mathfrak{s}_1^\nu p^\alpha n_{s_1r}^\beta(p)h_{s_2r_2}(p_2,\mathfrak{s}_2) \times \langle p,p_2;r,r_2|t+t^\dagger|p,p_2;s_1,s_2\rangle. \tag{68}$$

The term $\mathfrak{C}_{\mathrm{p+s}}[f]$ in Eq. (65) describes momentum- and spin-exchange interactions, while the term $\mathfrak{C}_{\mathrm{s}}[f]$ corresponds to spin exchange without momentum exchange. If the distribution functions do not depend on the spin variables, i.e., $f(x,p,\mathfrak{s}) \equiv f(x,p)$, we recover the collision term familiar from the Boltzmann equation, where averaging and summation over spins is done directly in the cross section. We do not obtain Pauli-blocking factors because of the low-density approximation [62]. Moreover, this implies that, in equilibrium, only Boltzmann instead of Fermi-Dirac distributions will appear. If the distribution functions depend on spin, the two terms on the right-hand side of Eq. (65) require further discussion. Considering $\mathfrak{C}_{\mathrm{p+s}}[f]$, the term proportional to $f(x,p_1,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2)$ has the form of a gain term for particles with momentum p and spin $\mathfrak{s}$, while the term proportional to $f(x,p,\mathfrak{s}_1')f(x,p',\mathfrak{s}')$ does not have an obvious interpretation as a loss term, because the spin variable is $\mathfrak{s}_1'$, not $\mathfrak{s}$.

However, it is possible to define a new distribution function and a new collision term such that we recover the standard interpretation of gain and loss terms in the latter without changing the physics. The underlying idea is that, since the phase-space spin variable is not observable, physical quantities are only obtained after integrating over the former; see, e.g., Eq. (35b). Therefore, we seek to replace the distribution function $f(x,p,\mathfrak{s})$ by another distribution function $\tilde{f}(x,p,\mathfrak{s})$ and likewise the collision term $\mathfrak{C}[f]$ by $\tilde{\mathfrak{C}}[\tilde{f}]$, such that

$$p\cdot\partial\tilde{f}(x,p,\mathfrak{s}) = \tilde{\mathfrak{C}}[\tilde{f}], \tag{69a}$$

$$\int dS(p)b\tilde{Q}(x,p,\mathfrak{s}) = \int dS(p)bQ(x,p,\mathfrak{s}) \tag{69b}$$

is fulfilled for $Q \in \{f,\mathfrak{C}[f]\}$, $\tilde{Q} \in \{\tilde{f},\tilde{\mathfrak{C}}[\tilde{f}]\}$, $b \in \{1,\mathfrak{s}^\mu\}$. In other words, after integration over the spin variable, the new distribution function $\tilde{f}(x,p,\mathfrak{s})$ and the collision term $\tilde{\mathfrak{C}}[\tilde{f}]$ are equivalent to the old ones. Moreover, they fulfill the same Boltzmann equation. Consequently, the new quantities $\tilde{f}(x,p,\mathfrak{s})$ and $\tilde{\mathfrak{C}}[\tilde{f}]$ describe the same physics as the old ones. Equations (69) constitute a "weak equivalence principle," stating that f and $\tilde{f}$ formally obey the same equation of motion and give identical results when integrating over the spin variable.

We now want to derive the collision term $\tilde{\mathfrak{C}}_{\mathrm{p+s}}[\tilde{f}]$, which satisfies the weak equivalence principle. The ultimate goal is to modify Eq. (66a) and the first line of Eq. (67) in such a way that $\mathfrak{s}_1$ is replaced by $\mathfrak{s}$ and the integration over $dS_1'(p)$ disappears, so that one obtains a collision term which has a standard gain and loss term. According to Eq. (43), we will show this separately for the (p + s) parts of C_F and C_A^μ, respectively. The calculation for C_F is actually straightforward, so we only present the somewhat more complicated case of C_A^μ. Considering $\int dS(p)\mathfrak{s}^\alpha\mathfrak{C}_{\mathrm{p+s}}[f]$, one encounters a term of the form

$$\begin{aligned}&\frac{1}{2}\sum_{s_1'}\int dS(p)dS_1'(p)\mathfrak{s}^\alpha\mathfrak{s}_\mu\mathfrak{s}_1'^\beta\mathfrak{s}_{1\nu}' \times [n_{s_1'r}^\mu(p)n_{ss_1'}^\nu(p) + n_{ss_1'}^\mu(p)n_{s_1'r}^\nu(p)] \\ &= -\int dS(p)dS_1'(p)\mathfrak{s}^\alpha\mathfrak{s}_\mu\mathfrak{s}_1'^\beta\mathfrak{s}_{1\nu}'\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)\delta_{rs} \\ &= 2\int dS(p)\mathfrak{s}^\alpha\mathfrak{s}^\beta\delta_{rs} = -4\left(g^{\alpha\beta}-\frac{p^\alpha p^\beta}{p^2}\right)\delta_{rs},\end{aligned} \tag{70}$$

where we used Eq. (60) in the first and Eq. (34c) in the last step. With this relation, and similar ones for the other terms, we see that, after integration over $dS(p)$, the replacement

$$\sum_{s_1'}\int dS_1'(p)[h_{ss_1'}(p,\mathfrak{s}_1')h_{s_1'r}(p,\mathfrak{s}) + h_{ss_1'}(p,\mathfrak{s})h_{s_1'r}(p,\mathfrak{s}_1')] \to 4h_{sr}(p,\mathfrak{s}) \tag{71}$$

fulfills the weak equivalence principle.

Furthermore, by definition, Eq. (32), f is linear in $\mathfrak{s}$, and, since $\mathcal{A}^{(0)\mu} = 0$, the $\mathfrak{s}$ dependence enters only at first order in $\hbar$, i.e., $f = f(\hbar\mathfrak{s})$. We assume that both f and $\tilde{f}$ can be Taylor expanded in terms of $\hbar\mathfrak{s}$. Inserting this Taylor expansion on the left- and right-hand sides of Eq. (69b), we conclude that, to order $\mathcal{O}(\hbar)$, the only choice for $\tilde{f}$ is $\tilde{f} \equiv f$, with deviations entering at order $\mathcal{O}(\hbar^2)$. With Eq. (71), we then find that, with the relations $\int dS(p)\mathfrak{s}^\mu = 0$ and $\int dS(p)\mathfrak{s}^\mu\mathfrak{s}^\nu\mathfrak{s}^\lambda = 0$, the choice

$$\tilde{\mathfrak{C}}_{\mathrm{p+s}}[f] \equiv \int d\Gamma_1 d\Gamma_2 d\Gamma'\tilde{\mathcal{W}} \times [f(x,p_1,\mathfrak{s}_1)f(x,p_2,\mathfrak{s}_2) - f(x,p,\mathfrak{s})f(x,p',\mathfrak{s}')], \tag{72}$$

with

$$\tilde{\mathcal{W}} \equiv \delta^{(4)}(p+p'-p_1-p_2)\frac{1}{8}\sum_{s,r} h_{sr}(p,\mathfrak{s}) \times \sum_{s',r',s_1,s_2,r_1,r_2} h_{s'r'}(p',\mathfrak{s}')h_{s_1r_1}(p_1,\mathfrak{s}_1)h_{s_2r_2}(p_2,\mathfrak{s}_2) \times \langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle\langle p_1,p_2;r_1,r_2|t^\dagger|p,p';s,s'\rangle, \tag{73}$$

satisfies the weak equivalence principle (69) up to $\mathcal{O}(\hbar)$.

Let us now focus on $\mathfrak{C}_s[f]$. We will argue that this term has already the expected structure with gain and loss terms. In order to see this, let us first note that $\mathfrak{C}_s[f]$ corresponds to collisions where the momentum of each particle is conserved, but the spin can change: $(p,\mathfrak{s}_1),(p_2,\mathfrak{s}_2)\to(p,\mathfrak{s}),(p_2,\mathfrak{s}')$ [62]. Here, the distribution functions $f(x,p,\cdot)$ and $f(x,p',\cdot)$ describe the particles before *and* after the collision, which means that they contribute to *both* the gain and the loss term. We see from Eq. (68) that the interchange of $\mathfrak{s}^\mu$ and $\mathfrak{s}_1^\nu$ flips the sign of $\mathfrak{W}$. This means that a net gain of particles with $(p,\mathfrak{s})$ corresponds to a net loss of particles with $(p,\mathfrak{s}_1)$. Thus, $\mathfrak{C}_s[f]$ contains both gain and loss terms.

V. NONLOCAL COLLISIONS

In order to calculate the nonlocal collision term, we focus on the second term in the second line of Eq. (15). Note that the $\partial^\mu_{q_j}\delta^{(4)}(q_j)$ term implies that the momentum p^μ of the Wigner function is not on shell anymore, which is in contrast to the local term. In fact, when we integrate by parts, the nonlocal kernel in Eq. (15) can be divided into two terms:

$$\mathfrak{C}^{(1)}_{nl} = \mathfrak{C}^{(1)}_{nl,1} + \mathfrak{C}^{(1)}_{nl,2}. \tag{74}$$

In the first term, the q_j-derivative acts on the spinors, i.e.,

$$\begin{aligned}\mathfrak{C}^{(1)}_{nl,1} = \frac{i}{8m^4}\sum_{r_1,r_2,s_1,s_2}\int d^4p_1d^4p_2d^4q_1d^4q_2\delta^{(4)}(q_1)\delta^{(4)}(q_2) \\ \times \mathrm{Tr}\left[\left(\frac{1}{p^2}p\cdot\gamma-\frac{1}{m}\mathfrak{s}\cdot\gamma\gamma^5\right)(2\pi\hbar)^6{}_{\mathrm{in}}\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|\Phi(p)\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\mathrm{in}}\right] \\ \times\left\{\bar{u}_{s_1}\left(p_1+\frac{q_1}{2}\right)W(x,p_1)u_{r_1}\left(p_1-\frac{q_1}{2}\right)\partial^\mu_{q_2}\left[\bar{u}_{s_2}\left(p_2+\frac{q_2}{2}\right)\partial_\mu W(x,p_2)u_{r_2}\left(p_2-\frac{q_2}{2}\right)\right]\right. \\ \left.+\,\partial^\mu_{q_1}\left[\bar{u}_{s_1}\left(p_1+\frac{q_1}{2}\right)\partial_\mu W(x,p_1)u_{r_1}\left(p_1-\frac{q_1}{2}\right)\right]\bar{u}_{s_2}\left(p_2+\frac{q_2}{2}\right)W(x,p_2)u_{r_2}\left(p_2-\frac{q_2}{2}\right)\right\},\end{aligned} \tag{75}$$

which is on shell because of $\delta^{(4)}(q_1)\delta^{(4)}(q_2)$, i.e.,

$$\mathfrak{C}^{(1)}_{nl,1} = \delta(p^2-m^2)\mathfrak{C}^{(1)}_{\text{on-shell},nl,1}. \tag{76}$$

In the second term in Eq. (74), the q_j-derivative acts on the matrix element of Φ, i.e.,

$$\begin{aligned}\mathfrak{C}^{(1)}_{nl,2} = \frac{i}{8m^4}\sum_{r_1,r_2,s_1,s_2}\int d^4p_1d^4p_2d^4q_1d^4q_2\delta^{(4)}(q_1)\delta^{(4)}(q_2) \\ \times\left\{\bar{u}_{s_1}\left(p_1+\frac{q_1}{2}\right)W(x,p_1)u_{r_1}\left(p_1-\frac{q_1}{2}\right)\bar{u}_{s_2}\left(p_2+\frac{q_2}{2}\right)\partial_\mu W(x,p_2)u_{r_2}\left(p_2-\frac{q_2}{2}\right)\partial^\mu_{q_2}\right. \\ \left.+\,\bar{u}_{s_1}\left(p_1+\frac{q_1}{2}\right)[\partial_\mu W(x,p_1)]u_{r_1}\left(p_1-\frac{q_1}{2}\right)\bar{u}_{s_2}\left(p_2+\frac{q_2}{2}\right)W(x,p_2)u_{r_2}\left(p_2-\frac{q_2}{2}\right)\partial^\mu_{q_1}\right\} \\ \times\mathrm{Tr}\left[\left(\frac{1}{p^2}p\cdot\gamma-\frac{1}{m}\mathfrak{s}\cdot\gamma\gamma^5\right)(2\pi\hbar)^6{}_{\mathrm{in}}\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|\Phi(p)\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\mathrm{in}}\right],\end{aligned} \tag{77}$$

which in general contains off-shell parts, cf. Appendix D. Note that the factor $(2\pi\hbar)^6$ in front of the matrix element in Eqs. (75) and (77) is part of the normalization of the latter, cf. Eq. (45), and does not participate in the $\hbar$-counting, which is obvious since it is not accompanied by any gradient ∂_μ. As discussed in relation to Eq. (17), in our expansion scheme, the zeroth-order distribution function is identified with the equilibrium distribution. Off-equilibrium contributions are at least of first order in gradients and will thus enter the nonlocal collision term

only at higher order. Thus, in the following discussion, we will always assume that the distribution functions in the nonlocal collision term are identical to the zeroth-order equilibrium distribution.

In Appendix D, it is shown that $\mathfrak{C}^{(1)}_{nl,2}$ can be divided into an on-shell and an off-shell part,

$$\begin{aligned}\mathfrak{C}^{(1)}_{nl,2} &= \mathfrak{C}^{(1)}_{\text{off-shell}} + \delta(p^2-m^2)\mathfrak{C}^{(1)}_{\text{on-shell},2} \\ &= \mathfrak{C}^{(1)}_{\text{off-shell}} + \delta(p^2-m^2)(\mathfrak{C}^{(1)}_{\text{on-shell},2,1} + \mathfrak{C}^{(1)}_{\text{on-shell},2,2}),\end{aligned} \tag{78}$$

and it is proved that the off-shell contribution $\mathfrak{C}^{(1)}_{\text{off-shell}}$ cancels the off-shell part of the left-hand side of the Boltzmann equation (42) when substituting Eq. (40). Furthermore, in Appendix D, we show that $\mathfrak{C}^{(1)}_{\text{on-shell},2,1} = 0$, if one inserts the zeroth-order equilibrium distribution [see Eq. (D18)]. The explicit expression for $\mathfrak{C}^{(1)}_{\text{on-shell},2,2}$ contains momentum derivatives of matrix elements and is computed in Appendix D; see Eq. (D16). This term is neglected as we assume the scattering amplitude to be constant over scales of order of the interaction range defining the scattering nonlocality. This is consistent with the low-density approximation; see e.g., Ref. [78]. Therefore, the Boltzmann equation contains only on-shell contributions and can be written in terms of the distribution function $f(x,p,\mathfrak{s})$ as

$$\delta(p^2-m^2)p\cdot\partial f(x,p,\mathfrak{s}) = \delta(p^2-m^2)\mathfrak{C}_{\text{on-shell}}[f], \tag{79}$$

with

$$\mathfrak{C}_{\text{on-shell}}[f] \equiv \mathfrak{C}_{\text{on-shell},l}[f] + \hbar\mathfrak{C}^{(1)}_{\text{on-shell},nl,1}[f], \tag{80}$$

where $\mathfrak{C}_{\text{on-shell},l}$ is the local term calculated in the previous section.

We are now ready to calculate $\mathfrak{C}^{(1)}_{nl,1}$. We note that, since in our scheme the lowest-order contribution to $\mathcal{A}^\mu$ and $\mathcal{S}^{\mu\nu}$ is of first order in gradients, these terms can be neglected in the nonlocal collision term, as the total contribution would be of second order. Using the spinor identities (C2), the relevant terms to compute in Eq. (75) are of the form

$$\begin{aligned}&i\partial^\mu_{q_j}\left[\bar u_{s_j}\left(p_j+\frac{q_j}{2}\right)\partial_\mu W(x,p_j)u_{r_j}\left(p_j-\frac{q_j}{2}\right)\right]_{q_j=0} \\ &\quad= i\left[\partial^\mu_{q_j}\bar u_{s_j}\left(p_j+\frac{q_j}{2}\right)u_{r_j}\left(p_j-\frac{q_j}{2}\right)\partial_\mu\mathcal{F}^{(0)}(x,p_j)+\partial^\mu_{q_j}\bar u_{s_j}\left(p_j+\frac{q_j}{2}\right)\gamma^\alpha u_{r_j}\left(p_j-\frac{q_j}{2}\right)\partial_\mu\mathcal{V}^{(0)}_\alpha(x,p_j)\right]_{q_j=0} \\ &\quad= \frac{1}{p_j^0+m}p_{j\nu}\Sigma^{\mu\nu}_{s_jr_j}(p_\star)\partial_\mu f^{(0)}(x,p_j) \\ &\quad= \frac{1}{p_j^0+m}[\mathbf{p}_j\times\mathbf{n}_{s_jr_j}(p_j)]\cdot\boldsymbol{\nabla}f^{(0)}(x,p_j) \\ &\quad= -\frac{1}{2(p_j^0+m)}\int dS_j(p_j)h_{s_jr_j}(p_j,\mathfrak{s}_j)(\mathbf{p}_j\times\boldsymbol{\mathfrak{s}}_j)\cdot\boldsymbol{\nabla}f^{(0)}(x,p_j),\end{aligned} \tag{81}$$

where $p^\mu_\star \equiv (m,\mathbf{0})$ is the 4-momentum in the rest frame of the particle. Using the result (81), defining the space-time shift

$$\Delta^\mu \equiv -\frac{\hbar}{2m(p\cdot\hat t+m)}\epsilon^{\mu\nu\alpha\beta}p_\nu\hat t_\alpha\mathfrak{s}_\beta, \tag{82}$$

where $\hat t^\mu$ is the timelike unit vector which is (1,0) in the frame where p^μ is measured, and applying similar steps as in the derivation of Eq. (66), we find that Eq. (75) becomes

$$\begin{aligned}\hbar\mathfrak{C}^{(1)}_{\text{on-shell},nl,1}[f] = \int d\Gamma_1 d\Gamma_2 d\Gamma' dS_1'(p)\mathcal{W}\,&[f(x,p_2,\mathfrak{s}_2)\Delta_1\cdot\partial f(x,p_1,\mathfrak{s}_1)+f(x,p_1,\mathfrak{s}_1)\Delta_2\cdot\partial f(x,p_2,\mathfrak{s}_2) \\ &- f(x,p',\mathfrak{s}')\Delta_1'\cdot\partial f(x,p,\mathfrak{s}_1')-f(x,p,\mathfrak{s}_1')\Delta'\cdot\partial f(x,p',\mathfrak{s}')] \\ &+\frac{\pi\hbar}{4m}\sum_{r,r_2,s_1,s_2}\int d\Gamma_2 dS_1(p)h_{s_2r_2}(p_2,\mathfrak{s}_2)\mathfrak{s}_\mu\mathfrak{s}_{1\nu}\epsilon^{\mu\nu\alpha\beta}p_\alpha n_{s_1r\beta}\langle p,p_2;r,r_2|t+t^\dagger|p,p_2;s_1,s_2\rangle \\ &\times[f(x,p,\mathfrak{s}_1)\Delta_2\cdot\partial f(x,p_2,\mathfrak{s}_2)+f(x,p_2,\mathfrak{s}_2)\Delta_1\cdot\partial f(x,p,\mathfrak{s}_1)].\end{aligned} \tag{83}$$

We now observe that $\Delta \cdot \partial f(x,p,\mathfrak{s})$ is the first-order contribution of the Taylor expansion of $f(x+\Delta,p,\mathfrak{s})$. Hence, after applying the weak equivalence principle (69) to Eq. (83), we can summarize the total collision term up to first order as

$$\begin{aligned}\tilde{\mathfrak{C}}_{\text{on-shell}}[f] = &\int d\Gamma_1 d\Gamma_2 d\Gamma' \tilde{\mathcal{W}} \\ &\times [f(x+\Delta_1,p_1,\mathfrak{s}_1) f(x+\Delta_2,p_2,\mathfrak{s}_2) \\ &- f(x+\Delta,p,\mathfrak{s}) f(x+\Delta',p',\mathfrak{s}')] \\ &+ \int d\Gamma_2 dS_1(p) \mathfrak{W} f(x+\Delta_1,p,\mathfrak{s}_1) \\ &\times f(x+\Delta_2,p_2,\mathfrak{s}_2). \end{aligned} \tag{84}$$

The interpretation of Eq. (84) is the following: incoming and outgoing particles are dislocated from the geometric center x of the collision by a spacelike distance Δ^μ. This leads to a finite difference between the incoming and outgoing orbital angular momentum. This angular momentum is converted into spin polarization through a collision, which leads to the alignment of spin with the direction of vorticity discussed in Ref. [34].

We remark that the nonlocality in Eq. (84) should be distinguished from the nonlocality of a collision due to the side-jump effect in the massless case without interactions as it was discussed in Refs. [79,80]. The latter arises due to the anomalous Lorentz transformation of the center of inertia of massless particles [36,81,82]. If a collision of massless particles is local in one reference frame, it will in general be nonlocal in a boosted reference frame, as the transformation behavior of the center of inertia leads to a position shift after the collision. On the other hand, for massive particles, it is always possible to define a space-time 4-vector associated with the center of mass, which properly transforms as a Lorentz vector as long as only local collisions are considered [34,36]. In other words, if the collision is local in one reference frame, it will stay local in all other reference frames. Hence, massive particles with local collisions will not experience any side-jump effect. The nonlocality in Eq. (84) is thus a nonlocality (in the sense of a finite impact parameter) in all reference frames, and, therefore, there is no "no-jump frame." In the massless case, this kind of nonlocality could be considered on top of the side-jump effect by introducing a collision which is not local even in the center-of-momentum frame. For a recent review about the difference between the centers of inertia and of mass and their connection to field theory, see Ref. [36].

VI. EQUILIBRIUM

In order to find the conditions necessary to reach equilibrium, we consider the standard form of the local equilibrium distribution function [8,25,30]

$$f_{\text{eq}}(x,p,\mathfrak{s}) = \frac{1}{(2\pi\hbar)^3} \exp\left[-\beta(x)\cdot p + \frac{\hbar}{4}\Omega_{\mu\nu}(x)\Sigma^{\mu\nu}_{\mathfrak{s}}\right]. \tag{85}$$

The exponent in Eq. (85) is a linear combination of the conserved quantities, which are momentum and total angular momentum, where the Lagrange multipliers $\beta^\mu(x) = u^\mu(x)/T(x)$ and $\Omega^{\mu\nu}(x)$ have the interpretation of fluid velocity over temperature and spin potential, respectively [25,29]. Here, we absorbed the orbital part of the angular momentum into the definition of $\beta^\mu(x)$ [8] and for the sake of simplicity considered the case of zero chemical potential, which corresponds to uncharged particles. We now insert Eq. (85) into Eq. (84) and obtain after expanding up to first order in $\hbar$

$$\begin{aligned}\tilde{\mathfrak{C}}_{\text{on-shell}}[f_{\text{eq}}] = &-\int d\Gamma' d\Gamma_1 d\Gamma_2 \tilde{\mathcal{W}} e^{-\beta\cdot(p_1+p_2)} \left[\partial_\mu\beta_\nu(\Delta_1^\mu p_1^\nu + \Delta_2^\mu p_2^\nu - \Delta^\mu p^\nu - \Delta'^\mu p'^\nu) - \frac{\hbar}{4}\Omega_{\mu\nu}(\Sigma^{\mu\nu}_{\mathfrak{s}_1} + \Sigma^{\mu\nu}_{\mathfrak{s}_2} - \Sigma^{\mu\nu}_{\mathfrak{s}} - \Sigma^{\mu\nu}_{\mathfrak{s}'})\right] \\ &- \int d\Gamma_2 dS_1(p) dS'(p_2) \mathfrak{W} e^{-\beta\cdot(p+p_2)} \left\{\partial_\mu\beta_\nu[(\Delta_1^\mu - \Delta^\mu)p^\nu + (\Delta_2^\mu - \Delta'^\mu)p_2^\nu] - \frac{\hbar}{4}\Omega_{\mu\nu}(\Sigma^{\mu\nu}_{\mathfrak{s}_1} + \Sigma^{\mu\nu}_{\mathfrak{s}_2} - \Sigma^{\mu\nu}_{\mathfrak{s}} - \Sigma^{\mu\nu}_{\mathfrak{s}'})\right\}. \end{aligned} \tag{86}$$

Here, we used that the zeroth-order contribution to the collision term vanishes for the distribution function (85). As the orbital angular momentum tensor of the particle with $(p,\mathfrak{s})$ is given by $L^{\mu\nu} = \Delta^{[\mu}p^{\nu]}$, the parentheses in the second and fourth lines can be interpreted as the balance of orbital angular momentum in the respective collision.

Defining the total angular momentum $J^{\mu\nu} = L^{\mu\nu} + \frac{\hbar}{2}\Sigma^{\mu\nu}_{\mathfrak{s}}$ of the particle, which is assumed to be conserved in a collision, $J^{\mu\nu} + J'^{\mu\nu} = J_1^{\mu\nu} + J_2^{\mu\nu}$, the conditions for the vanishing of the collision term are for any $\tilde{\mathcal{W}}$, $\mathfrak{W}$ given by

$$\partial_\mu\beta_\nu + \partial_\nu\beta_\mu = 0, \tag{87}$$

$$\Omega_{\mu\nu} = \varpi_{\mu\nu} \equiv -\frac{1}{2}\partial_{[\mu}\beta_{\nu]} = \text{const}. \tag{88}$$

This corresponds to global (and not just local) equilibrium.

We remark that the conditions for global equilibrium given in Eqs. (87) and (88) are necessary for the collision term to vanish if the standard ansatz for the equilibrium distribution function in Eq. (85) is used. In other words,

the standard concept of local equilibrium cannot be adopted when both spin effects and nonlocal collisions are considered. This result can be understood by looking at the ordering of scales. The usual way to derive hydrodynamics from kinetic theory is by assuming that there is a clear separation between some microscopic scale associated to the mean free path $\ell_{\rm mfp}$ and a macroscopic scale associated to the hydrodynamic gradients $L_{\rm hydro}$. One can then define the so-called Knudsen number Kn,

$$\mathrm{Kn} \equiv \frac{\ell_{\rm mfp}}{L_{\rm hydro}}, \tag{89}$$

and require that $\mathrm{Kn} \ll 1$. The expansion near equilibrium as defined in Eq. (17) is a measure of how important dissipative effects are and can be associated to an expansion in the Knudsen number. Qualitatively, we can write

$$\frac{\delta f}{f_{\rm eq}} \sim \mathcal{O}(\mathrm{Kn}), \tag{90}$$

where δf is a function which describes deviations of both the scalar and axial-vector part of the Wigner function from equilibrium. For systems made of particles with spin, in addition to the expansion in Eq. (90), we introduce an expansion in powers of $\hbar$. In our framework, spin effects and scattering nonlocality are treated as being of the same order, and it is natural to consider Δ as a new scale of the system. We can now relate a new parameter κ to the $\hbar$-expansion of the Wigner function, defined as

$$\kappa \equiv \frac{\Delta}{L_{\rm hydro}} \sim \frac{\hbar f^{(1)}}{f^{(0)}}. \tag{91}$$

The relation between κ and the Knudsen number is given by

$$\kappa = \frac{\Delta}{\ell_{\rm mfp}} \mathrm{Kn}. \tag{92}$$

In order for the assumption of molecular chaos to hold, and hence for particles to be considered as free between scatterings, we require that $\Delta \lesssim \ell_{\rm mfp}$, implying

$$\kappa \lesssim \mathrm{Kn}. \tag{93}$$

The physical implication of this condition is that a local-equilibrium description of a fluid with spin and nonlocal collisions in kinetic theory would be inconsistent with the power counting. In fact, if we consider spin as of first order in κ, we cannot neglect dissipative effects at first order in Kn. For related discussions, see also Refs. [26,41].

We can also express the condition in Eq. (93) in terms of the properties of the system. To this end, consider σ to be a cross section and n the particle density. The geometric area given by the cross section, $\sigma \equiv \pi r_{\rm int}^2$, defines a typical interaction range, $r_{\rm int} \equiv \sqrt{\sigma/\pi}$. On the other hand, the particle density n defines the typical interparticle distance $d \sim n^{-1/3}$. From Eq. (82), we conclude that $\Delta \sim \hbar/m$, i.e., Δ is of the order the Compton wavelength of the particle. Since $\ell_{\rm mfp} \sim (\sigma n)^{-1}$, the condition we need to satisfy reads

$$\frac{\Delta}{\ell_{\rm mfp}} \sim \frac{\Delta}{r_{\rm int}} \left(\frac{r_{\rm int}}{d}\right)^3 \lesssim 1. \tag{94}$$

As long as the interparticle distance is much larger than the interaction range, this condition is fulfilled, even if the Compton wavelength exceeds the interaction range.

The dissipative currents (bulk viscous pressure, diffusion current, and shear-stress tensor) can be expanded in terms of powers of gradients of temperature, chemical potential, and fluid velocity. Such a gradient expansion is, by virtue of Eq. (89), an expansion in powers of the Knudsen number. The first-order terms in this expansion correspond to the relativistic generalization of Navier-Stokes theory. However, it has been shown that this theory is acausal and unstable [83]. To remedy this shortcoming, transient theories of relativistic dissipative hydrodynamics have been developed [84], where the dissipative currents relax to their Navier-Stokes values on a timescale proportional to $\ell_{\rm mfp}$. Such theories effectively resum all orders of the gradient expansion and render relativistic dissipative hydrodynamics causal and stable. However, since the values of the dissipative currents can now differ from their asymptotic (Navier-Stokes) values, at least at early times, besides the Knudsen number, another, independent, dimensionless quantity enters, the inverse Reynolds number R^{-1}, which is defined as the ratio of a dissipative current over a quantity in thermodynamic equilibrium (e.g., pressure or particle density). Consequently, at early times $t \lesssim \ell_{\rm mfp}$, the inverse Reynolds number may differ from the Knudsen number. However, for late times $t \gtrsim \ell_{\rm mfp}$, all dissipative currents relax to their Navier-Stokes values, and we therefore do not need to differentiate between Kn and R^{-1}.

VII. CONCLUSIONS

In this paper, we provided a detailed derivation of the collision term in the Boltzmann equation starting from the Wigner-function formalism put forward in Ref. [34]. The main result of this work is to provide an explicit expression of the nonlocal collision kernel based on the framework developed in Ref. [62]. The advantage of this formalism is that it relates the collision kernel to vacuum scattering amplitudes, which can be computed using standard field-theory techniques. Employing the $\hbar$, or gradient, expansion to solve for the Wigner function, it follows that the nonlocal term enters at next-to-leading order. Enlarging the phase space to include a classical variable related to spin degrees of freedom allows one to write the equations of motion for the Clifford components of the Wigner function as a single

scalar Boltzmann equation. Furthermore, we proved that this scalar equation contains only on-shell contributions. An implication of this work is that the conditions for the collision term to vanish are those of global equilibrium [34]. Moreover, in Refs. [34,36], the relation between the scattering nonlocality and spin hydrodynamics is analyzed. An important question for phenomenological applications, which can be addressed using nonequilibrium spin dynamics, is whether spin equilibrates sufficiently fast on the timescale of the evolution of the hot and dense system created in heavy-ion collisions. Recent works addressing the spin-equilibration time can be found in Refs. [76,85–89].

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APPENDIX A: ENSEMBLE AVERAGE OF THE COLLISION TERM

Consider an arbitrary operator O. In kinetic theory, where dilute systems are considered, it is permissible to take the ensemble average $\langle O\rangle$ with respect to the initial, free n-particle states defined in Eq. (10). The ensemble average $\langle O\rangle$ is derived in Ref. [62] and reads

$$\begin{aligned}\langle O\rangle = \sum_{n=0}^{\infty}\frac{4^n}{n!}\sum_{r_1\cdots r_n}\sum_{r'_1\cdots r'_n}\int dP_1\cdots dP_n dP'_1\cdots dP'_n \\ \times {}_{\text{in}}\langle\!\langle p_1,\ldots,p_n;r_1,\ldots,r_n|O|p'_1,\ldots,p'_n;r'_1,\ldots,r'_n\rangle\!\rangle_{\text{in}} \\ \times \langle a^\dagger_{\text{in},r_1}(p_1)\cdots a^\dagger_{\text{in},r_n}(p_n)a_{\text{in},r'_1}(p'_1)\cdots a_{\text{in},r'_n}(p'_n)\rangle,\end{aligned} \tag{A1}$$

where $dP_i \equiv d^4p_i\delta(p_i^2-m^2)$, $dP'_i \equiv d^4p'_i\delta(p_i'^2-m^2)$ and where we defined

$$\begin{aligned}&{}_{\text{in}}\langle\!\langle p_1,\ldots,p_n;r_1,\ldots,r_n|O|p'_1,\ldots,p'_n;r'_1,\ldots,r'_n\rangle\!\rangle_{\text{in}} \\ &= \mathfrak{A}\sum_{m=0}^{n}(-1)^m\left(\frac{n!}{m!(n-m)!}\right)^2{}_{\text{in}}\langle p_1,\ldots,p_m;r_1,\ldots,r_m|p'_1,\ldots,p'_m;r'_1,\ldots,r'_n\rangle_{\text{in}} \\ &\quad\times {}_{\text{in}}\langle p_{m+1},\ldots,p_n;r_{m+1},\ldots,r_n|O|p'_{m+1},\ldots,p'_n;r'_{m+1},\ldots,r'_n\rangle_{\text{in}}.\end{aligned} \tag{A2}$$

Here, the symbol $\mathfrak{A}$ indicates the antisymmetrization with respect to all momenta and spin indices. Neglecting initial correlations, the expectation values of the creation and annihilation operators factorize according to

$$\begin{aligned}&\langle a^\dagger_{\text{in},r_1}(p_1)\cdots a^\dagger_{\text{in},r_n}(p_n)a_{\text{in},r'_1}(p'_1)\cdots a_{\text{in},r'_n}(p'_n)\rangle \\ &\quad=\sum_{\mathfrak{P}}(-1)^{\mathfrak{P}}\prod_{j=1}^{n}\langle a^\dagger_{\text{in},r_j}(p_j)a_{\text{in},r'_j}(p'_j)\rangle,\end{aligned} \tag{A3}$$

where $\mathfrak{P}$ denotes the sum over all permutations of primed and unprimed variables with $(-1)^{\mathfrak{P}}=1$ for even permutations and $(-1)^{\mathfrak{P}}=-1$ for odd permutations.

We are now interested in computing the ensemble average of the collision operator in Eq. (9). Using the relation for the field operator in the Heisenberg picture

$$\psi\left(x-\frac{y}{2}\right) = e^{\frac{i}{\hbar}P\cdot x}\psi\left(-\frac{y}{2}\right)e^{-\frac{i}{\hbar}P\cdot x}, \tag{A4}$$

where P^μ is the total 4-momentum operator, and applying a similar formula also to $\bar\psi,\rho,\bar\rho$, Eq. (9) reads

$$C_{\alpha\beta} = \langle e^{\frac{i}{\hbar}P\cdot x}\Phi_{\alpha\beta}(p)e^{-\frac{i}{\hbar}P\cdot x}\rangle, \tag{A5}$$

with $\Phi_{\alpha\beta}$ given by Eq. (12). At this point, we can calculate $C_{\alpha\beta}$ using Eq. (A1) with the factorization in Eq. (A3). For the scattering kernel $C_{\alpha\beta}$, Eq. (A3) corresponds to the assumption of molecular chaos. Furthermore, since we consider only binary scatterings, we restrict ourselves to two-particle states, i.e., $n=2$. Hence, after exploiting the fact that two-particle states are eigenstates of the total momentum, Eq. (A5) takes the form

$$\begin{aligned}C_{\alpha\beta} = 8\sum_{r_1,r_2,r'_1,r'_2}\int dP_1dP_2dP'_1dP'_2 \\ \times {}_{\text{in}}\langle p_1,p_2;r_1,r_2|\Phi(p)|p'_1,p'_2;r'_1,r'_2\rangle_{\text{in}} \\ \times\prod_{j=1}^{2}e^{\frac{i}{\hbar}(p_j-p'_j)\cdot x}\langle a^\dagger_{\text{in},r_j}(p_j)a_{\text{in},r'_j}(p'_j)\rangle.\end{aligned} \tag{A6}$$

The positive-energy part of the initial noninteracting field is

$$\psi_{\mathrm{in}}(x) = \sqrt{\frac{2}{(2\pi\hbar)^3}}\sum_r \int dP e^{-\frac{i}{\hbar}p\cdot x} u_r(p) a_{\mathrm{in},r}(p). \tag{A7}$$

Using the inverse relation

$$\frac{1}{m\sqrt{2(2\pi\hbar)^5}}\int d^4x e^{\frac{i}{\hbar}p\cdot x}\bar{u}_r(p)\psi_{\mathrm{in}}(x) = 2\delta(p^2-m^2)a_{\mathrm{in},r}(p), \tag{A8}$$

we can express Eq. (A6) in terms of the initial Wigner function

$$W_{\mathrm{in},\alpha\beta}(x,p) = \int \frac{d^4y}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p\cdot y}\langle :\bar{\psi}_{\mathrm{in},\beta}(x_1)\psi_{\mathrm{in},\alpha}(x_2):\rangle. \tag{A9}$$

The result is given in Eq. (11).

APPENDIX B: CALCULATION OF THE EXPECTATION VALUE OF Φ

We want to explicitly compute the scattering-matrix element in Eq. (15),

$$_{\mathrm{in}}\left\langle p_1-\frac{q_1}{2}, p_2-\frac{q_2}{2}; r_1, r_2\middle|\Phi(p)\middle|p_1+\frac{q_1}{2}, p_2+\frac{q_2}{2}; s_1, s_2\right\rangle_{\mathrm{in}}, \tag{B1}$$

where the operator $\Phi(p)$ is given in Eq. (12). Inserting a completeness relation of free out states and following similar steps as done in Ref. [62], we obtain after the y-integration

$$\begin{aligned}
&_{\mathrm{in}}\left\langle p_1-\frac{q_1}{2}, p_2-\frac{q_2}{2}; r_1, r_2\middle|\Phi(p)\middle|p_1+\frac{q_1}{2}, p_2+\frac{q_2}{2}; s_1, s_2\right\rangle_{\mathrm{in}} = i\sum_{r'}\int dP'\delta^{(4)}(p+p'-p_1-p_2)\\
&\times\left\{_{\mathrm{out}}\left\langle p'; r'\middle|\psi(0)\middle|p_1+\frac{q_1}{2}, p_2+\frac{q_2}{2}; s_1, s_2\right\rangle_{\mathrm{in\,in}}\left\langle p_1-\frac{q_1}{2}, p_2-\frac{q_2}{2}; r_1, r_2\middle|:\bar{\rho}(0):\middle|p'; r'\right\rangle_{\mathrm{out}}\left[\gamma\cdot\left(p-\frac{q_1+q_2}{2}\right)+m\right]\right.\\
&\left.-\left[\gamma\cdot\left(p+\frac{q_1+q_2}{2}\right)+m\right]_{\mathrm{out}}\left\langle p'; r'\middle|:\rho(0):\middle|p_1+\frac{q_1}{2}, p_2+\frac{q_2}{2}; s_1, s_2\right\rangle_{\mathrm{in\,in}}\left\langle p_1-\frac{q_1}{2}, p_2-\frac{q_2}{2}; r_1, r_2\middle|\bar{\psi}(0)\middle|p'; r'\right\rangle_{\mathrm{out}}\right\}.
\end{aligned} \tag{B2}$$

In deriving Eq. (B2), we also made use of the fact that one- and two-particle states are eigenstates of the total momentum operator; hence, the expectation value involving, e.g., $\psi(-y/2)$, is given by

$$\begin{aligned}
&_{\mathrm{out}}\left\langle p'; r'\middle|\psi\left(-\frac{y}{2}\right)\middle|p_1+\frac{q_1}{2}, p_2+\frac{q_2}{2}; s_1, s_2\right\rangle_{\mathrm{in}}\\
&= e^{\frac{i}{2\hbar}(p'-p_1-q_1/2-p_2-q_2/2)\cdot y}\\
&\times\,_{\mathrm{out}}\left\langle p'; r'\middle|\psi(0)\middle|p_1+\frac{q_1}{2}, p_2+\frac{q_2}{2}; s_1, s_2\right\rangle_{\mathrm{in}}.
\end{aligned} \tag{B3}$$

In order to compute the matrix element on the right-hand side of this equation, we write the field $\psi(0)$ as a general solution of the Dirac equation in the presence of interaction

$$\psi(0) = \psi_{\mathrm{in}}(0) + \int d^4x S_R(-x)\rho(x), \tag{B4}$$

where ψ_{in} is given in Eq. (A7) and $S_R(x)$ is the retarded Green's function, which we express as a Fourier transform

$$S_R(x) = \frac{1}{(2\pi\hbar)^4}\int d^4p\tilde{S}_R(p)e^{-\frac{i}{\hbar}p\cdot x}, \tag{B5}$$

with

$$\tilde{S}_R(p) = -\frac{1}{\hbar}(\gamma\cdot p+m)G(p), \tag{B6}$$

and $G(p)$ defined in Eq. (52). The matrix element of ψ is thus given by

$$\begin{aligned}
&_{\mathrm{out}}\left\langle p'; r'\middle|\psi(0)\middle|p_1+\frac{q_1}{2}, p_2+\frac{q_2}{2}; s_1, s_2\right\rangle_{\mathrm{in}}\\
&= \left[u_{s_1}\left(p_1+\frac{q_1}{2}\right)\delta^{(3)}\left(\mathbf{p}'-\mathbf{p_2}-\frac{\mathbf{q_2}}{2}\right)\delta_{r's_2}-(1\leftrightarrow 2)\right]\\
&\times\frac{p'^0}{\sqrt{2(2\pi\hbar)^3}}+\tilde{S}_R\left(p_1+\frac{q_1}{2}+p_2+\frac{q_2}{2}-p'\right)\\
&\times\,_{\mathrm{out}}\left\langle p', r'\middle|:\rho(0):\middle|p_1+\frac{q_1}{2}, p_2+\frac{q_2}{2}; s_1, s_2\right\rangle_{\mathrm{in}},
\end{aligned} \tag{B7}$$

where we made use of the orthogonality condition $\langle p, r|p', r'\rangle = p^0\delta^{(3)}(\mathbf{p}-\mathbf{p}')\delta_{rr'}$. Plugging Eq. (B7) into Eq. (B2) and using Eqs. (B6), (52), as well as the relation

$$(\gamma\cdot k+m)_{\alpha\beta} = \sum_r u_r(k)_\alpha\bar{u}_r(k)_\beta, \tag{B8}$$

we obtain

$$
\begin{aligned}
&{}_{\text{in}}\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|\Phi(p)\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\text{in}}\\
&=\frac{i}{2}\sum_{r,s}\Bigg\{\frac{1}{\sqrt{2(2\pi\hbar)^3}}\left[\delta^{(3)}\left(\mathbf{p}-\mathbf{p_1}+\frac{\mathbf{q_2}}{2}\right)\delta\left(p^0+\sqrt{\left(\mathbf{p_2}+\frac{\mathbf{q_2}}{2}\right)^2+m^2}-E_{p_1}-E_{p_2}\right)\delta_{s_1r}\right.\\
&\left.\times u_r\left(p+\frac{q_1+q_2}{2}\right)\bar{u}_s\left(p-\frac{q_1+q_2}{2}\right){}_{\text{in}}\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|:\bar{\rho}(0):\middle|p_2;s_2\right\rangle_{\text{out}}u_s\left(p-\frac{q_1+q_2}{2}\right)+(1\leftrightarrow 2)\right]\\
&-\frac{1}{\sqrt{2(2\pi\hbar)^3}}\left[\delta^{(3)}\left(\mathbf{p}-\mathbf{p_1}-\frac{\mathbf{q_2}}{2}\right)\delta\left(p^0+\sqrt{\left(\mathbf{p_2}-\frac{\mathbf{q_2}}{2}\right)^2+m^2}-E_{p_1}-E_{p_2}\right)\delta_{sr_1}\right.\\
&\left.\times u_r\left(p+\frac{q_1+q_2}{2}\right)\bar{u}_s\left(p-\frac{q_1+q_2}{2}\right)\bar{u}_r\left(p+\frac{q_1+q_2}{2}\right){}_{\text{out}}\left\langle p_2;r_2\middle|:\rho(0):\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\text{in}}+(1\leftrightarrow 2)\right]\\
&-\hbar\sum_{r'}\int dP'\delta^{(4)}(p+p'-p_1-p_2)\left[G\left(p+\frac{q_1+q_2}{2}\right)-G^{\star}\left(p-\frac{q_1+q_2}{2}\right)\right]\\
&\times u_r\left(p+\frac{q_1+q_2}{2}\right)\bar{u}_s\left(p-\frac{q_1+q_2}{2}\right)\bar{u}_r\left(p+\frac{q_1+q_2}{2}\right){}_{\text{out}}\left\langle p';r'\middle|:\rho(0):\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\text{in}}\\
&\times{}_{\text{in}}\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|:\bar{\rho}(0):\middle|p';r'\right\rangle_{\text{out}}u_s\left(p-\frac{q_1+q_2}{2}\right)\Bigg\},
\end{aligned}
\tag{B9}
$$

where we defined $E_p\equiv\sqrt{\mathbf{p}^2+m^2}$. Finally, we use Eq. (47) to write

$$
\begin{aligned}
&{}_{\text{in}}\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|\Phi(p)\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle_{\text{in}}\\
&\quad=\frac{1}{2(2\pi\hbar)^6}\sum_{r,s}u_r\left(p+\frac{q_1+q_2}{2}\right)\bar{u}_s\left(p-\frac{q_1+q_2}{2}\right)w^{rs}_{r_1r_2s_1s_2}(p_1,q_1,p_2,q_2,p),
\end{aligned}
\tag{B10}
$$

with

$$
\begin{aligned}
w^{rs}_{r_1r_2s_1s_2}(p_1,q_1,p_2,q_2,p)&=2\sum_{r'}\int dP'\frac{1}{i\pi\hbar^2}\left[G\left(p+\frac{q_1+q_2}{2}\right)-G^{\star}\left(p-\frac{q_1+q_2}{2}\right)\right]\\
&\times\delta^{(4)}(p+p'-p_1-p_1)\left\langle p+\frac{q_1+q_2}{2},p';r,r'\middle|t\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle\\
&\times\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|t^{\dagger}\middle|p-\frac{q_1+q_2}{2},p';s,r'\right\rangle\\
&-i2\pi\hbar\delta^{(3)}\left(\mathbf{p}-\mathbf{p_1}+\frac{\mathbf{q_2}}{2}\right)\delta\left(p^0+p_2^0+\frac{q_2^0}{2}-E_{p_1}-E_{p_2}\right)\\
&\times\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|t^{\dagger}\middle|p-\frac{q_1+q_2}{2},p_2;s,s_2\right\rangle\delta_{rs_1}+(1\leftrightarrow 2)\\
&+i2\pi\hbar\delta^{(3)}\left(\mathbf{p}-\mathbf{p_1}-\frac{\mathbf{q_2}}{2}\right)\delta\left(p^0+p_2^0-\frac{q_2^0}{2}-E_{p_1}-E_{p_2}\right)\\
&\times\left\langle p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};r,r_2\middle|t\middle|p+\frac{q_1+q_2}{2},p_2;s_1,s_2\right\rangle\delta_{r_1s}+(1\leftrightarrow 2).
\end{aligned}
\tag{B11}
$$

Here, we made use of the fact that to linear order in $\mathbf{q}_2$ we may replace $\sqrt{(\mathbf{p_2}\pm\mathbf{q_2}/2)^2+m^2}=p_2^0\pm q_2^0/2$. Working to linear order is sufficient since we only consider zeroth- and first-order terms of a Taylor expansion in q_i in Eq. (15). In this form, the scattering-matrix element is inserted into Eq. (15). Furthermore, for the sake of simplicity, we define

$$
w^{rs}_{r_1r_2s_1s_2}(p_1,p_2,p)\equiv w^{rs}_{r_1r_2s_1s_2}(p_1,q_1=0,p_2,q_2=0,p).
\tag{B12}
$$

APPENDIX C: SPINOR IDENTITIES

For any on-shell momentum $p^\mu = (E_p, \mathbf{p})$, we can write

$$u_r(p) = \frac{\gamma\cdot p + m}{\sqrt{2m(E_p+m)}} u_r(p_\star), \tag{C1}$$

where $p^\mu_\star = (m, \mathbf{0})$ is the 4-momentum in the rest frame of the particle. Then, the following identities hold to first order in q^μ,

$$\begin{aligned}
\bar{u}_s\left(p+\frac{q}{2}\right)u_r\left(p-\frac{q}{2}\right) &= 2m\delta_{sr} - \frac{i}{2(E_p+m)} q_\mu p_\nu \Sigma^{\mu\nu}_{sr}(p_\star),\\
\bar{u}_s\left(p+\frac{q}{2}\right)\gamma^\alpha u_r\left(p-\frac{q}{2}\right) &= 2p^\alpha\delta_{sr} + \frac{im}{2(E_p+m)} q_\mu \Sigma^{\alpha\mu}_{sr}(p_\star) - \frac{i}{E_p+m}\epsilon^{\alpha\mu\nu\rho} q_\mu p_\nu n_{sr\rho}(p_\star),\\
\bar{u}_s\left(p+\frac{q}{2}\right)\gamma^5\gamma^\alpha u_r\left(p-\frac{q}{2}\right) &= 2mn^\alpha_{sr}(p) - \frac{i}{E_p+m}\epsilon^{\alpha\mu\nu 0} q_\nu p_\mu \delta_{sr},
\end{aligned} \tag{C2}$$

where we used Eqs. (50), (54), and the identity

$$\gamma^\mu\gamma^\alpha\gamma^\nu = g^{\mu\alpha}\gamma^\nu + g^{\alpha\nu}\gamma^\mu - g^{\nu\mu}\gamma^\alpha - i\epsilon^{\mu\alpha\nu\rho}\gamma_\rho\gamma^5. \tag{C3}$$

APPENDIX D: CALCULATION OF NONLOCAL COLLISION TERM

The second contribution of the nonlocal term in Eq. (77) term is given by

$$\begin{aligned}
m\mathfrak{C}^{(1)}_{nl,2} &= i\frac{(2\pi\hbar)^6}{8m^4}\sum_{r_1,r_2,s_1,s_2}\int d^4p_1 d^4p_2 d^4q_1 d^4q_2 \delta^{(4)}(q_1)\delta^{(4)}(q_2)\\
&\times\left\{\partial^\mu_{q_1}\mathrm{Tr}\left[\left(\frac{m}{p^2}p\cdot\gamma - \mathfrak{s}\cdot\gamma\gamma^5\right)_{\mathrm{in}}\left\langle p_1-\frac{q_1}{2}, p_2-\frac{q_2}{2}; r_1, r_2\middle|\Phi(p)\middle|p_1+\frac{q_1}{2}, p_2+\frac{q_2}{2}; s_1, s_2\right\rangle_{\mathrm{in}}\right]\right.\\
&\times \bar{u}_{s_2}(p_2)W(x,p_2)u_{r_2}(p_2)\bar{u}_{s_1}(p_1)\partial_\mu W(x,p_1)u_{r_1}(p_1)\\
&+\partial^\mu_{q_2}\mathrm{Tr}\left[\left(\frac{m}{p^2}p\cdot\gamma - \mathfrak{s}\cdot\gamma\gamma^5\right)_{\mathrm{in}}\left\langle p_1-\frac{q_1}{2}, p_2-\frac{q_2}{2}; r_1, r_2\middle|\Phi(p)\middle|p_1+\frac{q_1}{2}, p_2+\frac{q_2}{2}; s_1, s_2\right\rangle_{\mathrm{in}}\right]\\
&\left.\times \bar{u}_{s_1}(p_1)W(x,p_1)u_{r_1}(p_1)\bar{u}_{s_2}(p_2)\partial_\mu W(x,p_2)u_{r_2}(p_2)\right\}\\
&= \frac{i}{16m^4}\sum_{r,s,r_1,r_2,s_1,s_2}\int d^4p_1 d^4p_2 d^4q_1 d^4q_2\delta^{(4)}(q_1)\delta^{(4)}(q_2)\\
&\times\left\{\partial^\mu_{q_1}\left[\bar{u}_s\left(p-\frac{q_1}{2}-\frac{q_2}{2}\right)\left(\frac{m}{p^2}p\cdot\gamma-\mathfrak{s}\cdot\gamma\gamma^5\right)u_r\left(p+\frac{q_1+q_2}{2}\right)w'^{rs}_{r_1r_2s_1s_2}(p_1,q_1,p_2,q_2,p)\right]\right.\\
&\times\bar{u}_{s_2}(p_2)W(x,p_2)u_{r_2}(p_2)\bar{u}_{s_1}(p_1)\partial_\mu W(x,p_1)u_{r_1}(p_1)\\
&+\partial^\mu_{q_2}\left[\bar{u}_s\left(p-\frac{q_1+q_2}{2}\right)\left(\frac{m}{p^2}p\cdot\gamma-\mathfrak{s}\cdot\gamma\gamma^5\right)u_r\left(p+\frac{q_1}{2}+\frac{q_2}{2}\right)w'^{rs}_{r_1r_2s_1s_2}(p_1,q_1,p_2,q_2,p)\right]\\
&\left.\times\bar{u}_{s_1}(p_1)W(x,p_1)u_{r_1}(p_1)\bar{u}_{s_2}(p_2)\partial_\mu W(x,p_2)u_{r_2}(p_2)\right\}\\
&= \frac{i}{16m^4}\sum_{r,s,r_1,r_2,s_1,s_2}\int d^4p_1 d^4p_2\left\{\frac{i}{2(p^0+m)}[p_\nu\Sigma^{\mu\nu}_{sr}(p_\star)+\epsilon^{\nu\lambda\mu 0}p_\nu\mathfrak{s}_\lambda\delta_{sr}]w'^{rs}_{r_1r_2s_1s_2}(p_1,p_2,p)\right.\\
&\left.\times\partial_\mu\bar{u}_{s_1}(p_1)W(x,p_1)u_{r_1}(p_1)\bar{u}_{s_2}(p_2)W(x,p_2)u_{r_2}(p_2)\right\}\\
&+\frac{i}{16m^4}\sum_{r,s,r_1,r_2,s_1,s_2}\int d^4p_1 d^4p_2\left[\frac{m}{p^2}\bar{u}_s(p)p\cdot\gamma u_r(p) - \mathfrak{s}_\mu\bar{u}_s(p)\gamma^\mu\gamma^5 u_r(p)\right]\\
&\times\{[\partial^\mu_{q_1}w'^{rs}_{r_1r_2s_1s_2}(p_1,q_1,p_2,q_2,p)]_{q_1=q_2=0}\bar{u}_{s_2}(p_2)W(x,p_2)u_{r_2}(p_2)\bar{u}_{s_1}(p_1)\partial_\mu W(x,p_1)u_{r_1}(p_1)\\
&+[\partial^\mu_{q_2}w'^{rs}_{r_1r_2s_1s_2}(p_1,q_1,p_2,q_2,p)]_{q_1=q_2=0}\bar{u}_{s_1}(p_1)W(x,p_1)u_{r_1}(p_1)\bar{u}_{s_2}(p_2)\partial_\mu W(x,p_2)u_{r_2}(p_2)\},
\end{aligned} \tag{D1}$$

where we used Eq. (B10) in the second step and, in the last step, Eq. (B12) and the relation

$$\partial^{\alpha}_{q_j}\left[\frac{m}{p^2}p_\mu \bar{u}_s\left(p-\frac{q_1+q_2}{2}\right)\gamma^\mu u_r\left(p+\frac{q_1+q_2}{2}\right)+\mathfrak{s}_\mu \bar{u}_s\left(p-\frac{q_1+q_2}{2}\right)\gamma^5\gamma^\mu u_r\left(p+\frac{q_1+q_2}{2}\right)\right]_{q_1=q_2=0}$$
$$=\frac{i}{2(p^0+m)}\left[\frac{m^2}{p^2}p_\nu \Sigma^{\alpha\nu}_{sr}(p_\star)+\epsilon^{\nu\mu\alpha 0}p_\nu \mathfrak{s}_\mu \delta_{sr}\right]. \tag{D2}$$

When inserting this equation into Eq. (D1), the term $m^2/p^2=1$, since $w^{rs}_{r_1r_2s_1s_2}(p_1,p_2,p)$ puts the 4-momentum p^μ on shell; see Eq. (46).

The q_j-derivatives acting on $w^{rs}_{r_1r_2s_1s_2}(p_1,q_1,p_2,q_2,p)$ in the last two lines of Eq. (D1) contain several terms. In order to calculate them, it is convenient to split $w^{rs}_{r_1r_2s_1s_2}(p_1,q_1,p_2,q_2,p)$, cf. Eq. (B11), into a gain term,

$$w^{rs}_{r_1r_2s_1s_2,\mathrm{gain}}(p_1,q_1,p_2,q_2,p)=2\sum_{r'}\int dP'\frac{1}{i\pi\hbar^2}\left[G\left(p+\frac{q_1+q_2}{2}\right)-G^\star\left(p-\frac{q_1+q_2}{2}\right)\right]\delta^{(4)}(p+p'-p_1-p_1)$$
$$\times\left\langle p+\frac{q_1+q_2}{2},p';r,r'\middle|t\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle$$
$$\times\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|t^\dagger\middle|p-\frac{q_1+q_2}{2},p';s,r'\right\rangle, \tag{D3}$$

and a loss term,

$$w^{rs}_{r_1r_2s_1s_2,\mathrm{loss}}(p_1,q_1,p_2,q_2,p)=-i2\pi\hbar\delta^{(3)}\left(\mathbf{p}-\mathbf{p_1}+\frac{\mathbf{q_2}}{2}\right)\delta\left(p^0+\sqrt{\left(\mathbf{p_2}+\frac{\mathbf{q_2}}{2}\right)^2+m^2}-E_{p_1}-E_{p_2}\right)$$
$$\times\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|t^\dagger\middle|p-\frac{q_1+q_2}{2},p_2;s,s_2\right\rangle\delta_{rs_1}+(1\leftrightarrow 2)$$
$$+i2\pi\hbar\delta^{(3)}\left(\mathbf{p}-\mathbf{p_1}-\frac{\mathbf{q_2}}{2}\right)\delta\left(p^0+\sqrt{\left(\mathbf{p_2}-\frac{\mathbf{q_2}}{2}\right)^2+m^2}-E_{p_1}-E_{p_2}\right)$$
$$\times\left\langle p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};r,r_2\middle|t\middle|p+\frac{q_1+q_2}{2},p_2;s_1,s_2\right\rangle\delta_{r_1s}+(1\leftrightarrow 2). \tag{D4}$$

Since we compute a contribution of order $\mathcal{O}(\hbar)$, the Wigner functions in Eq. (D1) can be approximated by their zeroth-order expression, such that the terms proportional to $W\partial_\mu W$ will give rise to terms proportional to $f^{(0)}\partial_\mu f^{(0)}$, with f being the zeroth-order contribution to $f(x,p,\mathfrak{s})$. To zeroth order, the $\mathfrak{s}$-dependence vanishes, such that $f^{(0)}(x,p,\mathfrak{s})\equiv f^{(0)}(x,p)$. Acting with the q_j-derivative on the gain part, Eq. (D3), the respective terms in Eq. (D1) lead to contributions of the form

$$[f^{(0)}(x,p_2)\partial^\mu f^{(0)}(x,p_1)\partial_{q_1\mu}+f^{(0)}(x,p_1)\partial^\mu f^{(0)}(x,p_2)\partial_{q_2\mu}]\left[G\left(p+\frac{q_1+q_2}{2}\right)-G^\star\left(p-\frac{q_1+q_2}{2}\right)\right]_{q_1=q_2=0}$$
$$=-\frac{1}{2}\hbar^2\partial^\mu f^{(0)}(x,p_1)f^{(0)}(x,p_2)\,\partial_{q\mu}\left[\frac{1}{(p+q)^2-m^2-i\epsilon(p^0+q^0)}-\frac{1}{(p-q)^2-m^2+i\epsilon(p^0-q^0)}\right]_{q=0}$$
$$=\partial^\mu f^{(0)}(x,p_1)f^{(0)}(x,p_2)\frac{p_\mu}{p^2-m^2}[G(p)+G^\star(p)]. \tag{D5}$$

Due to the factor p^2-m^2 in the denominator, this is an off-shell contribution to the Boltzmann equation. Further off-shell contributions also emerge when the q_j-derivatives act on the loss term, Eq. (D4), i.e., in terms of the form

$$
\begin{aligned}
&\frac{im}{2}\sum_{r,s,r_1,r_2,s_1,s_2}\int dP_1 dP_2 h_{sr}(p,\mathfrak{s}_1)[f^{(0)}(x,p_2)\partial_\nu f^{(0)}(x,p)\partial^\nu_{q_1}+f^{(0)}(x,p)\partial_\nu f^{(0)}(x,p_2)\partial^\nu_{q_2}]\\
&\quad\times\left[-i2\pi\hbar\delta^{(3)}\left(\mathbf{p}-\mathbf{p_1}+\frac{\mathbf{q_2}}{2}\right)\delta\left(p^0+p_2^0+\frac{q_2^0}{2}-E_{p_1}-E_{p_2}\right)\right.\\
&\quad\times\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\left|t^\dagger\right|p-\frac{q_1+q_2}{2},p_2-\frac{q_2}{2};s,s_2\right\rangle\delta_{rs_1}+(1\leftrightarrow 2)\\
&\quad+i2\pi\hbar\delta^{(3)}\left(\mathbf{p}-\mathbf{p_1}-\frac{\mathbf{q_2}}{2}\right)\delta\left(p^0+p_2^0-\frac{q_2^0}{2}-E_{p_1}-E_{p_2}\right)\\
&\quad\left.\times\left\langle p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};r,r_2\left|t\right|p+\frac{q_1+q_2}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle\delta_{r_1s}+(1\leftrightarrow 2)\right]_{q_1=q_2=0}\delta_{r_1s_1}\delta_{r_2s_2}\\
&=\frac{im}{2}\sum_{r,s,r_1,r_2,s_2}\int dP_2 h_{sr}(p,\mathfrak{s}_1)\partial_\nu[f^{(0)}(x,p_2)f^{(0)}(x,p)](\partial^\nu_{q_1}+\partial^\nu_{q_2})\\
&\quad\times\left[-\frac{i\pi\hbar}{E_{p+\frac{q_2}{2}}}\delta\left(p^0+\frac{q_2^0}{2}-E_{p+\frac{q_2}{2}}\right)\left\langle p-\frac{q_1-q_2}{2},p_2-\frac{q_2}{2};r_1,r_2\left|t^\dagger\right|p-\frac{q_1+q_2}{2},p_2-\frac{q_2}{2};s,s_2\right\rangle\delta_{rr_1}\delta_{r_2s_2}\right.\\
&\quad\left.+\frac{i\pi\hbar}{E_{p-\frac{q_2}{2}}}\delta\left(p^0-\frac{q_2^0}{2}-E_{p-\frac{q_2}{2}}\right)\left\langle p+\frac{q_1-q_2}{2},p_2+\frac{q_2}{2};r,r_2\left|t\right|p+\frac{q_1+q_2}{2},p_2+\frac{q_2}{2};r_1,s_2\right\rangle\delta_{r_1s}\delta_{r_2s_2}\right]_{q_1=q_2=0}. \qquad \text{(D6)}
\end{aligned}
$$

It is clear that both on- and off-shell contributions are present, since

$$
\partial^\mu_q\frac{1}{2E_{p+\frac{q}{2}}}\delta\left(p^0+\frac{q^0}{2}-E_{p+\frac{q}{2}}\right)\Big|_{q=0}=\partial^\mu_q\delta\left(\left(p^0+\frac{q^0}{2}\right)^2-E^2_{p+\frac{q}{2}}\right)\Big|_{q=0}=p^\mu\delta'(p^2-m^2). \qquad \text{(D7)}
$$

We can collect all the off-shell contributions to the collision term as

$$
\begin{aligned}
\mathfrak{C}^{(1)}_{\text{off-shell}}=&\frac{i}{2(p^2-m^2)}p\cdot\partial\sum_{r,s,r_1,r_2,s_1,s_2}dP_2 h_{sr}(p,\mathfrak{s})f^{(0)}(x,p_1)f^{(0)}(x,p_2)\\
&\times\left\{2\sum_{r'}\int dP_1 dP'\frac{1}{i\pi\hbar^2}[G(p)+G^\star(p)]\delta^{(4)}(p+p'-p_1-p_1)\right.\\
&\times\langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle\langle p_1,p_2;r_1,r_2|t^\dagger|p,p';s,r'\rangle\\
&\left.+i2\pi\hbar p^0\delta(p^2-m^2)[\langle p,p_2;r_1,r_2|t^\dagger|p,p_2;s,s_2\rangle\delta_{rr_1}\delta_{s_2r_2}+\langle p,p_2;r,r_2|t|p,p_2;r_1,s_2\rangle\delta_{r_1s}\delta_{r_2s_2}]\right\}. \qquad \text{(D8)}
\end{aligned}
$$

We now show that the off-shell part (D8) cancels with the off-shell part on the left-hand side of the Boltzmann equation (42). Using the quasiparticle approximation in Eq. (40), the left-hand side of Eq. (42) is given by

$$
\begin{aligned}
&mp\cdot\partial\delta(p^2-m^2-\hbar\delta m^2)f(x,p,\mathfrak{s})\\
&=m\delta(p^2-m^2)p\cdot\partial f(x,p,\mathfrak{s})+\hbar\frac{1}{p^2-m^2}p\cdot\partial\mathfrak{M}^{(0)}, \qquad \text{(D9)}
\end{aligned}
$$

where the correction to the mass shell at zeroth order reads

$$
\begin{aligned}
\mathfrak{M}^{(0)}=&\frac{im}{2}\sum_{r,s,r_1,r_2,s_1,s_2}\int dP_1 dP_2 h_{sr}(p,\mathfrak{s})\\
&\times\mathfrak{m}^{rs}_{r_1,r_2,s_1,s_2}(p_1,p_2,p)\prod_{j=1}^{2}\delta_{s_jr_j}f^{(0)}(x,p_j), \qquad \text{(D10)}
\end{aligned}
$$

with

$$
\begin{aligned}
&\mathfrak{m}^{r,s}_{r_1,r_2,s_1,s_2}(p_1,p_2,p)\\
&=2\sum_{r'}\int dP'\frac{1}{i\pi\hbar^2}[G(p)+G^\star(p)]\delta^{(4)}(p+p'-p_1-p_1)\\
&\times\langle p,p';r,r'|t|p_1,p_2;s_1,s_2\rangle\langle p_1,p_2;r_1,r_2|t^\dagger|p,p';s,r'\rangle\\
&+i2\pi\hbar p^0\delta(p^2-m^2)\\
&\times\{\delta^{(3)}(\mathbf{p}-\mathbf{p_1})[\langle p_1,p_2;r_1,r_2|t^\dagger|p,p_2;s,s_2\rangle\delta_{rs_1}\\
&+\langle p_1,p_2;r,r_2|t|p,p_2;s_1,s_2\rangle\delta_{r_1s}]+(1\leftrightarrow 2)\}.
\end{aligned}
\tag{D11}
$$

The steps to obtain Eq. (D10) are completely analogous to the calculation that leads to the local collision term, since we see from Eqs. (7) and (8) that δM is just the real part of the quantity of which C given by Eq. (15) is the imaginary part.

Comparing Eq. (D8) with Eq. (D10), we find up to first order

$$
m\mathfrak{C}^{(1)}_{\text{off-shell}}=\frac{1}{p^2-m^2}\,p\cdot\partial\mathfrak{M}^{(0)},
\tag{D12}
$$

which implies that all off-shell contributions cancel on the left- and right-hand sides and the Boltzmann equation involves only on-shell terms. Thus, we obtain the following kinetic equation for the distribution function $f(x,p,\mathfrak{s})$:

$$
\delta(p^2-m^2)p\cdot\partial f(x,p,\mathfrak{s})=\delta(p^2-m^2)\mathfrak{C}_{\text{on-shell}}[f],
\tag{D13}
$$

with

$$
\begin{aligned}
\mathfrak{C}_{\text{on-shell}}[f]\equiv{}&\mathfrak{C}_{\text{on-shell},l}[f]+\hbar\mathfrak{C}^{(1)}_{\text{on-shell},nl,1}[f]\\
&+\hbar\mathfrak{C}^{(1)}_{\text{on-shell},2,1}[f]+\hbar\mathfrak{C}^{(1)}_{\text{on-shell},2,2}[f].
\end{aligned}
\tag{D14}
$$

Here, we obtained from the first two lines in the last equality in Eq. (D1),

$$
\begin{aligned}
&\mathfrak{C}^{(1)}_{\text{on-shell},2,1}[f]\\
&=-\frac{1}{8m(p^0+m)}\sum_{r,s,r',r_1,r_2}(p_\nu\Sigma^{\mu\nu}_{sr}(p_\star)+\epsilon^{\nu\lambda\mu 0}p_\nu\mathfrak{s}_\lambda\delta_{sr})\\
&\times\int dP_1dP_2dP'\delta^{(4)}(p+p'-p_1-p_2)\\
&\times\langle p,p';r,r'|t|p_1,p_2;r_1,r_2\rangle\langle p_1,p_2;r_1,r_2|t^\dagger|p,p';s,r'\rangle\\
&\times[\partial_\mu f^{(0)}(x,p_1)f^{(0)}(x,p_2)-\partial_\mu f^{(0)}(x,p')f^{(0)}(x,p)],
\end{aligned}
\tag{D15}
$$

where we properly relabeled indices and applied the optical theorem (63). Furthermore, the on-shell contribution from the last three lines in Eq. (D1) is given by

$$
\begin{aligned}
\mathfrak{C}^{(1)}_{\text{on-shell},2,2}[f]={}&\frac{1}{4m}\sum_{r_1,r_2,s_1,s_2}\sum_{r,r',s}\int dP_1dP_2dP'h_{sr}(p,\mathfrak{s})\delta^{(4)}(p+p'-p_1-p_2)\delta(p^2-m^2)\\
&\times[f(x,p_2)\partial_\nu f(x,p_1)\partial^\nu_{q_1}+f(x,p_1)\partial_\nu f(x,p_2)\partial^\nu_{q_2}]\left\langle p+\frac{q_1}{2}+\frac{q_2}{2},p';r,r'\middle|t\middle|p_1+\frac{q_1}{2},p_2+\frac{q_2}{2};s_1,s_2\right\rangle\\
&\times\left\langle p_1-\frac{q_1}{2},p_2-\frac{q_2}{2};r_1,r_2\middle|t^\dagger\middle|p-\frac{q_1}{2}-\frac{q_2}{2},p';s,r'\right\rangle\delta_{s_1r_1}\delta_{s_2r_2}\\
&-\frac{m}{16\pi}\sum_{r_2,s_2}\int dP_2\sum_{r,s}h_{sr}(p,\mathfrak{s})\delta_{r_2s_2}\delta(p^2-m^2)\partial_\nu[f(x,p_2)f(x,p)](\partial^\nu_{q_1}+\partial^\nu_{q_2})\\
&\times i4\pi\hbar\left\langle p+\frac{q_2}{2}-\frac{q_1}{2},p_2-\frac{q_2}{2};r,r_2\middle|t+t^\dagger\middle|p-\frac{q_1}{2}-\frac{q_2}{2},p_2-\frac{q_2}{2};s,s_2\right\rangle.
\end{aligned}
\tag{D16}
$$

As discussed in Sec. V, in accordance with the low-density approximation, we neglect the momentum derivatives of the scattering amplitude, and, hence, all terms in Eq. (D16) vanish. We now show that the term in Eq. (D16) vanishes once the zeroth-order distribution function is inserted. The zeroth-order distribution function makes the zeroth-order collision term $\mathfrak{C}^{(0)}$ vanish and is given by the usual Boltzmann form

$$
f^{(0)}(x,p)=\frac{1}{(2\pi\hbar)^3}\,e^{-\beta(x)\cdot p}.
\tag{D17}
$$

(We consider here the simplest case of a neutral fluid. Adding a chemical potential is trivial and does not change the conclusion.) Inserting Eq. (D17) into Eq. (D15), we find

$$\begin{aligned}\mathfrak{C}^{(1)}_{\text{on-shell},2,1}[f] &= \frac{1}{(2\pi\hbar)^3}\frac{1}{8m(p^0+m)}\sum_{r,s,r',r_1,r_2}[p_\nu\Sigma^{\mu\nu}_{sr}(p_\star)+\epsilon^{\nu\lambda\mu 0}p_\nu\mathfrak{s}_\lambda\delta_{sr}]\\ &\times\int dP_1dP_2dP'\delta^{(4)}(p+p'-p_1-p_2)\\ &\times\langle p,p';r,r'|t|p_1,p_2;r_1,r_2\rangle\langle p_1,p_2;r_1,r_2|t^\dagger|p,p';s,r'\rangle\\ &\times\partial_\mu\beta_\lambda(p_1^\lambda+p_2^\lambda-p^\lambda-p'^\lambda)e^{-\beta\cdot(p_1+p_2)}\\ &=0.\end{aligned}\tag{D18}$$

Therefore, we proved the structure of the Boltzmann equation and the on-shell collision terms given in Eqs. (79) and (80).

[1] Z.-T. Liang and X.-N. Wang, Phys. Rev. Lett. **94**, 102301 (2005); **96**, 039901(E) (2006).
[2] S. A. Voloshin, arXiv:nucl-th/0410089.
[3] B. Betz, M. Gyulassy, and G. Torrieri, Phys. Rev. C **76**, 044901 (2007).
[4] F. Becattini, F. Piccinini, and J. Rizzo, Phys. Rev. C **77**, 024906 (2008).
[5] S. J. Barnett, Rev. Mod. Phys. **7**, 129 (1935).
[6] L. Adamczyk *et al.* (STAR Collaboration), Nature (London) **548**, 62 (2017).
[7] F. Becattini, L. Csernai, and D. J. Wang, Phys. Rev. C **88**, 034905 (2013); **93**, 069901(E) (2016).
[8] F. Becattini, V. Chandra, L. Del Zanna, and E. Grossi, Ann. Phys. (Amsterdam) **338**, 32 (2013).
[9] F. Becattini, G. Inghirami, V. Rolando, A. Beraudo, L. Del Zanna, A. De Pace, M. Nardi, G. Pagliara, and V. Chandra, Eur. Phys. J. C **75**, 406 (2015); **78**, 354(E) (2018).
[10] F. Becattini, I. Karpenko, M. Lisa, I. Upsal, and S. Voloshin, Phys. Rev. C **95**, 054902 (2017).
[11] I. Karpenko and F. Becattini, Eur. Phys. J. C **77**, 213 (2017).
[12] L.-G. Pang, H. Petersen, Q. Wang, and X.-N. Wang, Phys. Rev. Lett. **117**, 192301 (2016).
[13] Y. Xie, D. Wang, and L. P. Csernai, Phys. Rev. C **95**, 031901 (2017).
[14] J. Adam *et al.* (STAR Collaboration), Phys. Rev. Lett. **123**, 132301 (2019).
[15] F. Becattini and I. Karpenko, Phys. Rev. Lett. **120**, 012302 (2018).
[16] F. Becattini and M. A. Lisa, Annu. Rev. Nucl. Part. Sci. **70**, 395 (2020).
[17] W. Florkowski, A. Kumar, R. Ryblewski, and R. Singh, Phys. Rev. C **99**, 044910 (2019).
[18] W. Florkowski, A. Kumar, R. Ryblewski, and A. Mazeliauskas, Phys. Rev. C **100**, 054907 (2019).
[19] J.-J. Zhang, R.-H. Fang, Q. Wang, and X.-N. Wang, Phys. Rev. C **100**, 064904 (2019).
[20] F. Becattini, G. Cao, and E. Speranza, Eur. Phys. J. C **79**, 741 (2019).
[21] X.-L. Xia, H. Li, X.-G. Huang, and H. Z. Huang, Phys. Rev. C **100**, 014913 (2019).
[22] H.-Z. Wu, L.-G. Pang, X.-G. Huang, and Q. Wang, Phys. Rev. Research **1**, 033058 (2019).
[23] Y. Sun and C. M. Ko, Phys. Rev. C **99**, 011903 (2019).
[24] S. Y. F. Liu, Y. Sun, and C. M. Ko, Phys. Rev. Lett. **125**, 062301 (2020).
[25] W. Florkowski, B. Friman, A. Jaiswal, and E. Speranza, Phys. Rev. C **97**, 041901 (2018).
[26] D. Montenegro, L. Tinti, and G. Torrieri, Phys. Rev. D **96**, 056012 (2017); **96**, 079901(A) (2017).
[27] W. Florkowski, B. Friman, A. Jaiswal, R. Ryblewski, and E. Speranza, Phys. Rev. D **97**, 116017 (2018).
[28] W. Florkowski, E. Speranza, and F. Becattini, Acta Phys. Pol. B **49**, 1409 (2018).
[29] F. Becattini, W. Florkowski, and E. Speranza, Phys. Lett. B **789**, 419 (2019).
[30] W. Florkowski, R. Ryblewski, and A. Kumar, Prog. Part. Nucl. Phys. **108**, 103709 (2019).
[31] D. Montenegro, L. Tinti, and G. Torrieri, Phys. Rev. D **96**, 076016 (2017).
[32] K. Hattori, M. Hongo, X.-G. Huang, M. Matsuo, and H. Taya, Phys. Lett. B **795**, 100 (2019).
[33] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar, and R. Ryblewski, Phys. Lett. B **814**, 136096 (2021).
[34] N. Weickgenannt, E. Speranza, X.-l. Sheng, Q. Wang, and D. H. Rischke, companion Letter, Phys. Rev. Lett. **127**, 052301 (2021).
[35] S. Shi, C. Gale, and S. Jeon, Phys. Rev. C **103**, 044906 (2021).
[36] E. Speranza and N. Weickgenannt, Eur. Phys. J. A **57**, 155 (2021).
[37] S. Bhadury, W. Florkowski, A. Jaiswal, A. Kumar, and R. Ryblewski, Phys. Rev. D **103**, 014030 (2021).
[38] R. Singh, G. Sophys, and R. Ryblewski, Phys. Rev. D **103**, 074024 (2021).
[39] S. Bhadury, J. Bhatt, A. Jaiswal, and A. Kumar, Eur. Phys. J. Special Topics **230**, 655 (2021).
[40] D. Montenegro and G. Torrieri, Phys. Rev. D **100**, 056011 (2019).
[41] D. Montenegro and G. Torrieri, Phys. Rev. D **102**, 036007 (2020).
[42] A. D. Gallegos, U. Gürsoy, and A. Yarom, arXiv:2101.04759.

[43] K. Fukushima and S. Pu, Phys. Lett. B **817**, 136346 (2021).
[44] S. Li, M. A. Stephanov, and H.-U. Yee, arXiv:2011.12318.
[45] A. Gallegos and U. Gürsoy, J. High Energy Phys. 11 (2020) 151.
[46] M. Garbiso and M. Kaminski, J. High Energy Phys. 12 (2020) 112.
[47] S. Hess and L. Waldmann, Z. Naturforsch. A **21**, 1529 (1966).
[48] S. Hess, Z. Naturforsch. A **22**, 1871 (1967).
[49] K. Morawetz, Phys. Rev. E **96**, 032106 (2017).
[50] K. Morawetz, Phys. Rev. B **97**, 195142 (2018).
[51] K. Morawetz, V. Spicka, and P. Lipavsky, Ann. Phys. (Amsterdam) **294**, 135 (2001).
[52] D.-L. Yang, K. Hattori, and Y. Hidaka, J. High Energy Phys. 07 (2020) 070.
[53] Z. Wang, X. Guo, and P. Zhuang, arXiv:2009.10930.
[54] D. T. Son and N. Yamamoto, Phys. Rev. D **87**, 085016 (2013).
[55] Y. Hidaka, S. Pu, and D.-L. Yang, Phys. Rev. D **95**, 091901 (2017).
[56] Y. Hidaka, S. Pu, and D.-L. Yang, Phys. Rev. D **97**, 016004 (2018).
[57] A. Huang, S. Shi, Y. Jiang, J. Liao, and P. Zhuang, Phys. Rev. D **98**, 036010 (2018).
[58] J.-H. Gao, Z.-T. Liang, Q. Wang, and X.-N. Wang, Phys. Rev. D **98**, 036019 (2018).
[59] D.-L. Yang, Phys. Rev. D **98**, 076019 (2018).
[60] J.-h. Gao, J.-y. Pang, and Q. Wang, Phys. Rev. D **100**, 016008 (2019).
[61] S. Carignano, C. Manuel, and J. M. Torres-Rincon, Phys. Rev. D **100**, 016008 (2019).
[62] S. R. De Groot, W. A. Van Leeuwen, and C. G. Van Weert, *Relativistic Kinetic Theory. Principles and Applications* (North-Holland, Amsterdam, 1980).
[63] U. W. Heinz, Phys. Rev. Lett. **51**, 351 (1983).
[64] D. Vasak, M. Gyulassy, and H. T. Elze, Ann. Phys. (N.Y.) **173**, 462 (1987).
[65] N. Weickgenannt, X.-L. Sheng, E. Speranza, Q. Wang, and D. H. Rischke, Phys. Rev. D **100**, 056018 (2019).
[66] R.-H. Fang, L.-G. Pang, Q. Wang, and X.-N. Wang, Phys. Rev. C **94**, 024904 (2016).
[67] W. Florkowski, A. Kumar, and R. Ryblewski, Phys. Rev. C **98**, 044906 (2018).
[68] J.-H. Gao and Z.-T. Liang, Phys. Rev. D **100**, 056021 (2019).
[69] K. Hattori, Y. Hidaka, and D.-L. Yang, Phys. Rev. D **100**, 096011 (2019).
[70] Z. Wang, X. Guo, S. Shi, and P. Zhuang, Phys. Rev. D **100**, 014015 (2019).
[71] Y.-C. Liu, K. Mameda, and X.-G. Huang, Chin. Phys. C **44**, 094101 (2020).
[72] C. Manuel and J. M. Torres-Rincon, Phys. Rev. D **103**, 096022 (2021).
[73] J. Zamanian, M. Marklund, and G. Brodin, New J. Phys. **12**, 043019 (2010).
[74] R. Ekman, F. A. Asenjo, and J. Zamanian, Phys. Rev. E **96**, 023207 (2017).
[75] R. Ekman, H. Al-Naseri, J. Zamanian, and G. Brodin, Phys. Rev. E **100**, 023201 (2019).
[76] S. Li and H.-U. Yee, Phys. Rev. D **100**, 056022 (2019).
[77] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (Courier Corporation, New York, 2012).
[78] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Pergamon Press, Oxford, 1965).
[79] J.-Y. Chen, D. T. Son, and M. A. Stephanov, Phys. Rev. Lett. **115**, 021601 (2015).
[80] M. Stone, V. Dwivedi, and T. Zhou, Phys. Rev. D **91**, 025004 (2015).
[81] M. H. L. Pryce, Proc. R. Soc. A **195**, 62 (1948).
[82] C. Lorc, Eur. Phys. J. C **78**, 785 (2018).
[83] S. Pu, T. Koide, and D. H. Rischke, Phys. Rev. D **81**, 114039 (2010).
[84] W. Israel and J. M. Stewart, Ann. Phys. (N.Y.) **118**, 341 (1979).
[85] J. I. Kapusta, E. Rrapaj, and S. Rudaz, Phys. Rev. C **101**, 024907 (2020).
[86] J. I. Kapusta, E. Rrapaj, and S. Rudaz, Phys. Rev. C **101**, 031901 (2020).
[87] J. I. Kapusta, E. Rrapaj, and S. Rudaz, Phys. Rev. C **102**, 064911 (2020).
[88] A. Ayala, D. De La Cruz, S. Hernndez-Ortz, L. Hernndez, and J. Salinas, Phys. Lett. B **801**, 135169 (2020).
[89] A. Ayala, D. de la Cruz, L. Hernndez, and J. Salinas, Phys. Rev. D **102**, 056019 (2020).