Letter

$d > 2$ stress-tensor operator product expansion near a line

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We study the operator product expansion of stress tensors (TT OPE) in $d > 2$ conformal field theories whose bulk dual is Einstein gravity. Directly from the TT OPE, we obtain, in a certain null-like limit, an algebraic structure consistent with the Jacobi identity: $[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + Cm(m^2-1)\delta_{m+n,0}$. The dimensionless constant C is proportional to the central charge C_T. Transverse integrals in the definition of $\mathcal{L}_m$ play a crucial role. We comment on the corresponding limiting procedure and point out a curiosity related to the central term. A connection between the $d > 2$ near-lightcone stress-tensor conformal block and the $d = 2$ $\mathcal{W}$ algebra is observed. This note is motivated by the search for a field-theoretic derivation of $d > 2$ correlators in strong coupling critical phenomena.

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I. INTRODUCTION

The operator product expansions of stress tensors (TT OPE) in $d = 2$ conformal field theory (CFT)

$$T(z_1)T(z_2) = \frac{\mathbf{c}}{2s^4} + \frac{2}{s^2}T(z_2) + \frac{1}{s}\partial_{z_2}T(z_2) + \mathcal{O}(\partial^2 T),\qquad s = z_1 - z_2 \tag{1}$$

leads to the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{\mathbf{c}}{12}m(m^2-1)\delta_{m+n,0},\qquad L_m = \frac{1}{2\pi i}\oint dz z^{m+1}T(z). \tag{2}$$

The Virasoro algebra is omnipresent in two-dimensional critical phenomena [1] and has enormous implications; in particular, the algebra provides a nonperturbative derivation of $d = 2$ conformal correlators. In higher dimensions, the general TT OPE is contaminated by many model-dependent details. However, we ask the question; can one generalize the derivation (1)–(2) to $d > 2$ CFT in certain physical limits?

Over 27 years ago, Osborn and Petkos [2] computed the stress-tensor contribution to the $d > 2$ TT OPE, but we have not found any computation based on such an explicit TT OPE. A reason, presumably, is that the TT OPE is complicated. Given the recent developments of gauge/gravity correspondence and $d > 2$ strongly coupled field theories, we find it necessary to revisit the $d > 2$ TT OPE structure. In this paper, we adopt the following two simplifying limits to reduce the complexity of the TT OPE: (i) Infinitely large higher-spin gap, and (ii) Null/lightcone-like limit.[1]

As shown in [3–5], the gap $\Delta_{\rm gap}$ to the lightest spin > 2 single-trace primary controls the higher-order corrections to Einstein gravity; the limit $\Delta_{\rm gap} \to \infty$ then selects CFTs with an Einstein gravity bulk dual. We will focus on stress-tensor contribution to the TT OPE and suppress other primary operators.[2] On the other hand, the lightcone limit has been adopted in the recent computation of the multi-stress-tensor OPE data in $d > 2$ holographic CFTs [7–19]. The near-lightcone correlator at large central charge C_T is independent of higher-curvature terms in the purely gravitational action [7]; however, the correlator depends on certain nonminimal coupling bulk interactions which are suppressed at an infinite gap [9]. These results suggest that the simplest starting point is to impose the limits (i), and (ii) on the TT OPE.[3]

In general, the results depend on the order of limits (i.e., limiting procedure). A related motivation of this work is to help identify a lightcone-like limiting procedure that may be implemented to compute multi-stress-tensor OPE coefficients and near-lightcone correlators in $d > 2$ CFTs

[1] We use "-like" to distinguish our limiting procedure from similar limits used in the literature: the null-line limit is often defined by directly setting $x^+ = x_\perp^2 = 0$ in the Lorentzian signature, where $x^\pm = x^0 \pm x^1$ and $x_\perp$ denotes transverse directions.

[2] For instance, the three-point function $\langle TT\mathcal{O}\rangle$ is suppressed as $\Delta_{\rm gap} \to \infty$ [6].

[3] We should also assume the usual large N and large C_T limits but we will keep C_T in our expressions.

from the first principle via a Virasoro-like field-theoretic approach.

The main result of this work is that we find a structure similar to (2) in higher dimensions. Intuitively, one may expect that a Virasoro-like structure arises because the null-like limit brings stress tensors close to a line, a picture reminiscent of the two-dimensional case where $T = -\frac{\pi}{2} T^{\bar{z}\bar{z}}(z)$, $\bar{T} = -\frac{\pi}{2} T^{zz}(\bar{z})$ are holomorphic and anti-holomorphic functions, respectively. Note we are not introducing a physical line defect.

This paper is organized as follows. In Sec. II, we discuss the stress-tensor OPE structure. We focus on $d = 4$ for concreteness and expect that our results generalize to other dimensions. In Sec. III, we consider a null-line limit and obtain a Virasoro-like commutator via the stress-tensor OPE. A $d = 4$ single-stress-tensor-exchange derivation without explicitly using an algebra is discussed in Sec. IV, where we point out a curiosity related to the central term. We observe a connection between the $d > 2$ lightcone stress-tensor conformal block and the central-term of the $d = 2$ $\mathcal{W}$ algebra.

II. STRESS-TENSOR OPE

Our starting point is the stress-tensor contribution to the $d = 4$ TT OPE [2],

$$T^{\mu\nu}(x_1)T^{\sigma\rho}(x_2) = C_T \frac{\mathcal{I}^{\mu\nu,\sigma\rho}(s)}{s^8} + \hat{A}^{\mu\nu\sigma\rho}{}_{\alpha\beta}(s)T^{\alpha\beta}(x_2) + B^{\mu\nu\sigma\rho}{}_{\alpha\beta\lambda}(s)\partial^\lambda T^{\alpha\beta}(x_2) + \mathcal{O}(\partial^2 T), \quad (3)$$

where $s = x_1 - x_2$. The first term has the familiar form

$$\mathcal{I}^{\mu\nu,\sigma\rho}(s) = \frac{1}{2}(I^{\mu\sigma}(s)I^{\nu\rho}(s) + I^{\mu\rho}(s)I^{\nu\sigma}(s)) - \frac{1}{4}\delta^{\mu\nu}\delta^{\sigma\rho},$$
$$I^{\mu\sigma}(s) = \delta^{\mu\sigma} - 2\frac{s^\mu s^\sigma}{s^2}. \quad (4)$$

The structures of $\hat{A}_{\mu\nu\sigma\rho\alpha\beta}$ and $B_{\mu\nu\sigma\rho\alpha\beta\lambda}$ are cumbersome so we shall not list them here; see [2] for detailed expressions. As noted in [2], there are three undetermined coefficients in the TT OPE, denoted as a, b, c. The central charge C_T is given by[4]

$$C_T = \frac{\pi^2}{3}(14a - 2b - 5c). \quad (5)$$

In the lightcone limit, the relevant contribution is the lightcone component of the stress tensor, T^{++}. We will mostly work in the Euclidean space and adopt the line element $ds^2 = dz d\bar{z} + \sum_{i=1,2}(dx_\perp^{(i)})^2$ where z, $\bar{z}$, the Euclidean analogue of the lightcone coordinates, are complex coordinates. We will then focus on the $T^{zz}T^{zz}$ OPE.

The TT OPE simplifies significantly when one focuses on the T^{zz} component. Using (3), we obtain

$$T^{zz}(x_1)T^{zz}(x_2) = C_T \frac{\mathcal{I}^{zz,zz}(s)}{s^8} + \hat{A}^{zzzz}{}_{zz}(s)T^{zz}(x_2) + B^{zzzz}{}_{zz\lambda}(s)\partial^\lambda T^{zz}(x_2) + \mathcal{O}(\partial^2 T) \quad (6)$$

where $\mathcal{I}^{zz,zz} = \frac{4(s^z)^4}{s^4}$,

$$\hat{A}^{zzzz}{}_{zz} = \frac{4(s^z)^2}{C_T s^{10}}((2b+c)(s^\perp)^4 - 2s^z s^{\bar{z}}((8a-b-3c)(s^\perp)^2 + (6a-b-2c)s^z s^{\bar{z}})), \quad (7)$$

$$B^{zzzz}{}_{zzz} = \frac{s^{\bar{z}}}{4}\hat{A}^{zzzz}{}_{zz}, \quad (8)$$

$$B^{zzzz}{}_{zz\perp} = \frac{s^\perp}{2}\hat{A}^{zzzz}{}_{zz}, \quad (9)$$

$$B^{zzzz}{}_{zz\bar{z}} = \frac{(s^z)^3}{9C_T s^{10}}((64a + 18b - 11c)(s^\perp)^4 - 2s^z s^{\bar{z}}(4(3a - b - 2c)(s^\perp)^2 + (26a - 4b - 9c)s^z s^{\bar{z}})). \quad (10)$$

We will argue that higher-order pieces, $\mathcal{O}(\partial^2 T)$, are irrelevant when imposing the null-like limit considered in Sec. III. Observe that, from (7), $(8a - b - 3c) + (6a - b - 2c) = \frac{3}{\pi^2} C_T$. While this combination is interesting, we here consider a large N, large-gap condition which places strong constraints on the flux parameters "t_2" and "t_4" of the energy flux escaping to null infinity [20–22],

$$t_2 = \frac{30(13a + 4b - 3c)}{(14a - 2b - 5c)} = 0,$$
$$t_4 = \frac{-15(81a + 32b - 20c)}{2(14a - 2b - 5c)} = 0. \quad (11)$$

It is worth mentioning that two trace-anomaly central charges become the same under these conditions. By imposing $t_2 = t_4 = 0$ without first requiring a strictly infinite C_T, we can reduce three parameters to one parameter.

III. STRESS-TENSOR OPE NEAR A LINE AND A VIRASORO-LIKE COMMUTATOR

Consider the following operator in $d = 4$,

$$\mathcal{L}_m = \frac{\kappa}{2\pi i}\oint d\bar{z}\bar{z}^{m+1}\int d^2x_\perp T^{zz}(z, \bar{z}, x_\perp^{(1)}, x_\perp^{(2)}),$$
$$\text{where} \int d^2 x_\perp = \int_0^l dx_\perp^{(1)}\int_0^l dx_\perp^{(2)} \quad (12)$$

[4]The parameter c here should not be confused with the central charge in two dimensions where $C_T = \frac{\mathbf{c}}{2\pi^2}$.

in the null-like limit $z \to 0$, $l \to 0$.[5] We will determine the overall normalization factor κ later. The interpretation of the small l limit is that we consider the stress-tensor contribution near a two-dimensional plane. We are interested in computing the commutator $[\mathcal{L}_m, \mathcal{L}_n]$. The transverse integrals are crucial, as we will see, for extracting a central extension consistent with a Witt-like algebra.[6]

Let us first consider the c-number term which is controlled by the stress-tensor two-point function. After performing the transverse integrations, we consider a small s^z expansion,

$$\lim_{s^z\to\delta}\int d^4x_\perp \frac{C_T \mathcal{I}^{zz,zz}(s)}{s^8} = \frac{4\pi C_T}{5(s^{\bar z})^5}\frac{l^2}{\delta} - \frac{7\pi C_T}{16(s^{\bar z})^{9/2}}\frac{l}{\sqrt{\delta}} + \frac{C_T}{5(s^{\bar z})^4} + \frac{(356+315\pi)C_T}{14400}\frac{\delta^4}{l^8} + \cdots. \quad (13)$$

We would like to extract the cutoff-independent piece. We do so by next imposing a $l \to 0$ limit such that the first two terms are suppressed. The last piece of (13) and higher-order terms, although divergent as $l \to 0$, do not have a $\bar z$ pole and thus do not contribute to the commutator. The $\frac{C_T}{(s^{\bar z})^4}$ term shares the same form as the c-number term in the $d = 2$ TT OPE (1). The transverse integrals compensate for the additional dimensions of the $d > 2$ TT OPE. This c-number term derivation does not require a large-gap condition. The Cauchy's integral formula now leads to[7]

$$[\mathcal{L}_m, \mathcal{L}_n]|_{C_T} = \left(\frac{\kappa}{2\pi i}\right)^2 \oint_{\mathcal{C}(0)} d\bar z_2 \bar z_2^{n+1} \oint_{\mathcal{C}(\bar z_2)} d\bar z_1 \bar z_1^{m+1} \frac{C_T}{5(s^{\bar z})^4} = \kappa^2 \frac{C_T}{30} m(m^2-1)\delta_{m+n,0}. \quad (14)$$

We next turn to the operator part of the TT OPE, keeping explicit a, b, c parameters and imposing the conditions (11) at the end. We evaluate

$$\lim_{s^z\to\delta}\int d^2(x_1)_\perp (\hat A^{zzzz}{}_{zz}(x_1-x_2)T^{zz}(x_2)) = f(a,b,c)\frac{T^{zz}(x_2)}{\pi(s^{\bar z})^2} + \mathcal{O}(\delta) \quad (15)$$

where $f(a,b,c) = \frac{-52a+10b+19c}{14a-2b-5c}$. The leading-order term is cutoff-independent and only depends on $(x_2)_\perp$ through the stress tensor. To take the small s^z limit, we may assume $T^{zz}(x_2)$ is a suitable test function having a finite contribution only near $y_2 = z_2 = 0$, and then perform all the transverse integrations before imposing the small s^z limit. But we find it simpler, as we did above, to take $s^z \to \delta$ right after performing the integrations over the first set of transverse coordinates $(x_1)_\perp$.[8] It is now straightforward to complete the rest of the integrations,

$$[\mathcal{L}_m, \mathcal{L}_n]|_{A\text{-term}} = \left(\frac{\kappa}{2\pi i}\right)^2 \frac{f(a,b,c)}{\pi} \oint_{\mathcal{C}(0)} d\bar z_2 \bar z_2^{n+1} \oint_{\mathcal{C}(\bar z_2)} d\bar z_1 \times \bar z_1^{m+1} \int d^2(x_2)_\perp \frac{T^{zz}(x_2)}{(s^{\bar z})^2} = \frac{\kappa f(a,b,c)}{\pi}(m+1)\mathcal{L}_{m+n}. \quad (16)$$

Next, we have

$$\lim_{s^z\to\delta}\int d^2(x_1)_\perp (B^{zzzz}{}_{zzz}(s)\partial^z T^{zz}(x_2)) = f(a,b,c)\frac{\partial_{\bar z} T^{zz}(x_2)}{2\pi s^{\bar z}} + \mathcal{O}(\delta), \quad (17)$$

$$\lim_{s^z\to\delta}\int d^2(x_1)_\perp (B^{zzzz}{}_{zz\perp}(s)\partial_\perp T^{zz}(x_2)) = \mathcal{O}(\delta). \quad (18)$$

Since we only focus on the T^{zz} component in the null-like limit and effectively turn off other components of the stress tensor, the conservation of the stress tensor implies that we also drop $\partial_z T^{zz}$. Observe that the structures (including the relative coefficients) of (15) and (17), are the same as the two-dimensional case (1). From (17), we get

$$[\mathcal{L}_m, \mathcal{L}_n]|_{B\text{-term}} = -\frac{\kappa f(a,b,c)}{2\pi}(m+n+2)\mathcal{L}_{m+n}. \quad (19)$$

Similar to the corresponding $d = 2$ computation, we have performed an integration by parts to evaluate the $\partial_{\bar z} T^{zz}$ term.

We will not include the higher-order corrections $\mathcal{O}(\partial^2 T)$ in the TT OPE, but, based on the pattern from (15)

[5]One may perform a Wick-rotation to Lorentzian space and impose the lightcone limit, and then Wick-rotate back to the Euclidean space to carry out the $\bar z$ integral via the residue theorem. One may also formally impose a small z limit directly in Euclidean space, which is what we will do here. For two stress tensors, we take a small s^z. A similar analysis applies to the $T^{\bar z\bar z}T^{\bar z\bar z}$ OPE if one instead chooses a small $s^{\bar z}$ limit.

[6]This construction is essentially the same as the mode operator introduced in [23], but in that work the author adopts a different limiting procedure. See also [24–27] for related discussions.

[7]In general d, we find

$$[\mathcal{L}_m, \mathcal{L}_n]|_{C_T} = (-1)^d \kappa^2 \frac{4C_T}{\Gamma(d+2)} m(m^2-1)\delta_{m+n,0}.$$

[8]In the process of simplifying (15), we formally assume $l > (x_2)_\perp > 0$ to adopt the identity $\tan^{-1}(X) + \tan^{-1}(1/X) = \pi/2$ with $X > 0$. This, strictly speaking, means the end points of the $(x_2)_\perp$ integrals should be removed.

and (17), it seems reasonable to assume that the higher-order terms do not have a relevant pole in the null-like limit.

Combining (19), (16), and (14), the result is

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \kappa^2 \frac{C_T}{30} m(m^2-1)\delta_{m+n,0}. \quad (20)$$

where $\kappa = \frac{2\pi}{f(a,b,c)}$.

We choose a normalization κ such that the noncentral term has a simple coefficient. If we now impose the conditions listed in (11), we find the normalization factor to be $\kappa = -\frac{90\pi}{181}$.[9]

To summarize, we have described a null-line-like limiting procedure that allows us to extract an algebraic structure from the TT OPE. The result (20) is strikingly similar to the two-dimensional Virasoro algebra. We do not use holographic duality here, but it would be nice to find a potential connection to the AdS/CFT computation discussed some time ago [28,29] where a higher-dimensional generalization to the Brown-Henneaux symmetry [30] was identified in a certain infinite momentum frame. Most likely, whether or not there is a Virosoro-like structure at infinity depends on boundary conditions.[10]

It would certainly be of great interest to extend the two-dimensional CFT analysis to higher dimensions in the null/lightcone-like limit, where one expects to find relatively robust structures. By first focusing on a special class of higher-dimensional CFTs with an Einstein gravity dual, we would like to know if there is an effective algebraic derivation of the multi-stress-tensor OPE coefficients and conformal correlators. Considering perturbative corrections due to a large but finite higher-spin gap could be interesting as well.

IV. A $d=4$ SINGLE-STRESS-TENSOR-EXCHANGE DERIVATION

We conclude this paper by presenting some observations, which hopefully shed light on more general cases. In the following, we point out a simple derivation of the $d=4$ near-lightcone conformal scalar correlator via a mode summation.[11] This derivation does *not* explicitly rely on an algebra. In fact, as we will see, this derivation presents a central-term curiosity.

The scalar four-point conformal correlator can be written in terms of the conformal block decomposition [31],

$$\begin{aligned}&\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_L(0)\rangle \\ &\quad = \sum_{\Delta_T,J} c_{\mathrm{OPE}}(\Delta_T, J)\frac{B(z,\bar{z},\tau,J)}{(z\bar{z})^{\Delta_L}} \end{aligned} \quad (21)$$

where the twist of an operator is its dimension minus its spin, $\tau = \Delta_T - J$. We formally name $\mathcal{O}_H$ the "heavy" scalar and $\mathcal{O}_L$ the "light" scalar although the heavy-light limit [i.e., $\Delta_H, C_T \to \infty$ with Δ_H/C_T fixed and $\Delta_L \sim \mathcal{O}(1)$] does not play a special role in the single-stress-tensor-exchange computation. We adopt this notation as an example which is useful to compare with the literature that discusses multi-stress-tensor contributions to the heavy-light correlator. The Ward identity fixes the stress-tensor OPE coefficient to be $c_{\mathrm{OPE}}(4,2) = \frac{\Delta_H \Delta_L}{9\pi^4 C_T}$ in the convention of (3). The conformal block is[12]

$$\begin{aligned}B(z,\bar{z},\tau,J) = &\frac{z\bar{z}}{z-\bar{z}}\left[z^{\frac{\tau+2J}{2}}\bar{z}^{\frac{\tau-2}{2}}{}_2F_1\left(\frac{\tau+2J}{2},\frac{\tau+2J}{2};\tau+2J;z\right)\right. \\ &\left.\times {}_2F_1\left(\frac{\tau-2}{2},\frac{\tau-2}{2};\tau-2;\bar{z}\right) - (z\leftrightarrow\bar{z})\right]. \end{aligned} \quad (22)$$

In the limit $z \to 0$, the stress-tensor contribution in $d=4$ reads[13]

$$\begin{aligned}&\lim_{z\to 0}((z\bar{z})^{\Delta_L}\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_L(0)\rangle|_T) \\ &\quad = \frac{1}{9\pi^4}\frac{\Delta_H\Delta_L}{C_T}\bar{z}^3{}_2F_1(3,3,6,\bar{z})z + \mathcal{O}(z^2) \\ &\quad = \frac{10}{3\pi^4}\frac{\Delta_H\Delta_L}{C_T}\frac{3(\bar{z}-2)\bar{z} - (6+(\bar{z}-6)\bar{z})\ln(1-\bar{z})}{\bar{z}^2}z \\ &\qquad + \mathcal{O}(z^2). \end{aligned} \quad (23)$$

The higher-order pieces represent multi-stress-tensor contributions to the correlator.

It is instructive if we temporarily forget about the algebra and instead adopt the following operator,

$$\tilde{\mathcal{L}}_m = \lim_{z_T\to\delta}\oint \frac{d\bar{z}_T}{2\pi i}\bar{z}_T^{m+2}T^{zz}(z_T,\bar{z}_T,x^\perp = 0). \quad (24)$$

Notice we directly set $x^\perp = 0$ in this definition. The $z_T \to \delta$ represents the null-line limit for the stress tensor. The notation "$m+2$" will result in slightly more symmetric expressions in the following computation. The mode

[9] An overall rescaling of the mode operator $\mathcal{L}_m$ should not affect a scalar correlator computation. But one might wonder if the "right" proportionality constant should instead be $\kappa = -\frac{90\pi}{180} = -\frac{\pi}{2}$. If we formally adopt free-theory values of a, b, c [2], we notice that $\kappa = -\frac{\pi}{2}$ for both a fermion and a $U(1)$ gauge field, but $\kappa = -\frac{18\pi}{37}$ for a scalar. In fact, $\kappa = -\frac{\pi}{2}$ is true only under the condition $4a+2b-c=0$, which holds for both a free fermion and a $U(1)$ gauge field, but a free scalar has $4a+2b-c = -\frac{1}{9\pi^6}$. (In $d=2$, on the other hand, $4a+2b-c=0$ for both a free scalar and a free fermion.)

[10] I thank Gary Gibbons for related remarks.

[11] The derivation presented here is simpler than previous work [23] and we can avoid an arbitrary parameter introduced in that paper.

[12] Our convention differs by an overall factor of $(-\frac{1}{2})^J$ from the convention used in Dolan and Osborn [31].

[13] One can also choose $\bar{z} \to 0$ as the lightcone limit.

operator (24) is essentially the same as the lightray operator which does not contain transverse integrals [24,26,27,32]. We here use the Euclidean signature with complex coordinates $z, \bar{z}$. Similar to the $d=2$ case, we may expect that the stress-tensor-exchange contribution can be computed via the following mode summation,

$$\mathcal{V}_T = \lim_{z\to 0}\sum_{m=m^*}^{\infty}\frac{\langle\mathcal{O}_H(\infty)\mathcal{O}_H(1)\tilde{\mathcal{L}}_m^\dagger\rangle\langle\tilde{\mathcal{L}}_m\mathcal{O}_L(z,\bar{z})\mathcal{O}_L(0)\rangle}{\langle\mathcal{O}_H(\infty)\mathcal{O}_H(1)\rangle\mathcal{N}_m\langle\mathcal{O}_L(z,\bar{z})\mathcal{O}_L(0)\rangle} \tag{25}$$

where the normalization factor is $\mathcal{N}_m = \langle\tilde{\mathcal{L}}_m\tilde{\mathcal{L}}_m^\dagger\rangle$. We will find $m^*=3$.

Using the three-point function

$$\langle T^{\mu\nu}(x_1)\mathcal{O}_\Delta(x_2)\mathcal{O}_\Delta(x_3)\rangle = \frac{c_{T\mathcal{O}\mathcal{O}}}{x_{12}^4x_{13}^4x_{23}^{2\Delta-4}}\left(\frac{X^\mu X^\nu}{X^2}-\frac{\delta^{\mu\nu}}{4}\right) \tag{26}$$

with $X^\mu = x_{12}^\mu/x_{12}^2 - x_{13}^\mu/x_{13}^2$ and $c_{T\mathcal{O}\mathcal{O}} = -\frac{2\Delta}{3\pi^2}$, we first obtain

$$\begin{aligned}
&\lim_{z_T\to\delta}\lim_{z\to 0}\frac{\langle\tilde{L}_m\mathcal{O}_L(z,\bar{z})\mathcal{O}_L(0)\rangle}{\langle\mathcal{O}_L(z,\bar{z})\mathcal{O}_L(0)\rangle}\\
&\quad = -\frac{2\Delta_L}{3\pi^2}\lim_{z_T\to\delta}\lim_{z\to 0}\oint_{\mathcal{C}(\bar{z})}\frac{d\bar{z}_T}{2\pi i}\bar{z}_T^{m+2}\frac{\bar{z}^3 z}{\bar{z}_T^3 z_T(\bar{z}_T-\bar{z})^3(z_T-z)}\\
&\quad = -\frac{2\Delta_L}{3\pi^2}\lim_{z_T\to\delta}\lim_{z\to 0}\frac{(m-1)(m-2)\bar{z}^m}{2z_T(z_T-z)}z\\
&\quad = -\frac{\Delta_L}{3\pi^2}\frac{(m-1)(m-2)\bar{z}^m}{\delta^2}z,
\end{aligned} \tag{27}$$

where we introduce a short-distance cutoff δ. We shall find that the final four-point scalar correlator is independent of the UV cutoff. It is important to adopt a proper order of limits.

On the other hand, by taking $\tilde{\mathcal{L}}_m^\dagger = \tilde{\mathcal{L}}_{-m}$, we find

$$\begin{aligned}
\frac{\langle\mathcal{O}_H(\infty)\mathcal{O}_H(1)\tilde{\mathcal{L}}_m^\dagger\rangle}{\langle\mathcal{O}_H(\infty)\mathcal{O}_H(1)\rangle} &= \frac{2\Delta_H}{3\pi^2}\lim_{z_T\to\delta}\oint_{\mathcal{C}(1)}\frac{d\bar{z}_T}{2\pi i}\frac{(\bar{z}_T)^{-m+2}}{(\bar{z}_T-1)^3(z_T-1)}\\
&= \frac{2\Delta_H}{3\pi^2}\lim_{z_T\to\delta}\frac{(m-1)(m-2)}{2(z_T-1)}\\
&= -\frac{\Delta_H}{3\pi^2}(m-1)(m-2).
\end{aligned} \tag{28}$$

The normalization factor can be computed using the stress-tensor two-point function,

$$\begin{aligned}
\mathcal{N}_m &= \langle\tilde{\mathcal{L}}_m\tilde{\mathcal{L}}_m^\dagger\rangle\\
&= \lim_{s^z\to\delta}\oint_{\mathcal{C}(0)}\frac{d\bar{z}_{2T}}{2\pi i}\oint_{\mathcal{C}(\bar{z}_{2T})}\frac{d\bar{z}_{1T}}{2\pi i}\\
&\quad\times(\bar{z}_{1T})^{m+2}(\bar{z}_{2T})^{-m+2}\langle T^{zz}(z_{1T},\bar{z}_{1T})T^{zz}(z_{2T},\bar{z}_{2T})\rangle\\
&= \lim_{s^z\to\delta}\oint_{\mathcal{C}(0)}\frac{d\bar{z}_{2T}}{2\pi i}\oint_{\mathcal{C}(\bar{z}_{2T})}\frac{d\bar{z}_{1T}}{2\pi i}\frac{4C_T(\bar{z}_{1T})^{m+2}(\bar{z}_{2T})^{-m+2}}{(\bar{z}_{1T}-\bar{z}_{2T})^6(z_{1T}-z_{2T})^2}\\
&= \frac{C_T}{30}\frac{(m+2)(m+1)m(m-1)(m-2)}{\delta^2}.
\end{aligned} \tag{29}$$

The UV-cutoff dependencies cancel out in the final mode summation and we obtain exactly the $d=4$ stress-tensor-exchange structure (23),

$$\begin{aligned}
\mathcal{V}_T &= \frac{10}{3\pi^4}\frac{\Delta_H\Delta_L}{C_T}\sum_{m=3}^{\infty}\frac{(m-1)(m-2)\bar{z}^m}{m(m+1)(m+2)}z\\
&= \frac{1}{9\pi^4}\frac{\Delta_H\Delta_L}{C_T}\bar{z}^3{}_2F_1(3,3,6,\bar{z})z.
\end{aligned} \tag{30}$$

This computation does not require a large gap.

It is peculiar that we are able to reproduce the $d=4$ near-lightcone correlator, including the correct OPE coefficient, via a mode summation. The final result is finite and cutoff independent. Although the above single-stress-tensor computation does not rely on knowing an algebra, we would like to ask why such a derivation exists. Recall that, in two-dimensions, a similar derivation exists because of the Virasoro symmetry. Given the above $d>2$ computation, one may speculate that a certain symmetry emerges near the lightcone. An underlining algebra would provide a precise interpretation of the modes counting in (30). Since we have extracted a Virasoro-like commutator from the stress-tensor OPE (20), it seems natural to link the correlator computation to the algebra.

However, we find a curiosity related to the central term, or more generally, to the limiting procedure. In the above correlator computation, we emphasize that we take $x^\perp\to 0$ before imposing $z\to 0$.[14] The resulting "central" term has the following m-dependence (in the notation of $\tilde{\mathcal{L}}_m \sim \oint d\bar{z}\bar{z}^{m+2}T$),

$$\text{Type A:}\quad C_T(m+2)(m+1)m(m-1)(m-2)\delta_{m+n,0}. \tag{31}$$

Such an m-dependence is quite different from the central term in the Virasoro-like commutator, which has the structure,

$$\text{Type B:}\quad C_T(m+1)m(m-1)\delta_{m+n,0}. \tag{32}$$

[14]Using this order of limits, one can include transverse integrals but the correlator result is unchanged.

As shown above, in this case, we take $z \to 0$ before imposing $x^{\perp} \to 0$. The Type B central term has the familiar form fixed by the Jacobi identity, but the correlator derivation suggests that the Type A structure plays a nontrivial role in recovering the scalar correlator. The Type A structure, however, is incompatible with the Witt algebra.

Ideally, we would like to also compute $[\tilde{\mathcal{L}}_m, \tilde{\mathcal{L}}_n]$ via the $d = 4$ TT OPE, but we find that the commutator computation using $\tilde{\mathcal{L}}_m$ requires a higher-order term in the TT OPE. Such a computation will not be included in this paper.

On the other hand, we observe that, up to an overall coefficient, the $d = 4$ Type A structure (31) is identical to the central term of the $\mathcal{W}_3$ algebra in $d = 2$ CFTs [33] (see [34] for a review),

$$[W_m, W_n] = \frac{\mathbf{c}}{360}(m+2)(m+1)m(m-1)(m-2)\delta_{m+n,0} + (m-n)\left(\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)\right)L_{m+n} + \frac{16}{22+5\mathbf{c}}(m-n)\Lambda_{m+n}, \tag{33}$$

$$[L_m, W_n] = (2m-n)W_{m+n}, \tag{34}$$

where $\Lambda_m = \sum_n (L_{m-n}L_n) - \frac{3}{10}(m+3)(m+2)L_m$. W_m is the Laurent modes of a spin-three primary current. Note that closure of the $\mathcal{W}_3$ algebra requires first knowing the operator L_m that satisfies the Virasoro algebra. In general (even) d, we find the Type A structure is $\sim C_T m(m^2-1)(m^2-4)\cdots(m^2-(\frac{d}{2})^2)$ in the notation of $\tilde{\mathcal{L}}_m \sim \oint d\bar{z}\bar{z}^{(m+\frac{d}{2})}T$.

To our knowledge, a connection between $d > 2$ CFT near-lightcone correlators and the $\mathcal{W}$ (-like) symmetry has not been mentioned before.[15] This central-term curiosity needs to be better understood. Perhaps exploring more general structures involving multi-stress-tensor exchanges in $d > 2$ CFTs can help clarify its algebraic underpinnings.

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[15] In the context of supersymmetric CFTs in 3, 4, and 6 dimensions (with at least 8 real supercharges), it was shown in [35] that there is a subsector of the theory which is described by a $d = 2$ chiral algebra. In [36], it was argued that, in $d = 6$ maximally supersymmetric CFTs, the algebra is the $\mathcal{W}_N$ algebra. (See also [37] for an alternative viewpoint.) These results use very different methods and apply to a distinct class of CFTs.

[1] A. Belavin, A. Polyakov, and A. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, Nucl. Phys. **B241**, 333 (1984).
[2] H. Osborn and A. C. Petkou, Implications of conformal invariance in field theories for general dimensions, Ann. Phys. (N.Y.) **231**, 311 (1994).
[3] X. O. Camanho, J. D. Edelstein, J. Maldacena, and A. Zhiboedov, Causality constraints on corrections to the graviton three-point coupling, J. High Energy Phys. 02 (2016) 020.
[4] I. Heemskerk, J. Penedones, J. Polchinski, and J. Sully, Holography from conformal field theory, J. High Energy Phys. 10 (2009) 079.
[5] N. Afkhami-Jeddi, T. Hartman, S. Kundu, and A. Tajdini, Einstein gravity 3-point functions from conformal field theory, J. High Energy Phys. 12 (2017) 049.
[6] D. Meltzer and E. Perlmutter, Beyond $a = c$: Gravitational couplings to matter and the stress tensor OPE, J. High Energy Phys. 07 (2018) 157.
[7] A. L. Fitzpatrick and K.-W. Huang, Universal lowest-twist in CFTs from holography, J. High Energy Phys. 08 (2019) 138.
[8] A. L. Fitzpatrick, K.-W. Huang, and D. Li, Probing universalities in $d > 2$ CFTs: From black holes to shockwaves, J. High Energy Phys. 11 (**2019**) 139.
[9] A. L. Fitzpatrick, K.-W. Huang, D. Meltzer, E. Perlmutter, and D. Simmons-Duffin, Model-dependence of minimal-twist OPEs in $d > 2$ holographic CFTs, J. High Energy Phys. 11 (2020) 060.
[10] Y.-Z. Li, Z.-F. Mai, and H. Lu, Holographic OPE coefficients from AdS black holes with matters, J. High Energy Phys. 09 (2019) 001.
[11] M. Kulaxizi, G. S. Ng, and A. Parnachev, Subleading eikonal, AdS/CFT and double stress tensors, J. High Energy Phys. 10 (2019) 107.
[12] R. Karlsson, M. Kulaxizi, A. Parnachev, and P. Tadić, Leading multi-stress tensors and conformal bootstrap, J. High Energy Phys. 01 (**2020**) 076.
[13] R. Karlsson, M. Kulaxizi, A. Parnachev, and P. Tadić, Stress tensor sector of conformal correlators operators in the Regge limit, J. High Energy Phys. 07 (2020) 019.
[14] Y.-Z. Li, Heavy-light bootstrap from Lorentzian inversion formula, J. High Energy Phys. 07 (2020) 046.
[15] Y.-Z. Li and H.-Y. Zhang, More on heavy-light bootstrap up to double-stress-tensor, J. High Energy Phys. 10 (2020) 055.
[16] R. Karlsson, Multi-stress tensors and next-to-leading singularities in the Regge limit, J. High Energy Phys. 08 (2020) 037.

[17] A. Parnachev, Near lightcone thermal conformal correlators and holography, J. Phys. A **54**, 155401 (2021).

[18] A. Parnachev and K. Sen, Notes on AdS-Schwarzschild eikonal phase, J. High Energy Phys. 03 (2021) 289.

[19] R. Karlsson, A. Parnachev, and P. Tadić, Thermalization in large-N CFTs, arXiv:2102.04953.

[20] D. M. Hofman and J. Maldacena, Conformal collider physics: Energy and charge correlations, J. High Energy Phys. 05 (2008) 012.

[21] M. Henningson and K. Skenderis, The holographic Weyl anomaly, J. High Energy Phys. 07 (1998) 023.

[22] A. Belin, D. M. Hofman, and G. Mathys, Einstein gravity from ANEC correlators, J. High Energy Phys. 08 (2019) 032.

[23] K.-W. Huang, Lightcone commutator and stress-tensor exchange in $d > 2$ CFTs, Phys. Rev. D **102**, 021701 (2020).

[24] H. Casini, E. Teste, and G. Torroba, Modular Hamiltonians on the null plane and the Markov property of the vacuum state, J. Phys. A **50**, 364001 (2017).

[25] K.-W. Huang, Stress-tensor commutators in conformal field theories near the lightcone, Phys. Rev. D **100**, 061701 (2019).

[26] M. Beşken, J. De Boer, and G. Mathys, On local and integrated stress-tensor commutators, arXiv:2012.15724.

[27] A. Belin, D. M. Hofman, G. Mathys, and M. T. Walters, On the stress tensor light-ray operator algebra, J. High Energy Phys. 05 (2021) 033.

[28] M. Banados, A. Chamblin, and G. W. Gibbons, Branes, AdS gravitons and Virasoro symmetry, Phys. Rev. D **61**, 081901 (2000).

[29] D. Brecher, A. Chamblin, and H. S. Reall, AdS/CFT in the infinite momentum frame, Nucl. Phys. **B607**, 155 (2001).

[30] J. D. Brown and M. Henneaux, Central charges in the canonical realization of asymptotic symmetries: An example from three-dimensional gravity, Commun. Math. Phys. **104**, 207 (1986).

[31] F. A. Dolan and H. Osborn, Conformal four point functions and the operator product expansion, Nucl. Phys. **B599**, 459 (2001).

[32] P. Kravchuk and D. Simmons-Duffin, Light-ray operators in conformal field theory, J. High Energy Phys. 11 (2018) 102.

[33] A. B. Zamolodchikov, Infinite additional symmetries in two-dimensional conformal quantum field theory, Theor. Math. Phys. **65**, 1205 (1985).

[34] P. Bouwknegt and K. Schoutens, W symmetry in conformal field theory, Phys. Rep. **223**, 183 (1993).

[35] C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli, and B. C. van Rees, Infinite chiral symmetry in four dimensions, Commun. Math. Phys. **336**, 1359 (2015).

[36] C. Beem, L. Rastelli, and B. C. van Rees, $\mathcal{W}$ symmetry in six dimensions, J. High Energy Phys. 05 (2015) 017.

[37] N. Bobev, P. Bomans, and F. F. Gautason, Comments on chiral algebras and Ω-deformations, J. High Energy Phys. 04 (2021) 132.