Remarks on single trace $T\bar{T}$ and other current-current deformations

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String theory on AdS_3 with NS-NS fluxes admits a solvable irrelevant deformation which is close to the $T\bar{T}$ deformation of the dual conformal field theory (CFT_2). This consists of deforming the world sheet action, namely the action of the $SL(2,\mathbb{R})$ Wess-Zumino-Witten model, by adding to it the operator $J^-\bar{J}^-$, constructed with two Kac-Moody currents. The geometrical interpretation of the resulting theory is that of strings on a conformally flat background that interpolate between anti–de Sitter (AdS_3) in the IR and a flat linear dilaton spacetime with its Hagedorn spectrum in the UV, having passed through a transition region of positive curvature. Here, we study the properties of this string background both from the point of view of the low-energy effective theory and of the world sheet CFT. We first study the geometrical properties of the semiclassical geometry, then we revise the computation of correlation functions and of the spectrum of the $J^-\bar{J}^-$ deformed world sheet theory, and finally we discuss how to extend this type of current-current deformation to other conformal models.

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I. INTRODUCTION

In the context of $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence, it was shown in [1] that certain type of $T\bar{T}$-deformation of the boundary dual conformal field theory (CFT_2), which can be regarded as a single trace version of the one originally introduced in [2–4], gives rise in the bulk to a string theory background that interpolates between anti–de Sitter (AdS_3) in the infrared limit and a flat linear dilaton background in the ultraviolet. This construction was argued in [1] to provide a family of holographic pairs, including a large class of string theory vacua with asymptotically linear dilaton. The solvable irrelevant deformation of $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence studied in [1] was further studied in [5], where in particular its spectrum was studied. It was observed that this type of deformation leads in the ultraviolet to a theory with Hagedorn spectrum. This has been studied in [6–11] and references therein and thereof; see also [12–14].

In [15,16], the correlation functions in the deformed theory were studied, and it provided an alternative way of studying the spectrum; the insertion of an operator that realizes the deformation produces a logarithmic divergence in the correlation functions, leading to the renormalization of the primary operators. This yields an anomalous dimension that can be computed explicitly. From this, one may determine the spectrum of the theory from the world sheet computation. The form of the correlation functions, on the other hand, permits us to investigate the properties of the dual theory [15].

The model studied in [1] was later investigated in many different contexts. The entanglement entropy was first studied in [17]; in [18,19] the theory was studied in presence of boundaries and the $J\bar{T}$ analog of it has also been studied [20–23]. Here, we study the properties of this string background both from the point of view of the low-energy effective theory and of the world sheet CFT. In Sec. II, we study the geometrical properties of the semiclassical geometry. We study the geometry as a solution to the low-energy effective field theory, its T-dual background, the main properties of this specific deformation of AdS_3, and the field probes in such an spacetime. In Sec. III, we revise the computation of correlation functions of [15,16] and how it provides a direct way of studying the spectrum. Finally, in Sec. IV we discuss how to extend this type of deformation to other conformal models.

II. LOW ENERGY THEORY

A. Interpolating background

Let us start by considering the effective theory describing the low-energy limit of bosonic string theory. This is given by the field equations

$$R_{\alpha\beta} + 2\nabla_\alpha\nabla_\beta\Phi - \frac{1}{4}H_{\alpha\mu\nu}H^{\mu\nu}_{\beta} = 0, \tag{1}$$

$$\nabla_\alpha(e^{-2\Phi}H^{\alpha\mu\nu}) = 0, \tag{2}$$

$$\nabla^\alpha\nabla_\alpha\Phi - 2\nabla_\alpha\Phi\nabla^\alpha\Phi + 2\alpha' + \frac{1}{12}H_{\alpha\mu\nu}H^{\alpha\mu\nu} = 0, \tag{3}$$

where $H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$ is the field strengh associated to the Kalb-Ramond B-field, and Φ is the dilaton. These equations admit locally AdS_3 solutions [24],

$$ds^2 = -\frac{r^2}{\ell^2}dt^2 + \frac{\ell^2}{r^2}dr^2 + r^2 d\theta^2, \tag{4}$$

provided the other backgrounds fields take the form

$$\Phi = \Phi_0, \qquad H_{\mu\nu\rho} = \frac{2r}{\ell}\epsilon_{\mu\nu\rho}. \tag{5}$$

The dilaton receives quantum (i.e., finite-α') corrections. Here, we will consider the convention $\alpha' = 1$, so that the semiclassical limit corresponds to large $k = \ell^2/\alpha' = \ell^2$.

As (4) describes the universal covering of AdS_3, we have $t \in \mathbb{R}$. The radial coordinate is $r \in \mathbb{R}_{\geq 0}$, with the boundary of the space being located at $r \to \infty$. If we take θ to be periodic with a period 2π, the metric above corresponds to that of the massless Bañados-Teitelboim-Zanelli geometry [25,26]. It will be convenient to consider coordinates $r = \ell e^\phi$ and $x = \ell\theta$. In these variables, the metric and the field strength take the form

$$\begin{aligned} ds^2 &= e^{2\phi}(-dt^2 + dx^2) + \ell^2 d\phi^2, \\ H_{\mu\nu\rho} &= \partial_{[\mu}B_{\nu\rho]} = 2e^{2\phi}\epsilon_{\mu\nu\rho}, \end{aligned} \tag{6}$$

where we now consider the covering $x \in \mathbb{R}$. That is, the nonvanishing component of the Kalb-Ramond field is $B_{xt} = e^{2\phi}$ and grows when approaching the boundary at $\phi \to \infty$.

Now, let us consider a deformation of (6), given by

$$ds^2 = \frac{e^{2\phi}}{\lambda e^{2\phi} + 1}(-dt^2 + dx^2) + \ell^2 d\phi^2, \tag{7}$$

with λ being a real parameter. This metric solves the field equations (1) for arbitrary λ provided the Kalb-Ramond field and the dilaton are given by

$$B_{xt} = \frac{2e^{2\phi}}{\lambda e^{2\phi} + 1}, \qquad \Phi = \Phi_0 - \phi - \frac{1}{2}\log(\lambda + e^{-2\phi}), \tag{8}$$

respectively. Near the boundary, the dilaton becomes linear in ϕ. We are mostly interested in the case $\lambda \geq 0$, as for $\lambda < 0$ the geometry exhibits a singularity at $\phi = -\frac{1}{2}\log|\lambda|$.

In terms of the double null coordinates $u = (x + t)/\ell$ and $\bar{u} = (x - t)/\ell$, the fields take the following form

$$ds^2 = \ell^2 d\phi^2 + \frac{\ell^2 du d\bar{u}}{\lambda + e^{-2\phi}}, \qquad B = \frac{\ell^2 du \wedge d\bar{u}}{\lambda + e^{-2\phi}}. \tag{9}$$

This solution has recently attracted much attention [1,5,8–11,15–23] as it appears as an exact string background that corresponds to a marginal deformation of the world sheet theory on $\mathrm{AdS}_3 \times \mathcal{N}$ which is closely related to the $T\bar{T}$-deformation of the dual CFT.

Let us go back for a moment to the more familiar coordinates $r = \ell e^\phi$, namely

$$ds^2 = -\frac{r^2}{r^2\lambda + \ell^2}dt^2 + \frac{\ell^2}{r^2}dr^2 + \frac{r^2}{r^2\lambda + \ell^2}dx^2, \tag{10}$$

in which it becomes evident that the geometry interpolates between AdS_3 and Minkowski space; while in the limit $r \ll \ell/\sqrt{\lambda}$ one recovers the metric (4), in the limit $r \gg \ell/\sqrt{\lambda}$ one gets $ds^2 = -d\hat{t}^2 + d\hat{x}^2 + d\hat{y}^2$ where $\hat{t} = t/\sqrt{\lambda}$, $\hat{x} = x/\sqrt{\lambda}$, $\hat{y} = \ell\phi$. The local isometry group of spacetime (10) for arbitrary value of λ is $ISO(1,1)$ and is generated by the Killing vectors ∂_t, ∂_x and $x\partial_t + t\partial_x$. It gets enhanced to the full $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ for $\lambda = 0$ and to $ISO(2,1)$ in the limit $\lambda \to \infty$

The interpolating geometry (9) has very interesting properties. Apart from being fascinating in that it describes the transition between AdS_3 and the type of linear dilaton background that appears in little string theory, its geometry (9) exhibits peculiar features; it admits a supersymmetric embedding in type IIB SUGRA, since it appears in the S-dual frame of the D1/D5 system [27]. Besides, it is solvable in different limits: On the one hand, despite being a geometry of nonconstant curvature, it turns out that the probe fields are integrable on it, and thus enables us to gain intuition from the semiclassical analysis. On the other hand, the string world sheet σ-model is an *exact* string solution, being a marginal deformation of a WZW (Wess-Zumino-Witten) model, and can be solved explicitly in the sense that analytic expressions for the correlation functions can be obtained and the spectrum can be written down.

B. T-duality

Let us study some of the properties of (9), starting by noticing that it is dual to pp-waves on AdS_3. Spacetime (6) happens to be invariant under t and x translations, and so we can apply T-duality transformations along the direction generated by ∂_t and ∂_x in order to obtain new solutions $\tilde{g}_{\mu\nu}, \tilde{B}_{\mu\nu}, \tilde{\Phi}$ to the field equations (1)–(3). At the level of the low-energy effective action, this amounts to applying the Buscher rules [28,29]. For the t-direction, these transformation rules are

$$\tilde{g}_{tt} = \frac{1}{g_{tt}}, \quad \tilde{g}_{ti} = \frac{B_{ti}}{g_{tt}}, \quad \tilde{g}_{ij} = g_{ij} - \frac{g_{ti}g_{tj}}{g_{tt}} - \frac{B_{ti}B_{tj}}{g_{tt}}, \tag{11}$$

together with

$$\tilde{B}_{ti} = \frac{g_{ti}}{g_{tt}}, \quad \tilde{B}_{ij} = B_{ij} - 2\frac{g_{t[i}B_{j]t}}{g_{tt}}, \quad \tilde{\Phi} = \Phi - \frac{1}{2}\log(g_{tt}), \tag{12}$$

where i, j correspond to the coordinates other than t. After performing these transformations and renaming the variables as $u = x$, $v = t$, and $y = \ell\phi$, one obtains

$$d\tilde{s}^2 = -F(y)dv^2 + 2dudv + dy^2, \quad \text{with} \quad F(y) = \lambda + e^{-2y/\ell}, \tag{13}$$

together with $\tilde{B}_{\mu\nu} = 0$, and $\tilde{\Phi} = \tilde{\Phi}_0 - \phi$. Analogously, by applying similar transformations to (7) in the x direction, one gets

$$d\tilde{s}^2 = F(y)du^2 + 2dudv + dy^2. \tag{14}$$

This geometry describes a 3-dimensional version of a pp-wave solution in Brinkmann type coordinates with a wave profile F. The nonvanishing component of the Riemann tensor for this geometry is

$$R^{vy}{}_{uy} = -\frac{2}{\ell^2}e^{-2y/\ell}. \tag{15}$$

This solution represents an *exact* string background. In order to apply the T-duality transformation to (7) along the $\xi = x\partial_t + t\partial_x$ direction, we perform the change of coordinates $t = \sqrt{v}\sinh(u)$ and $x = \sqrt{v}\cosh(u)$ which maps $x\partial_t + t\partial_x \to \partial_u$. Using this new coordinate system, the metric of the interpolating background is

$$ds^2 = \frac{e^{2\phi}}{\lambda e^{2\phi} + 1}\left(-vdu^2 + \frac{1}{4v}dv^2\right) + \ell^2 d\phi^2, \tag{16}$$

while the Kalb-Ramond field changes as $B_{tx} \to B_{uv} = B_{tx}/2$, and the dilaton remains as in (8). By applying the Busher's rules in the ∂_u we get

$$ds^2 = -\frac{F(y)}{v}du^2 - \frac{1}{v}dudv + d\phi^2, \tag{17}$$

which under the change of coordinates $v = e^w$ takes the form

$$ds^2 = -e^{-w}F(y)du^2 - dudw + d\phi^2. \tag{18}$$

This spacetime also admits the interpretation of a pp-wave.

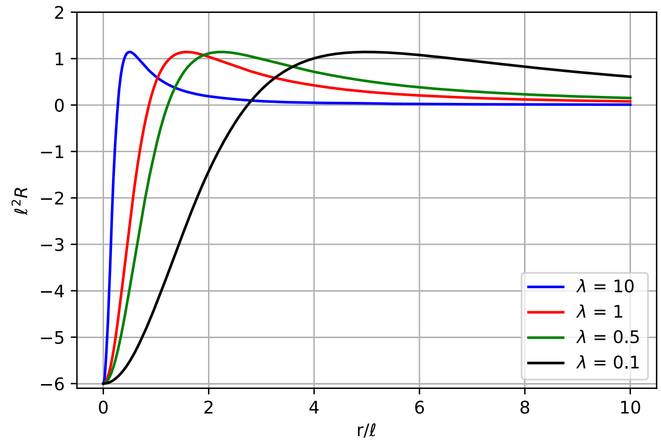

FIG. 1. Ricci scalar for different values of λ.

C. Geometric properties of the interpolating spacetime

As said, spacetime (9) interpolates between AdS_3 in the limit $\phi \to -\infty$ and a flat linear dilaton background in the opposite limit. The geometry thus has nonconstant curvature. In fact, it can be shown to have an infinite region of positive curvature. To see this, we can compute the scalar curvature

$$R = \frac{2(4\lambda r^2 - 3\ell^2)}{(\lambda r^2 + \ell^2)^2}, \tag{19}$$

which, indeed, happens to be positive for $r > \ell\sqrt{3/(4\lambda)}$, all the way to infinity. R has a global maximum at $r_{\max} = \ell\sqrt{5/(2\lambda)}$ with a maximum value $R_{\max} = 8/(7\ell^2)$ that does not depend on the deformation parameter λ. Figure 1 depicts the function R as a function of the radial coordinate r for different values of λ.

Other curvature invariants of (9) are

$$R^{\mu\nu}R_{\mu\nu} = \frac{4(6\lambda^2 r^4 - 8\ell^2\lambda r^2 + 3\ell^4)}{(\lambda r^2 + \ell^2)^4}, \tag{20}$$

$$R^{\mu}{}_{\nu}R^{\nu}{}_{\rho}R^{\rho}{}_{\mu} = \frac{-8(-10\lambda^3 r^6 + 18\ell^2\lambda^2 r^4 - 12\ell^4\lambda r^2 + 3\ell^6)}{(\lambda r^2 + \ell^2)^6}, \tag{21}$$

and we see from these, and from (19), that the geometry is actually singular at $r = \ell/\sqrt{-\lambda}$ when $\lambda < 0$. It is important noticing that it is sufficient to give the three curvature invariant R, $\mathrm{Tr}(R^2_{\mu\nu})$, and $\mathrm{Tr}(R^3_{\mu\nu})$ to characterize them all, since any higher curvature scalar can be obtained as a combination of powers of the latter three quantities by virtue of the three-dimensional identities

$$\delta^{\mu_1\cdots\mu_n}_{\nu_1\cdots\nu_n}\tilde{R}^{\nu_1}{}_{\mu_1}\cdots\tilde{R}^{\nu_n}{}_{\mu_n} \equiv 0, n > 3, \tag{22}$$

where $\tilde{R}^{\nu_1}{}_{\mu_1}$ is the traceless part of the Ricci tensor. Despite being of nonconstant curvature, geometry (9) yields vanishing Cotton tensor

$$C_{\mu\nu} = \epsilon_\mu{}^{\alpha\beta}\nabla_\alpha\left(R_{\nu\beta} - \frac{1}{4}Rg_{\nu\beta}\right) = 0, \tag{23}$$

which implies that it is locally conformally flat. This property makes Weyl invariant probes integrable on this background, as we will show below. This permits us to gain a semiclassical intuition. The conformal factor that allows to write the interpolating background (10) in a manifestly conformally flat form, defines an improper Weyl transformations and, therefore, imposing boundary conditions and asymptotic behaviors on probe fields is nontrivially related to their flat counterpart.

D. Probes on the deformed geometry

1. Conformally coupled scalar field

Consider a conformally coupled scalar field on the geometry (9). The corresponding equation is

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi) - \frac{1}{8}\sqrt{-g}R\Phi = 0. \tag{24}$$

We consider the separable ansatz

$$\Phi(t,r,x) = e^{-i\omega t}e^{i\kappa x}\varphi(r). \tag{25}$$

This problem is exactly solvable. However, we can first gain intuition from the well-known small and large r regimes, where it reduces to the AdS_3 and to the flat space computation, respectively. This does not mean that the solution to the complete problem will be a simple junction of the two constant curvature problems. The transmission coefficients may actually change due to the different boundary conditions that have to be satisfied in the λ-deformed background.

Let us consider first the case $\kappa^2 - \omega^2 < 0$. In this case, the solution for $\varphi(r)$ takes the form

$$\varphi(r) = \left(\frac{r^2\lambda + 1}{r^2}\right)^{1/4}\left(A_1 e^{i\sqrt{\omega^2-\kappa^2}\chi(r)} + B_1 e^{-i\sqrt{\omega^2-\kappa^2}\chi(r)}\right), \tag{26}$$

where

$$\chi(r) = \sqrt{\lambda}\log\left(\lambda r + \sqrt{(r^2\lambda + 1)\lambda}\right) - \frac{\sqrt{r^2\lambda + 1}}{r}, \tag{27}$$

and where A_1 and B_1 are two constants to be determined by requiring appropriate boundary conditions. In order to impose conditions at infinity, it is convenient to solve (24) on the nearly flat metric

$$ds^2 \simeq -\frac{1}{\lambda}dt^2 + \frac{dr^2}{r^2} + \frac{1}{\lambda}dx^2 \tag{28}$$

which is the large r limit of (10) (here, we set $\ell = 1$ for short). The radial dependence of the conformal scalar on this metric is $\varphi(r) \propto A_1 e^{i\sqrt{\lambda}\sqrt{\omega^2-\kappa^2}\log(r)} + B_2 e^{-i\sqrt{\lambda}\sqrt{\omega^2-\kappa^2}\log(r)}$. We want to impose outgoing boundary conditions at infinity. Since for $\lambda \neq 0$ we have[1]

$$\Phi(t,r,x) \sim A_1 e^{-i\omega(t-\sqrt{\lambda}\log(r))} + B_1 e^{-i\omega(t+\sqrt{\lambda}\log(r))}, \tag{29}$$

by expanding at infinity we find that imposing outgoing boundary conditions corresponds to $B_1 = 0$. This is confirmed by considering the flux of particles defined by the $U(1)$ current, $j^\mu = -i(\Phi^*\partial^\mu\Phi - \Phi\partial^\mu\Phi^*)$. Thus,

$$\varphi(r) = A_1\left(\frac{r^2\lambda + 1}{r^2}\right)^{1/4} e^{i\sqrt{\omega^2-\kappa^2}\left(\sqrt{\lambda}\log\left(\lambda r + \sqrt{(r^2\lambda+1)\lambda}\right) - \frac{\sqrt{r^2\lambda+1}}{r}\right)}. \tag{30}$$

Now, let us study the behavior near $r = 0$. To do so, we expand the expression around the origin, where we find that the dominant part goes like $\varphi(r) \sim r^{-1/2}e^{-i\sqrt{\omega^2-\kappa^2}/r}$. At this point, we are interested in making connection between this result and the well-known result for AdS_3 (i.e., $\lambda = 0$) when $\omega^2 > \kappa^2$; namely

$$\varphi_{\lambda=0}(z) = A_2 z J_{\frac{1}{2}}(\sqrt{\omega^2-\kappa^2}z) + B_2 z J_{-\frac{1}{2}}(\sqrt{\omega^2-\kappa^2}z), \tag{31}$$

where $z = 1/r$ and where $J_{\pm 1/2}$ are Bessel functions.[2] In order to relate (31) with the complex exponentials in (26) one can use

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\frac{\sin(x)}{\sqrt{x}}, \qquad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}}\frac{\cos(x)}{\sqrt{x}} \tag{32}$$

and, therefore, in terms of r this becomes

$$\varphi_{\lambda=0}(r) = (A_2 + B_2)\frac{e^{i\frac{\sqrt{\omega^2-\kappa^2}}{r}}}{r^{1/2}} - (A_2 - B_2)\frac{e^{-i\frac{\sqrt{\omega^2-\kappa^2}}{r}}}{r^{1/2}}. \tag{33}$$

We see that the asymptotic behavior of (30) is a particular linear combination of the solutions in (33), namely with $B_2 = -A_2$. This is equivalent to setting very special mixing

[1] As for analyzing the ingoing or outgoing behavior, it is necessary to inspect the solution in its form $t \pm f(r)$. To do so, it is sufficient to look at the case $\kappa = 0$.

[2] These Bessel functions actually reduce to elementary functions, due to the fact that on AdS our problem reduces to that of a scalar with the conformal mass $m^2\ell^2 = m^2_{\mathrm{conf}}\ell^2 = -3/4$. For arbitrary values of the mass the Bessel functions are replaced by $J_{\pm\nu}$ with $\nu = \sqrt{1 + m^2}$.

boundary conditions in the AdS_3 region $r \ll 1/\sqrt{\lambda}$. This makes a difference with respect to the AdS_3 case $\lambda = 0$.

Now, let us see what happens in the case $\kappa^2 - \omega^2 > 0$, where the solution is

$$\varphi(r) = \left(\frac{r^2\lambda + 1}{r^2}\right)^{1/4} (C_1 e^{-\sqrt{\kappa^2-\omega^2}\chi(r)} + D_1 e^{\sqrt{\kappa^2-\omega^2}\chi(r)}). \tag{34}$$

As the wave function is now confined, we want the solution to vanish at infinity; therefore, we set $D_1 = 0$. As before, one can first take a look at the solution in the case $\lambda = 0$, namely

$$\varphi_{\lambda=0}(r) = \frac{C_2}{r} K_{\frac{1}{2}}\left(\frac{\sqrt{\kappa^2-\omega^2}}{r}\right) + \frac{D_2}{r} I_{\frac{1}{2}}\left(\frac{\sqrt{\kappa^2-\omega^2}}{r}\right), \tag{35}$$

and then rewrite the modified Bessel functions $K_{1/2}, I_{1/2}$ as

$$K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}} \qquad I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi}} \frac{\sinh(x)}{\sqrt{x}} \tag{36}$$

to make contact with the solution for $\lambda = 0$,

$$\varphi_{\lambda=0}(r) = \frac{C_2}{r^{1/2}} e^{-\frac{\sqrt{\kappa^2-\omega^2}}{r}} + \frac{D_2}{r^{1/2}} \sinh\left(\frac{\sqrt{\kappa^2-\omega^2}}{r}\right). \tag{37}$$

Expanding the $\lambda \neq 0$ solution near $r = 0$, one obtains

$$\varphi(r) = C_1 e^{\frac{\sqrt{\kappa^2-\omega^2}}{r}} \left(r^{-1/2} e^{-\frac{1}{2}\sqrt{\kappa^2-\omega^2}\sqrt{\lambda}\log(\lambda)} + \mathcal{O}(r^{1/2})\right), \tag{38}$$

so we see that (38) corresponds to the linear combination $C_2 = D_2$ above. Notice that, while (30) diverges as $\sim r^{-1/2}$ when r tends to zero, (38) is exponentially divergent in that limit. Condition $C_2 = D_2$ also looks from the AdS_3 viewpoint ($r \ll 1/\sqrt{\lambda}$) as mixed boundary conditions which are induced by the presence of the unusual asymptotic of the deformed theory.

2. Free fermion

As the conformally coupled scalar, the Dirac action is also Weyl invariant; therefore, it is natural to study a spin-1/2 probe on the deformed, conformally flat background (10). Explicitly, the Dirac equation is

$$\left(\gamma^a e_a^\mu \partial_\mu + \frac{1}{2}\omega_\mu^{ab}\gamma^c e_c^\mu J_{ab}\right)\Psi = 0. \tag{39}$$

The spinor Ψ will split in two components $\Psi^T = (\Psi_1, \Psi_2)$. The defomed metric (10) is of the form

$$ds^2 = -f^2(r)dt^2 + g^2(r)dr^2 + f^2(r)dx^2, \tag{40}$$

for which we can choose dreibein and compute the spin connections, leading respectively to

$$e^0 = f dt, \qquad e^1 = g dr \quad \text{and} \quad e^2 = f dx, \tag{41}$$

$$\omega^{01} = \frac{f'}{g} dt \quad \text{and} \quad \omega^{12} = -\frac{f'}{g} dx. \tag{42}$$

It is useful to use the explicit, real representation of the Dirac matrices $\gamma^0 = i\sigma^2, \gamma^1 = \sigma^1$, and $\gamma^2 = \sigma^3$. With these expressions at hand, the Dirac equation leads to the coupled system

$$\frac{1}{f}\partial_t\Psi_2 + \frac{1}{g}\partial_r\Psi_2 + \frac{1}{f}\partial_x\Psi_1 + \frac{f'}{fg}\Psi_2 = 0, \tag{43}$$

$$\frac{1}{f}\partial_t\Psi_1 - \frac{1}{g}\partial_r\Psi_1 + \frac{1}{f}\partial_x\Psi_2 - \frac{f'}{fg}\Psi_1 = 0. \tag{44}$$

Defining $\Psi_i(t, r, x) = e^{-i\omega t + i\kappa x}\psi_i(r)$ with $i = 1, 2$, we can integrate the radial profiles for the spinor as

$$\psi_1(r) = \left(\frac{r^2\lambda + 1}{r^2}\right)^{1/4} \varphi(r) \quad \text{and}$$

$$\psi_2(r) = \frac{\omega}{\kappa}\psi_1(r) - \frac{i}{\kappa}\frac{1}{g(r)}\frac{d}{dr}(f(r)\psi_1(r)), \tag{45}$$

where $\varphi(r)$ is given by (26) or (34) depending on the sign of $\kappa^2 - \omega^2$, as before. Consequently, the asymptotic behavior for the fermion is inherited by that of the scalar. When the momentum along the direction x vanishes, namely when $\kappa = 0$ the integration for the Dirac field is simpler and leads to

$$\psi_1(r) = C_1 \left(\frac{1 + r^2\lambda}{r^2}\right)^{1/2} (r\sqrt{\lambda} + \sqrt{1 + r^2\lambda})^{-i\omega\sqrt{\lambda}} e^{\frac{i\omega\sqrt{1+r^2\lambda}}{r}}, \tag{46}$$

$$\psi_2(r) = C_2 \left(\frac{1 + r^2\lambda}{r^2}\right)^{1/2} (r\sqrt{\lambda} + \sqrt{1 + r^2\lambda})^{i\omega\sqrt{\lambda}} e^{-\frac{i\omega\sqrt{1+r^2\lambda}}{r}}. \tag{47}$$

The two independent solutions $C_1 = 0$ or $C_2 = 0$, respectively describe an ingoing or outgoing flux of particles. Again, from the perspective of AdS_3, these would correspond to mixed boundary conditions.

III. STRING THEORY

Now, let us study the string world sheet theory. The world sheet action on the background (9) takes the form

$$S=\frac{1}{2\pi}\int d^2z\left(\frac{1}{2}\partial\phi\bar{\partial}\phi+\frac{1}{2}\partial\bar{u}\,\bar{\partial}\,u(\lambda+e^{-\sqrt{2/(k-2)}\phi})^{-1}\right), \tag{48}$$

together with an extra linear dilaton term $-(1/2\pi)\int d^2z\sqrt{2/(k-2)}R\phi$ with R being here the world sheet curvature. We see here that, provided $\lambda\neq 0$, in the limit $\phi\to\infty$ one recovers the free theory with a background charge term. In the case of $\lambda=0$, in contrast, the theory at $\phi\to\infty$ exhibits a nontrivial coupling between ϕ and the $u,\bar{u}$ dependence. This can be regarded as an effective potential in the ϕ-direction. This potential vanishes for holomorphic configurations such that $\partial\bar{u}\,\bar{\partial}\,u=0$. These configurations are closely related to the so-called long strings, which form a continuum in the spectrum in AdS_3.

A. Strings on $AdS_3\times\mathcal{N}$

Let us begin by reviewing the undeformed theory ($\lambda=0$), namely bosonic string theory on $AdS_3\times\mathcal{N}$. This theory corresponds to the level-k WZW model on $SL(2,\mathbb{R})$, and so it has $\widehat{sl}(2)_k$ affine Kac-Moody symmetry, which is generated by local currents whose modes are usually denoted $J_n^{\pm}$, J_n^3, along with their anti-holomorphic counterparts. Virasoro symmetry follows from the Sugawara construction. We consider primary operators of the form

$$V_h(p|z)=Z_0|p|^{2-2h}e^{ipu(z)+i\bar{p}\bar{u}(\bar{z})}e^{\sqrt{2/(k-2)}(h-1)\phi(z,\bar{z})}\times\ldots \tag{49}$$

where the ellipsis stand for the contributions of internal part $\mathcal{N}$. These are the vertex operators of the theory. p and $\bar{p}$ are the momenta conjugate to directions u and $\bar{u}$, while h is related to the radial momentum. The factor $Z_0|p|^{2-2h}$ stands for a normalization. The world sheet conformal dimension of these operators are

$$\Delta_{\lambda=0}=\frac{h(1-h)}{k-2}+\Delta_{\mathcal{N}}+N. \tag{50}$$

An analogous expression holds for $\bar{\Delta}_{\lambda=0}$ with $\bar{\Delta}_{\mathcal{N}}$ and $\bar{N}$. $\Delta_{\mathcal{N}}$ stand for the conformal dimension of the operators of the CFT on the internal space $\mathcal{N}$, and N is the string excitation number. As just said, p and $\bar{p}$ represent the momentum in the boundary, and they relate to the momentum in (25) as follows:

$$\kappa=\frac{p+\bar{p}}{\ell},\qquad \omega=\frac{\bar{p}-p}{\ell}. \tag{51}$$

In the Euclidean theory, $t\to it$ and $\bar{p}$ is the complex conjugate of p. The index h labels the representations of $SL(2,\mathbb{R})$. We focus on the long string states, which belong to the continuous series representations, having

$$h=\frac{1}{2}+is,\quad \text{with}\quad s\in\mathbb{R}. \tag{52}$$

These long strings can reach the boundary due to the coupling to the B-field. They have a continuous energy spectrum, which depends on the spectral flow variable $w\in\mathbb{Z}_{\geq 0}$ that accounts for the winding number of the string around the boundary. To analyze the spectrum of the theory on $AdS_3\times\mathcal{N}$ in the momentum space, it is convenient to consider the operator basis

$$V_{h,m,\bar{m}}(z)=\frac{\Gamma(h+m)}{\Gamma(1-h-\bar{m})}\int\frac{d^2p}{|p|^2}p^{-m}\bar{p}^{-\bar{m}}V_h(p|z). \tag{53}$$

Performing spectral flow transformation on the states created by these operators, one obtains the states of the sector w, whose conformal dimensions are

$$\Delta_{\lambda=0}=\frac{h(1-h)}{k-2}-mw-\frac{k}{2}w^2+\Delta_{\mathcal{N}}+N. \tag{54}$$

where the energy is given by $m+\bar{m}+kw$ and the angular momentum by $m-\bar{m}$.

The 2-point function in the theory on $AdS_3\times\mathcal{N}$ is well-known. For long strings in the basis $V_h(p|z)$, this takes the form

$$\begin{aligned}&\langle V_{\frac{1}{2}+is_1}(p_1|z_1=0)V_{\frac{1}{2}+is_2}(p_2|z_2=1)\rangle_{\lambda=0}\\&=\frac{2s_1}{\pi k}Z_0^2\nu(k)^{2is_1}|p_1|^{4is_1}\delta^{(2)}(p_1+p_2)\delta(s_1-s_2)e^{2i\varphi},\end{aligned} \tag{55}$$

where

$$e^{2i\varphi}=\Gamma/\Gamma^*,\quad \text{with}\quad \Gamma=\Gamma(-2is)\Gamma(-2is/(k-2)) \tag{56}$$

and

$$\nu(k)=\frac{\Gamma(\frac{1}{k-2})}{\Gamma(1-\frac{1}{k-2})}. \tag{57}$$

The subscript $\lambda=0$ in (55) refers to the fact that the quantity corresponds to the undeformed $AdS_3\times\mathcal{N}$ background. In the deformed theory, the 2-point function has been computed in [15,16], yielding

$$\begin{aligned}&\langle V_{h_1}(p_1|z_1=0)V_{h_2}(p_2|z_2=1)\rangle_\lambda\\&=\delta^{(2)}(p_1+p_2)\delta_{h_1-h_2}|p_1|^{4h_1-2}B(h_1)\end{aligned} \tag{58}$$

with

$$B(h_1) = \frac{\nu(k)^{2h_1-1}}{\pi} \frac{\Gamma(1-2h_1)\Gamma(1-\frac{2h_1-1}{k-2})}{\Gamma(2h_1-1)\Gamma(\frac{2h_1-1}{k-2})}, \quad (59)$$

and where the spectrum of the theory is given by

$$h = \frac{1}{2} \pm \frac{1}{2}\sqrt{8(k-2)\lambda|p|^2 - s^2}. \quad (60)$$

This reduces to the 2-point function of the $SL(2,\mathbb{R})_k$ WZW model in the limit $\lambda = 0$. Equation (60) follows from imposing the Virasoro constraint $\Delta_{\lambda=0} = 1$ on (50).

Equation (60) for the expression can be regarded as a string theory analog of the mode mixing discussed when we analyzed the scalar probes in the deformed background. Of course, spectrum (60) exhibits purely stringy phenomena; in particular, for $\lambda = 0$ it reduces to the long string theory spectrum (52). The long string states form the continuous part of the spectrum and, being states that escape to the asymptotic region, permit to define a scattering matrix in AdS_3. This is due to the coupling to the B-field, and so have no particle analog. Still, we see in (60) that a value $\lambda > 0$ change the spectrum of such states, introducing, on the one hand, a p-dependence on the world sheet dimension h, and, on the other hand, a λ-dependent threshold for the continuous part of the spectrum. This is the string theory analog to the mode mixing discussed in Sec. II.

B. Turning on the deformation

To see in detail how to obtain the correlator (58)–(59) and (60), we may first rewrite action (48) by adding auxiliary fields $v, \bar{v}$, yielding the equivalent action

$$S_\lambda = \frac{1}{2\pi}\int d^2z\left(\frac{1}{2}\partial\phi\bar{\partial}\phi - v\bar{\partial}u - \bar{v}\partial\bar{u} - 2v\bar{v}e^{-\sqrt{2/(k-2)}\phi} - 2\lambda v\bar{v}\right), \quad (61)$$

where now the pair (v, u) forms a commuting, dimension-(1,0) (β, γ) ghost system. As we work in the conformal gauge, we are omitting here a background charge that represents the dilaton term. For $\lambda = 0$, equation (61) is, indeed, the WZW model written in Wakimoto variables [30]. Nevertheless, we prefer to keep the notation v, u to make contact with the spacetime interpretation (9). The action of the λ-deformed theory is thus given by $S_{SL(2,\mathbb{R})\mathrm{WZW}} - 2\lambda\int dz^2 v\bar{v}$. This corresponds to a current-current deformation of the WZW model, with the deformation being realized by the operator $\lambda v\bar{v}$. This is consistent with the fact that the specific Kac-Moody current in these variables reads $J^-(z) = v(z)$.

When trying to compute a correlation function such as $\langle V_{h_1}(p_1|z_1)V_{h_2}(p_2|z_2)\rangle_\lambda$ in the path integral approach, namely

$$\langle V_{h_1}(p_1|z_1)V_{h_2}(p_2|z_2)\rangle_\lambda = \int \mathcal{D}\phi\mathcal{D}^2u\mathcal{D}^2v e^{-S_\lambda}V_{h_1}(p_1|z_1)V_{h_2}(p_2|z_2), \quad (62)$$

the presence of the operator $\lambda\int d^2z v\bar{v}$ induces a UV divergent term in the effective action after integrating the fields $u, \bar{u}$ (see [16] for more details). This makes the contribution of the correlators coming from the undeformed theory to factorize, and the deformation ends up contributing with an exponential that contains the conformal integral

$$I_0 = \int d^2z|z - z_1|^{-2}|z - z_2|^{-2}. \quad (63)$$

This divergent integral appears frequently in quantum field theory calculations. For instance, it appears in the one-loop computation of the anomalous dimension of the composite operator $\bar{\psi}\psi$ in the Thirring model. The result of it is logarithmically divergent and it can be regularized using different methods. By introducing a regulator ϵ, this can be resolved as[3]

$$I_\epsilon = (1 + 2\epsilon\log|z_1 - z_2| + \mathcal{O}(\epsilon^2))\left(\frac{2\pi}{\epsilon} + \mathcal{O}(\epsilon^0)\right), \quad (64)$$

and, after renormalizing the vertex operators by choosing $Z_\epsilon = e^{-2\lambda|p|^2/\epsilon}$, one obtains

$$\langle V_{h_1}(p_1|z_1)V_{h_2}(p_2|z_2)\rangle_\lambda \sim |z_1 - z_2|^{-4\Delta_0 - 4\lambda|p_1|^2} \quad (65)$$

From this, it is possible to read the anomalous dimension induced by the deformation; namely

$$\Delta_{\lambda=0} \to \Delta_\lambda = \Delta_{\lambda=0} + \lambda|p|^2. \quad (66)$$

Finally, imposing the Virasoro constraint $\Delta_\lambda = 1$ and writing it in terms of the quantities of the undeformed theory that satisfied $\Delta_{\lambda=0} = 1$, one gets (60), which reduces to (52) in the case $\lambda = 0$. We observe that $h \in \mathbb{R}$ provided $-4|p|\sqrt{k\lambda} \geq s \geq +4|p|\sqrt{k\lambda}$; and $h \in \frac{1}{2} + i\mathbb{R}$ provided $|s| > 4|p|\sqrt{k\lambda_0}$. The overall factor $|p_1|^{4h_1-2}$ in the 2-point function and the dependence of h_1 on λ has been studied in detail in [15] to investigate the properties of the dual theory, especially its nonlocality encoded in a branch cut that the 2-point function of the λ-deformed theory exhibits.

IV. GENERALIZATIONS

The advantage of the computation of the anomalous dimension described above is that it admits a

[3] Here $\mathcal{O}(\epsilon^0)$ stands for all constant terms that are independent of the insertion points, as we can eliminate such terms by the normalization of the vertex operators.

straightforward generalization to other models, such as the $SL(N,\mathbb{R})$ WZW models or their supersymmetric extensions. Despite not in all such cases one has a string σ-model interpretation of the CFT, this is still interesting from the CFT point of view as it provides a set of solvable nonrational models. The simplest extension of this sort is the $SL(N,\mathbb{R})$ WZW model. In that case, the action can in principle be written as a sum of a Gaussian piece and an interaction piece S_I; namely

$$S_{SL(N,\mathbb{R})\mathrm{WZW}} = \frac{1}{2\pi}\int d^2z\left((\partial\phi,\bar{\partial}\phi) - \sum_{a=1}^{N(N-1)/2}(v_a\bar{\partial}u_a + \bar{v}_a\partial\bar{u}_a)\right) + S_I. \tag{67}$$

This involves a set of $N-1$ scalars and $N(N-1)$ copies of β, γ systems, which here we keep denoting by v_a, u_a with $a = 1,2,\ldots N(N-1)/2$. The scalars, ϕ_i, with $i = 1,2,\ldots N-1$ form a vector in the space of roots of $sl(N)$. We denote (.,.) the product in this space of roots, which is defined in terms of the Cartan matrix $K_{ij} = (e_i, e_j)$ with $e_1, e_2, \ldots, e_{N-1}$ being the simple roots, with the $N-1$ fundamental weights w_i satisfying $(w_i, e_j) = \delta_{ij}$. ρ is the Weyl vector, i.e., the half-sum of all positive roots. The Lagrangian also includes a background charge term $\int d^2z(\rho,\phi)R/\sqrt{k-N}$.

As before, in the appropriate basis a solvable family of current-current deformation of the theory is given by the addition of the marginal operator

$$\sum_{i=1}^{N-1}\frac{\lambda_i}{\pi}\int d^2z v_i\bar{v}_i, \tag{68}$$

where the field $J_i^-(z) = v_i(z)$ with $i = 1,2,\ldots N-1$ correspond to the Abelian subalgebra formed by $N-1$ lowering operators. If we denote $H = (J_1^3, J_2^3, \ldots J_{N-1}^3)$ the generators of the Cartan subalgebra, and $E = (J_1^+, J_2^+, \ldots J_{N(N-1)/2}^+)$ and $F = (J_1^-, J_2^-, \ldots J_{N(N-1)/2}^-)$ the raising and lowering operators, respectively, then there exists an ordering such that the first $N-1$ elements $F \supset (J_1^-, J_2^-, \ldots J_{N-1}^-)$ form an Abelian subalgebra. More importantly, there exists a free field representation such that these $N-1$ fields are given by $J_i^- = v_i$ with $i = 1,2,\ldots N-1$. For the case A_{N-1} with $N = 2$, 3, 4, 5 these representations have been explicitly constructed in the literature [31–33], and the generic case has been extensively discussed [34–36]. Let us first show how the argument goes for generic N and then consider an illustrating particular case.

Consider the operators

$$V_h(p,z) = Z_0 e^{\sqrt{2/(k-N)}(h,\phi(z))} e^{i\sum_{a=1}^{N(N-1)/2}(p^a u_a(z) + \bar{p}^a\bar{u}_a(z))} \tag{69}$$

where $h = (h_1, h_2, \ldots h_{N-1})$ is the vector of the space of roots, and $p = (p^1, p^2, \ldots p^{N(N-1)/2})$ are the momentum associated to the directions u_a; and consider the correlation functions

$$\begin{aligned}\langle V_h(p_1|z_1)V_h(p_2|z_2)\rangle_{\lambda_1,\ldots\lambda_{N-1}} &= \int\prod_{i=1}^{N-1}\mathcal{D}\phi_i\prod_{a=1}^{N(N-1)/2}\mathcal{D}^2u_a\mathcal{D}^2v_a e^{-S_{\lambda_1,\ldots,\lambda_{N-1}}} \\ &\quad\times V_h(p_1|z_1)V_h(p_2|z_2).\end{aligned} \tag{70}$$

After integrating in u_i for some i (those that correspond to the fields u_i that do not appear other than in the kinetic term) the action (67)–(68) being linear in these fields, one obtains

$$\bar{\partial}v_i = 2\pi i(p_1^i\delta^2(z-z_1) + p_2^i\delta^2(z-z_2)), \tag{71}$$

The solution is

$$v_i(z) = \frac{ip_1^i}{z-z_1} - \frac{ip_1^i}{z-z_2}, \tag{72}$$

where we have used that, on the sphere, $p_1 + p_2 = 0$ in virtue of the Riemann-Roch theorem. This can now be inserted back in (68). When doing so, one observes that a logarithmically divergent integral similar to (63) appears, yielding an anomalous correction to the conformal dimension of operators (69). To see this in detail, let us consider the case $N = 3$, in which the undeformed theory is given by the WZW model on $SL(3,\mathbb{R})$, whose action reads

$$\begin{aligned}S_{SL(3,\mathbb{R})\mathrm{WZW}} = \frac{1}{2\pi}\int d^2z\Bigg((\partial\phi,\bar{\partial}\phi) &- \sum_{a=1}^{3}(v_a\bar{\partial}u_a + \bar{v}_a\partial\bar{u}_a) \\ &+ |v_2 + v_1u_3|^2 e^{\sqrt{2/(k-3)}(e_2,\phi)} \\ &- v_3\bar{v}_3 e^{\sqrt{2/(k-3)}(e_3,\phi)} - v_1\bar{v}_1 e^{\sqrt{2/(k-3)}(\rho,\phi)}\Bigg),\end{aligned} \tag{73}$$

together with a background charge term $\int d^2z(\rho,\phi)R/\sqrt{k-3}$. As before, this action can be written in terms of the Wakimoto variables in such a way that two commuting currents take a simple form $J_1^-(z) = v_1(z)$ and $J_2^-(z) = v_2(z)$. Therefore, in the spirit of the deformation for $SL(2,\mathbb{R})$, we deform the $SL(3,\mathbb{R})$ WZW model by adding to it two quadratic operators, for v_1 and v_2; namely

$$S_{\lambda_1,\lambda_2} = S_{SL(3,\mathbb{R})\mathrm{WZW}} - \frac{\lambda_1}{\pi}\int d^2z v_1\bar{v}_1 - \frac{\lambda_2}{\pi}\int d^2z v_2\bar{v}_2. \tag{74}$$

We consider operators (69) with $N=3$, $h=(h_1,h_2)$ and $p=(p_1,p_2,p_3)$, and the correlation function $\langle V_h(p,z_1)V_h(-p,z_2)\rangle_{\lambda_1,\lambda_2}$. After integrating on u_i, one finds the solutions for v_1 and v_2 to be

$$v_1(z)=\frac{ip^1(z_1-z_2)}{(z-z_1)(z-z_2)},\quad v_2(z)=\frac{ip^2(z_1-z_2)}{(z-z_1)(z-z_2)}. \tag{75}$$

which, when replaced in the action, yields

$$\begin{aligned} S_{\lambda_1,\lambda_2}=&\frac{1}{2\pi}\int d^2z((\partial\phi,\bar{\partial}\phi)-v_3\bar{\partial}u_3-\bar{v}_3\partial\bar{u}_3\\ &-v_3\bar{v}_3e^{(\phi_2-\sqrt{3}\phi_3)/\sqrt{k-3}}+|p^2+p^1u_3|^2e^{(\phi_2+\sqrt{3}\phi_3)/\sqrt{k-3}}\\ &-|p^1|^2e^{2\phi_2/\sqrt{k-3}})+\frac{1}{\pi}(\lambda_1|p^1|^2+\lambda_2|p^2|^2)|z_1-z_2|^2I_0, \end{aligned} \tag{76}$$

with I_0 given by (63). Regularizing as in (64) and renormalizing the vertices accordingly, one obtains the corrected conformal dimension

$$\Delta_{\lambda_1,\lambda_2}=\Delta_{\lambda_1=\lambda_2=0}+\lambda_1|p^1|^2+\lambda_2|p^2|^2. \tag{77}$$

This follows from the last line in (76), which contains the logarithmic dependence in (64). This manifestly shows that the method of [16] can be straightforwardly adapted to higher rank.

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