Laplacians on fuzzy Riemann surfaces

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We consider the matrix regularization of scalar fields on a Riemann surface with a general gauge-field background. We propose a construction of the fuzzy version of the Laplacian.

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I. INTRODUCTION

The concept of noncommutative geometry naturally arises in superstring theory [1] and is expected to give a wider framework of geometry admitting also theories of quantum gravity. The matrix models [2,3], which are conjectured to be nonperturbative formulations of M-theory and superstring theories, also involve noncommutative geometry and various objects such as membranes or D-branes are described in terms of fuzzy (finite noncommutative) geometry in the matrix models.

The main purpose of this paper lies in understanding the fuzzy geometry by investigating the so-called matrix regularization [4]. In particular, for an arbitrary fuzzy Riemann surface with (or without) a general gauge-field background, we give a construction of the fuzzy version of the Laplacian, which has rich information on the geometry and is needed to study scalar field theories on the fuzzy surface.

The matrix regularization is a method of constructing a fuzzy space from a given ordinary commutative space. This method is very useful, because it enables us to understand elusive fuzzy geometry in terms of well-established differential geometry of commutative spaces. For a given compact Riemann surface M with a symplectic form ω, the matrix regularization is defined as a linear map $T_N : C^\infty(M) \to M_N(\mathbb{C})$ which satisfies [5]

$$\lim_{N\to\infty} |T_N(f)T_N(g) - T_N(fg)| = 0, \tag{1.1}$$

$$\lim_{N\to\infty} |i\hbar_N^{-1}[T_N(f), T_N(g)] - T_N(\{f,g\})| = 0, \tag{1.2}$$

$$\lim_{N\to\infty} \hbar_N \mathrm{Tr} T_N(f) - \frac{1}{2\pi}\int_M \omega f = 0 \tag{1.3}$$

for any $f, g \in C^\infty(M)$. Here, $\hbar_N = V/N$, $V = \frac{1}{2\pi}\int_M \omega$, $\{,\}$ is the Poisson bracket defined by ω and $|\cdot|$ is a matrix norm. Equation (1.1) states that the algebraic structure of functions is well approximated by using the noncommutative matrix algebra and the approximation becomes more precise as the matrix size N goes to infinity. Equation (1.2) shows that the Poisson bracket is approximated by the matrix commutator, and thus the matrix regularization can be seen as a generalization of the canonical quantization of classical mechanics such that the phase space is not just a plane but the general compact surface M. Equation (1.3) is needed to avoid the trivial case, $T_N(f) = 0$ for any f, and is essential to derive the actions of the matrix models from the worldvolume theories of a membrane or a string [4].

The matrix regularization can be explicitly constructed by the Berezin-Toeplitz quantization [6–9]. In this quantization, as we will describe in more detail in the next section, one starts from a suitably constructed Dirac operator D with totally N normalizable zero modes. Then, one obtains the map T_N satisfying (1.1)–(1.3) as the restriction of the algebra $C^\infty(M)$ onto the space of the zero modes. The map can be written as $T_N(C^\infty(M)) = \Pi C^\infty(M)\Pi$ with the projection operator Π onto the Dirac zero modes.[1] The $N \times N$ matrix $T_N(f)$ for $f \in C^\infty(M)$ is called the Toeplitz operator of f.

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[1]It is notable that this mathematical framework naturally arises in the context of the Tachyon condensation on non-BPS D-branes [10,11]. See also [12,13].

The Berezin-Toeplitz quantization was further generalized in [14,15] and applied to $U(1)$ charged scalar fields on M [16], toward understanding the fuzzy description of D-branes.[2] When M has a nontrivial magnetic flux, charged scalar fields cannot be globally defined. They are defined on each local coordinate patch and glued together by a gauge transformation on any overlap of two patches. Such fields (mathematically called local sections of a complex line bundle) are naturally mapped to rectangular $N \times N'$ matrices, where the difference $N - N'$ is kept fixed to be the charge of the fields. For a charged field φ with charge Q, let us write its Toeplitz operator as $T_{NN'}(\varphi)$, which is $N \times N'$ matrix with $N - N' = Q$. With an appropriate construction, which we will review later, it was shown that the operator satisfies [14,15]

$$\lim_{N\to\infty} |T_N(f)T_{NN'}(\varphi) - T_{NN'}(f\varphi)| = 0 \tag{1.4}$$

for any $f \in C^\infty(M)$ and a similar equation also holds for the left action of $T_{N'}(f)$ onto the rectangular matrix $T_{NN'}(\varphi)$. This is a generalization of Eq. (1.2) and shows that the $C^\infty(M)$-module structure of charged fields can be approximated by the $M_N(\mathbb{C})$- and $M_{N'}(\mathbb{C})$-module structures of the rectangular matrices.

In this paper, we further investigate the Berezin-Toeplitz quantization by extending the work [16]. We consider a more general setup than [16], such that the scalar fields to be regularized take values in a general representation of an arbitrary gauge group. We will show that the regularization for such fields can also be achieved by rectangular matrices. We will then derive a general large-N asymptotic expansion of the product of two Toeplitz operators up to the second order in $1/N$. This expansion basically contains all important information of the quantization map, and the fundamental relations such as (1.1), (1.2), and (1.4) can also be derived from this expansion. By using the asymptotic expansion, we then construct an operator acting on the rectangular matrices such that its spectrum approaches in the commutative limit to that of the continuum Laplacian on M with an arbitrary configuration of the background gauge field.

This paper is organized as follows. In Sec. II, we first review the Berezin-Toeplitz quantization for scalar fields in a general gauge-field background and then derive the asymptotic expansion. In Sec. III, we construct the fuzzy Laplacian and show some examples of this construction. In Sec. IV, we summarize our results.

II. BEREZIN-TOEPLITZ QUANTIZATION

In this section, we consider the Berezin-Toeplitz quantization of scalar fields in the presence of nontrivial background gauge fields [8,9,14,15,20] (see also [16]). After defining the quantization map, we derive the large-N asymptotic expansion for Toeplitz operators.

A. Berezin-Toeplitz quantization of scalar fields

Let M be a closed Riemann surface with a metric g. We denote by ω the volume form of g. Since ω is a nondegenerate closed 2-form, it is also a symplectic form on M.

We denote by L a complex line bundle with a particular $U(1)$ connection A such that its field strength F is proportional to the symplectic form as

$$F = dA = \omega/V. \tag{2.1}$$

Here, V is the volume, $V = \frac{1}{2\pi}\int_M \omega$, so that $\frac{1}{2\pi}\int_M F = 1$. The line bundle L becomes very important below and will be used to realize the desired large-N expansion satisfying (1.1)–(1.3) or (1.4). The gauge field A may be different from the physical background gauge field introduced below.[3]

We next introduce physical gauge fields coupling to the scalar fields, to which we apply the Berezin-Toeplitz quantization. We regard the scalar fields as sections of the vector bundle, $\mathrm{Hom}(E, E')$, and the gauge fields as its connection. Here, E and E' are arbitrary finite-rank vector bundles on M with Hermitian inner products and Hermitian connections, and $\mathrm{Hom}(E, E')$ is the vector bundle on M such that its fiber is given by a set of all linear maps from the fiber of E to that of E'.[4] If the dimensions of the fibers of E and E' are n and n', respectively, the fiber of $\mathrm{Hom}(E, E')$ is just a set of all $n' \times n$ matrices. This definition of scalar fields covers all physically interesting cases. For example, when E and E' are given by $E = \tilde{L}^{\otimes n}$ and $E' = \tilde{L}^{\otimes m}$ with a certain complex line bundle $\tilde{L}$ with a $U(1)$ connection $\tilde{A}$, $\mathrm{Hom}(E, E')$ reduces to $\tilde{L}^{\otimes(m-n)}$. Sections of $\tilde{L}^{\otimes(m-n)}$ are just complex scalar fields coupled to the gauge field $\tilde{A}$ with the charge $m - n$. Another example is scalars fields in the adjoint representation of a non-Abelian gauge group. By taking both E and E' to be the same as a vector bundle of the fundamental representation space of a given gauge group, sections of $\mathrm{Hom}(E, E')$ correspond to the adjoint scalars. This definition of scalar fields in terms of $\mathrm{Hom}(E, E')$ is suitable for defining the quantization map, since there is a natural product of two scalar fields given by the composition of linear maps. For two scalar fields $\varphi \in \Gamma(\mathrm{Hom}(E, E'))$ and $\varphi' \in \Gamma(\mathrm{Hom}(E', E''))$, where $\Gamma(E)$ denotes a set of all sections of E, the pointwise

[2]See [17] for a generalization to matrix valued scalar fields and [18,19] for the quantization using instanton configurations.

[3]The work [16] treats the special case in which A is identical to the physical gauge field.

[4]In this paper, we are mainly interested in the case where E and E' are bundles of representation spaces of a given gauge group. Another interesting case, which will be studied elsewhere, is such that E and E' are given as tensor products of TM or T^*M. In this case, the sections of $\mathrm{Hom}(E, E')$ are not scalar but tensor fields.

composition of the linear maps on M gives $\varphi'\varphi \in \Gamma(\mathrm{Hom}(E, E''))$. This is the product that is to be promoted to the matrix product through the quantization map.

The quantization map is given in terms of the projection to Dirac zero modes as briefly mentioned in the previous section. So let us introduce spinor fields on M. We consider the twisted spinor bundle, $S \otimes L^{\otimes N} \otimes E$, where S is the two-component spinor bundle on M, N is a positive integer, and E is any Hermitian vector bundle. We equip an inner product on $\Gamma(S \otimes L^{\otimes N} \otimes E)$ by

$$(\psi', \psi) \coloneqq \int_M \omega(\psi')^\dagger \cdot \psi \tag{2.2}$$

for $\psi, \psi' \in \Gamma(S \otimes L^{\otimes N} \otimes E)$. Here, $\cdot$ is the inner product (contraction) of the all indices. The norm on $\Gamma(S \otimes L^{\otimes N} \otimes E)$ is defined by $|\psi| = \sqrt{(\psi, \psi)}$. We denote by $L^2(S \otimes L^{\otimes N} \otimes E)$ the subset of $\Gamma(S \otimes L^{\otimes N} \otimes E)$ given by all elements with finite norms. Note that a scalar field $\varphi \in \Gamma(\mathrm{Hom}(E, E'))$ can be seen as a map from $\psi \in \Gamma(S \otimes L^{\otimes N} \otimes E)$ to $\varphi\psi \in \Gamma(S \otimes L^{\otimes N} \otimes E')$, where the latter is defined as the pointwise product on M. The quantization map is essentially given by the restriction of this action onto the Dirac zero modes, which we will discuss shortly.

We define the (twisted) Dirac operator $D^{(E)}$ as an elliptic differential operator on $\Gamma(S \otimes L^{\otimes N} \otimes E)$ given by

$$D^{(E)}\psi = i\gamma^\alpha \nabla_\alpha \psi, \tag{2.3}$$

where $\{\gamma^\alpha\}$ are the gamma matrices in curved space satisfying $\{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta}$, namely, for the constant gamma matrices $\{\gamma^a\}_{a=1,2}$ on a local orthogonal frame satisfying $\{\gamma^a, \gamma^b\} = 2\delta^{ab}$, γ^α are given by $\gamma^\alpha = e^\alpha_a \gamma^a$ with e^α_a the inverse of the zweibein for the metric g. The covariant derivative ∇_α acts on $\psi \in \Gamma(S \otimes L^{\otimes N} \otimes E)$ as

$$\nabla_\alpha \psi = (\partial_\alpha + \Omega_\alpha - iNA_\alpha - iA^{(E)}_\alpha)\psi, \tag{2.4}$$

where Ω_α is the spin connection and $A^{(E)}_\alpha$ is the connection for the bundle E, which takes values in square matrices acting on the fiber of E. We denote by $\mathrm{Ker}D^{(E)}$ the set of all normalizable zero modes of D with respect to the inner product (2.2). As shown in Appendix A, $\mathrm{Ker}D^{(E)}$ becomes a $(d^{(E)}N + c^{(E)})$-dimensional vector space for sufficiently large N, where $d^{(E)}$ and $c^{(E)}$ are the rank and the first Chern number of E, respectively.

By using the above structures, we can define the Berezin-Toeplitz quantization for scalar fields. For any scalar field $\varphi \in \Gamma(\mathrm{Hom}(E, E'))$, which gives a map $\Gamma(S \otimes L^{\otimes N} \otimes E) \to \Gamma(S \otimes L^{\otimes N} \otimes E')$, the quantization map is defined by

$$T_N^{(E',E)}(\varphi) = \Pi'\varphi\Pi. \tag{2.5}$$

Here, $\Pi : \Gamma(S \otimes L^{\otimes N} \otimes E) \to \mathrm{Ker}D^{(E)}$ is the projection operator onto $\mathrm{Ker}D^{(E)}$ and Π' is the similar projection for E'. $T_N^{(E',E)}(\varphi)$ can be represented as a rectangular matrix with size $(d^{(E')}N + c^{(E')}) \times (d^{(E)}N + c^{(E)})$ and is called the Toeplitz operator for φ. As we will see below, the Toeplitz operator (2.5) enjoys a nice large-N asymptotic behavior, from which one can derive (1.1), (1.2), and (1.4).

From (2.5), we notice that the quantization map preserves the Hermitian conjugation as

$$T_N^{(E,E')}(\varphi^\dagger) = (T_N^{(E',E)}(\varphi))^\dagger, \tag{2.6}$$

where $\varphi^\dagger \in \Gamma(\mathrm{Hom}(E', E))$ is the Hermitian conjugate of φ defined by the inner product (2.2) and $\dagger$ on the right-hand side is the Hermitian conjugate for the rectangular matrices.

B. Asymptotic expansion of Toeplitz operators

For any scalar fields $\varphi \in \Gamma(\mathrm{Hom}(E, E'))$ and $\varphi' \in \Gamma(\mathrm{Hom}(E', E''))$, let us consider their Toeplitz operators, $T(\varphi) = \Pi'\varphi\Pi$ and $T(\varphi) = \Pi''\varphi'\Pi'$. Here and hereafter, we will omit all subscripts of the Toeplitz operators as it is obvious from their arguments, and we will recover the subscripts only when it may cause confusion. The product $T(\varphi')T(\varphi)$ is a $(d^{(E'')}N + c^{(E'')}) \times (d^{(E)}N + c^{(E)})$ matrix and has the following asymptotic expansion in $\hbar_N = V/N$:

$$T(\varphi')T(\varphi) = \sum_{i=0}^{\infty} \hbar_N^i T(C_i(\varphi', \varphi)), \tag{2.7}$$

where $C_i : \Gamma(\mathrm{Hom}(E', E'')) \otimes \Gamma(\mathrm{Hom}(E, E')) \to \Gamma(\mathrm{Hom} \times (E, E''))$ represent bilinear differential operators such that the order of the derivatives in C_i is at most i for each argument. We find that the first three C_i's are explicitly given by

$$\begin{aligned}
C_0(\varphi', \varphi) &= \varphi'\varphi, \\
C_1(\varphi', \varphi) &= -\frac{1}{2}(g^{\alpha\beta} + iW^{\alpha\beta})(\nabla_\alpha\varphi')(\nabla_\beta\varphi), \\
C_2(\varphi', \varphi) &= \frac{1}{8}(g^{\alpha\beta} + iW^{\alpha\beta})(\nabla_\alpha\varphi')(R + 4F^{(E')}_{12})(\nabla_\beta\varphi) \\
&\quad + \frac{1}{8}(g^{\alpha\beta} + iW^{\alpha\beta})(g^{\gamma\delta} + iW^{\gamma\delta}) \\
&\quad \times (\nabla_\alpha\nabla_\gamma\varphi')(\nabla_\beta\nabla_\delta\varphi).
\end{aligned} \tag{2.8}$$

Here, R is the Ricci scalar and $W^{\alpha\beta} \coloneqq \epsilon^{ab} e^\alpha_a e^\beta_b$, which is the Poisson tensor induced by the symplectic structure. $F^{(E')}_{12} = e^\alpha_1 e^\beta_2 F^{(E')}_{\alpha\beta} = e^\alpha_1 e^\beta_2 (\partial_\alpha A^{(E')}_\beta - \partial_\beta A^{(E')}_\alpha - i[A^{(E')}_\alpha, A^{(E')}_\beta])$ is the curvature of E' in the orthonormal frame. The covariant derivatives in (2.8) act on the scalar fields as

$$\nabla_\alpha \varphi = \partial_\alpha \varphi - iA_\alpha^{(E')}\varphi + i\varphi A_\alpha^{(E)},$$
$$\nabla_\alpha \varphi' = \partial_\alpha \varphi' - iA_\alpha^{(E'')}\varphi' + i\varphi' A_\alpha^{(E')}. \tag{2.9}$$

We leave the proof of (2.7) to Appendix B (see also Appendix C for a consistency check of our calculation) and discuss here some important corollaries of (2.7). From the leading term in (2.7), we first notice that

$$\lim_{N\to\infty} |T(\varphi')T(\varphi) - T(\varphi'\varphi)| = 0. \tag{2.10}$$

When both E' and E'' are the trivial line bundle and $E = L^{\otimes(-Q)}$, the relation (2.10) reduces to (1.4), as $\varphi' \in C^\infty(M)$ and $\varphi \in \Gamma(L^{\otimes Q})$. When E is also taken to be the trivial line bundle, it further reduces to (1.1).

Next, suppose that four fields $\varphi_1 \in \Gamma(\mathrm{Hom}(E, E'))$, $\varphi_2 \in \Gamma(\mathrm{Hom}(E', E''))$, $\varphi_3 \in \Gamma(\mathrm{Hom}(E, \tilde{E}'))$, and $\varphi_4 \in \Gamma(\mathrm{Hom}(\tilde{E}', E''))$ satisfy $\varphi_2\varphi_1 = \varphi_4\varphi_3$. Then, from (2.7), we find that

$$\lim_{N\to\infty} |\hbar_N^{-1}(T(\varphi_2)T(\varphi_1) - T(\varphi_4)T(\varphi_3)) + \frac{1}{2}T((g^{\alpha\beta} + iW^{\alpha\beta})((\nabla_\alpha\varphi_2)(\nabla_\beta\varphi_1) - (\nabla_\alpha\varphi_4)(\nabla_\beta\varphi_3)))| = 0. \tag{2.11}$$

We further consider a special case in which $E' = E''$, $\tilde{E}' = E$, $\varphi_1 = \varphi_4 =: \varphi \in \mathrm{Hom}(E, E')$, $\varphi_2 = f\mathbf{1}_{E'} \in \mathrm{Hom}(E', E')$, and $\varphi_3 = f\mathbf{1}_E \in \mathrm{Hom}(E, E)$, where $f \in C^\infty(M)$ and $\mathbf{1}_{E'}$ and $\mathbf{1}_E$ are the identity matrices acting on the fibers of E' and E, respectively. Then, (2.11) reduces to

$$\lim_{N\to\infty} |\hbar_N^{-1}[T(f\mathbf{1}), T(\varphi)]_N^{(E',E)} + iT_N^{(E',E)}(\{f, \varphi\})| = 0. \tag{2.12}$$

Here, we defined the generalized commutator,

$$[T(f\mathbf{1}), T(\varphi)]_N^{(E',E)} := T_N^{(E',E')}(f\mathbf{1}_{E'})T_N^{(E',E)}(\varphi) - T_N^{(E',E)}(\varphi)T_N^{(E,E)}(f\mathbf{1}_E), \tag{2.13}$$

and the generalized Poisson bracket,

$$\{f, \varphi\} := W^{\alpha\beta}(\partial_\alpha f)(\nabla_\beta \varphi). \tag{2.14}$$

If we put both E and E' to be the trivial line bundle and consider φ as an ordinary function, Eq. (2.12) reduces to the second equation in (1.2).

Equations (2.10)–(2.12) for general vector bundles are our new result. In particular, (2.12) shows a new correspondence between the generalized Poisson bracket (2.14) and the generalized commutator (2.13). This correspondence is very useful in constructing the matrix Laplacian in the next section.

Before closing this section, we discuss a correspondence between the trace of matrices and the integration on M. For $\varphi \in \Gamma(\mathrm{Hom}(E, E))$, the Toeplitz operator $T(\varphi)$ is a square matrix. Its trace, $\mathrm{Tr}T(\varphi)$, is related to the integral of the trace part of φ as

$$\lim_{N\to\infty} \hbar_N \mathrm{Tr}T(\varphi) = \frac{1}{2\pi}\int_M \omega \mathrm{Tr}_E \varphi, \tag{2.15}$$

where Tr_E stands for the trace over the fiber of E. See Appendix D for a proof of (2.15). Note that, when E is the trivial line bundle, the relation (2.15) reduces to (1.3). The relation (2.15) also implies a correspondence for the inner product of scalar fields, as follows. For $\varphi, \varphi' \in \Gamma(\mathrm{Hom}(E, E'))$, there is the natural inner product,

$$(\varphi, \varphi') := \frac{1}{2\pi}\int_M \omega \mathrm{Tr}_E(\varphi^\dagger \varphi'). \tag{2.16}$$

On the other hand, the Toeplitz operators behave as

$$T(\varphi^\dagger)T(\varphi') = \sum_{i=0}^{\infty} \hbar_N^i T(C_i(\varphi^\dagger, \varphi')) = T(\varphi^\dagger\varphi') + O(1/N). \tag{2.17}$$

By taking the matrix trace on both sides and using (2.6) and (2.15), we find that

$$\lim_{N\to\infty} \hbar_N \mathrm{Tr}(T(\varphi)^\dagger T(\varphi')) = (\varphi, \varphi'). \tag{2.18}$$

Thus, the inner product of the scalar fields is related to the Frobenius inner product of their Toeplitz operators.

III. LAPLACIAN FOR RECTANGULAR MATRICES

In this section, we construct the matrix Laplacian, which is related, via the Berezin-Toeplitz quantization, to the continuum Laplacian with a general background gauge field. We will first show that the continuum Laplacian for a Kähler metric can be written in terms of isometric embedding functions and the generalized Poisson bracket (2.14). Then, by using the relation (2.12), we will find the corresponding operator on the matrix side. We will also consider two examples, the fuzzy sphere and the fuzzy torus, and show explicit forms of the matrix Laplacians.

A. Laplacian and isometric embedding

The Nash embedding theorem states that any Riemannian manifold can be isometrically embedded in the Euclidean space $\mathbb{R}^d$ for sufficiently large d. Thus, for a closed Riemann surface M with a metric g, there exists an isometric embedding

$$X : M \to \mathbb{R}^d \tag{3.1}$$

for sufficiently large d. We denote the embedding coordinate functions as $\{X^A\}_{A=1,2,\ldots,d}$. The word *isometric* means that the induced metric of the embedding is equal to the intrinsic metric g on M,

$$(\partial_\alpha X^A)(\partial_\beta X^A) = g_{\alpha\beta}, \tag{3.2}$$

where the repeated index $A = 1, 2, \ldots, d$ is summed over.

Now, let us consider the Laplacian for the metric g. For a scalar field $\varphi \in \Gamma(\mathrm{Hom}(E, E'))$, the Laplacian is defined by

$$\Delta\varphi := -g^{\alpha\beta}\nabla_\alpha\nabla_\beta\varphi, \tag{3.3}$$

where the covariant derivatives act on φ as (2.9). This Laplacian is a positive semidefinite Hermitian operator with respect to the inner product (2.16). Below, we will prove that this operator can also be written by using the isometric embedding as

$$\Delta\varphi = -\{X^A, \{X^A, \varphi\}\}, \tag{3.4}$$

where $\{,\}$ is the generalized Poisson bracket (2.14). We start from the right-hand side of (3.4) and calculate it as follows:

$$\begin{aligned}
-\{X^A, \{X^A, \varphi\}\} &= -W^{\alpha\beta}W^{\gamma\delta}(\partial_\alpha X^A)\nabla_\beta[(\partial_\gamma X^A)(\nabla_\delta\varphi)] \\
&= -W^{\alpha\beta}W^{\gamma\delta}(\partial_\alpha X^A)[(\nabla_\beta\partial_\gamma X^A)(\nabla_\delta\varphi) + (\partial_\gamma X^A)(\nabla_\beta\nabla_\delta\varphi)] \\
&= -W^{\alpha\beta}W^{\gamma\delta}[\nabla_\beta((\partial_\alpha X^A)(\partial_\gamma X^A))(\nabla_\delta\varphi) - (\nabla_\beta\partial_\alpha X^A)(\partial_\gamma X^A)(\nabla_\delta\varphi) + (\partial_\alpha X^A)(\partial_\gamma X^A)(\nabla_\beta\nabla_\delta\varphi)] \\
&= -W^{\alpha\beta}W^{\gamma\delta}[(\nabla_\beta g_{\alpha\gamma})(\nabla_\delta\varphi) - (\nabla_\beta\partial_\alpha X^A)(\partial_\gamma X^A)(\nabla_\delta\varphi) + g_{\alpha\gamma}(\nabla_\beta\nabla_\delta\varphi)] \\
&= -W^{\alpha\beta}W^{\gamma\delta}g_{\alpha\gamma}(\nabla_\beta\nabla_\delta\varphi) \\
&= -g^{\beta\delta}(\nabla_\beta\nabla_\delta\varphi).
\end{aligned} \tag{3.5}$$

To obtain the first equality, we used the fact that $W^{\gamma\delta}$ is covariantly constant in two dimensions. In the fifth equality, we also used $\nabla_\beta g_{\alpha\gamma} = 0$ and $W^{\alpha\beta}\nabla_\beta\partial_\alpha X^A = W^{\alpha\beta}(\partial_\beta\partial_\alpha X^A - \Gamma^\gamma_{\alpha\beta}\partial_\gamma X^A) = 0$, where $\Gamma^\gamma_{\alpha\beta}$ is the Christoffel symbol. The last equality follows from the identity $W^{\alpha\beta}W^{\gamma\delta}g_{\alpha\gamma} = g^{\beta\delta}$, which follows from $W^{\alpha\beta} = \epsilon^{ab}e^\alpha_a e^\beta_b$. The last expression in (3.5) is just the Laplacian and thus, we have shown Eq. (3.4).

B. Laplacians on fuzzy surfaces

Now, let us consider the matrix counterpart of the Laplacian (3.3). For $\varphi \in \Gamma(\mathrm{Hom}(E, E'))$, the Toeplitz operator $T(\varphi)$ is a rectangular matrix with size $(d^{(E')}N + c^{(E')}) \times (d^{(E)}N + c^{(E)})$. Let B be any matrix of this size. From (2.12) and (3.4), we find that the continuum Laplacian is mapped to

$$\hat{\Delta}B := \hbar_N^{-2}[T(X^A\mathbf{1}), [T(X^A\mathbf{1}), B]]. \tag{3.6}$$

Here, $[,] = [,]_N^{(E',E)}$ is the generalized commutator (2.13), and we again omit the subscripts for simplicity. Note that the operator (3.6) is a positive semidefinite Hermitian operator with respect to the Frobenius inner product. Below, we will argue that the spectra of the original and the regularized Laplacians agree with each other in the large-N limit.

Let $\{B_n\}$ be exact eigenstates of $\hat{\Delta}$ which satisfy

$$\hat{\Delta}B_n = E_n B_n, \qquad \hbar_N \mathrm{Tr}(B_n^\dagger B_m) = \delta_{mn}. \tag{3.7}$$

The indices m, n run from 1 to $(d^{(E')}N + c^{(E')}) \times (d^{(E)}N + c^{(E)})$. On the other hand, let $\{a_n \in \Gamma(\mathrm{Hom}(E, E'))\}$ be exact eigenstates of Δ which satisfy

$$\Delta a_n = e_n a_n, \qquad (a_n, a_m) = \delta_{mn}, \tag{3.8}$$

where the inner product is given by (2.16). Here, the indices run from 1 to infinity. We focus on the eigenstates of $\hat{\Delta}$ which have eigenvalues of $O(N^0)$. For such eigenstates, we write $E_n = \tilde{E}_n + \epsilon_n$, where $\tilde{E}_n = \lim_{N\to\infty} E_n$ and ϵ_n is the $1/N$ correction of E_n satisfying $\lim_{N\to\infty}\epsilon_n = 0$. We will show that such eigenstates of $\hat{\Delta}$ are in one-to-one correspondence with those of Δ in the large-N limit.

First, we take a specific eigenstate B_n with the eigenvalue $O(N^0)$ and write it as $B_n = T(b_n)$ by using a local section $b_n \in \Gamma(\mathrm{Hom}(E, E'))$. This is always possible since the quantization map is surjective. From (2.12), we have

$$\hat{\Delta}B_n = T\left(\Delta b_n + \frac{1}{N}c_n\right), \tag{3.9}$$

where $c_n \in \Gamma(\mathrm{Hom}(E, E'))$ is another section of $O(1)$ (the section c_n is explicitly given as a combination consisting of $C_i(\cdot,\cdot)$, X^A and b_n). Since the left-hand side of (3.9) is equal to $E_n M_n$, we obtain

$$T\left(E_n b_n - \Delta b_n - \frac{1}{N}c_n\right) = 0. \tag{3.10}$$

Here, notice that if $T(b_0) = 0$ for a certain section b_0 of $O(1)$, b_0 goes to zero in the large-N limit. This follows from the mapping between the trace and integral (2.15). If $T(b_0) = 0$, we have

$$\begin{aligned} 0 &= \hbar_N \mathrm{Tr}(T(b_0)^\dagger T(b_0)) \\ &= \hbar_N \mathrm{Tr} T\left(b_0^\dagger b_0 + \frac{1}{N} C_1(b_0^\dagger, b_0) + \cdots\right) \\ &= \frac{1}{2\pi}\int_M \omega \mathrm{Tr}_E(b_0^\dagger b_0) + O(1/N). \end{aligned} \tag{3.11}$$

In order for this equation to hold, b_0 has to vanish in the large-N limit. Thus, (3.10) implies that

$$\lim_{N\to\infty} |E_n b_n - \Delta b_n - \frac{1}{N} c_n| = 0. \tag{3.12}$$

Here, note also that b_n is nontrivial and finite in the large-N limit. This is because we have

$$\begin{aligned} \frac{1}{2\pi}\int_M \omega \mathrm{Tr}_E(b_n^\dagger b_n) &= \hbar_N \mathrm{Tr}(B_n^\dagger B_n) + O(1/N) \\ &= 1 + O(1/N), \end{aligned} \tag{3.13}$$

but this equation contradicts if $b_n = 0$ or $\lim_{N\to\infty} |b_n| = \infty$. Thus, b_n should converge to a certain section $\tilde{b}_n$ in the large-N limit. Furthermore, if we consider several different n's, the sections $\tilde{b}_n$ satisfy the orthonormality condition. In fact, the large-N limit of the second equation in (3.7) gives $(\tilde{b}_m, \tilde{b}_n) = \delta_{mn}$. Equation (3.12) then implies that

$$\Delta \tilde{b}_n = \tilde{E}_n \tilde{b}_n. \tag{3.14}$$

Thus, there exists an eigenstate of Δ with the eigenvalue $\tilde{E}_n = \lim_{N\to\infty} E_n$. What we have shown above can be summarized as follows. Let I be any index set such that if $n \in I$, the eigenvalue E_n is of $O(1)$. Then, for the set of orthonormal eigenstates $\{(E_n, B_n) | n \in I\}$ of $\hat{\Delta}$, there always exists a corresponding set of orthonormal eigenstates $\{(\tilde{E}_n, \tilde{b}_n) | n \in I\}$ of Δ. The two set of eigenvalues are related by $\tilde{E}_n = \lim_{N\to\infty} E_n$.

We next focus on the converse of the above statement. Namely, we start from the eigenstates $\{a_n\}$ of Δ and try to construct a corresponding eigenstate of $\hat{\Delta}$. We define the Toeplitz operator of a_n as

$$B'_n := T(a_n). \tag{3.15}$$

By applying $\hat{\Delta}$ on this equation and using (2.12), we obtain

$$\hat{\Delta} B'_n = T\left(\Delta a_n + \frac{1}{N} c'_n\right) = e_n B'_n + \frac{1}{N} T(c'_n), \tag{3.16}$$

where c'_n is a section of $O(1)$. This equation shows that in the large-N limit, B'_n becomes an eigenstate of $\hat{\Delta}$ with the eigenvalue e_n.[5] The orthonormality of B'_n in the large-N limit can also be shown in a similar way as we described above for $\tilde{b}_n$. Thus, for any index set I' and a set of orthonormal eigenstates $\{(e_n, a_n) | n \in I'\}$ of Δ, we can construct corresponding orthonormal eigenstates $\{(e_n, B'_n) | n \in I'\}$ of $\hat{\Delta}$ in the large-N limit.

The above arguments show that, in the large-N limit, the $O(1)$ eigenvalues of $\hat{\Delta}$ are in one-to-one correspondence with those of Δ.

It is intriguing that the form of the matrix Laplacian (3.6) naturally appears in the context of emergence of noncommutative Yang-Mills theories from matrix models [21–25]. In fact, if the matrices have a block diagonal background (see, e.g., [26]), the Laplacian (3.6) appears in the quadratic part of the fluctuations of off-diagonal blocks, which are generally rectangular matrices.

C. Laplacian on fuzzy S^2

In this section, we consider the regularized Laplacian on fuzzy S^2 in a monopole background [27]. We consider the case in which $E = L^{\otimes(-Q)}$ and E' is the trivial line bundle. In this case, $\Gamma(\mathrm{Hom}(E, E')) = \Gamma(L^{\otimes Q})$ and $(c^{(E)}, d^{(E)}, c^{(E')}, d^{(E')}) = (-Q, 1, 0, 1)$. The Toeplitz operator $T(\varphi)$ for $\varphi \in \Gamma(L^{\otimes Q})$ is thus a rectangular matrix of size $N \times (N - Q)$.

Let us consider S^2 in the standard polar coordinate $(\theta, \phi) \in [0, \pi] \times [0, 2\pi)$. We will focus on the chart $\mathcal{C}$ that does not include the north pole $\theta = 0$ and the south pole $\theta = \pi$. On $\mathcal{C}$, the standard metric and the symplectic form are defined by

$$\begin{aligned} g &:= d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi, \\ \omega &:= \sin\theta d\theta \wedge d\phi. \end{aligned} \tag{3.17}$$

In this convention, the symplectic volume is $V = 2$. The connection of the line bundle L satisfying (2.1) is given by

$$A = \frac{1 - \cos\theta}{2} d\phi. \tag{3.18}$$

This is nothing but the Wu-Yang monopole configuration. The standard isometric embedding of S^2 into $\mathbb{R}^3$ is given by

[5] A little more rigorous statement may be made as follows. We first expand B'_n by using B_n as $B'_n = \sum_{n'} q_{nn'} B_{n'}$. By substituting this into (3.16), multiplying $B_m^\dagger$ and taking the trace and the large-N limit, we obtain $\lim_{N\to\infty} q_{nm}(e_n - E_m) = 0$ for any m. If $e_n \neq \lim_{N\to\infty} E_m$ for all m, it leads to $q_{nm} \to 0$ for all m. This means $B'_n \to 0$, which contradicts with the orthonormality of a_n. Thus, there exists at least one E_m such that $\lim_{N\to\infty} E_m = e_n$.

$$X^1 = \sin\theta\cos\phi, \quad X^2 = \sin\theta\sin\phi, \quad X^3 = \cos\theta. \tag{3.19}$$

Now, let us consider a Laplacian acting on $\varphi \in \Gamma(L^{\otimes Q})$. As mentioned above, this is the case where $E = L^{\otimes(-Q)}$ and E' is the trivial line bundle. This means that $A^{(E)} = -QA$ and $A^{(E')} = 0$. Then, the Laplacian can be explicitly be written as

$$\Delta\varphi = -\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\varphi) - \frac{1}{\sin^2\theta}\partial_\phi^2\varphi + iQ\frac{1-\cos\theta}{\sin^2\theta}\partial_\phi\varphi + \frac{Q^2}{2}\frac{1-\cos\theta}{\sin^2\theta}\varphi - \frac{Q^2}{4}\varphi. \tag{3.20}$$

The spectrum of this operator is exactly solvable using the monopole harmonics [28,29]. Let us define the following operators on $\mathcal{C}$:

$$\begin{aligned}\mathcal{L}_1^{(Q)} &= i(\sin\phi\partial_\theta + \cot\theta\cos\phi\partial_\phi) - \frac{Q}{2}\frac{1-\cos\theta}{\sin\theta}\cos\phi,\\ \mathcal{L}_2^{(Q)} &= i(-\cos\phi\partial_\theta + \cot\theta\sin\phi\partial_\phi) - \frac{Q}{2}\frac{1-\cos\theta}{\sin\theta}\sin\phi,\\ \mathcal{L}_3^{(Q)} &= -i\partial_\phi - \frac{Q}{2}.\end{aligned} \tag{3.21}$$

These operators correspond to the angular momentum operators in the presence of a magnetic monopole with charge $Q/2$ located at the origin of a sphere. They form a representation of the $su(2)$ algebra,

$$[\mathcal{L}_A^{(Q)}, \mathcal{L}_B^{(Q)}] = i\epsilon_{ABC}\mathcal{L}_C^{(Q)}, \tag{3.22}$$

on the representation space $\Gamma(L^{\otimes Q})$. A unitary irreducible representation of the $\mathfrak{su}(2)$ algebra is constructed by the highest weight method,

$$\begin{aligned}(\mathcal{L}_A^{(Q)})^2 Y_{lm}^{(Q)} &= l(l+1)Y_{lm}^{(Q)},\\ \mathcal{L}_3^{(Q)} Y_{lm}^{(Q)} &= mY_{lm}^{(Q)}.\end{aligned} \tag{3.23}$$

Here, $\{Y_{lm}^{(Q)} | l = |Q|/2, |Q|/2+1, \ldots, \infty; m = -l, -l+1, \ldots, l\}$ are the monopole harmonics [28,29] and they form an orthonormal basis of the representation space $\Gamma(L^{\otimes Q})$. By the direct calculation, we can show that the Laplacian is equal to the quadratic Casimir operator plus a constant,

$$\Delta = (\mathcal{L}_A^{(Q)})^2 - \frac{Q^2}{4}. \tag{3.24}$$

Thus, the eigenvalues of Δ are $l(l+1) - \frac{Q^2}{4}$ and the eigenfunctions are given by $Y_{lm}^{(Q)}$.

Now, let us consider the regularized Laplacian (3.6). A direct calculation (e.g., in [16,30]) shows that the embedding functions are mapped to

$$T_N^{(E',E')}(X^A\mathbf{1}_{E'}) = \frac{1}{J+1}L_A^{(J)}, \quad T_N^{(E,E)}(X^A\mathbf{1}_E) = \frac{1}{\tilde{J}+1}L_A^{(\tilde{J})}, \tag{3.25}$$

where $J = (N-1)/2$, $\tilde{J} = (N-Q-1)/2$, and $L_A^{(J)}$ are the $(2J+1)$-dimensional representation of the $\mathfrak{su}(2)$ generators satisfying the Lie algebra

$$[L_A^{(J)}, L_B^{(J)}] = i\epsilon_{ABC}L_C^{(J)}. \tag{3.26}$$

The matrix configuration (3.25) is known as the fuzzy sphere [27]. For any $N \times (N-Q)$ matrix B, the regularized Laplacian (3.6) in this case is given by

$$\begin{aligned}\hat{\Delta}B &= \frac{N^2}{4}\left(\frac{1}{(J+1)^2}(L_A^{(J)})^2 B - \frac{2}{(J+1)(\tilde{J}+1)}L_A^{(J)}BL_A^{(\tilde{J})} + \frac{1}{(\tilde{J}+1)^2}B(L_A^{(\tilde{J})})^2\right)\\ &= \frac{N^2}{4}\left(\frac{J}{J+1}B + \frac{\tilde{J}}{\tilde{J}+1}B - \frac{2}{(J+1)(\tilde{J}+1)}L_A^{(J)}BL_A^{(\tilde{J})}\right),\end{aligned} \tag{3.27}$$

where we used $(L_A^{(J)})^2 = J(J+1)$.

We then test whether the spectrum of $\hat{\Delta}$ agrees with that of the continuum Laplacian in the large-N limit. Let us first introduce an operation

$$L_A \circ B \coloneqq L_A^{(J)}B - BL_A^{(\tilde{J})}. \tag{3.28}$$

Note that the operation $L_A\circ$ also forms $N(N-Q)$-dimensional representation of $\mathfrak{su}(2)$,

$$[L_A\circ, L_B\circ] = i\epsilon_{ABC}L_C\circ. \tag{3.29}$$

It is known that there exist $N \times (N-Q)$ matrices called fuzzy spherical harmonics [26,31–34], denoted by $\{\hat{Y}_{lm(J\tilde{J})} | l = |J-\tilde{J}|, |J-\tilde{J}|+1, \ldots, J+\tilde{J}; m = -l, -l+1, \ldots, l\}$, which satisfy

$$\begin{aligned}(L_A\circ)^2\hat{Y}_{lm(J\tilde{J})} &= l(l+1)\hat{Y}_{lm(J\tilde{J})},\\ L_3\circ\hat{Y}_{lm(J\tilde{J})} &= m\hat{Y}_{lm(J\tilde{J})}.\end{aligned} \tag{3.30}$$

These matrices are indeed the Toeplitz map of the monopole harmonics [16]. They are also a complete orthonormal basis of complex $N \times (N-Q)$ matrices. The first equation of (3.30) implies that

$$L_A^{(J)}\hat{Y}_{lm(J\tilde{J})}L_A^{(\tilde{J})}=\frac{J(J+1)+\tilde{J}(\tilde{J}+1)-l(l+1)}{2}\hat{Y}_{lm(J\tilde{J})}. \tag{3.31}$$

From (3.27) and (3.31), we find that $\{\hat{Y}_{lm(J\tilde{J})}|l=|J-\tilde{J}|, |J-\tilde{J}|+1,\ldots,J+\tilde{J};m=-l,-l+1,\ldots,l\}$ are complete eigen modes of the operator $\hat{\Delta}$ and the eigenvalues are given as

$$\begin{aligned}\hat{\Delta}\hat{Y}_{lm(J\tilde{J})}&=\frac{N^2}{4(J+1)(\tilde{J}+1)}\left(l(l+1)-\frac{Q^2}{4}\right)\hat{Y}_{lm(J\tilde{J})}\\&=\left(l(l+1)-\frac{Q^2}{4}+O(N^{-1})\right)\hat{Y}_{lm(J\tilde{J})}.\end{aligned} \tag{3.32}$$

Therefore, the spectrum indeed approaches the continuum spectrum as N goes to infinity.

D. Laplacian on fuzzy T^2

In this section, we consider the Laplacian on the fuzzy T^2 [35]. We again consider the case in which $E=L^{\otimes(-Q)}$ and E' is the trivial line bundle.

Let us consider a flat plane $\mathbb{R}^2$. We define the metric and the symplectic form on $\mathbb{R}^2$ by

$$\begin{aligned}g&:=dx^1\otimes dx^1+dx^2\otimes dx^2,\\ \omega&:=dx^1\wedge dx^2.\end{aligned} \tag{3.33}$$

By introducing equivalence relations

$$x^\alpha\sim x^\alpha+2\pi\quad(\alpha=1,2), \tag{3.34}$$

we define two-dimensional torus T^2 as the quotient space,

$$T^2=\mathbb{R}^2/\sim. \tag{3.35}$$

This space inherits the flat metric and the symplectic form on $\mathbb{R}^2$. The symplectic volume of T^2 is then given by $V=2\pi$. The $U(1)$ gauge field A satisfying (2.1) is given by

$$A=\frac{1}{4\pi}(-x^2dx^1+x^1dx^2). \tag{3.36}$$

The embedding functions,

$$\begin{aligned}X^1&=\cos x^1, & X^2&=\sin x^1,\\ X^3&=\cos x^2, & X^4&=\sin x^2,\end{aligned} \tag{3.37}$$

give an isometric embedding of T^2 into $\mathbb{R}^4$.

We then consider a Laplacian acting on $\Gamma(L^{\otimes Q})$, where the background gauge fields are again taken to be $A^{(E)}=-QA$ and $A^{(E')}=0$. By employing the complex coordinate $z=\frac{x^1+ix^2}{\sqrt{2}}$, the Laplacian can be written as

$$\Delta\varphi=-(\nabla_z\nabla_{\bar{z}}+\nabla_{\bar{z}}\nabla_z)\varphi \tag{3.38}$$

for $\varphi\in\Gamma(L^{\otimes Q})$. The commutator of ∇_z and $\nabla_{\bar{z}}$ produces the constant field strength multiplied by the charge Q. For $Q\neq0$, this commutation relation is identical to that of the creation and annihilation operators, up to some rescalings. Indeed, if we introduce the creation and annihilation operators by

$$\hat{a}:=i\sqrt{\frac{2\pi}{Q}}\nabla_{\bar{z}},\qquad \hat{a}^\dagger:=i\sqrt{\frac{2\pi}{Q}}\nabla_z, \tag{3.39}$$

they satisfy the algebra $[\hat{a},\hat{a}^\dagger]=1$ on $\Gamma(L^{\otimes Q})$. In this case, we can write the Laplacian as

$$\Delta\varphi=\frac{Q}{\pi}\left(\hat{N}+\frac{1}{2}\right)\varphi, \tag{3.40}$$

where $\hat{N}:=\hat{a}\hat{a}^\dagger$ is the number operator. Therefore, the eigenvalues of Δ are the same as those of the one-dimensional harmonic oscillator, $\frac{Q}{\pi}(n+\frac{1}{2})(n=0,1,\ldots)$. The eigenfunctions are explicitly computed in [16] and they can be expressed in terms of the Jacobi-theta function and the Hermite polynomials. On the other hand, for $Q=0$, the spectrum of the Laplacian is given by a sum of two integers which correspond to the momenta for the x^1 and x^2 directions. Thus, the spectrum for $Q=0$ is completely different from those for $Q\neq0$.

Let us next consider the matrix Laplacian (3.6). The explicit calculation in [16] shows that the Toeplitz operators of the embedding functions are given by

$$\begin{aligned}T_N^{(E',E')}(X^1\mathbf{1}_{E'})&=\frac{U^{(N)}+U^{(N)\dagger}}{2},\\ T_N^{(E',E')}(X^2\mathbf{1}_{E'})&=\frac{U^{(N)}-U^{(N)\dagger}}{2i},\\ T_N^{(E',E')}(X^3\mathbf{1}_{E'})&=\frac{V^{(N)}+V^{(N)\dagger}}{2},\\ T_N^{(E',E')}(X^4\mathbf{1}_{E'})&=\frac{V^{(N)}-V^{(N)\dagger}}{2i},\end{aligned} \tag{3.41}$$

where

$$\begin{aligned}U^{(N)}&=e^{-\frac{\pi}{2N}}\begin{pmatrix} & & & & 1\\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix},\\ V^{(N)}&=e^{-\frac{\pi}{2N}}\begin{pmatrix} q^{-1} & & & \\ & q^{-2} & & \\ & & \ddots & \\ & & & q^{-N}\end{pmatrix}\end{aligned} \tag{3.42}$$

are the N-dimensional clock and shift matrices with $q = e^{i2\pi/N}$. The Toeplitz operators $T_N^{(E,E)}(X^A \mathbf{1}_E)$ are given by replacing N with $N - Q$ in the above expressions. The matrices (3.42) satisfy the well-known algebra $U^{(N)}V^{(N)} = qV^{(N)}U^{(N)}$, which characterizes the fuzzy torus [35]. The Laplacian (3.6) is then given by

$$\hat{\Delta}B = \frac{N^2}{4\pi^2}(U \circ U^\dagger \circ + V \circ V^\dagger \circ)B \tag{3.43}$$

for any $N \times (N - Q)$ matrix B, where $A \circ B \coloneqq A^{(N)}B - BA^{(N-Q)}$. It is easy to see that for $Q = 0$, the exact eigen modes of the Laplacian are given by $(U^{(N)})^m (V^{(N)})^n$, where m, n are integers. The corresponding eigenvalues approach to $m^2 + n^2$ in the large-N limit, which agree with the continuum spectrum. On the other hand, for $Q \neq 0$, we could not obtain exact eigen modes for finite N. However, in [16], it is shown that the eigenvalue problem of the regularized Laplacian is equivalent to a class of Hofstadter problem [36] and the problem was numerically solved. The result shows that the spectrum of the regularized Laplacian indeed agrees with the continuum Laplacian in the large-N limit.

IV. SUMMARY AND DISCUSSION

In this paper, we proposed a general construction of Laplacians for scalar fields on fuzzy Riemann surfaces with a general background gauge field. Our construction is based on the so-called Berezin-Toeplitz quantization, which was first considered as a method of mapping commutative function algebra to noncommutative matrix algebra in such way that two algebraic structures of functions (the ordinary function algebra and the Poisson algebra) are well approximated in terms of the matrix algebra. We used a generalized form of the Berezin-Toeplitz quantization, which can also be applied to fields in various representations of any gauge group. The quantization map is given by (2.5) and the fields are mapped to rectangular matrices in this quantization. The Laplacian we constructed in this paper acts on those rectangular matrices and reproduces the continuum spectrum in the large-N limit.

In order to construct the matrix Laplacian, we first showed that the Toeplitz operators (2.5) satisfy the asymptotic expansion (2.7). In particular, this expansion implies the relation (2.12), which shows a mapping between the generalized Poisson bracket and the commutatorlike operation for the Toeplitz operators.

We then showed that any Laplacian for a Kähler metric on a Riemann surface with an arbitrary background gauge field can be written in terms of the isometric embedding function and the generalized Poisson bracket. By using (2.12), we mapped the continuum Laplacian on the Riemann surface to the matrix side. Thus, we obtained the general form of the matrix Laplacian (3.6). We also argued that its spectrum indeed agrees with the original Laplacian in the large-N limit. We finally checked our construction for two examples of the fuzzy S^2 and the fuzzy T^2.

It is straightforward to generalize our formulation to higher (even) dimensions. We expect that, in that case, the results of earlier work (e.g., [34]) can also be naturally understood in our framework. We will study this generalization elsewhere.

Our results give a formulation of scalar field theories on fuzzy Riemann surfaces, in which the scalar fields couple to arbitrary external gauge fields. It is an interesting future work to study such theories to see the structure of the UV/IR mixing [37,38] or to understand the problem of the renormalization [39–41].

It will also be possible to formulate gauge-field theories on fuzzy spaces by extending this work. When the vector bundles E and E' contain tensor products of TM's and T^*M's, the fields to be quantized, which are elements of $\mathrm{Hom}(E, E')$, correspond to tensor fields on M. Thus, it will be possible to regularize vector fields with this method. It should be verified whether general gauge-field theories on fuzzy spaces can be constructed based on this quantization map and this approach reproduces known examples of gauge theories on fuzzy spaces [21–25].

Finally, one of the most interesting and challenging problems in this context is to formulate gravitational theories on noncommutative spaces [42–44]. The above-mentioned quantization map for tensor fields may provide a quantization of the metric field and thus may give a new formulation of gravitational theories on fuzzy spaces. We hope to challenge this problem in the near future.

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APPENDIX A: VANISHING THEOREM AND INDEX THEOREM

In this Appendix, for the Dirac operator $D^{(E)}$ on $\Gamma(S \otimes L^{\otimes N} \otimes E)$, we will show that $\mathrm{Ker}D^{(E)}$ is spanned by spinors with positive chirality and $\dim \mathrm{Ker}D^{(E)} = d^{(E)}N + c^{(E)}$ for sufficiently large N, where $d^{(E)}$ and $c^{(E)}$ are the rank and the first Chern number of the vector bundle E. The former statement is known as the vanishing theorem and the latter is a consequence of the index theorem. We also show that nonzero eigenvalues of $D^{(E)}$ have a large gap of $O(\sqrt{N})$. Below, we simply denote the Dirac operator by D, making the E dependence implicit.

In two dimension, spinors can be decomposed according to their chirality: $\Gamma(S \otimes L^{\otimes N} \otimes E) = \Gamma^+(S \otimes L^{\otimes N} \otimes E) \oplus \Gamma^-(S \otimes L^{\otimes N} \otimes E)$. If we take the gamma matrices in the orthonormal frame as the two Pauli matrices σ^1 and σ^2, then the chirality operator is given by σ^3. By adopting a basis where the chirality operator becomes diagonal, we can decompose D as

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}. \tag{A1}$$

Here, $\pm$ indicates the chirality of the space on which the operators are acting. This decomposition is always possible, since the Dirac operator anticommutes with the chirality operator.

We first show that $\mathrm{Ker} D^- = \{0\}$ for sufficiently large N, which means that $\mathrm{Ker} D^{(E)}$ is spanned by spinors with positive chirality. We consider the square of D,

$$D^2 = \begin{pmatrix} D^-D^+ & 0 \\ 0 & D^+D^- \end{pmatrix}. \tag{A2}$$

We also use the Weitzenböck formula

$$D^2 = -\nabla^a \nabla_a - (\hbar_N^{-1} + F_{12}^{(E)})\sigma_3 + \frac{1}{4}R, \tag{A3}$$

where $\nabla_a = e_a^\alpha \nabla_\alpha$, $\hbar_N = V/N$, R is the scalar curvature, and $F_{12}^{(E)} = e_1^\alpha e_2^\beta F_{\alpha\beta}^{(E)}$ is the curvature of E in the orthonormal frame. By comparing (A2) and (A3), we find that

$$D^+D^- = -\nabla^a \nabla_a + \hbar_N^{-1} + F_{12}^{(E)} + \frac{1}{4}R. \tag{A4}$$

By using this relation and also $(D^+)^\dagger = D^-$, which follows from the Hermiticity of D, we obtain the following inequalities for all $\psi^- \in \Gamma^-(S \otimes L^{\otimes N} \otimes E)$:

$$\begin{aligned} |D^-\psi^-|^2 &= |\nabla_a \psi^-|^2 + \hbar_N^{-1}(\psi^-, \psi^-) \\ &\quad + \left(\psi^-, \left(F_{12}^{(E)} + \frac{1}{4}R\right)\psi^-\right) \geq (\hbar_N^{-1} - C)|\psi^-|^2. \end{aligned} \tag{A5}$$

Here, we introduced $C \coloneqq |F_{12}^{(E)} + \frac{1}{4}R|$. From the above inequalities, we conclude that $\mathrm{Ker} D^- = \{0\}$ for $\hbar_N^{-1} > C$ and this is indeed the case in the large-N limit.

We next show that $\dim \mathrm{Ker} D = d^{(E)}N + c^{(E)}$ for sufficiently large N. Note that, for sufficiently large N, since $\mathrm{Ker} D^- = \{0\}$ as we saw above, we have the following relations:

$$\dim \mathrm{Ker} D = \dim \mathrm{Ker} D^+ = \mathrm{Ind} D, \tag{A6}$$

where $\mathrm{Ind} D \coloneqq \dim \mathrm{Ker} D^+ - \dim \mathrm{Ker} D^-$ is the analytical index of D. By using the Atiyah-Singer index theorem, we obtain

$$\begin{aligned} \dim \mathrm{Ker} D = \mathrm{Ind} D &= \frac{1}{2\pi}\int_M (NF \mathrm{Tr}_E(\mathbf{1}_E) + \mathrm{Tr}_E F^{(E)}) \\ &= d^{(E)}N + c^{(E)}, \end{aligned} \tag{A7}$$

where Tr_E is the trace for the fiber of E and $\mathbf{1}_E$ is the identity matrix on the fiber of E. The coefficients are explicitly given by $d^{(E)} = \mathrm{Tr}_E(\mathbf{1}_E)$ and $c^{(E)} = \frac{1}{2\pi}\int_M \mathrm{Tr}_E F^{(E)}$.

Finally, we prove that nonzero eigenvalues of D have a large gap of $O(\sqrt{N})$. Let λ be a nonzero eigenvalue of D with the eigen spinor $\psi \in \Gamma(S \otimes L^{\otimes N} \otimes E)$. We make the chirality decomposition as $\psi = \psi^+ \oplus \psi^-$, where $\psi^\pm \in \Gamma^\pm(S \otimes L^{\otimes N} \otimes E)$. In terms of the expression (A1), ψ^+ and ψ^- are the upper and the lower components of ψ, respectively. The eigenvalue equation for D^2 is then equivalent to

$$\begin{cases} D^-D^+\psi^+ &= \lambda^2\psi^+, \\ D^+D^-\psi^- &= \lambda^2\psi^-. \end{cases} \tag{A8}$$

If $\psi^- \neq 0$, (A5) implies that $\lambda^2 \geq \hbar_N^{-1} - C$. If $\psi^- = 0$, we have $\psi^+ \neq 0$ in order for ψ to be nonzero. By using the relation $D^+D^-(D^+\psi^+) = \lambda^2(D^+\psi^+)$, we again find that (A5) implies $\lambda^2 \geq \hbar_N^{-1} - C$. Thus, in any case, we have $\lambda^2 \geq \hbar_N^{-1} - C$. This shows that λ^2 is of $O(N)$ and thus, the nonzero eigenvalues of D indeed have a gap of $O(\sqrt{N})$.

APPENDIX B: ASYMPTOTIC EXPANSION FOR TOEPLITZ OPERATORS

In this Appendix, we derive the large-N asymptotic expansion (2.7). The computation technique used in this Appendix is based on [20].

For $\varphi \in \Gamma(\mathrm{Hom}(E, E'))$ and $\varphi' \in \Gamma(\mathrm{Hom}(E', E''))$, let $T(\varphi) = \Pi'\varphi\Pi$ and $T(\varphi) = \Pi''\varphi'\Pi'$ be their Toeplitz operators. The product $T(\varphi')T(\varphi)$ can be written as

$$\begin{aligned} T(\varphi')T(\varphi) &= \Pi''\varphi'\Pi'\varphi\Pi \\ &= T(\varphi'\varphi) - \Pi''\varphi'(1 - \Pi')\varphi\Pi. \end{aligned} \tag{B1}$$

We will compute the second term in the following.

In order to compute $1 - \Pi'$, let us consider the following Hermitian operator on $\Gamma(S \otimes L^{\otimes N} \otimes E')$:

$$P^{(E')} \coloneqq \begin{pmatrix} 0 & D^-(D^+D^-)^{-1} \\ (D^+D^-)^{-1}D^+ & 0 \end{pmatrix}, \tag{B2}$$

where $D^\pm$ are the off-diagonal elements of $D^{(E')}$ in the chiral decomposition (A1). Note that, since $\mathrm{Ker} D^- = \mathrm{Ker} D^+D^- = \{0\}$ for sufficiently large N as shown in Appendix A, the inverse $(D^+D^-)^{-1}$ always exists. Hereafter, we will omit the subscript (E') and if we simply write P or D, it shall mean $P^{(E')}$ or $D^{(E')}$, respectively. The operator P has the following properties:

$$DP = PD, \qquad PDP = P. \tag{B3}$$

The first identity implies that $\mathrm{Ker}(DP) = \mathrm{Ker}(PD) = \mathrm{Ker}D$. The second identity implies that $(DP)^2 = DP$, which together with the Hermiticity of DP, shows that DP is a projection onto $(\mathrm{Ker}D)^\perp$, which is the orthogonal complement of $\mathrm{Ker}D$. This projection is nothing but $1-\Pi'$ and thus, we find the expression

$$1-\Pi' = DP = DP^2D. \tag{B4}$$

We substitute (B4) into (B1) and act it onto an arbitrary zero mode $\chi \in \mathrm{Ker}D^{(E)}$. By taking the inner product with another zero mode $\psi \in \mathrm{Ker}D^{(E'')}$, we obtain

$$\begin{aligned}(\psi, T(\varphi')T(\varphi)\chi) &= (\psi, T(\varphi'\varphi)\chi) - (\psi, \varphi' DP^2 D\varphi\chi) \\ &= (\psi, T(\varphi'\varphi)\chi) + (\psi, \dot{\varphi}' P^2 \dot{\varphi}\chi).\end{aligned} \tag{B5}$$

Here, we introduced the notation $\dot{\varphi} \coloneqq i\sigma^a(\nabla_a\varphi)$. Because the Pauli matrices in $\dot{\varphi}$ flip the chirality, $\dot{\varphi}\chi$ has the negative chirality and accordingly $\dot{\varphi}\chi \in (\mathrm{Ker}D)^\perp$. On $(\mathrm{Ker}D)^\perp$, the operator $1-\Pi' = DP$ acts as the identity operator. This means that P is the inverse of D on $(\mathrm{Ker}D)^\perp$. Consequently, (B5) can be written as

$$(\psi, T(\varphi')T(\varphi)\chi) = (\psi, T(\varphi'\varphi)\chi) + (\psi, \dot{\varphi}' D^{-2} \dot{\varphi}\chi). \tag{B6}$$

We compute the operator D^{-2} on $(\mathrm{Ker}D)^\perp$ as follows. First, from the Weitzenböck formula (A3), we have

$$D^2 = -2\nabla_-\nabla_+ + (1-\sigma_3)\left(\hbar_N^{-1} + \frac{1}{2}R_1\right), \tag{B7}$$

where $\nabla_\pm \coloneqq \frac{1}{\sqrt{2}}(\nabla_1 \pm i\nabla_2)$ and $R_1 \coloneqq 2F_{12}^{(E')} + \frac{R}{2}$. By taking the inverse of this on the negative chirality modes, we obtain

$$\begin{aligned}D^{-2}|_- &= (-2\nabla_-\nabla_+ + 2\hbar_N^{-1} + R_1)^{-1} \\ &= \frac{\hbar_N}{2} - \frac{\hbar_N}{2}(-2\nabla_-\nabla_+ + R_1)D^{-2}|_-.\end{aligned} \tag{B8}$$

Here, we used the elementary identity, $(a+b)^{-1} = a^{-1} - a^{-1}b(a+b)^{-1}$. The term $\nabla_-\nabla_+D^{-2}|_-$ can be further evaluated by using the following commutation relation:

$$\begin{aligned}[\nabla_+, D^2|_-] &= -2[\nabla_+, \nabla_-]\nabla_+ + (\nabla_+R_1) \\ &= (2\hbar_N^{-1} + R_2)\nabla_+ + (\nabla_+R_1),\end{aligned} \tag{B9}$$

where $R_2 \coloneqq R - \frac{R}{2}\sigma_3 + 2F_{12}^{(E')}$. This commutation relation is equivalent to

$$(D^2|_- + 2\hbar_N^{-1} + R_2)\nabla_+ = \nabla_+ D^2|_- - (\nabla_+R_1). \tag{B10}$$

By multiplying $(D^2|_- + 2\hbar_N^{-1} + R_2)^{-1}$ from the left and $D^{-2}|_-$ from the right, we obtain

$$\begin{aligned}\nabla_+D^{-2}|_- &= (D^2|_- + 2\hbar_N^{-1} + R_2)^{-1}\nabla_+ \\ &\quad - (D^2|_- + 2\hbar_N^{-1} + R_2)^{-1}(\nabla_+R_1)D^{-2}|_-.\end{aligned} \tag{B11}$$

Plugging this into (B8), we obtain

$$\begin{aligned}D^{-2}|_- =& \frac{\hbar_N}{2} - \frac{\hbar_N}{2}R_1D^{-2}|_- + \hbar_N\nabla_-(D^2|_- + 2\hbar_N^{-1} + R_2)^{-1}\nabla_+ \\ &- \hbar_N\nabla_-(D^2|_- + 2\hbar_N^{-1} + R_2)^{-1}(\nabla_+R_1)D^{-2}|_-.\end{aligned} \tag{B12}$$

By using $\nabla_+\psi = 0$ and $\nabla_+\chi = 0$, we then obtain

$$(\psi, T(\varphi')T(\varphi)\chi) = (\psi, T(\varphi'\varphi)\chi) + \frac{\hbar_N}{2}(\psi, \dot{\varphi}'\dot{\varphi}\chi) + \epsilon, \tag{B13}$$

where

$$\begin{aligned}&\epsilon \coloneqq \epsilon_1 + \epsilon_2 + \epsilon_3, \\ &\epsilon_1 \coloneqq -\frac{\hbar_N}{2}(\psi, \dot{\varphi}'R_1D^{-2}|_-\dot{\varphi}\chi), \\ &\epsilon_2 \coloneqq -\hbar_N(\psi, (\nabla_-\dot{\varphi}')(D^2|_- + 2\hbar_N^{-1} + R_2)^{-1}(\nabla_+\dot{\varphi})\chi), \\ &\epsilon_3 \coloneqq \hbar_N(\psi, (\nabla_-\dot{\varphi}')(D^2|_- + 2\hbar_N^{-1} + R_2)^{-1}(\nabla_+R_1)D^{-2}|_-\dot{\varphi}\chi).\end{aligned} \tag{B14}$$

Let us estimate the order of ϵ with respect to $\hbar_N$. From general properties of the inner product and the norm, we find that

$$\begin{aligned}&|\epsilon_1| \le \frac{\hbar_N}{2}|\psi||\dot{\varphi}'||R_1||D^{-2}|_-||\dot{\varphi}||\chi|, \\ &|\epsilon_2| \le \hbar_N|\psi||\nabla_-\dot{\varphi}'||(D^2|_- + 2\hbar_N^{-1} + R_2)^{-1}||\nabla_+\dot{\varphi}||\chi|, \\ &|\epsilon_3| \le \hbar_N|\psi||\nabla_-\dot{\varphi}'||(D^2|_- \\ &\qquad + 2\hbar_N^{-1} + R_2)^{-1}||\nabla_+R_1||D^{-2}|_-||\dot{\varphi}||\chi|.\end{aligned} \tag{B15}$$

Note that $\dot{\varphi}', \dot{\varphi}, \nabla_-\dot{\varphi}', \nabla_+\dot{\varphi}, R_1$, and ∇_+R_1 are all N independent and hence their norms are finite in the large-N limit. In addition, we can normalize ψ and χ in such a way that their norms are N independent. The only objects with nontrivial N dependence are $D^{-2}|_-$ and $(D^2|_- + 2\hbar_N^{-1} + R_2)^{-1}$. As we discussed in Appendix A, all eigenvalues of $D^2|_-$ are in the range $[\hbar_N^{-1} - C, \infty)$, where C is an N-independent constant. Hence, the eigenvalues of $D^{-2}|_-$ are in $(0, (\hbar_N^{-1} - C)^{-1}]$. From this property and the fact that the norm of a positive operator is equal to its maximum eigenvalues, we find that

$$|D^{-2}|_-| = O(\hbar_N). \tag{B16}$$

A similar analysis also leads to

$$|(D^2|_- + 2\hbar_N^{-1} + R_2)^{-1}| = O(\hbar_N). \tag{B17}$$

From these estimations, it follows that

$$|\epsilon_1| = O(\hbar_N^2), \quad |\epsilon_2| = O(\hbar_N^2), \quad |\epsilon_3| = O(\hbar_N^3). \tag{B18}$$

Then, since $\epsilon \leq |\epsilon| \leq |\epsilon_1| + |\epsilon_2| + |\epsilon_3|$, we conclude that ϵ is $O(\hbar_N^2)$ and we can write Eq. (B13) as

$$(\psi, T(\varphi')T(\varphi)\chi) = (\psi, T(\varphi'\varphi)\chi) + \frac{\hbar_N}{2}(\psi, \dot{\varphi}'\dot{\varphi}\chi) + O(\hbar_N^2). \tag{B19}$$

This is nothing but the first two terms of the asymptotic expansion (2.7). By using the relation $\gamma^a\gamma^b = \delta^{ab} + i\epsilon^{ab}\sigma^3$, we find that $C_0(\varphi', \varphi)$ and $C_1(\varphi', \varphi)$ in this expansion are indeed given by those in (2.8).

We can further obtain $C_2(\varphi', \varphi)$ in the following manner. The contribution of $O(\hbar_N^2)$ comes from ϵ_1 and ϵ_2. As for ϵ_1, the operator $D^{-2}|_-$ in (B14) can be again expanded as in (B12) and only the first term of the right-hand side of (B12) contributes to $C_2(\varphi_1, \varphi_2)$. Similarly, in estimating ϵ_2, the operator $(D^2|_- + 2\hbar_N^{-1} + R_2)^{-1}$ is expanded as $\frac{\hbar_N}{4} + O(\hbar_N^2)$. After a short calculation, one finds that $C_2(\varphi_1, \varphi_2)$ is exactly given by the expression in (2.8). Note that by applying this calculation recursively, one can in principle obtain arbitrary higher order contributions of the asymptotic expansion.

APPENDIX C: CONSISTENCY CHECK OF THE ASYMPTOTIC EXPANSION

In this Appendix, we give a consistency check of the asymptotic expansion (2.7) with (2.8), derived in Appendix B.

Our consistency check is about the associativity of the matrix product. For $\varphi \in \Gamma(\mathrm{Hom}(E, E'))$, $\varphi' \in \Gamma(\mathrm{Hom}(E', E''))$, and $\varphi'' \in \Gamma(\mathrm{Hom}(E'', E'''))$, we must have

$$(T(\varphi'')T(\varphi'))T(\varphi) = T(\varphi'')(T(\varphi')T(\varphi)). \tag{C1}$$

By substituting the expansion (2.7), the associativity imposes the condition,

$$\sum_{i,j=0}^{\infty} \hbar_N^{i+j} T(C_j(C_i(\varphi'', \varphi'), \varphi) - C_i(\varphi'', C_j(\varphi', \varphi))) = 0. \tag{C2}$$

At each order of $\hbar_N$, the summand should be separately vanishing. Furthermore, (2.18) implies that, if $T(\varphi) = 0$ in the large-N limit, we have $\varphi = 0$. Thus, Eq. (C2) provides an infinite tower of constraints for $C_i's$,

$$\sum_{i=0}^{n} C_{n-i}(C_i(\varphi'', \varphi'), \varphi) - C_i(\varphi'', C_{n-i}(\varphi', \varphi)) = 0, \tag{C3}$$

for $n = 0, 1, 2, \ldots$.

We will check that our C_0, C_1, C_2 in (2.8) indeed satisfy the conditions (C3) up to $n = 2$, which corresponds to the second order of $\hbar_N^2$ in (C2). First, the left-hand side of (C3) for $n = 0$ is given by

$$\begin{aligned} &C_0(C_0(\varphi'', \varphi'), \varphi) - C_0(\varphi'', C_0(\varphi', \varphi)) \\ &\quad = (\varphi''\varphi')\varphi - \varphi''(\varphi'\varphi). \end{aligned} \tag{C4}$$

This is vanishing because of the associativity of the linear maps on the fiber vector spaces. Next, for $n = 1$, the left-hand side of (C3) is given by

$$\begin{aligned} &\sum_{i=0}^{1} C_{1-i}(C_i(\varphi'', \varphi'), \varphi) - C_i(\varphi'', C_{1-i}(\varphi', \varphi)) \\ &\quad = -(\nabla_-(\varphi''\varphi'))(\nabla_+\varphi) + \varphi''(\nabla_-\varphi')(\nabla_+\varphi) \\ &\qquad - (\nabla_-\varphi'')(\nabla_+\varphi')\varphi + (\nabla_-\varphi'')(\nabla_+(\varphi'\varphi)). \end{aligned} \tag{C5}$$

Here, we used the relation $(g^{\alpha\beta} + iW^{\alpha\beta})(\nabla_\alpha A)(\nabla_\beta B) = 2(\nabla_- A)(\nabla_+ B)$. This is again vanishing because of the derivation property of the covariant derivatives. Finally, for $n = 2$, a long but straightforward calculation leads to

$$\begin{aligned} &\sum_{i=0}^{2} C_{2-i}(C_i(\varphi'', \varphi'), \varphi) - C_i(\varphi'', C_{2-i}(\varphi', \varphi)) \\ &\quad = (\nabla_-\varphi'')([\nabla_-, \nabla_+]\varphi')(\nabla_+\varphi) \\ &\qquad - (\nabla_-\varphi'')(F_{12}^{(E'')}\varphi' - \varphi' F_{12}^{(E')})(\nabla_+\varphi). \end{aligned} \tag{C6}$$

This is also vanishing because $[\nabla_-, \nabla_+]\varphi' = F_{12}^{(E'')}\varphi' - \varphi' F_{12}^{(E')}$. Thus, our asymptotic expansion (2.7) with C_i's given by (2.8) is consistent with the associativity condition (C1) up to the second order of $\hbar_N^2$.

APPENDIX D: TRACE OF TOEPLITZ OPERATORS

In this Appendix, we prove Eq. (2.15).

Let $\{\psi_I | I = 1, 2, \ldots, d^{(E)}N + c^{(E)}\}$ be an orthonormal basis of $\mathrm{Ker}D^{(E)}$ satisfying $(\psi_I, \psi_J) = \delta_{IJ}$. For $\varphi \in \Gamma(\mathrm{Hom}(E, E))$, we write

$$\mathrm{Tr}T(\varphi) = \mathrm{Tr}(\Pi\varphi\Pi) = \sum_I (\psi_I, \varphi\psi_I) = \int_M \omega \mathrm{Tr}_{S\otimes E}(K^{(E)}\varphi). \tag{D1}$$

Here, $\mathrm{Tr}_{S\otimes E}$ is the trace on the fiber of $S\otimes E$ and $K^{(E)}$ is defined by

$$K^{(E)}_{st}(x)=\sum_I(\psi_I(x))_s(\psi_I^\dagger(x))_t, \tag{D2}$$

where $x\in M$ and s,t are collective labels for the indices of $S\otimes E$. $K^{(E)}$ corresponds to the diagonal elements of the so-called Bergmann kernel of the Dirac operator $D^{(E)}$. It is known that the Bergmann kernel has the following large-N asymptotic expansion [45]:

$$K^{(E)}=(2\pi\hbar_N)^{-1}P_+\mathbf{1}_E+O(N^0), \tag{D3}$$

where $\mathbf{1}_E$ is the identity matrix on the fiber of E and $P_+ := (1+\sigma_3)/2$ is the projection onto the positive chirality modes of S. By substituting this into (D1), we can obtain (2.15).

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