Elliptic genus, anomaly cancellation and heterotic M-theory

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We derive global consistency condition for strongly coupled heterotic string in the presence of M5-branes. Its elliptic genus is interpretable as generating functional of anomaly polynomials and so, on anomaly-free vacua, the genus is both holomorphic and modular invariant. In the holomorphic basis, we identify the modular properties by calculating the phase. By interpreting the refinement parameters as background curvature of tangent and vector bundles, we show that the extended Bianchi identity for Kalb–Ramond field of heterotic M-theory is satisfied in the presence of arbitrary numbers of M5-branes.

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I. INTRODUCTION

The anomaly of gauge and flavor symmetries played an important role as a unique window to physics at short distances, encompassing neutral pion decay, baryon number violation, matter contents of the Standard Model, quantum Hall edge states, topological insulators, and so on. For chiral gauge theories, anomaly *structure* is elegantly organized, while anomalous *field contents* lead to quantum inconsistency. Anomaly cancellation restricts possible consistent vacua of the theory purely in terms of low-energy degrees of freedom, i.e., spectrum of massless fields. At high-energy scale, the anomaly structure is embedded to *global consistency condition* of ultraviolet completion of string theory: modular invariance of closed strings and tadpole cancellation of open strings. These conditions, which lead to a specific form of anomalies, can be read off from the partition functions and their behavior in the complexified parameter plane. One expects anomaly structure severely constrains the functional form of partition function.

In this paper, we study global consistency condition of heterotic M-theory [1,2] or strongly coupled heterotic string theory [3], whose new feature is the presence of M5-branes in the M-theory bulk. The elliptic genus [4,5] is known to be the generating function for anomaly polynomial, and tells us that anomaly cancellation occurs when it is both holomorphic and modular invariant [6,7]. The fluctuations of M5-branes are described by M2-branes attached to M5-branes. Thus, to identify consistent vacua in the presence of M5-branes, we may analyze the corresponding elliptic genera as a probe for anomaly cancellation. Such elliptic genera are most elegantly computed by the topological vertex formalism [8,9].

II. ELLIPTIC GENUS

The elliptic genus is defined by trace over the Ramond sector of heterotic string worldsheet with $\mathcal{N}=(0,2)$ supersymmetry [5]

$$Z(q,\mathbf{x}) = \mathrm{Tr}_{\mathrm{R}} q^{H} \bar{q}^{\bar{H}} (-1)^{F} \prod_{a} x_a^{Q_a}. \qquad (1)$$

Here, $q = e^{2\pi i\tau}$ with τ the modular parameter of torus, F is the fermion number, Q_a are the set of global charges, and $H, (\bar{H})$ are the (anti)holomorphic Hamiltonians. For compact target space, their spectra are discrete, rendering the sum over states well defined. The $\mathcal{N}=(0,2)$ supersymmetry ensures that the elliptic genus is independent of $\bar{q}$.

For noncompact target space, the elliptic genus is afflicted by infrared divergence due to infinite target space volume. We regularize it so that the heterotic string is localized at a point. We do so by introducing Ω-deformation and extract anomaly structure from the associated global symmetry. It is formally analogous to orbifolding the ambient target space. Denoting eight transverse coordinates as $z_m \equiv x_{2m} + i x_{2m+1}$, $m = 1, 2, 3, 4$, we set the Ω-deformation of the target space $\mathbb{C}^4$ [10] by twisting

$$z_m \to e^{2\pi i \epsilon_m} z_m, \quad m = 1,2,3,4, \qquad (2)$$

and by simultaneously shifting the vectors $\vec{p} = \vec{p}_L \oplus \vec{p}_R$ in the internal $E_8 \times E_8$ lattice $\Gamma_8 \oplus \Gamma_8$ by

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$$\vec{p}_A \to \vec{p}_A + \vec{m}_A, \quad (A = L, R) \tag{3}$$

where $\vec{m}_A$ is an 8-component vector in Γ_8.

We further compactify longitudinal x^0, x^1 directions on a torus. The Ω-deformation has the effect that, whenever we go around the cycles of the torus, we have the above twisting (2) and (3). One-eighth of 32 supercharges survive for $\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 = 0$. Supercharges are further reduced in the presence of M9 and M5 branes.

In the case of weakly coupled heterotic string, viz. no M5 branes in the bulk, we obtain the corresponding elliptic genera using the Hirzebruch–Riemann–Roch index theorem [5,7]. For a single heterotic string, we can decompose the elliptic genus as

$$Z_1^{\text{het}} = \frac{1}{16\pi^4}\hat{A}(R)P(\tau, F)P_B(\tau, R)\text{vol}(\mathbb{C}^4), \tag{4}$$

where, under the Ω-deformation (2),

$$\hat{A}(R) \equiv \prod_{j=1}^{4} \frac{\pi\epsilon_j}{\sin(\pi\epsilon_j)}, \tag{5}$$

$$P(q, F) \equiv \frac{A_1(\vec{m}_L)A_1(\vec{m}_R)}{\eta(\tau)^{16}}, \tag{6}$$

$$P_B(q, R) \equiv \prod_{j=1}^{4} \frac{2\sin(\pi\epsilon_j)\eta(\tau)}{\vartheta_1(\epsilon_j)}, \tag{7}$$

$$\text{vol}\mathbb{C}^4 = \frac{1}{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}. \tag{8}$$

The Dirac genus (5) counts the "the number of fixed points" under the twisting (2). The next factor $P(q, F)$ is the generating function for Chern character $A_1(\vec{m}_L)A_1(\vec{m}_R) \equiv \sum_{\vec{p}\in\Gamma_8\oplus\Gamma_8} q^{\vec{p}^2/2} e^{ip\cdot\vec{m}}$ that comes from the lattice $\Gamma_8 \oplus \Gamma_8$. The contribution of spacetime twist is encoded in $P_B(q, R)$ in (7). As ϵ_i is the eigenvalue of the Riemann curvature tensor in the ith direction in (2) [10], Eq. (8) corresponds to the regularized volume localized at the fixed point.

The partition function of a consistent string theory should be *both* holomorphic and modular invariant. We may track the reason if we use the *holomorphic* elliptic genus (1) as a particular kind of the partition function, because it is also the generating function for anomaly polynomials [7]. It is and invariant under the shift $T: \tau \to \tau + 1$ of modular transformation $SL(2, \mathbb{Z})$, but not in general under the inversion $S: \tau \to -1/\tau$ [6]. Regarding ϵ_i and m_I as eigenvalues of the Riemann curvatures and field strengths of Cartan subalgebra of $E_8 \times E_8$, the phase under the S transformation [7],

$$Z_1^{\text{het}}(-1/\tau) = Z_1^{\text{het}}(\tau)\exp\left[\frac{\pi i}{\tau}(\text{tr}R \wedge R - \text{tr}F \wedge F)\right], \tag{9}$$

reveals the Bianchi identity for H, the field strength of Kalb–Ramond field B. Here F is the field strength of the $E_8 \times E_8$ and R is Riemann curvature tensor, all in the adjoint representation. In defining the trace tr is done for an adjoint representation divided by dual Coxeter number, so this normalization gives us an integral instanton number. We can interpret the parameters in the elliptic genus (4) as skew-eigenvalues of the vector bundles and the tangent bundles

$$\text{tr}F \wedge F = \sum_{I=1}^{8} m_{L,I}^2 + \sum_{I=1}^{8} m_{R,I}^2, \quad \text{tr}R \wedge R = \sum_{m=1}^{4} \epsilon_m^2. \tag{10}$$

The latter agrees with the relationship between the curvature and the volume (8).

The above discussion shows that the failure of modular invariance of *holomorphic* elliptic genus is related to the failure of the Bianchi identity, yielding nonvanishing anomaly polynomials. In what follows, we extend this condition in the presence of M5-branes, by calculating an additional nonperturbative contribution to the phase of (9) (we also discuss the reason for holomorphicity shortly).

III. M5- AND M9-BRANES

We describe strongly coupled heterotic string theory by M-theory compactified on an interval [11]. We have two M9-branes with E_8 gauge theories at the ends of the interval in, say, the x^{10}-direction, $0 \le x^{10} \le L_M$. An M2-brane stretched between two M9-branes gives rise to heterotic string [2,12].

We may put additionally a number of M5-branes at various places in the interval, away from M9-branes. Their locations are $z^3 = z^4 = 0$ with x^{10} arbitrary within $[0, L_M]$. The setup gives rise to so-called M- and E-strings of variable tension, obtained from M2-branes connected between different M9/M5 branes [13,14]. M-strings come from M2-branes stretched between two M5-branes, and describe interbrane fluctuations [13,14]. E-strings come from M2-branes stretched between M9- and M5-branes, and describe fluctuation of M5-brane relative to the M9-brane.

The elliptic genus corresponding to this setup again contains information on anomaly structure and hence on global consistency conditions, but now including new contributions from M5-branes. Their presence is a source of technical as well as conceptual complications but, as we show below, the new elliptic genus is still computable for arbitrary number of M5-branes and heterotic strings. To fully probe non-Abelian structure of M- and E-strings, one would further need to uplift to the F-theory dual, as analyzed in [15].

More specifically, the presence of M5-branes affect the modular transform (9), modifying the anomaly structure. In this paper, we extract this information from the corresponding elliptic genus, which we calculate from the refined topological vertex method [8,9].

IV. ELLIPTIC GENUS FROM REFINED TOPOLOGICAL VERTEX

We can calculate the elliptic genera of nonperturbative heterotic string in the presence of n M5-branes, using the refined topological vertex method. The calculation boils down to the product of defect operators

$$Z_n(\tau,\vec{\epsilon},\vec{m}) = \sum_{\{\nu_a\}} D^{M9}_{L,\nu_1}\left(\prod_{a=1}^{n} D^{M5}_{\nu_a^t\nu_{a+1}}\right) D^{M9}_{\nu_{n+1}^t,R} + \sum_{\{\nu_a\}} D^{M9,c}_{L,\nu_1}\left(\prod_{a=1}^{n} D^{M5}_{\nu_a^t\nu_{a+1}}\right) D^{M9,c}_{\nu_{n+1}^t,R}. \quad (11)$$

The defect operator $D^{M5}_{\nu_a^t\nu_{a+1}}$ for ath M5-brane connected by M2-branes with the Young diagram ν_a on the left and ν_{a+1} on the right was computed in Ref. [13]. Here, we take the convention that a Young diagram λ encodes the configuration of M2-branes by descending ordered set of numbers $\lambda = (\lambda_1, \lambda_2, \ldots), \lambda_1 \geq \lambda_2 \geq \ldots$. The superscript t refers to transpose. The size of λ is $|\lambda| = \sum_i \lambda_i = \sum_j \lambda_j^t$.

We also have two operators for M9-branes, D^{M9}_{L,ν_1} and $D^{M9}_{\nu^t_{n+1},R}$. For our foregoing analysis, however, we do not need detailed form of them (they can be found in [14,16,17]) except for the followings. First, we have exchange symmetry $D^{M9}_{L,\nu^t} = D^{M9}_{L,\nu}(\epsilon_1 \leftrightarrow \epsilon_2)$. Second, operationally, these defect operators are obtainable from the elliptic genus of E-strings by assuming that M5-branes are located at $(z^3, z^4) = (0, 0)$. This choice, however, explicitly breaks the $SO(8)$ symmetry of M9-brane world volume. To restore $SO(8)$, we may symmetrize the orientation of M5-brane world volume. Equivalently, we may fix the M5-branes orientation as above and then symmetrize M9-brane world volume coordinates (z_1, z_2, z_3, z_4). The net effect is to introduce additional defect operators $D^{M9,c}_{L,\nu} \equiv D^{M9}_{L,\nu}(\epsilon_1 \leftrightarrow \epsilon_3)$, which implies $D^{M9,c}_{L,\nu^t} \equiv D^{M9}_{L,\nu}(\epsilon_1 \leftrightarrow \epsilon_4)$ [14]. This is how we expressed the partition function in the form (11).

V. MODULAR ANOMALY AND HOLOMORPHIC ANOMALY

A consistent field content must give rise to modular invariant and holomorphic elliptic genus. In general, it is *not* possible to maintain *both* of them. A basic building block of elliptic genera is the Jacobi ϑ-function ϑ_1. We can check that $\vartheta_1(\frac{a\tau+b}{c\tau+d}; \frac{z}{c\tau+d}) = (c\tau+d)^{1/2} e^{\pi i z^2/(c\tau+d)} \vartheta_1(\tau; z)$. We can understand the reason why the phase is quadratic in z. Expanding it,

$$\vartheta_1(\tau;z) = \eta(\tau)^3 (2\pi z) \exp\left(\sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)(2k)!} E_{2k}(\tau)(2\pi i z)^{2k}\right),$$

where E_{2k} are $2k$th Eisenstein series and B_{2k} are Bernoulli numbers. All the E_{2k} for $k \geq 2$ are holomorphic modular form and generated by E_4 and E_6. The exception is E_2 which transforms under $SL(2,\mathbb{Z})$ as

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) - \frac{6ci}{\pi}(c\tau+d),$$

where $a, b, c, d \in \mathbb{Z}, ad - bc = 1$. We may redefine this to be modular at the price of giving up holomorphy,

$$\hat{E}_2(\tau,\bar{\tau}) = E_2(\tau) - \frac{6i}{\pi(\tau-\bar{\tau})},$$

such that $\hat{E}_2(\frac{a\tau+b}{c\tau+d}, \frac{a\bar{\tau}+b}{c\bar{\tau}+d}) = (c\tau+d)^2 \hat{E}_2(\tau,\bar{\tau})$. Thus, anomalous phase of the elliptic genus (11) is only up to quadratic because the only nonholomorphic part in ϑ_1 is the coefficient of E_2:

$$\frac{3}{\pi^2}\frac{\delta \log Z_n}{\delta E_2}. \quad (12)$$

If the net phase vanishes for a given field content, anomaly cancellation is ensured for the corresponding vacuum. For generic $\vec{\epsilon}$ and $\vec{m}$, we have noninvariant phase under S for the $Z_n(E_2, E_4, \cdots)$ in the holomorphic basis. Although the complete expression for (11) is unknown, for extracting information on anomalies, it suffices to study the phases under the S modular transformation. Being additive, we separate the phase of each term of (11) into two separate pieces.

First, the transformation of M9 defect operators in (11) is the same as that of the elliptic genus of weakly coupled k heterotic strings. The latter can be obtained by Hecke transformation of single string [14,18]. We found that this is a modular form provided $\nu_1 = \nu_{n+1}$, so that

$$Z_k^{\rm het} \equiv \sum_{|\nu_1|=k} D^{M9}_{L,\nu_1} D^{M9}_{\nu_1^t,R} + \sum_{|\nu_1|=k} D^{M9,c}_{L,\nu_1} D^{M9,c}_{\nu_1^t,R} = \frac{1}{k}\sum_{a,d>0}\sum_{b (\mathrm{mod}\ d)} Z_1^{\rm het}\left(\frac{a\tau+b}{d}, a\vec{\epsilon}, a\vec{m}\right), \quad (13)$$

where the sum is over positive a, d satisfying $ad = k$. Here $Z_1^{\rm het}$ is the elliptic genus of a single heterotic string (4). It is not invariant under S by a phase factor,

$$Z_k^{\rm het}(-1/\tau) = Z_k^{\rm het}(\tau) \exp\left[\frac{\pi i k}{\tau}\left(\sum_{m=1}^{4}\epsilon_m^2 - \sum_{I=1}^{16} m_I^2\right)\right]. \quad (14)$$

Second, the S modular transformation of M5 defect operators in (11), $\prod_{a=1}^n D^{M5}_{\nu_a^t\nu_{a+1}} \equiv \mathcal{D}$, is again a quasimodular form provided $\nu_1 = \nu_{n+1}$, while each individual factor is not. This can be seen from the relation between quantum dilogarithmic function and Jacobi ϑ-function [13].

The phase of $\mathcal{D}^2$ is equal to the phase of $\prod_{a=1}^n D^{M5}_{\nu_a^t \nu_{a+1}} D^{M5}_{\nu_{a+1}^t \nu_a}$. Each factor

$$D^{M5}_{\nu^t \mu} D^{M5}_{\mu^t \nu} = \prod_{(i,j)\in\nu} \frac{\vartheta^{\epsilon_2+\epsilon_3}_{ij,\nu\mu} \vartheta^{\epsilon_2+\epsilon_4}_{ij,\nu\mu}}{\vartheta^{\epsilon_2}_{ij,\nu\nu} \vartheta^{-\epsilon_1}_{ij,\nu\nu}} \prod_{(k,l)\in\mu} \frac{\vartheta^{\epsilon_2+\epsilon_3}_{kl,\mu\nu} \vartheta^{\epsilon_2+\epsilon_4}_{kl,\mu\nu}}{\vartheta^{\epsilon_2}_{kl,\mu\mu} \vartheta^{-\epsilon_1}_{kl,\mu\mu}} \qquad (15)$$

where

$$\vartheta^{\epsilon}_{ij,\nu\mu} = \vartheta_1(\epsilon - \epsilon_1(\nu_i - j) + \epsilon_2(\mu^t_j - i)) \qquad (16)$$

is quasi-modular form and only depends on the types of Young diagrams ν and μ (We neglect overall phase which does not affect the phase change under S).

We find (see Appendix) that the phase of $\mathcal{D}$ under the S transformation is given by

$$\prod_{a=1}^n D^{M5}_{\nu_a^t \nu_{a+1}}(-1/\tau) = \prod_{a=1}^n D^{M5}_{\nu_a^t \nu_{a+1}}(\tau) \exp\left[\frac{\pi i}{\tau}((|\nu_a| - |\nu_{a+1}|)^2 \epsilon_1\epsilon_2 - (|\nu_a| + |\nu_{a+1}|)\epsilon_3\epsilon_4)\right]. \qquad (17)$$

It is remarkable that, despite stack of M-strings *cannot* be understood as Hecke transform of a single M-string, the net phase depends only on the sizes of Young diagrams $|\nu_a|$, not on their shapes. For instance, in the case of two M5-branes ($n = 2$) with $\nu_1 = \varnothing = \nu_3, |\nu_2| \equiv k$, we have the overall phase $k^2\epsilon_1\epsilon_2 - k\epsilon_3\epsilon_4$ in unit of π/τ. Previously, this was derived from the holomorphic anomaly equation of M-strings [13,15,19,20].

Hereafter, we require the coefficient of $\epsilon_1\epsilon_2$ to vanish, viz. $|\nu_a| = |\nu_{a+1}| \equiv k$ for all a. Physically, this amounts to forbidding any leakage of M2-brane charge on M5-brane world volume. The M2-brane charge simply flows from ν_{a+1} to ν_a^t as a local process in (z_1, z_2) space. Indeed, $k$strongly coupled heterotic strings chopped by M5-branes give rise to k M-strings in each interval. Under the S transform, each M5 defect operator generates an equal amount of phase, so

$$\prod_{a=1}^n D^{M5}_{\nu_a^t \nu_{a+1}}(-1/\tau) = \prod_{a=1}^n D^{M5}_{\nu_a^t \nu_{a+1}}(\tau) \times e^{-\frac{2\pi i}{\tau} k\epsilon_3\epsilon_4}. \qquad (18)$$

Putting together, we achieve the modular invariance by demanding that the phase (18) from M5-branes cancels off the phase (14) from strongly coupled heterotic strings. It is straightforward to generalize this cancellation mechanism to include the contribution proportional to $\epsilon_1\epsilon_2$ in (z_1, z_2)-space.

VI. BIANCHI IDENTITY INCLUDING M5-BRANES

The result above catches only the local contribution at the singular locus $(z_3, z_4) = (0, 0)$. Each M5-brane sources Kalb-Ramond magnetic flux H. From localization, the volume of (z_3, z_4) space is concentrated at the point $(z_3, z_4) = (0, 0)$

$$\mathrm{vol}\mathbb{C}^2 = \int d^2z_3 d^2z_4 e^{-\epsilon_3|z_3|^2 - \epsilon_4|z_4|^2} = \frac{1}{\epsilon_3\epsilon_4}. \qquad (19)$$

Thus, from

$$\int dz_3^2 dz_4^2 \epsilon_3\epsilon_4 e^{-\epsilon_3|z_3|^2 - \epsilon_4|z_4|^2} \simeq 1, \qquad (20)$$

in the limit of small ϵ_3, ϵ_4, we interpret the phase in Eq. (18) as Dirac δ-function

$$\epsilon_3\epsilon_4 \simeq \delta^2(z_3)\delta^2(z_4) \equiv \delta^4(z_{3,4}), \qquad (21)$$

near $(z_3, z_4) = (0, 0)$. The M5-branes are located at this locus. This is the result of localization in noncompact Ω-background.

Adding (14) to (18) and using (21), we obtain the total phase in the background of n M5-branes

$$-\sum_{a=1}^n \delta^4(z_{3,4}) + \frac{1}{2}\sum_{m=1}^4 \epsilon_m^2 - \frac{1}{2}\sum_{I=1}^{16} m_I^2, \qquad (22)$$

for each string. This phase gives anomaly polynomial, if we write it in a covariant form using (10), and is to be cancelled by a local counterterm involving the Kalb–Ramond field B [21]. Therefore, it provides the Bianchi identity for the B, now in the presence of the M5-branes in the bulk

$$-\sum_{a=1}^n \delta^4(z_{3,4}) - \frac{1}{2}\mathrm{tr}F\wedge F + \frac{1}{2}\mathrm{tr}R\wedge R = dH, \qquad (23)$$

which is precisely the anomaly cancellation condition [22]. Integrating over a compact manifold, for instance K3, we may constrain the number of M5-branes.

We have obtained anomaly cancellation condition for arbitrary number of tensor multiplets in six-dimensional nonperturtabative heterotic string. The key idea behind our derivation is the requirement that the elliptic genus must satisfy modular invariance and holomorphy simultaneously. It would be interesting to generalize the analysis to orbifolded M-strings [23–26] and also to classify all possible globally consistent string configurations.

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APPENDIX: PROOF OF RELATION (17)

In this Appendix, we prove the relation (17). The phase is

$$\sum_{(i,j)\in\nu}\sum_{(k,l)\in\mu}[-\epsilon_3\epsilon_4+\epsilon_2^2(\mu_j^t-2i\mu_j^t+(\mu_j^t)^2-\mu_l^t+2k\mu_l^t-(\mu_l^t)^2-\nu_j^t+2i\nu_j^t-(\nu_j^t)^2+\nu_l^t-2k\nu_l^t+(\nu_l^t)^2)$$
$$+\epsilon_1\epsilon_2(\nu_j^t-2j\nu_j^t+2\nu_i\nu_j^t+\mu_l^t-2l\mu_l^t+2\mu_k\mu_l^t-\mu_j^t+2j\mu_j^t-2\mu_j^t\nu_i-\nu_l^t+2l\nu_l^t-2\mu_k\nu_l^t)].$$

We may note that, in the summand, the coefficient of ϵ_2^2 is the repetition of the pattern $-\nu_j^t+2i\nu_j^t-(\nu_j^t)^2$ with $\mu=\varnothing$, up to a possible exchange $\mu\leftrightarrow\nu$. We rewrite this as

$$(i-1)\nu_j^t-\nu_j^t(\nu_j^t-i). \tag{A1}$$

First consider the sum coming from the minus of the last term

$$\sum_{(i,j)\in\nu}(\nu_j^t-i)\nu_j^t.$$

We draw the Young tableau ν and put each summand

$(\nu_1^t-1)\nu_1^t$	$(\nu_2^t-1)\nu_2^t$	$\dots$	$(\nu_{\ell'}^t-1)\nu_{\ell'}^t$
$(\nu_1^t-2)\nu_1^t$	$(\nu_2^t-2)\nu_2^t$	$\dots$	
$(\nu_1^t-3)\nu_1^t$	$(\nu_2^t-3)\nu_2^t$	$\dots$	
$\vdots$			
$(\nu_1^t-\ell')\nu_1^t$	$\dots$		

where $\ell'\equiv\ell(\nu^t)$. The sum of the first column is $\nu_1^t\sum_{i=1}^{\nu_1^t-1}i=\nu_j^t\frac{\nu_j^t(\nu_j^t-1)}{2}$ and this pattern is repeated in the remaining columns. Thus we can sum up column by column to obtain

$$\sum_{j=1}^{\ell'}\nu_j^t\frac{\nu_j^t(\nu_j^t-1)}{2}.$$

Next, we may calculate the first term by drawing a similar tableau

$$\sum_{(i,j)\in\nu}\nu_j^t(i-1)=\nu_1^t\cdot 0+\nu_2^t\cdot 0+\cdots+\nu_{\ell'-1}^t\cdot 0$$
$$+\nu_1^t\cdot 1+\nu_2^t\cdot 1+\dots$$
$$+\nu_1^t\cdot 2+\nu_2^t\cdot 2+\dots$$
$$\vdots$$
$$+\nu_1^t\cdot(\nu_1^t-1)+\dots,$$

whose sum is

$$\sum_{j=1}^{\ell'}\nu_j^t\frac{\nu_j^t(\nu_j^t-1)}{2}.$$

Therefore the difference of the above two vanishes. This means that the coefficient of ϵ_2^2 is zero.

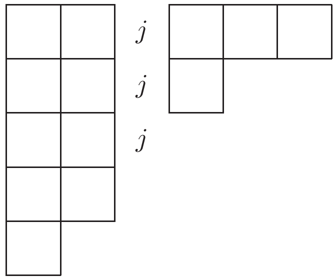
Now consider the coefficient of $\epsilon_1\epsilon_2$. We first focus on the first six terms of the form

$$\sum_{(i,j)\in\nu}(\nu_j^t-2j\nu_j^t+2\nu_i\nu_j^t). \tag{A2}$$

We first calculate the combination

$$\sum_{(i,j)\in\nu}\nu_j^t(\nu_i-j),$$

by drawing the following Young tableau. For each column with j fixed, we can sum over i.



In fact the sum $\sum_{i=1}^{\nu_j^t}(\nu_i - j)$ counts the total number of boxes of the subtableau if we cut the original tableau at the jth column. Thus

$$\sum_{i=1}^{\nu_j^t}(\nu_i - j) = \sum_{k=j+1}^{\ell'} \nu_k^t$$

Then we sum over j

$$\begin{aligned}\sum_{(i,j)\in\nu} \nu_j^t(\nu_i - j) &= \sum_{j=1}^{\ell'} \nu_j^t \sum_{k=j+1}^{\ell'} \nu_k^t \\ &= \frac{1}{2}\left(\sum_{j=1}^{\ell'} \nu_j^t\right)^2 - \frac{1}{2}\sum_{j=1}^{\ell'} (\nu_j^t)^2 \\ &= \frac{1}{2}|\nu|^2 - \frac{1}{2}\sum_{(i,j)\in\nu} \nu_j^t \end{aligned} \tag{A3}$$

Therefore we have

$$\sum_{(i,j)\in\nu} (\nu_j^t - 2j\nu_j^t + 2\nu_i\nu_j^t) = |\nu|^2. \tag{A4}$$

This will make the first six terms of the coefficient of $\epsilon_1\epsilon_2$ to be $|\mu|^2 + |\nu|^2$. Finally we consider the last six terms. We compute

$$\sum_{(i,j)\in\nu} \mu_j^t(\nu_i - j) = \sum_{j=1}^{\ell'} \mu_j^t \sum_{k=j+1}^{\ell'} \nu_k^t,$$

and

$$\sum_{(k,l)\in\mu} \nu_l^t(\mu_k - l) = \sum_{k=1}^{\ell''} \nu_j^t \sum_{k=j+1}^{\ell''} \mu_k^t,$$

where $\ell'' = \ell(\mu)$. If we have $\ell' = \ell''$ the sum is

$$\sum_{(i,j)\in\nu} \mu_j^t(\nu_i - j) + \sum_{(k,l)\in\mu} \nu_l^t(\mu_k - l) = |\mu||\nu| - \sum_{j=1}^{\ell'} \nu_j^t\mu_j^t.$$

However also we have

$$\sum_{(i,j)\in\nu} \mu_j^t + \sum_{(k,l)\in\mu} \nu_l^t = 2\sum_{j=1}^{\ell'} \nu_j^t\mu_j^t,$$

which cancels the twice of the last term. So that the coefficient of $\epsilon_1\epsilon_2$ is

$$|\mu|^2 + |\nu|^2 - 2|\mu||\nu| = (|\mu| - |\nu|)^2.$$

In fact this relation holds for $\ell' \neq \ell''$, because if $\ell' > \ell''$, without loss of generality, we have

$$\mu_j^t = 0, \quad \text{for } j > \ell'',$$

and the summation $\sum^{\ell''}$ is replaced by $\sum^{\ell'}$.

With the trivial sum of $\epsilon_3\epsilon_4$, we have the overall phase

$$(|\mu| - |\nu|)^2\epsilon_1\epsilon_2 - (|\mu| + |\nu|)\epsilon_3\epsilon_4,$$

completing the proof.

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